Numer. Math. 62, 49–73 (1992)

Numerische
Mathematik

On error estimates of some higher order projection and penalty-projection methods for Navier-Stokes equations*

Jie Shen

Institute for Applied Mathematics and Scientific Computing, Indiana University, Department of Mathematics, Penn State University, 618 East 3rd Street, Bloomington, IN 47405, USA

Received September 17, 1991

Summary. This paper is a continuation of our previous work [10] on projection methods. We study first existing "higher order" projection schemes in the semi-discretized form for the Navier-Stokes equations. One error analysis suggests that the precision of these schemes is most likely plagued by the inconsistent Neumann boundary condition satisfied by the pressure approximations. We then propose a penalty-projection scheme for which we obtain improved error estimates.

Mathematics Subject Classification (1991): 35A40, 35Q10, 65J15

1 Introduction

As in [10], we consider the numerical integration of the time dependent Navier-Stokes equations in the primitive variable formulation:

$$(1.1)\quad \begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f, \quad \forall (x,t) \quad \text{in } Q = \Omega\times[0,T] \ , \\ \operatorname{div} u = 0, \quad \forall (x,t) \quad \text{in } Q \ , \\ u(0) = u_0, \quad \int\limits_\Omega p(x,t)\,dx = 0, \quad \forall t\in[0,T] \ , \end{cases}$$

subject to the homogeneous boundary condition (for the sake of simplicity): $u(t)|_\Gamma = 0$, $\forall t\in[0,T]$. Ω is an open bounded domain in $\mathbb{R}^d$ ($d = 2$ or 3) with a sufficiently smooth boundary Γ.

The coupling between the velocity u and the pressure p by the incompressibility condition "$\operatorname{div} u = 0$" is one of the main concern in designing efficient time integration schemes for (1.1). The projection methods, initially proposed by Chorin [2] and Temam [12], are designed to decouple the velocity and the pressure. Another way to overcome the difficulties caused by $\operatorname{div} u = 0$ is to use a penalty formulation, proposed by Temam in [11], in which the pressure is totally eliminated.

* This work is partially supported by NSF grant DMS-8802596.

In a previous work [10], we analyzed the classical projection scheme as well as a modified projection scheme. For the classical scheme, we established error estimates of weakly first order for the velocity and of weakly order $\frac{1}{2}$ for the pressure. For the modified scheme, we improved error estimates to strongly first order for the velocity and weakly first order for the pressure. In this paper, we intend to investigate several higher order projection and penalty-projection schemes and to provide precise error estimates for them. We recall that our error estimates would automatically imply the stability and convergence of the schemes.

Given the author's knowledge, there exist essentially three types of higher order projection schemes. Namely, schemes via improved intermediate velocity boundary condition (Kim and Moin [6]), schemes via pressure-correction (Van Kan [17], Bell et al. [1], Gresho [3]) and schemes via improved pressure boundary condition (Orszag et al. [7]). We will study below the schemes in [6] and [17] and indicate an intrinsic relation between them. The schemes via improved pressure boundary condition will not be addressed in this paper.

The scheme proposed by Kim and Moin [6] in semi-discretized form can be written as follows:

$$\text{(1.2)}\quad \begin{cases} \frac{1}{k}(\hat{u}^{n+1} - u^n) - \frac{\nu}{2}\Delta(\hat{u}^{n+1} + u^n) \\ \qquad = \frac{3}{2}(f(t_n) - (u^n\cdot\nabla)u^n) - \frac{1}{2}(f(t_{n-1}) - (u^{n-1}\cdot\nabla)u^{n-1}) \ , \\ \hat{u}^{n+1}|_\Gamma = k\nabla\phi^n \ , \end{cases}$$

and

$$\text{(1.3)}\quad \begin{cases} \frac{1}{k}(u^{n+1} - \hat{u}^{n+1}) + \nabla\phi^{n+1} = 0 \ , \\ \operatorname{div} u^{n+1} = 0 \ , \\ u^{n+1}\cdot \boldsymbol{n}|_\Gamma = 0 \ . \end{cases}$$

where k is the time step, $t_{n+1} = (n+1)k$ and $\boldsymbol{n}$ is the normal vector to Γ. Note that the non physical boundary condition for the intermediate velocity $\hat{u}^{n+1}$ is introduced to reduce the tangential boundary error of the final velocity approximation u^{n+1}, for we can derive from (1.2)–(1.3) that

$$u^{n+1}\cdot\tau|_\Gamma = k\frac{\partial(\phi^{n+1} - \phi^n)}{\partial\tau} ,$$

while the no-slip condition $\hat{u}^{n+1}|_\Gamma = 0$ would lead to $y^{n+1}\cdot\tau|_\Gamma = k\frac{\partial\phi^{n+1}}{\partial\tau}$.

Another scheme proposed by Van Kan [17] takes the following form:

$$\text{(1.4)}\quad \begin{cases} \frac{1}{k}(\tilde{u}^{n+1} - u^n) - \frac{\nu}{2}\Delta(\tilde{u}^{n+1} + u^n) + \nabla\phi^n \\ \qquad = \frac{3}{2}(f(t_n) - (u^n\cdot\nabla)u^n) - \frac{1}{2}(f(t_{n-1}) - (u^{n-1}\cdot\nabla)u^{n-1}) \ , \\ \tilde{u}^{n+1}|_\Gamma = 0 \ , \end{cases}$$

and

$$(1.5)\qquad \begin{cases} \dfrac{1}{k}(u^{n+1}-\tilde{u}^{n+1})+\tfrac{1}{2}\nabla(\phi^{n+1}-\phi^n)=0\ , \\ \operatorname{div} u^{n+1}=0\ , \\ u^{n+1}\cdot \boldsymbol{n}|_\Gamma=0\ . \end{cases}$$

We recall that for both schemes, at each time step, we only have to solve a cascade of Helmholtz equations, while for a conventional coupled scheme, we have to solve a Stokes equation. Hence, using schemes of projection type might reduce considerably CPU time as well as programming complexity. Because of their efficiency, the schemes (1.2)–(1.3), (1.4)–(1.5) and their various variations have been widely used in practice (see among others [1, 6, 8, 17]). It has been also believed that both schemes would lead to second order approximations, at least for the velocity. However, to the author's knowledge, rigorous theoretical justifications for these schemes are not yet available.

One can easily derive that for both schemes, we have

$$(1.6)\qquad \left.\frac{\partial\phi^n}{\partial \boldsymbol{n}}\right|_\Gamma=\cdots=\left.\frac{\partial\phi^0}{\partial \boldsymbol{n}}\right|_\Gamma,\quad \forall n\ ,$$

which is not generally satisfied by the exact pressure. However, it is now a well-known fact that despite of (1.6), the velocity approximations $\tilde{u}^n$ and u^n still converge to the exact velocity $u(t_n)$ (see for instance [2, 15, 16]). In addition, we showed in [10] that (1.6) did not affect the precision of first order schemes. However, how it would affect the precision of higher order schemes is not clear. Our error analyses reveal that in case

$$(1.7)\qquad \left.\frac{\partial p(\cdot,t)}{\partial \boldsymbol{n}}\right|_\Gamma=g(\cdot)|_\Gamma,\quad \forall t\in[0,T]\ ,$$

we do obtain "essentially" (to be specified later) second order error estimates for the velocity. However, if (1.7) is violated, we are only able to obtain error estimates of order $\frac{3}{2}$ for the velocity.

We will also consider the following totally implicit semi-discrete penalty scheme:

$$(1.8)\qquad \begin{cases} \dfrac{1}{k}(u^{n+1}-u^n)-\dfrac{\nu}{2}\Delta(u^{n+1}+u^n) \\ \qquad+\tfrac{1}{4}\tilde{B}(u^{n+1}+u^n,u^{n+1}+u^n)-\dfrac{1}{\varepsilon}\nabla\operatorname{div}u^{n+1}=f(t_{n+\frac{1}{2}}) \\ u^{n+1}|_\Gamma=0\ , \end{cases}$$

where $\tilde{B}(u,v)=(u\cdot\nabla)v+\frac{1}{2}(\operatorname{div}u)v$. The extra term $\frac{1}{2}(\operatorname{div}u)v$ in $\tilde{B}(u,v)$, introduced by Temam [11], is crucial for preserving the dissipativity of the discrete system when the approximate velocities are not divergence free.

We notice that in the scheme (1.8), the pressure is totally eliminated so that at each time step we only have to solve a nonlinear elliptic system. The main disadvantage of penalty schemes is that in order to get adequate accuracy, we have to use a very small ε for which (1.8) would become a very stiff system to solve numerically, especially when ν is also small because of the two different scales ν and

$1/\varepsilon$ in the system (1.8). In fact, we will prove that to get second order accuracy, we have to choose $\varepsilon = O(k^4)$. In order to relax the stiffness of (1.8), we will propose a penalty-projection scheme which is second order accurate with $\varepsilon = O(k^2)$. The price we pay for the improvement is to add a projection step for which we have to solve an additional scalar Poisson equation.

To simplify our presentation, we will only consider the case $\tau < \infty$. Related uniform (in time) stability and convergence analysis for $\tau = +\infty$ can be done as in [9].

The paper is organized in the following way. In the next section, we recall some preliminary results and lay out some assumptions on the data and the solutions of (1.1) which are required for our error analyses. In Sect. 3, we analyze a pressure correction scheme similar to (1.4)–(1.5). In Sect. 4, we show briefly how the results of Sect. 3 also apply to Kim and Moin's scheme (1.2)–(1.3). Then in Sect. 5, we provide an error estimates for the penalty scheme (1.8). Finally, in Sect. 6, we propose a penalty-projection scheme which leads to improved error estimates.

2 Preliminaries

We recall below some of the notations and inequalities which will be used frequently in this paper.

Let us denote

- $H = \{u \in L^2(\Omega)^d \colon \operatorname{div} u = 0, u \cdot \boldsymbol{n}|_\Gamma = 0\}$,
- $V = \{v \in H_0^1(\Omega)^d \colon \operatorname{div} v = 0\}$,
- P_H: the orthogonal projector in $L^2(\Omega)^d$ onto H.

We define the Stokes operator

$$Au = -P_H \Delta u, \quad \forall u \in D(A) = V \cap H^2(\Omega)^d .$$

The Stokes operator A is an unbounded positive self-adjoint closed operator in H with domain $D(A)$ and its inverse A^{-1} is compact in H.

Let $|\cdot|$, $\|\cdot\|$ denote respectively the norms in $L^2(\Omega)$ and $H_0^1(\Omega)$, i.e.

$$|u|^2 = \int_\Omega |u(x)|^2 dx \qquad \text{and} \qquad \|u\|^2 = \int_\Omega |\nabla u(x)|^2 dx .$$

The norm in $H^s(\Omega)$ ($\forall s$) will be simply denoted by $\|\cdot\|_s$. We will use respectively $(\cdot,\cdot)$ to denote the inner product in $L^2(\Omega)$ and $\langle\cdot,\cdot\rangle$ to denote the duality between $H^{-s}(\Omega)$ and $H_0^s(\Omega)$, $\forall s > 0$.

We define the trilinear form $b(\cdot,\cdot,\cdot)$ by

$$b(u, v, w) = \int_\Omega (u\cdot\nabla)v\cdot w dx .$$

It is an easy matter to verify that

$$b(u, v, w) = -b(u, w, v), \quad \forall u \in H, v, w \in H_0^1(\Omega)^d . \tag{2.1}$$

In particular, we have

$$b(u, v, v) = 0, \quad \forall u \in H, \quad v \in H_0^1(\Omega)^d . \tag{2.2}$$

We define $\tilde{b}(\cdot,\cdot,\cdot)$ by

$$\tilde{b}(u, v, w) = (\tilde{B}(u, v), w);, \quad \forall u, v \in H^1(\Omega)^d,, w \in H_0^1(\Omega)^d .$$

One can readily check that

(2.3) $$\tilde{b}(u, v, v) = 0, \quad \forall u \in H^1(\Omega)^d, \quad v \in H_0^1(\Omega)^d .$$

(2.4) $$\tilde{b}(u, v, w) = \tfrac{1}{2}(b(u, v, w) - b(u, w, v)), \quad \forall u, v \in H^1(\Omega)^d, \quad w \in H_0^1(\Omega)^d .$$

The following inequalities will be used repeatedly in the sequel.

If $d \leqq 4$, then (see for instance [13])

(2.5) $$\left\{\begin{matrix} \tilde{b}(u, v, w) \\ b(u, v, w) \end{matrix}\right. \leqq c \|u\| \, \|v\| \, \|w\|, \quad \forall u, v, w \in H^1(\Omega)^d .$$

(2.6) $$|b(u, v, w)| \leqq \begin{cases} c|u| \, \|v\| \, \|w\|_2 \\ c|u| \, \|v\|_2 \, \|w\| \\ c\|u\| \, \|v\|_2 |w| \\ c\|u\|_2 \, \|v\| \, |w| \end{cases}, \quad \forall u, v, w \in H^1(\Omega)^d .$$

We recall (see [10]) that

(2.7) $$\exists c_1, c_2 > 0, \quad \text{such that } \forall u \in H: \begin{cases} \|A^{-1}u\|_s \leqq c_1 \|u\|_{s-2}, & \text{for } s = 1, 2 ; \\ c_2 \|u\|_{-1}^2 \leqq (A^{-1}u, u) \leqq c_1^2 \|u\|_{-1}^2 . \end{cases}$$

Hence, we can use $(A^{-1}u, u)^{\frac{1}{2}}$ as an equivalent norm of $H^{-1}(\Omega)^d$ for $u \in H$.

We will use the terminology in [10] to classify the precision of a scheme.

Definition. Let X be a Banach space equipped with norm $\|\cdot\|_X$ and $f: [0, T] \to X$ is continuous. Let $\{t_n^{(k)}\}_{n=0}^{n=T/k}$ be a family of discretization of $[0, T]$ such that

$$0 = t_0^{(k)} < \cdots < t_n^{(k)} < t_{n+1}^{(k)} < \cdots < t_{T/k}^{(k)} = T; \text{ and}$$

$$\max_{0 \leqq n \leqq T/k - 1} |t_{n+1}^{(k)} - t_n^{(k)}| \leqq \delta_k \to 0 \quad (\text{as } k \to 0) .$$

Then, we say f_k is a weakly order α approximation of f in X if there exists a constant c independent of k such that

$$k \sum_{n=0}^{T/k} \|f_k(t_n^{(k)}) - f(t_n^{(k)})\|_X^2 \leqq ck^{2\alpha} ;$$

and we say f_k is a strongly order α approximation of f in X if there exists a constant c independent of k and n such that

$$\|f_k(t_n^{(k)}) - f(t_n^{(k)})\|_X^2 \leqq ck^{2\alpha}, \quad \forall 0 \leqq n \leqq T/k .$$

Now we lay out some assumptions, which will be used throughout the rest of the paper, on the data and the solutions of (1.1).

We assume u_0 and f satisfy

(A1) $$u_0 \in H^2(\Omega)^d \cap V, \quad f \in L^\infty(0, T; L^2(\Omega)^d) \cap L^2(0, T; H^1(\Omega)^d) .$$

in the three dimensional case, we assume additionally there exists a global strong solution, i.e.

(A2) $$\sup_{t \in [0, T]} \|u(t)\| \leqq M_1 .$$

Under the assumptions (A1)–(A2), we can prove (see for instance [4])

$$\text{(2.8)} \qquad \sup_{t\in(0,T]} \{\|u(t)\|_2 + |u'(t)| + |\nabla p|\} \leqq M_2 .$$

Notice that (2.8) is automatically satisfied when $d = 2$.

To simplify our presentation, we assume the solution $\{u, p\}$ of (1.1) satisfies the following additional regularity condition:

$$\text{(A3)} \qquad \int_0^T \{\|u_{ttt}\|_{-1}^2 + \|u_{tt}\|^2 + |p_{tt}|^2\} \leqq M_3 .$$

Remark 1. The verification of (A3) involves some compatibility conditions on the data which are generally not satisfied (see for instance [4, 14]). We assume (A3) merely for simplifying the presentation. In fact, with the assumptions (A1)–(A2), we can prove (see for instance [4])

$$\int_0^T t^2\{\|u_{ttt}\|_{-1}^2 + \|u_{tt}\|^2 + |p_{tt}|^2\} \leqq M_4 .$$

Hence, all the results presented later can be accordingly modified by taking into account the "smoothing" property at $t = 0$ (see also [5] and Remark 3 in [10]).

Hereafter, we will use c to denote a generic positive constant which depends only on Ω, ν, T and constants from various Sobolev inequalities. We will use M as a generic positive constant which may additionally depend on u_0, f and the solution u through the constant M_i, $i = 1, 2, 3$.

3 A pressure correction scheme

As in [10], to avoid technicalities for handling different spatial discretizations, we will study directly schemes in semi-discretized form for which a totally implicit version must be used to ensure the stability. Although schemes in semi-implicit form as (1.2)–(1.3) and (1.4)–(1.5) are often implemented in practice, especially when a fast Helmholtz solver is available.

In this section, we study the following version of the pressure correction scheme:

$$\text{(3.1)} \qquad \begin{cases} \frac{1}{k}(\tilde{u}^{n+1} - u^n) - \nu\Delta\tilde{u}^{n+\frac{1}{2}} + \tilde{B}(\tilde{u}^{n+\frac{1}{2}}, \tilde{u}^{n+\frac{1}{2}}) + \nabla\phi^n = f(t_{n+\frac{1}{2}}) , \\ \tilde{u}^{n+\frac{1}{2}}|_\Gamma = 0 , \end{cases}$$

and

$$\text{(3.2)} \qquad \begin{cases} \frac{1}{k}(u^{n+1} - \tilde{u}^{n+1}) + \alpha\nabla(\phi^{n+1} - \phi^n) = 0 , \\ \operatorname{div} u^{n+1} = 0 , \\ u^{n+1}\cdot \boldsymbol{n}|_\Gamma = 0 , \end{cases}$$

where $\tilde{u}^{n+\frac{1}{2}} = \frac{1}{2}(\tilde{u}^{n+1} + u^n)$ and α can be any constant $>\frac{1}{2}$.

Equation (3.1) is a nonlinear equation for $\tilde{u}^{n+1}$ very similar to the stationary Navier-Stokes equation. The existence of at least one solution $\tilde{u}^{n+1}$ satisfying (3.1) can be carried out exactly as the existence for stationary Navier-Stokes equation by using Galerkin procedure (see Theorem 2.1.2 in [13]), while u^{n+1} in (4.2) is uniquely defined by the relation $u^{n+1} = P_H\tilde{u}^{n+1}$. This argument is also valid for schemes appearing later, so it will not be repeated.

Remark 2. Apart from the implicit treatment of the nonlinear term, the scheme is still different to (1.4)–(1.5) in the following aspects:

- $\tilde{u}^{n+1}$ does not satisfy the homogeneous boundary condition but $\tilde{u}^{n+\frac{1}{2}}$ does.
- A flexible constant $\alpha > \frac{1}{2}$ is used instead of the constant $\frac{1}{2}$ before the pressure correction term.

Let $t_n = kn$ and $t_{n+\frac{1}{2}} = (n+\frac{1}{2})k$. For any function $w(t)$ and any series $\{a^n\}$ and $\{\tilde{a}^n\}$, we denote

$$\tilde{w}(t_{n+\frac{1}{2}}) = \tfrac{1}{2}(w(t_{n+1}) + w(t_n)),$$

$$a^{n+\frac{1}{2}} = \tfrac{1}{2}(a^{n+1} + a^n), \qquad \tilde{a}^{n+\frac{1}{2}} = \tfrac{1}{2}(\tilde{a}^{n+1} + a^n).$$

We denote also

$$e^{n+1} = u(t_{n+1}) - u^{n+1}, \qquad \tilde{e}^{n+1} = u(t_{n+1}) - \tilde{u}^{n+1}, \qquad q^{n+1} = p(t_{n+1}) - \phi^{n+1}.$$

The above notations will be used throughout the paper.

To fix the idea, we will impose hereafter $\int_\Omega \phi^n dx = 0$, $\forall n$. We denote in this section

$$\phi_1^{n+\frac{1}{2}} = \phi^n + \alpha(\phi^{n+1} - \phi^n), \qquad \phi_2^{n+\frac{1}{2}} = \phi^n + \alpha(\phi^{n+1} - \phi^n) - \frac{k\nu\alpha}{2}\Delta(\phi^{n+1} - \phi^n).$$

Our main results in this section is

Theorem 1. *$\forall \alpha > \frac{1}{2}$, $\tilde{u}^{n+\frac{1}{2}}$ and $u^{n+\frac{1}{2}}$ are weakly order $\frac{3}{2}$ approximations to $u(t_{n+\frac{1}{2}})$ in $L^2(\Omega)^d$, $\phi_1^{n+\frac{1}{2}}$ as well as $\phi_2^{n+\frac{1}{2}}$ are weakly first order approximation to $\tilde{p}(t_{n+\frac{1}{2}})$ in $L^2(\Omega)^d$. Namely,*

$$k\sum_{n=0}^{T/k-1}\{|\tilde{e}^{n+\frac{1}{2}}|^2 + |e^{n+\frac{1}{2}}|^2\} \leq Mk^3, \tag{3.3}$$

$$k\sum_{n=0}^{T/k-1}\{|\phi_1^{n+\frac{1}{2}} - \tilde{p}(t_{n+\frac{1}{2}})|^2 + |\phi_2^{n+\frac{1}{2}} - \tilde{p}(t_{n+\frac{1}{2}})|^2\} \leq Mk^2. \tag{3.4}$$

We will prove Theorem 1 by proving a series of lemmas which will be frequently used later. We begin with a preliminary lemma for the truncation error defined by

$$R^n = \frac{1}{k}(u(t_{n+1}) - u(t_n)) - \nu\Delta\tilde{u}(t_{n+\frac{1}{2}}) + (\tilde{u}(t_{n+\frac{1}{2}})\cdot\nabla)\tilde{u}(t_{n+\frac{1}{2}}) + \nabla\tilde{p}(t_{n+\frac{1}{2}}) - f(t_{n+\frac{1}{2}}). \tag{3.5}$$

Lemma 1.

$$k \sum_{n=0}^{T/k-1} \|R^n\|_{-1}^2 \leqq ck^4 \cdot \int_0^T (\|u_{ttt}\|_{-1}^2 + \|u_{tt}\|^2 + |p_{tt}|^2)dt \leqq Mk^4 .$$

Proof. Adding (3.5) to (1.1) at $t = t_{n+\frac{1}{2}}$, we obtain

$$\text{(3.6)}\quad R^n = \left(\frac{1}{k}(u(t_{n+1}) - u(t_n)) - u'(t_{n+\frac{1}{2}})\right) - (\nu\Delta\tilde{u}(t_{n+\frac{1}{2}}) - \nu\Delta u(t_{n+\frac{1}{2}}))$$
$$+ ((\tilde{u}(t_{n+\frac{1}{2}})\cdot\nabla)\tilde{u}(t_{n+\frac{1}{2}}) - (u(t^{n+\frac{1}{2}})\cdot\nabla)u(t_{n+\frac{1}{2}})) + (\nabla(\tilde{p}(t_{n+\frac{1}{2}}) - p(t_{n+\frac{1}{2}})))$$
$$= A_1^n - A_2^n + A_3^n + A_4^n .$$

We majorize the four terms on the right hand side as follows.

Using the integral residual formula of Taylor series for $u(t_{n+1})$ and $u(t_n)$, we derive

$$\text{(3.7)}\qquad A_1^n = \frac{1}{2k}\left(\int_{t_{n+\frac{1}{2}}}^{t_{n+1}} (t_{n+1} - s)^2 u_{ttt}(s)ds - \int_{t_n}^{t_{n+\frac{1}{2}}} (s - t_n)^2 u_{ttt}(s)ds\right) .$$

Applying Schwarz inequality to the above integrals, we obtain

$$\|A_1^n\|_{-1}^2 \leqq \frac{1}{k^2}\int_{t_n}^{t_{n+\frac{1}{2}}} (s - t_n)^4\, ds \int_{t_n}^{t_{n+\frac{1}{2}}} \|u_{ttt}(s)\|_{-1}^2\, ds$$
$$+ \frac{1}{k^2}\int_{t_{n+\frac{1}{2}}}^{t_{n+1}} (t_{n+1} - s)^4\, ds \int_{t_{n+\frac{1}{2}}}^{t_{n+1}} \|u_{ttt}(s)\|_{-1}^2\, ds$$
$$\leqq ck^3 \int_{t_n}^{t_{n+1}} \|u_{ttt}(s)\|_{-1}^2\, ds .$$

Therefore

$$\text{(3.8)}\qquad k \sum_{n=0}^{T/k-1} \|A_1^n\|_{-1}^2 \leqq ck^4 \int_0^T \|u_{ttt}\|_{-1}^2\, ds .$$

Let us denote

$$E_u^n = \tilde{u}(t_{n+\frac{1}{2}}) - u(t_{n+\frac{1}{2}}), \qquad E_p^n = \tilde{p}(t_{n+\frac{1}{2}}) - p(t_{n+\frac{1}{2}}) .$$

Using again the integral residual of the Taylor series, we have

$$E_u^n = \tfrac{1}{4}\left(\int_{t_n}^{t_{n+\frac{1}{2}}} (s - t_n)u''(s)\, ds + \int_{t_{n+\frac{1}{2}}}^{t_{n+1}} (t_{n+1} - s)u''(s)\, ds\right),$$
$$E_p^n = \tfrac{1}{4}\left(\int_{t_n}^{t_{n+\frac{1}{2}}} (s - t_n)p''(s)\, ds + \int_{t_{n+\frac{1}{2}}}^{t_{n+1}} (t_{n+1} - s)p''(s)\, ds\right).$$

As above, we can derive by using Schwarz inequality,

$$\|E_u^n\|^2 \leqq ck^3 \int_{t_n}^{t_{n+1}} \|u''(s)\|^2\, ds, \qquad |E_p^n|^2 \leqq ck^3 \int_{t_n}^{t_{n+1}} |p''(s)|^2\, ds .$$

Therefore,

$$(3.9)\qquad k\sum_{n=0}^{T/k-1}\|A_2^n\|_{-1}^2 \leqq cvk\sum_{n=0}^{T/k-1}\|E_u^n\|^2 \leqq ck^4\int_0^T\|u''(s)\|^2\,ds\,,$$

$$(3.10)\qquad k\sum_{n=0}^{T/k-1}\|A_4^n\|_{-1}^2 \leqq ck\sum_{n=0}^{T/k-1}\max_{v\in H_0^1(\Omega)^d}\frac{(\langle\nabla E_p^n, v\rangle)^2}{\|v\|^2}$$

$$\leqq ck\sum_{n=0}^{T/k-1}|E_p^n|^2 \leqq ck^4\int_0^T|p''(s)|^2\,ds\,.$$

Next, we can rewrite A_3^n as

$$A_3^n = (E_u^n\cdot\nabla)\tilde{u}(t_{n+\frac{1}{2}}) + (\tilde{u}(t_{n+\frac{1}{2}})\cdot\nabla)E_u^n\,.$$

Since $\|\tilde{u}(t_{n+\frac{1}{2}})\| \leqq M$, by using (2.5), we have

$$\langle A_3^n, v\rangle = b(E_u^n, \tilde{u}(t_{n+\frac{1}{2}}), v) + b(\tilde{u}(t_{n+\frac{1}{2}}), E_u^n, v)$$

$$\leqq c\|E_u^n\|\,\|v\|\,\|\tilde{u}(t_{n+\frac{1}{2}})\| \leqq M\|E_u^n\|\,\|v\|\,.$$

Hence

$$(3.11)\qquad k\sum_{n=0}^{T/k-1}\|A_3^n\|_{-1}^2 \leqq ck\sum_{n=0}^{T/k-1}\max_{v\in H_0^1(\Omega)^d}\frac{(\langle A_3^n, v\rangle)^2}{\|v\|^2}$$

$$\leqq Mk\sum_{n=0}^{T/k-1}\|E_u^n\|^2 \leqq Mk^4\,.$$

Thanks to (3.8), (3.9), (3.10) and (3.11), the proof of Lemma 1 is complete. □

Now we are going to derive a first error estimate for the velocity approximations.

Lemma 2.

$$|e^{N+1}|^2 + |\tilde{e}^{N+1}|^2 + \frac{2\alpha-1}{2\alpha}\sum_{n=0}^{N}|e^{n+1}-\tilde{e}^{n+1}|^2$$

$$+ kv\sum_{n=0}^{N}\{\|\tilde{e}^{n+\frac{1}{2}}\|^2 + \|e^{n+\frac{1}{2}}\|^2\} \leqq Mk^2, \quad \forall 0 \leqq N \leqq T/k-1\,.$$

Proof. Subtracting (3.1) from (3.5), we obtain

$$(3.12)\qquad \frac{1}{k}(\tilde{e}^{n+1}-e^n) - v\Delta\tilde{e}^{n+\frac{1}{2}} + \nabla(\tilde{p}(t_{n+\frac{1}{2}}) - \phi^n)$$

$$= \tilde{B}(\tilde{u}^{n+\frac{1}{2}}, \tilde{u}^{n+\frac{1}{2}}) - (\tilde{u}(t_{n+\frac{1}{2}})\cdot\nabla)\tilde{u}(t_{n+\frac{1}{2}}) + R^n \equiv NLT + R^n\,.$$

Since $\operatorname{div}\tilde{u}(t_{n+\frac{1}{2}}) = 0$, we can rearrange the nonlinear terms on the right-hand side as

$$(3.13)\qquad NLT = \tilde{B}(\tilde{u}^{n+\frac{1}{2}}, \tilde{u}^{n+\frac{1}{2}}) - (\tilde{u}(t_{n+\frac{1}{2}})\cdot\nabla)\tilde{u}(t_{n+\frac{1}{2}})$$

$$= \tilde{B}(\tilde{u}^{n+\frac{1}{2}}, \tilde{u}^{n+\frac{1}{2}}) - \tilde{B}(\tilde{u}(t_{n+\frac{1}{2}}), \tilde{u}(t_{n+\frac{1}{2}}))$$

$$= -\tilde{B}(\tilde{u}^{n+\frac{1}{2}}, \tilde{e}^{n+\frac{1}{2}}) - \tilde{B}(\tilde{e}^{n+\frac{1}{2}}, \tilde{u}(t_{n+\frac{1}{2}}))\,.$$

We now take the inner product of (3.12) with $2k\tilde{e}^{n+\frac{1}{2}}$.

$$2k\left(\frac{1}{k}(\tilde{e}^{n+1}-e^n)-\nu\Delta\tilde{e}^{n+\frac{1}{2}},\tilde{e}^{n+\frac{1}{2}}\right)=|\tilde{e}^{n+1}|^2-|e^n|^2+2k\nu\|\tilde{e}^{n+\frac{1}{2}}\|^2\,.$$

$$2k\langle R^n,\tilde{e}^{n+\frac{1}{2}}\rangle\leqq\frac{k\nu}{2}\|\tilde{e}^{n+\frac{1}{2}}\|^2+\frac{2k}{\nu}\|R^n\|_{-1}^2\,.$$

By using (2.3) and (2.4),

$$\begin{aligned}2k(NLT,\tilde{e}^{n+\frac{1}{2}})&=2k(\tilde{B}(\tilde{e}^{n+\frac{1}{2}},\tilde{u}(t_{n+\frac{1}{2}})),\tilde{e}^{n+\frac{1}{2}})\\&=2kb(\tilde{e}^{n+\frac{1}{2}},\tilde{u}(t_{n+\frac{1}{2}}),\tilde{e}^{n+\frac{1}{2}})-2kb(\tilde{e}^{n+\frac{1}{2}},\tilde{e}^{n+\frac{1}{2}},\tilde{u}(t_{n+\frac{1}{2}}))\,.\end{aligned}$$

Since $\|u(t)\|_2\leqq M$, using appropriate inequalities in (2.6) for the two terms above, we derive

$$2k(NLT,\tilde{e}^{n+\frac{1}{2}})\leqq Mk|\tilde{e}^{n+\frac{1}{2}}|\,\|\tilde{e}^{n+\frac{1}{2}}\|\leqq\frac{\nu k}{2}\|\tilde{e}^{n+\frac{1}{2}}\|^2+Mk|\tilde{e}^{n+\frac{1}{2}}|^2\,. \tag{3.14}$$

Combining the above inequalities, we obtain

$$\begin{aligned}&|\tilde{e}^{n+1}|^2-|e^n|^2+2k\nu\|\tilde{e}^{n+\frac{1}{2}}\|^2\\&\quad\leqq 2k\langle R^n,\tilde{e}^{n+\frac{1}{2}}\rangle+2k(NLT,\tilde{e}^{n+\frac{1}{2}})+2k(\nabla(\phi^n-\tilde{p}(t_{n+\frac{1}{2}})),\tilde{e}^{n+\frac{1}{2}})\\&\quad\leqq k\nu\|\tilde{e}^{n+\frac{1}{2}}\|^2+Mk|\tilde{e}^{n+\frac{1}{2}}|^2+Mk\|R^n\|_{-1}^2\\&\qquad+k(\nabla(\phi^n-\tilde{p}(t_{n+\frac{1}{2}})),\tilde{e}^{n+1})\,.\end{aligned} \tag{3.15}$$

On the other hand, we derive from (3.2) that

$$\frac{1}{k}(e^{n+1}-\tilde{e}^{n+1})=\alpha\nabla(\phi^{n+1}-\phi^n)\,. \tag{3.16}$$

Taking the inner product of (3.16) with $\frac{(2\alpha-1)k}{2\alpha}e^{n+1}$, we obtain

$$\frac{2\alpha-1}{2\alpha}\{|e^{n+1}|^2-|\tilde{e}^{n+1}|^2+|e^{n+1}-\tilde{e}^{n+1}|^2\}=0\,. \tag{3.17}$$

Now, taking the inner product of (3.16) with $\frac{k}{2\alpha}(e^{n+1}+\tilde{e}^{n+1})$, since

$$(\nabla p,v)=0,\quad\forall p\in H^1(\Omega),\quad v\in H\,, \tag{3.18}$$

we derive

$$\frac{1}{2\alpha}(|e^{n+1}|^2-|\tilde{e}^{n+1}|^2)=\frac{k}{2}(\nabla(\phi^{n+1}-\phi^n),\tilde{e}^{n+1})\,. \tag{3.19}$$

Adding (3.15), (3.17) and (3.19), we arrive to

$$(3.20)\qquad |e^{n+1}|^2 - |e^n|^2 + kv\|\tilde{e}^{n+\frac{1}{2}}\|^2 + \frac{2\alpha-1}{2\alpha}|e^{n+1} - \tilde{e}^{n+1}|^2$$

$$\leqq Mk(|\tilde{e}^{n+\frac{1}{2}}|^2 + \|R^n\|_{-1}^2) + \frac{k}{2}(\nabla(\phi^{n+1} + \phi^n - 2\tilde{p}(t_{n+\frac{1}{2}})), \tilde{e}^{n+1})$$

$$= Mk(|\tilde{e}^{n+\frac{1}{2}}|^2 + \|R^n\|_{-1}^2) - \frac{k}{2}(\nabla(q^{n+1} + q^n), \tilde{e}^{n+1})\,.$$

The pressure term on the right-hand side can be dealt as follows.
From (3.16), we have

$$\tilde{e}^{n+1} = e^{n+1} - k\alpha\nabla(\phi^{n+1} - \phi^n)\,.$$

Therefore

$$(3.21)\qquad -\frac{k}{2}(\nabla(q^{n+1} + q^n), \tilde{e}^{n+1}) = \frac{\alpha k^2}{2}(\nabla(q^{n+1} + q^n), \nabla(\phi^{n+1} - \phi^n))$$

$$= -\frac{\alpha k^2}{2}(\nabla(q^{n+1} + q^n), \nabla(q^{n+1} - q^n)$$

$$- \nabla(p(t_{n+1}) - p(t_n)))$$

$$= -\frac{\alpha k^2}{2}(|\nabla q^{n+1}|^2 - |\nabla q^n|^2) + \frac{\alpha k^2}{2} I_p^n\,,$$

where

$$(3.22)\qquad I_p^n = (\nabla(q^{n+1} + q^n), \nabla(p(t_{n+1}) - p(t_n))) = \left(\nabla(q^{n+1} + q^n), \int_{t_n}^{t_{n+1}} \nabla p'(s)\,ds\right)$$

$$\leqq |\nabla(q^{n+1} + q^n)| \int_{t_n}^{t_{n+1}} |\nabla p'(s)|\,ds$$

$$\leqq k(|\nabla q^{n+1}|^2 + |\nabla q^n|^2) + \int_{t_n}^{t_{n+1}} |\nabla p'(s)|^2\,ds\,.$$

Hence, adding (3.20) to (3.21), because of (3.22), we arrive to

$$(3.23)\qquad |e^{n+1}|^2 - |e^n|^2 + \frac{\alpha k^2}{2}(|\nabla q^{n+1}|^2 - |\nabla q^n|^2) + kv\|\tilde{e}^{n+\frac{1}{2}}\|^2$$

$$+ \frac{2\alpha-1}{2\alpha}|e^{n+1} - \tilde{e}^{n+1}|^2 \leqq Mk(|\tilde{e}^{n+\frac{1}{2}}|^2 + \|R^n\|_{-1}^2)$$

$$+ k^3(|\nabla q^{n+1}|^2 + |\nabla q^n|^2) + \frac{\alpha k^2}{2}\int_{t_n}^{t_{n+1}} |\nabla p'(s)|^2\,ds\,.$$

Now, taking the sum of (3.23) for n from 0 to N, since $|\tilde{e}^{n+\frac{1}{2}}|^2 \leqq |\tilde{e}^{n+1}|^2 + |e^n|^2$, we arrive to

$$|e^{N+1}|^2 + \frac{\alpha k^2}{2}|\nabla q^{N+1}|^2 + \sum_{n=0}^{N}\left\{kv\|\tilde{e}^{n+\frac{1}{2}}\|^2 + \frac{2\alpha-1}{2\alpha}|e^{n+1} - \tilde{e}^{n+1}|^2\right\}$$
$$\leqq Mk\sum_{n=1}^{N+1}\left\{|\tilde{e}^n|^2 + |e^n|^2 + \frac{\alpha k^2}{2}|\nabla q^n|^2\right\} + \frac{\alpha k^2}{2}\left(|\nabla q^0|^2 + \int_0^T |\nabla p'(s)|^2\,ds\right).$$

Since $|\tilde{e}^{n+1}|^2 = |e^{n+1}|^2 + |e^{n+1} - \tilde{e}^{n+1}|^2$, for $\alpha > \frac{1}{2}$, we can rewrite the above inequality as

$$\text{(3.24)}\quad \frac{2\alpha+1}{4\alpha}|e^{N+1}|^2 + \frac{2\alpha-1}{4\alpha}|\tilde{e}^{N+1}|^2 + \frac{\alpha k^2}{2}|\nabla q^{N+1}|^2$$
$$+ \sum_{n=0}^{N}\left\{kv\|\tilde{e}^{n+\frac{1}{2}}\|^2 + \frac{2\alpha-1}{4\alpha}|e^{n+1} - \tilde{e}^{n+1}|^2\right\}$$
$$\leqq Mk\sum_{n=1}^{N+1}\left\{|\tilde{e}^n|^2 + |e^n|^2 + \frac{\alpha k^2}{2}|\nabla q^n|^2\right\} + Mk^2.$$

By applying the discrete Gronwall lemma to the last inequality, we derive

$$\text{(3.25)}\quad |e^{N+1}|^2 + |\tilde{e}^{N+1}|^2 + \sum_{n=0}^{N}\{kv\|\tilde{e}^{n+\frac{1}{2}}\|^2 + |e^{n+1} - \tilde{e}^{n+1}|^2\}$$
$$\leqq Mk^2, \quad \forall\, 0 \leqq N \leqq T/k - 1.$$

Thanks to (3.25) and the following inequality (see [13])

$$\text{(3.26)}\quad \|P_H u\|_{H^1(\Omega)^d} \leqq c(\Omega)\|u\|_{H^1(\Omega)^d}, \quad \forall\, u \in H^1(\Omega)^d,$$

the proof of Lemma 2 is complete. □

A side product of Lemma 2 is that we have

$$\text{(3.27)}\quad \|\tilde{u}^{n+\frac{1}{2}}\| \leqq 2(\|\tilde{e}^{n+\frac{1}{2}}\| + \|\tilde{u}(t_{n+\frac{1}{2}})\|) \leqq M, \quad \forall\, 0 \leqq n \leqq T/k - 1.$$

In order to derive an error estimate for the pressure approximation, we need the following

Lemma 3.

$$\sum_{n=0}^{T/k-1}\|e^{n+1} - e^n\|_{-1}^2 \leqq Mk^2\sum_{n=0}^{T/k-1}\{\|\tilde{e}^{n+\frac{1}{2}}\|^2 + \|R^n\|_{-1}^2\}.$$

Proof. Taking the sum of (3.1) and (3.2), we obtain

$$\text{(3.28)}\quad \frac{1}{k}(u^{n+1} - u^n) - v\Delta\tilde{u}^{n+\frac{1}{2}} + \tilde{B}(u^{\overline{n+\frac{1}{2}}}, u^{n+\frac{1}{2}}) + \nabla\phi^{n+\frac{1}{2}} = f(t_{n+\frac{1}{2}}).$$

Taking the sum of (3.28) and (3.5), we obtain

$$\text{(3.29)}\quad \frac{1}{k}(e^{n+1} - e^n) - v\Delta\tilde{e}^{n+\frac{1}{2}} + \nabla(\tilde{p}(t_{n+\frac{1}{2}}) - \phi^{n+\frac{1}{2}}) = NLT + R^n.$$

Since $e^{n+1} - e^n \in H = D(A^{-1})$, we can take the inner product of (3.29) with $kA^{-1}(e^{n+1} - e^n)$. From the property (2.7) of A^{-1}, we derive

$$(\nabla(\tilde{p}(t_{n+\frac{1}{2}}) - \phi^{n+\frac{1}{2}}), A^{-1}(e^{n+1} - e^n)) = 0 ,$$

$$(e^{n+1} - e^n, A^{-1}(e^{n+1} - e^n)) = \| e^{n+1} - e^n \|_{-1}^2 ,$$

$$k(R^n, A^{-1}(e^{n+1} - e^n)) \leqq \tfrac{1}{8} \| e^{n+1} - e^n \|_{-1}^2 + ck^2 \| R^n \|_{-1}^2 ,$$

$$\begin{aligned} - k\nu \langle \Delta \tilde{e}^{n+\frac{1}{2}}, A^{-1}(e^{n+1} - e^n) \rangle &= k\nu(\nabla \tilde{e}^{n+\frac{1}{2}}, \nabla A^{-1}(e^{n+1} - e^n)) \\ &\leqq \tfrac{1}{8} \| e^{n+1} - e^n \|_{-1}^2 + ck^2 \| \tilde{e}^{n+\frac{1}{2}} \|^2 . \end{aligned}$$

We recall that (see (3.13))

$$NLT = - \tilde{B}(\tilde{u}^{n+\frac{1}{2}}, \tilde{e}^{n+\frac{1}{2}}) - \tilde{B}(\tilde{e}^{n+\frac{1}{2}}, \tilde{u}(t_{n+\frac{1}{2}})) .$$

Then we derive from (2.4) that

$$\begin{aligned} k(NLT, A^{-1}(e^{n+1} - e^n)) = \frac{k}{2}(b(\tilde{u}^{n+\frac{1}{2}}, \tilde{e}^{n+\frac{1}{2}}, A^{-1}(e^{n+1} - e^n)) \\ - b(\tilde{u}^{n+\frac{1}{2}}, A^{-1}(e^{n+1} - e^n), \tilde{e}^{n+\frac{1}{2}})) \\ + \frac{k}{2}(b(\tilde{e}^{n+\frac{1}{2}}, \tilde{u}(t_{n+\frac{1}{2}}), A^{-1}(e^{n+1} - e^n)) \\ - b(\tilde{e}^{n+\frac{1}{2}}, A^{-1}(e^{n+1} - e^n), \tilde{u}(t_{n+\frac{1}{2}}))) \end{aligned}$$

Hence, by using (2.7), (2.5) and (3.27), we obtain

$$\begin{aligned} k(NLT, A^{-1}(e^{n+1} - e^n)) &\leqq ck \| \tilde{u}^{n+\frac{1}{2}} \| \, \| \tilde{e}^{n+\frac{1}{2}} \| \, \| e^{n+1} - e^n \|_{-1} \\ &\quad + ck \| \tilde{u}(t_{n+\frac{1}{2}}) \| \, \| \tilde{e}^{n+\frac{1}{2}} \| \, \| e^{n+1} - e^n \|_{-1} \\ &\leqq Mk \| \tilde{e}^{n+\frac{1}{2}} \| \, \| e^{n+1} - e^n \|_{-1} \\ &\leqq \tfrac{1}{4} \| e^{n+1} - e^n \|_{-1}^2 + Mk^2 \| \tilde{e}^{n+\frac{1}{2}} \|^2 . \end{aligned} \tag{3.30}$$

Therefore, combining the above inequalities, we obtain

$$\frac{1}{2} \sum_{n=0}^{T/k-1} \| e^{n+1} - e^n \|_{-1}^2 \leqq Mk^2 \sum_{n=0}^{T/k-1} \{ \| \tilde{e}^{n+\frac{1}{2}} \|^2 + \| R^n \|_{-1}^2 \} . \quad \square$$

We rearrange (3.29) to

$$\nabla q_*^{n+\frac{1}{2}} = \frac{1}{k}(e^{n+1} - e^n) - \alpha \Delta e_*^{n+\frac{1}{2}} + NLT - R^n , \tag{3.31}$$

where $\{q_*^{n+\frac{1}{2}}, e_*^{n+\frac{1}{2}}\} = \{\tilde{p}(t_{n+\frac{1}{2}}) - \phi_1^{n+\frac{1}{2}}, \tilde{e}^{n+\frac{1}{2}}\}$.

We notice that (3.31) is also true for $\{q_*^{n+\frac{1}{2}}, e_*^{n+\frac{1}{2}}\} = \{\tilde{p}(t_{n+\frac{1}{2}}) - \phi_2^{n+\frac{1}{2}}, e^{n+\frac{1}{2}}\}$. So we can consider the two pressure approximations simultaneously.

Lemma 4.

$$k \sum_{n=0}^{T/k-1} |q_*^{n+\frac{1}{2}}|^2 \leqq Mk \sum_{n=0}^{T/k-1} \{ \| \tilde{e}^{n+\frac{1}{2}} \|^2 + \| R^n \|_{-1}^2 \} + \frac{1}{k} \sum_{n=0}^{T/k-1} \| e^{n+1} - e^n \|_{-1}^2 .$$

Proof. Replacing $A^{-1}(e^{n+1}-e^n)$ by v in (3.30), we obtain

$$(NLT, v) \leqq M\|\tilde{e}^{n+\frac{1}{2}}\| \, \|v\|, \quad \forall v \in H_0^1(\Omega)^d . \tag{3.32}$$

Using Schwarz inequality, we have also $\forall v \in H_0^1(\Omega)^d$,

$$\left(\frac{1}{k}(e^{n+1}-e^n) - \nu\Delta e_*^{n+\frac{1}{2}} - R^n, v\right) \leqq \frac{c}{k}(\|e^{n-1}-e^n\|_{-1} + \|R^n\|_{-1} + \nu\|e_*^{n+\frac{1}{2}}\|)\|v\| . \tag{3.33}$$

Using the inequality

$$|p| \leqq c \sup_{v \in H_0^1(\Omega)^d} \frac{(\nabla p, v)}{\|v\|}, \tag{3.34}$$

we derive from (3.31), (3.32) and (3.33) that

$$\begin{aligned} |q_*^{n+\frac{1}{2}}| &\leqq c \sup_{v \in H_0^1(\Omega)^d} \frac{(\nabla q_*^{n+\frac{1}{2}}, v)}{\|v\|} \\ &\leqq \frac{M}{k}\|e^{n+1}-e^n\|_{-1} + M(\|R^n\|_{-1} + \|\tilde{e}^{n+\frac{1}{2}}\| + \|e^{n+\frac{1}{2}}\|) . \end{aligned}$$

The lemma is then a direct consequence of the last inequality and (3.26). □

Lemma 5. $\forall 0<\delta<1, \exists c_\delta > 0$ *such that*

$$-\langle \Delta u, A^{-1}P_H u\rangle \geqq \begin{cases} \delta|P_H u|^2 - c_\delta|u-P_H u|^2 \\ \delta|P_H u|^2 - c_\delta|\operatorname{div} u|^2 \end{cases}, \quad \forall u \in H_0^1(\Omega)^d . \tag{3.35}$$

Proof. $\forall u \in H_0^1(\Omega)^d$, let $\{v, p\}$ be the solution of the following Stokes equation

$$-\Delta v + \nabla p = P_H u, \qquad \operatorname{div} v = 0, \qquad \int_\Omega p\,dx = 0, \qquad v|_\Gamma = 0 .$$

Hence, $v = A^{-1}P_H u$ and

$$\|v\|_s + \|p\|_{s-1} \leqq c\|P_H u\|_{s-2}, \quad s = 1, 2 . \tag{3.36}$$

Since $u, A^{-1}P_H u \in H_0^1(\Omega)^d$, integrating by parts twice and using (3.36), we obtain

$$\begin{aligned} -\langle \Delta u, A^{-1}P_H u\rangle &= \langle u, -\Delta A^{-1}P_H u\rangle = (u, P_H u - \nabla p) \\ &= |P_H u|^2 + (u, \nabla p) = |P_H u|^2 + (u - P_H u, \nabla p) \\ &\geqq |P_H u|^2 - |u-P_H u\|\nabla p| \geqq |P_H u|^2 - c|u - P_H u\|P_H u| \\ &\geqq \delta|P_H u|^2 - c_\delta|u-P_H u|^2 . \end{aligned}$$

Likewise

$$\begin{aligned} -\langle \Delta u, A^{-1}P_H u\rangle &= |P_H u|^2 + (u, \nabla p) = |P_H u|^2 + (\operatorname{div} u, p) \\ &\geqq |P_H u|^2 - |\operatorname{div} u||p| \geqq |P_H u|^2 - c|\operatorname{div} u||P_H u| \\ &\geqq \delta|P_H u|^2 - c_\delta|u-P_H u|^2 . \quad \square \end{aligned}$$

Lemma 6.

$$kv\sum_{n=0}^{T/k-1}\{|\tilde{e}^{n+\frac{1}{2}}|^2+|e^{n+\frac{1}{2}}|^2\}\leqq k\sum_{n=0}^{T/k-1}\{\|R^n\|_{-1}^2+|e^{n+1}-\tilde{e}^{n+1}|^2+k^2\|\tilde{e}^{n+\frac{1}{2}}\|^2\}.$$

Proof. We rearrange the nonlinear term NLT as follows.

$$NLT=-\tilde{B}(\tilde{u}^{n+\frac{1}{2}},\tilde{e}^{n+\frac{1}{2}})-\tilde{B}(\tilde{e}^{n+\frac{1}{2}},\tilde{u}(t_{n+\frac{1}{2}}))$$
$$=\tilde{B}(\tilde{e}^{n+\frac{1}{2}},\tilde{u}(t_{n+\frac{1}{2}}))-\tilde{B}(\tilde{e}^{n+\frac{1}{2}},\tilde{e}^{n+\frac{1}{2}})+\tilde{B}(\tilde{u}(t_{n+\frac{1}{2}}),\tilde{e}^{n+\frac{1}{2}}). \tag{3.37}$$

We take the inner product of (3.29) with $2kA^{-1}e^{n+\frac{1}{2}}$. By Lemma 5, we have

$$-\langle\Delta\tilde{e}^{n+\frac{1}{2}},A^{-1}e^{n+\frac{1}{2}}\rangle\geqq\frac{15}{16}|e^{n+\frac{1}{2}}|^2-c|e^{n+1}-\tilde{e}^{n+1}|^2. \tag{3.38}$$

Using (2.7), and noticing that $(A^{-1}u,\nabla p)=0(\forall u\in H)$, we obtain

$$\|e^{n+1}\|_{-1}^2-\|e^n\|_{-1}^2+\frac{15kv}{8}|e^{n+\frac{1}{2}}|^2\leqq 2k\langle R^n,A^{-1}e^{n+\frac{1}{2}}\rangle$$
$$+c|e^{n+1}-\tilde{e}^{n+1}|^2+B_1^n-B_2^n+B_3^n, \tag{3.39}$$

where we have denoted $2k(NLT,A^{-1}\tilde{e}^{n+\frac{1}{2}})$ by $B_1^n-B_2^n+B_3^n$.

Using (2.7) and the fact that $\|u\|_{-1}\leqq c|u|(\forall u\in L^2(\Omega))$, we have

$$2k\langle R^n,A^{-1}e^{n+\frac{1}{2}}\rangle\leqq ck\|R^n\|_{-1}\|A^{-1}e^{n+\frac{1}{2}}\|\leqq\frac{vk}{4}|e^{n+\frac{1}{2}}|^2+Mk\|R^n\|_{-1}^2. \tag{3.40}$$

Using (2.4), we have

$$B_1^n=2k\tilde{b}(\tilde{e}^{n+\frac{1}{2}},\tilde{u}(t_{n+\frac{1}{2}}),A^{-1}e^{n+\frac{1}{2}})$$
$$=k(b(\tilde{e}^{n+\frac{1}{2}},\tilde{u}(t_{n+\frac{1}{2}}),A^{-1}e^{n+\frac{1}{2}})-b(\tilde{e}^{n+\frac{1}{2}},A^{-1}e^{n+\frac{1}{2}},\tilde{u}(t_{n+\frac{1}{2}})))$$
$$=\frac{k}{2}b(\tilde{e}^{n+1}-e^{n+1},\tilde{u}(t_{n+\frac{1}{2}}),A^{-1}e^{n+\frac{1}{2}})+b(e^{n+\frac{1}{2}},\tilde{u}(t_{n+\frac{1}{2}}),A^{-1}e^{n+\frac{1}{2}})$$
$$+\frac{k}{2}b(e^{n+1}-\tilde{e}^{n+1},A^{-1}e^{n+\frac{1}{2}},\tilde{u}(t_{n+\frac{1}{2}}))+b(e^{n+\frac{1}{2}},A^{-1}e^{n+\frac{1}{2}},\tilde{u}(t_{n+\frac{1}{2}})),$$

we then use (2.6) to derive

$$B_1^n\leqq Mk(|\tilde{e}^{n+1}-e^{n+1}|+\|e^{n+\frac{1}{2}}\|_{-1})|e^{n+\frac{1}{2}}|$$
$$\leqq Mk(|\tilde{e}^{n+1}-e^{n+1}|^2+\|e^{n+\frac{1}{2}}\|_{-1}^2)+\frac{vk}{8}|e^{n+\frac{1}{2}}|^2;$$

Similarly, we can derive

$$B_3^n\leqq Mk(|\tilde{e}^{n+1}-e^{n+1}|+\|e^{n+\frac{1}{2}}\|_{-1})|e^{n+\frac{1}{2}}|$$
$$\leqq Mk(|\tilde{e}^{n+1}-e^{n+1}|^2+\|e^{n+\frac{1}{2}}\|_{-1}^2)+\frac{vk}{8}|e^{n+\frac{1}{2}}|^2;$$

As for B_2^n, we have

$$\begin{aligned}B_2^n &= 2k\tilde{b}(\tilde{e}^{n+\frac12}, \tilde{e}^{n+\frac12}, A^{-1}e^{n+\frac12})\\ &= k(b(\tilde{e}^{n+\frac12}, \tilde{e}^{n+\frac12}, A^{-1}e^{n+\frac12}) - b(\tilde{e}^{n+\frac12}, A^{-1}e^{n+\frac12}, \tilde{e}^{n+\frac12}))\\ &\leqq Mk|\tilde{e}^{n+\frac12}|\,\|\tilde{e}^{n+\frac12}\|\,|e^{n+\frac12}| \leqq Mk^2\|\tilde{e}^{n+\frac12}\|\,|e^{n+\frac12}| \quad \text{(by Lemma 2)}\\ &\leqq Mk^3\|\tilde{e}^{n+\frac12}\|^2 + \frac{\nu k}{8}|e^{n+\frac12}|^2 .\end{aligned}$$

Combining these inequalities into (3.39), we arrive to

$$\begin{aligned}\|e^{n+1}\|_{-1}^2 - \|e^n\|_{-1}^2 + \nu k|e^{n+\frac12}|^2 \leqq Mk\{\|e^{n+\frac12}\|_{-1}^2 + \|R^n\|_{-1}^2\\ + |e^{n+1} - \tilde{e}^{n+1}|^2\} + Mk^3\|\tilde{e}^{n+\frac12}\|^2 .\end{aligned}$$

Taking the sum of the last inequality for n from 0 to $N(\forall 0 \leqq N \leqq T/k - 1)$, we derive from Lemma 2 that

$$\begin{aligned}\|e^{N+1}\|_{-1}^2 + \sum_{n=0}^{N} k\nu|e^{n+\frac12}|^2 \leqq Mk\sum_{n=0}^{N}\{\|R^n\|_{-1}^2 + |e^{n+1} - \tilde{e}^{n+1}|^2\\ + k^2\|\tilde{e}^{n+\frac12}\|^2\} + Mk\sum_{n=1}^{N+1}\|e^n\|_{-1}^2 .\end{aligned}$$

By applying the discrete Gronwall lemma to the last inequality, we obtain

$$\text{(3.41)}\quad \begin{aligned}\|e^{N+1}\|_{-1}^2 + \sum_{n=0}^{N} k\nu|e^{n+\frac12}|^2 \leqq Mk\sum_{n=0}^{N}\{\|R^n\|_{-1}^2\\ + |e^{n+1} - \tilde{e}^{n+1}|^2 + k^2\|\tilde{e}^{n+\frac12}\|^2\} .\end{aligned}$$

Since

$$2\tilde{e}^{n+\frac12} = \tilde{e}^{n+1} + e^n = 2e^{n+\frac12} + \tilde{e}^{n+1} - e^{n+1} ,$$

we derive from Lemma 2 that

$$k\sum_{n=0}^{T/k-1}|\tilde{e}^{n+\frac12}|^2 \leqq k\sum_{n=0}^{T/k-1}\{|e^{n+\frac12}|^2 + |\tilde{e}^{n+1} - e^{n+1}|^2\} .$$

The proof of the lemma is complete thanks to the last inequality and (3.41). □

Proof of Theorem 1. (3.4) is a direct consequence of Lemmas 2, 3 and 4. (3.3) is a direct consequence of Lemmas 2 and 6. □

In proving Lemma 2, we notice that it is actually the term I_p^n, defined in (3.22), which prevents us from obtaining second order error estimates. The reason for which we failed to obtain second order error estimates is directly linked to the inconsistent Neumann boundary condition

$$\text{(3.42)}\qquad \left.\frac{\partial\phi_1^{n+\frac12}}{\partial \boldsymbol{n}}\right|_\Gamma = \left.\frac{\partial\phi^n}{\partial \boldsymbol{n}}\right|_\Gamma = \cdots = \left.\frac{\partial\phi^0}{\partial \boldsymbol{n}}\right|_\Gamma, \quad \forall n .$$

However, in case that (1.7) is satisfied, the Neumann boundary condition for the pressure is well represented by (3.42). Hence, we expect that (1.7) would lead to second order error estimates. In fact we have the following

Theorem 2. *We assume that* $|\nabla(\phi^0 - p(0))|^2 \leqq O(k)$ *and that the pressure* p *in* (1.1) *verifies* (1.7) *and* $\int_0^T |\Delta p'(s)|^2\, ds \leqq M$. *Then* $\forall\, \varepsilon > 0$, $\exists\, M_\varepsilon > 0$ *such that the following inequalities hold*:

$$k \sum_{n=0}^{T/k-1} \{|e^{n+\frac{1}{2}}|^2 + |\tilde{e}^{n+\frac{1}{2}}|^2\} \leqq M_\varepsilon k^{4-\varepsilon}, \tag{3.43}$$

$$|e^{N+1}|^2 + |\tilde{e}^{N+1}|^2 + k^2|\nabla q^{N+1}|^2 + k \sum_{n=0}^{N} \{\|e^{n+\frac{1}{2}}\|^2 + \|\tilde{e}^{n+\frac{1}{2}}\|^2\} \leqq M_\varepsilon k^{3-\varepsilon}, \quad \forall\, 0 \leqq N \leqq T/k - 1, \tag{3.44}$$

$$k \sum_{n=0}^{T/k-1} \{|\phi_1^{n+\frac{1}{2}} - \tilde{p}(t_{n+\frac{1}{2}})|^2 + |\phi_2^{n+\frac{1}{2}} - \tilde{p}(t_{n+\frac{1}{2}})|^2\} \leqq M_\varepsilon k^{3-\varepsilon}. \tag{3.45}$$

We will begin by proving the following recursive result.

Lemma 7. *Under the assumptions of Theorem* 2, $\forall\, 0 \leqq \beta \leqq 1$, *if*

$$\sum_{n=0}^{T/k-1} \{k|\phi_1^{n+\frac{1}{2}} - \tilde{p}(t_{n+\frac{1}{2}})|^2 + |e^{n+1} - \tilde{e}^{n+1}|^2\} \leqq M k^{2+\beta}, \tag{3.46}$$

then

$$|e^{N+1}|^2 + k^2|\nabla q^{N+1}|^2 + \sum_{n=0}^{N} \{k\|\tilde{e}^{n+\frac{1}{2}}\|^2 + |e^{n+1} - \tilde{e}^{n+1}|^2\} \leqq M k^{2+\frac{1+\beta}{2}}, \quad \forall\, 0 \leqq N \leqq T/k - 1, \tag{3.47}$$

$$k \sum_{n=0}^{T/k-1} \{|\phi_1^{n+\frac{1}{2}} - \tilde{p}(t_{n+\frac{1}{2}})|^2 + |\phi_2^{n+\frac{1}{2}} - \tilde{p}(t_{n+\frac{1}{2}})|^2\} \leqq M k^{2+\frac{1+\beta}{2}}. \tag{3.48}$$

Proof. Since

$$\phi_1^{n+\frac{1}{2}} = \phi^n + \alpha(\phi^{n+1} - \phi^n) = \tfrac{1}{2}(\phi^{n+1} + \phi^n) + (\alpha - \tfrac{1}{2})(\phi^{n+1} - \phi^n),$$

we have

$$\tfrac{1}{2}(q^{n+1} + q^n) = \tilde{p}(t_{n+\frac{1}{2}}) - \phi_1^{n+\frac{1}{2}} + (\tfrac{1}{2} - \alpha)(\phi^{n+1} - \phi^n).$$

On the other hand, we derive from (3.16) and the Poincaré inequality that

$$|\phi^{n+1} - \phi^n| \leqq c|\nabla(\phi^{n+1} - \phi^n)| \leqq \frac{c}{k}|e^{n+1} - \tilde{e}^{n+1}|.$$

We then derive from the last two relations and (3.46) that

$$\begin{aligned} k \sum_{n=0}^{T/k-1} |q^{n+1} + q^n|^2 &\leqq 4k \sum_{n=0}^{T/k-1} \{|\tilde{p}(t_{n+\frac{1}{2}}) - \phi_1^{n+\frac{1}{2}}|^2 + (\alpha - \tfrac{1}{2})^2|\phi^{n+1} - \phi^n|^2\} \\ &\leqq 4k \sum_{n=0}^{T/k-1} \left\{|\tilde{p}(t_{n+\frac{1}{2}}) - \phi_1^{n+\frac{1}{2}}|^2 - \frac{c}{k^2}|e^{n+1} - \tilde{e}^{n+1}|^2\right\} \\ &\leqq M k^{1+\beta}. \end{aligned} \tag{3.49}$$

Now we will treat I_p^n differently as we did in the proof of Lemma 2. In fact, thanks to (1.7), we can integrate by parts for I_p^n so that

$$\begin{aligned} I_p^n &= (\nabla(q^{n+1} + q^n) . \nabla(p(t_{n+1}) - p(t_n))) \\ &= -(q^{n+1} + q^n, \Delta(p(t_{n+1}) - p(t_n))) \\ &\leqq \sqrt{k}|q^{n+1} + q^n| \left(\int_{t_n}^{t_{n+1}} |\Delta p'(s)|^2 \right)^{\frac{1}{2}} \\ &\leqq k^{1-\frac{1+\beta}{2}} |q^{n+1} + q^n|^2 + k^{\frac{1+\beta}{2}} \int_{t_n}^{t_{n+1}} |\Delta p'(s)|^2 \, ds . \end{aligned}$$

Therefore, as in the proof of Lemma 2, we can derive the following inequality similar to (3.24):

$$\begin{aligned} &|e^{N+1}|^2 + k^2 |\nabla q^{N+1}|^2 + \sum_{n=0}^{N} \{k \| \tilde{e}^{n+\frac{1}{2}} \|^2 + |e^{n+1} - \tilde{e}^{n+1}|^2\} \\ &\leqq k^2 |\nabla q^0|^2 + k^{2+\frac{1+\beta}{2}} \int_0^T |\Delta p'(s)|^2 \, ds \\ &\quad + M \sum_{n=0}^{N} \{k^{3-\frac{1+\beta}{2}} |q^{n+1} + q^n|^2 + k \| R^n \|_{-1}^2\} \\ &\leqq M k^{2+\frac{1+\beta}{2}} \quad \text{(from (3.49))} . \end{aligned}$$

Thanks to Lemmas 3, 4 and the above inequality, we can obtain (3.48). □

Proof of Theorem 2. Thanks to Lemmas 2 and 3, the condition of Lemma 7 is satisfied with $\beta = 0$. Then $\forall \varepsilon > 0$, we can apply successively Lemma 7 with $\beta = 0$ and $\beta = \beta_m = \sum_{i=1}^{m} 1/2^i$, for $m = 1, 2, \ldots$. For m large enough such that $\beta_m \geqq 1 - \varepsilon$, Lemma 7 implies (3.44) and (3.45). (3.43) is then a direct consequence of (3.44), (3.45) and Lemma 6. The proof of Theorem 2 is complete. □

Remark 3. Theorem 2 tells that if (1.7) is satisfied, then the scheme (4.1)–(4.2) is essentially (i.e. to within an arbitrary small $\varepsilon > 0$) weakly second order in $L^2(\Omega)^d$ for the velocity.

It is interesting to observe that the choice of α does not affect the precision of the scheme. The restriction on α being strictly larger than $\frac{1}{2}$ seems purely technical, we do expect comparable results in practice with $\alpha = \frac{1}{2}$. In fact, no essential difference in results was observed by Gresho in [3] when using different α.

Schemes similar to (3.1)–(3.2) can be easily constructed. For instance, the following scheme is based on another second order time discretization scheme with pressure correction:

$$(3.50) \quad \begin{cases} \dfrac{1}{2k}(3\tilde{u}^{n+1} - 4u^n + u^{n-1}) - \nu \Delta \tilde{u}^{n+1} + \tilde{B}(\tilde{u}^{n+1}, \tilde{u}^{n+1}) + \nabla \phi^n = f(t_{n+1}) \ , \\ \tilde{u}^{n+1}|_\Gamma = 0 \ , \end{cases}$$

and

$$(3.51)\qquad \begin{cases} \frac{1}{k}(u^{n+1} - \tilde{u}^{n+1}) + \alpha\nabla(\phi^{n+1} - \phi^n) = 0 \ , \\ \operatorname{div} u^{n+1} = 0 \ , \\ u^{n+1} \cdot \boldsymbol{n}|_\Gamma = 0 \ , \end{cases}$$

By using essentially the same technique, we can prove similar results as in Theorems 1 and 2 for the scheme (3.50)–(3.51) with $\alpha > 1$.

4 Kim and Moin's scheme

Since the semi-implicit scheme (1.2)–(1.3) is only conditionally stable and it seems that the totally implicit form of (1.2)–(1.3) is inconsistent, we will only consider the scheme (1.2)–(1.3) applied to the unsteady Stokes equations, i.e. without the nonlinear term. After all, the explicit treatment of the nonlinear term in (1.2)–(1.3) would only affect the stability of the scheme and would not spoil the time discretization accuracy which is the main concern of this paper.

If we introduce a new intermediate velocity $\tilde{u}^{n+1}$ and let $\hat{u}^{n+1} = \tilde{u}^{n+1} + k\nabla\phi^n$, we can rewrite (1.2)–(1.3) without the nonlinear term as

$$(4.1)\qquad \begin{cases} \frac{1}{k}(\tilde{u}^{n+1} - u^n) - \frac{\nu}{2}\Delta(\tilde{u}^{n+1} + u^n) + \nabla\left(\phi^n - \frac{\nu k}{2}\Delta\phi^n\right) = f(t_{n+\frac{1}{2}}) \ , \\ \tilde{u}^{n+1}|_\Gamma = 0 \ , \end{cases}$$

and

$$(4.2)\qquad \begin{cases} \frac{1}{k}(u^{n+1} - \tilde{u}^{n+1}) + \nabla(\phi^{n+1} - \phi^n) = 0 \ , \\ \operatorname{div} u^{n+1} = 0 \ , \\ u^{n+1} \cdot \boldsymbol{n}|_\Gamma = 0 \ . \end{cases}$$

We notice that (4.1)–(4.2) is very similar to (3.1)–(3.2) with $\alpha = 1$ and without the nonlinear term. In fact, their only difference is a small extra potential $\frac{\nu k}{2}\nabla\Delta\phi^n$ in (4.1). We then expect they lead to approximations with comparable accuracy. Actually, we can prove the following analog of Theorem 1 for the scheme (4.1)–(4.2).

Theorem 3. *$\tilde{u}^{n+\frac{1}{2}}$ and $u^{n+\frac{1}{2}}$ are weakly order $\frac{3}{2}$ approximations to $u(t_{n+1})$ in $L^2(\Omega)^d$, $\phi^{n+1} - \frac{k\nu}{2}\Delta\phi^{n+1}$ as well as $\phi^{n+1} - \frac{k\nu}{2}\Delta\phi^n$ are weakly first order approximation to $\tilde{p}(t_{n+\frac{1}{2}})$ in $L^2(\Omega)^d$. Namely,*

$$(4.3)\qquad k\sum_{n=0}^{T/k-1}\{|\tilde{e}^{n+\frac{1}{2}}|^2 + |e^{n+\frac{1}{2}}|^2\} \leqq Mk^3 \ ,$$

$$(4.4)\qquad k\sum_{n=0}^{T/k-1}\left\{\left|\phi^{n+1} - \frac{k\nu}{2}\Delta\phi^n - \tilde{p}(t_{n+\frac{1}{2}})\right|^2 + \left|\phi^n - \frac{k\nu}{2}\Delta\phi^{n+1} - \tilde{p}(t_{n+\frac{1}{2}})\right|^2\right\} \leqq Mk^2 \ .$$

Sketch of the proof. The proof of this Theorem is very similar to that of Theorem 1, so we will only point out how to handle the extra potential term in (4.1).

The corresponding error equations for (4.1)–(4.2) are:

$$(4.5)\qquad \frac{1}{k}(\tilde{e}^{n+1}-e^n)-\frac{\nu}{2}\Delta(\tilde{e}^{n+1}+e^n)+\nabla\left(\tilde{p}(t_{n+\frac{1}{2}})-\phi^n+\frac{\nu k}{2}\Delta\phi^n\right)=R_1^n$$

$$(4.6)\qquad \frac{1}{k}(e^{n+1}-\tilde{e}^{n+1})=\nabla(\phi^{n+1}-\phi^n)\,,$$

where R_1^n is R^n excluding the nonlinear terms.

As in the proof of Lemma 2, we take respectively the inner product of (4.5) with $2k\tilde{e}^{n+\frac{1}{2}}$ and that of (4.6) with $\frac{k}{2}(e^{n+1}+\tilde{e}^{n+1})$, we can derive exact as before the following inequality similar to (3.23) with $\alpha=1$,

$$(4.7)\quad |e^{n+1}|^2-|e^n|^2+\frac{k^2}{4}(|\nabla q^{n+1}|^2-|\nabla q^n|^2)+k\nu\|\tilde{e}^{n+\frac{1}{2}}\|^2$$
$$+\tfrac{1}{2}|e^{n+1}-\tilde{e}^{n+1}|^2+D^n$$
$$\leqq Mk(|\tilde{e}^{n+\frac{1}{2}}|^2+\|R^n\|_{-1}^2)+k^3(|\nabla q^{n+1}|^2+|\nabla q^n|^2)+\frac{k^2}{4}\int_{t_n}^{t_{n+1}}|\nabla p'(s)|^2\,ds\,,$$

where $D^n=\nu k^2(\nabla\Delta\phi^n,\tilde{e}^{n+\frac{1}{2}})$ is the only extra term on the left hand side. Using the relation

$$\operatorname{div}\tilde{e}^{n+1}=-k\Delta(\phi^{n+1}-\phi^n)\,,$$

We can rearrange D^n as follows.

(4.8)

$$D^n=\frac{\nu k^2}{2}(\nabla\Delta\phi^n,\tilde{e}^{n+1})=-\frac{\nu k^2}{2}(\Delta\phi^n,\operatorname{div}\tilde{e}^{n+1})$$
$$=\frac{\nu k^3}{2}(\Delta\phi^n,\Delta(\phi^{n+1}+\phi^n))=\frac{\nu k^3}{4}\{|\Delta\phi^{n+1}|^2-|\Delta\phi^n|^2-|\Delta\phi^{n+1}-\Delta\phi^n|^2\}$$
$$=\frac{\nu k^3}{4}\{|\Delta\phi^{n+1}|^2-|\Delta\phi^n|^2\}-\frac{\nu k}{4}|\operatorname{div}\tilde{e}^{n+1}|^2\,.$$

We recall that (see [13]) $|\operatorname{div}u|\leqq\|u\|,\ \forall u\in H_0^1(\Omega)^d$. Therefore, replacing D^n in (4.7) by (4.8), we obtain

$$(4.9)\qquad |e^{n+1}|^2-|e^n|^2+\frac{k^2}{4}(|\nabla q^{n+1}|^2-|\nabla q^n|^2)+\frac{\nu k^3}{4}(|\Delta\phi^{n+1}|^2-|\Delta\phi^n|^2)$$
$$+\frac{3k\nu}{4}\|\tilde{e}^{n+\frac{1}{2}}\|^2+\tfrac{1}{2}|e^{n+1}-\tilde{e}^{n+1}|^2\leqq Mk(|\tilde{e}^{n+\frac{1}{2}}|^2+\|R^n\|_{-1}^2)$$
$$+k^3(|\nabla q^{n+1}|^2+|\nabla q^n|^2)+\frac{k^2}{4}\int_{t_n}^{t_{n+1}}|\nabla p'(s)|^2\,ds\,.$$

Taking the sum of the last inequality for n from 0 to N, we obtain

$$(4.10)\qquad \begin{aligned}&|e^{N+1}|^2 + |\tilde{e}^{N+1}|^2 + k^2|\nabla q^{N+1}|^2 + k^3|\Delta\phi^{N+1}|^2 \\ &+ \sum_{n=0}^{N} \{kv\|\tilde{e}^{n+\frac{1}{2}}\|^2 + |e^{n+1} - \tilde{e}^{n+1}|^2\} \\ &\leqq M \sum_{n=0}^{N} \{k|e^n|^2 + k^2|\nabla q^{n+1}|^2 + k^3|\Delta\phi^{n+1}|^2\} + Mk^2 .\end{aligned}$$

Applying the discrete Gronwall lemma, we can derive the following result similar to Lemma 2:

$$(4.11)\qquad \begin{aligned}&|e^{N+1}|^2 + |\tilde{e}^{N+1}|^2 + k^2|\nabla q^{N+1}|^2 + k^3|\Delta\phi^{N+1}|^2 \\ &+ \sum_{n=0}^{N} \{kv\|\tilde{e}^{n+\frac{1}{2}}\|^2 + kv\|e^{n+\frac{1}{2}}\|^2 \\ &+ |e^{n+1} - \tilde{e}^{n+1}|^2\} \leqq Mk^2 .\end{aligned}$$

The rest of the proof is basically identical to that of Theorem 1 with

$$\phi_1^{n+\frac{1}{2}} = \phi^{n+1} - \frac{kv}{2}\Delta\phi^n, \qquad \phi_1^{n+\frac{1}{2}} = \phi^n - \frac{kv}{2}\Delta\phi^{n+1} . \quad \square$$

Remark 4. With $\phi_1^{n+\frac{1}{2}}$ and $\phi_2^{n+\frac{1}{2}}$ defined as above, the results in Theorem 2 also apply to the scheme (4.1)–(4.2).

5 Penalty method

With the framework we established in the previous sections, it is now an easy matter to prove the following result for the penalty scheme (1.8).

Theorem 4. *Let* $e^n = u(t_n) - u^n$, *where* u^n *is defined in* (1.8). *Then*

$$(5.1)\qquad \begin{aligned}&|e^{n+1}|^2 + kv \sum_{n=0}^{N} \left\{ \|e^{n+1}\|^2 + \frac{1}{\varepsilon}|\mathrm{div}\, e^{n+1}|^2 \right\} \\ &\leqq M(k^4 + \varepsilon), \quad \forall 0 \leqq N \leqq T/k - 1 .\end{aligned}$$

$$(5.2)\qquad k \sum_{n=0}^{T/k-1} \left| \frac{1}{\varepsilon}\mathrm{div}\, u^{n+1} - \tilde{p}(t_{n+\frac{1}{2}}) \right|^2 \leqq M(k^4 + \varepsilon) .$$

Proof. Taking the difference of (1.1) and (1.8), we obtain the following error equation

$$(5.3)\qquad \frac{1}{k}(e^{n+1} - e^n) - v\Delta e^{n+\frac{1}{2}} - \frac{1}{\varepsilon}\nabla \mathrm{div}\, u^{n+1} = NLT' + R^n - \nabla\tilde{p}(t_{n+\frac{1}{2}}) ,$$

where $NLT' = -\tilde{B}(e^{n+\frac{1}{2}}, \tilde{u}(t_{n+\frac{1}{2}})) - \tilde{B}(u^{n+\frac{1}{2}}, e^{n+\frac{1}{2}})$.

Taking the inner product of (5.3) with $2ke^{n+\frac{1}{2}}$, as in (3.14), we can derive

$$2k(NLT', e^{n+\frac{1}{2}}) \leqq \frac{kv}{2}\|e^{n+\frac{1}{2}}\|^2 + Mk|e^{n+\frac{1}{2}}|^2.$$

Therefore, using Schwarz inequality, we obtain

$$|e^{n+1}|^2 - |e^n|^2 + 2k\nu\|e^{n+\frac{1}{2}}\|^2 + \frac{k}{\varepsilon}|\operatorname{div} u^{n+1}|^2$$
$$\leqq 2k(R^n, e^{n+\frac{1}{2}}) + k(\tilde{p}(t_{n+\frac{1}{2}}), \operatorname{div} u^{n+1}) + 2k(NLT', e^{n+\frac{1}{2}})$$
$$\leqq k\nu\|e^{n+\frac{1}{2}}\|^2 + ck\|R^n\|_{-1}^2 + \frac{k}{2\varepsilon}|\operatorname{div} u^{n+1}|^2$$
$$+ \frac{k\varepsilon}{2}|\tilde{p}(t_{n+\frac{1}{2}})|^2 + Mk|e^{n+\frac{1}{2}}|^2 .$$

Taking the sum of the last inequality for n from 0 to N and using discrete Gronwall lemma, we recover (5.1).

The proof of (5.2) is now standard. In fact, we can prove (5.2) exactly as we proved Lemmas 3 and 4. □

Remark 5. It is clear from Theorem 3 that in order to get second order accuracy, we have to choose $\varepsilon = O(k^4)$ for which the system (1.8) would become seriously ill conditioned. This might be the primary reason for which the success of Penalty methods is somewhat limited.

6 A penalty-projection scheme

We notice that the precision of the projection schemes partially depends on how well the incompressibility condition is satisfied by the intermediate velocity $\tilde{u}^{n+1}$. In order to reduce $\operatorname{div}\tilde{u}^{n+1}$, we can introduce a penalty function as in (1.8) to the first step of the scheme (3.1)–(3.2), hoping that we can improve the error estimates to second order with a much relaxed parameter ε. So we consider here the following penalty-projection scheme

(6.1)
$$\begin{cases} \frac{1}{k}(\tilde{u}^{n+1} - u^n) - \nu\Delta\tilde{u}^{n+\frac{1}{2}} + \tilde{B}(\tilde{u}^{n+\frac{1}{2}}, \tilde{u}^{n+\frac{1}{2}}) - k^{-\beta}\nabla \operatorname{div}\tilde{u}^{n+1} + \nabla\phi^n = f(t_{n+\frac{1}{2}}) , \\ \tilde{u}^{n+\frac{1}{2}}|_\Gamma = 0 , \end{cases}$$

and

$$\begin{cases} \frac{1}{k}(u^{n+1} - \tilde{u}^{n+1}) + \alpha\nabla(\phi^{n+1} - \phi^n) = 0 , \\ \operatorname{div} u^{n+1} = 0 , \\ u^{n+1}\cdot \boldsymbol{n}|_\Gamma = 0 , \end{cases} \tag{6.2}$$

where β is some constant > 0.

The corresponding error equations are

$$\frac{1}{k}(\tilde{e}^{n+1} - e^n) - \nu\Delta\tilde{e}^{n+\frac{1}{2}} - k^{-\beta}\nabla \operatorname{div}\tilde{u}^{n+1} + \nabla(\tilde{p}(t_{n+\frac{1}{2}}) - \phi^n) = NLT + R^n , \tag{6.3}$$

$$\frac{1}{k}(e^{n+1} - \tilde{e}^{n+1}) = \alpha\nabla(\phi^{n+1} - \phi^n) . \tag{6.4}$$

Let us denote

$$\phi_1^{n+\frac{1}{2}} = \phi^n + \alpha(\phi^{n+1} - \phi^n) - k^{-\beta} \operatorname{div} \tilde{u}^{n+1} ,$$

$$\phi_2^{n+\frac{1}{2}} = \phi^n + \alpha(\phi^{n+1} - \phi^n) - \frac{k\nu\alpha}{2} \Delta(\phi^{n+1} - \phi^n) - k^{-\beta} \operatorname{div} \tilde{u}^{n+1} .$$

We have the following

Theorem 5. *If* $\beta = 2$ *and* $\alpha > \frac{1}{2}$, *then the following error estimates hold*:

(6.5) $$k \sum_{n=0}^{T/k-1} |e^{n+\frac{1}{2}}|^2 \leqq Mk^4 ,$$

(6.6) $$|e^{N+1}|^2 + k \sum_{n=0}^{N} \{ \| e^{n+\frac{1}{2}} \|^2 + \| \tilde{e}^{n+\frac{1}{2}} \|^2 \} \leqq Mk^3, \quad \forall 0 \leqq N \leqq T/k - 1 ,$$

(6.7) $$k \int_{n=0}^{T/k-1} \{ |\phi_1^{n+\frac{1}{2}} - \tilde{p}(t_{n+\frac{1}{2}})|^2 + |\phi_2^{n+\frac{1}{2}} - \tilde{p}(t_{n+\frac{1}{2}})|^2 \} \leqq Mk^3 .$$

We begin with a first error estimate for the velocity and the pressure approximations.

Lemma 8. $\forall \beta > 0$, *we have*

(6.8) $$|e^{N+1}|^2 + \sum_{n=0}^{N} \{ k \| \tilde{e}^{n+\frac{1}{2}} \|^2 + |e^{n+1} - \tilde{e}^{n+1}|^2 + k^{1-\beta} |\operatorname{div} \tilde{e}^{n+1}|^2 \} \leqq Mk^2, \quad \forall 0 \leqq N \leqq T/k - 1 ,$$

(6.9) $$k \sum_{n=0}^{T/k-1} \{ |\phi_1^{n+\frac{1}{2}} - \tilde{p}(t_{n+\frac{1}{2}})|^2 + |\phi_2^{n+\frac{1}{2}} - \tilde{p}(t_{n+\frac{1}{2}})|^2 \} \leqq Mk^2 .$$

Sketch of the proof. The proof of (6.8) is essentially identical to the proof of Lemma 2. We only have to take care of the extra term $-k^{-\beta} \nabla \operatorname{div} \tilde{u}^{n+1}$. Since $\operatorname{div} u(t_{n+1}) = 0$, we have $-k^{-\beta} \nabla \operatorname{div} \tilde{u}^{n+1} = -k^{-\beta} \nabla \operatorname{div} \tilde{e}^{n+1}$. Taking the inner product of (6.3) with $2k\tilde{e}^{n+\frac{1}{2}}$, using integration by parts, the extra term becomes

(6.10) $$(-k^{-\beta} \nabla \operatorname{div} \tilde{e}^{n+\frac{1}{2}}, 2k\tilde{e}^{n+\frac{1}{2}}) = \frac{k^{1-\beta}}{2} (\operatorname{div} \tilde{e}^{n+1}, \operatorname{div} \tilde{e}^{n+1}) = \frac{k^{1-\beta}}{2} |\operatorname{div} \tilde{e}^{n+1}|^2 .$$

Keeping in mind the above term and repeating the proof of Lemma 2, we can obtain (6.8). With $\phi_1^{n+\frac{1}{2}}$ and $\phi_2^{n+\frac{1}{2}}$ defined as above, (6.9) is a direct consequence of (6.8) and Lemmas 3 and 4. □

Next, we prove a first improvement to the last lemma.

Lemma 9. $\forall \beta > 0$, *we have*

(6.11) $$|e^{N+1}|^2 + k \sum_{n=0}^{N} \{ \| \tilde{e}^{n+\frac{1}{2}} \|^2 + \| e^{n+\frac{1}{2}} \|^2 \} \leqq Mk^{2+\frac{\beta}{2}} + Mk^3, \quad \forall 0 \leqq N \leqq T/k - 1 ,$$

(6.12) $$k \sum_{n=0}^{T/k-1} \{ |\phi_1^{n+\frac{1}{2}} - \tilde{p}(t_{n+\frac{1}{2}})|^2 + |\phi_2^{n+\frac{1}{2}} - \tilde{p}(t_{n+\frac{1}{2}})|^2 \} \leqq Mk^{2+\frac{\beta}{2}} + Mk^3 .$$

Proof. Taking the inner product of (6.3) with $2k\tilde{e}^{n+\frac{1}{2}}$ and that of (6.4) with $k(e^{n+1} + \tilde{e}^{n+1})$, using (3.14), we can obtain

(6.13)
$$\begin{aligned} |e^{n+1}|^2 - |e^n|^2 + k\nu \|\tilde{e}^{n+\frac{1}{2}}\|^2 &\leq Mk(|\tilde{e}^{n+\frac{1}{2}}|^2 + \|R^n\|_{-1}^2) - k(\nabla(\tilde{p}(t_{n+\frac{1}{2}}) - \phi_1^{n+\frac{1}{2}}), \tilde{e}^{n+1}) \\ &\leq Mk(|e^{n+1}|^2 + |e^n|^2 + |e^{n+1} - \tilde{e}^{n+1}|^2 + \|R^n\|_{-1}^2) \\ &\quad - k(\nabla(\tilde{p}(t_{n+\frac{1}{2}}) - \phi_1^{n+\frac{1}{2}}), \tilde{e}^{n+1}) . \end{aligned}$$

We treat the pressure term on the right hand side as follows.

Integrating by parts and using Schwarz inequality, we derive

$$\begin{aligned} - k(\nabla(\tilde{p}(t_{n+\frac{1}{2}}) - \phi_1^{n+\frac{1}{2}}), \tilde{e}^{n+1}) &= k(\tilde{p}(t_{n+\frac{1}{2}}) - \phi_1^{n+\frac{1}{2}}), \operatorname{div} \tilde{e}^{n+1} \\ &\leq k^{1+\frac{\beta}{2}}|\tilde{p}(t_{n+\frac{1}{2}}) - \phi_1^{n+\frac{1}{2}}|^2 + k^{1-\frac{\beta}{2}}|\operatorname{div} \tilde{e}^{n+1}|^2 . \end{aligned}$$

Therefore, taking the sum of the last two inequalities for n from 0 to N, using the results of Lemma 8, we obtain

$$\begin{aligned} |e^{N+1}|^2 + k\nu \sum_{n=0}^{N} \|\tilde{e}^{n+\frac{1}{2}}\|^2 &\leq Mk \sum_{n=0}^{N} \{|e^{n+1}|^2 + \|R^n\|_{-1}^2 + |e^{n+1} - \tilde{e}^{n+1}|^2\} \\ &\quad + k^{\frac{\beta}{2}} \sum_{n=0}^{N} \{k|q_1^{n+\frac{1}{2}} - \tilde{p}(t_{n+\frac{1}{2}})|^2 + k^{1-\beta}|\operatorname{div} \tilde{e}^{n+1}|^2\} \\ &\leq Mk \sum_{n=0}^{N} |e^{n+1}|^2 + Mk^{2+\frac{\beta}{2}} + Mk^3 . \end{aligned}$$

Applying the discrete Gronwall lemma to the above inequality, we obtain (6.11). The proof of (6.12) is again essentially identical to the proof of Lemmas 3 and 4. □

Proof of Theorem 4. In case $\beta = 2$, the previous lemma implies (6.6) and (6.7). So it remains to prove (6.5). To this end, we will proceed as in the proof of Lemma 6. Taking the sum of (6.3) and (6.4), we obtain

(6.14)
$$\frac{1}{k}(e^{n+1} - e^n) - \nu\Delta\tilde{e}^{n+\frac{1}{2}} - k^{-\beta}\nabla \operatorname{div} \tilde{u}^{n+1} + \nabla(\tilde{p}(t_{n+\frac{1}{2}}) - \phi_1^{n+\frac{1}{2}}) = NLT + R^n .$$

We now take the inner product of (6.14) with $2kA^{-1}e^{n+\frac{1}{2}}$. From Lemma 5, we have

$$- \langle \Delta\tilde{e}^{n+\frac{1}{2}}, A^{-1}e^{n+\frac{1}{2}} \rangle \geq \frac{15}{16}|e^{n+\frac{1}{2}}|^2 - c|\operatorname{div} \tilde{e}^{n+1}|^2 .$$

We can then obtain the following analog of Lemma 6:

$$k\nu \sum_{n=0}^{T/k-1} |e^{n+\frac{1}{2}}|^2 \leq k \sum_{n=0}^{T/k-1} \{\|R^n\|_{-1}^2 + |\operatorname{div} \tilde{e}^{n+1}|^2 + k^2\|\tilde{e}^{n+\frac{1}{2}}\|^2\}.$$

(6.5) is then a direct consequence of and the last inequality and Lemmas 8 and 9. The proof of Theorem 3 is complete. □

Remark 6. We notice that the pressure approximation $\phi_1^{n+\frac{1}{2}}$ here is no longer plagued by the inconsistent Neumann boundary condition (3.42), so that the improvement over the scheme (3.1)–(3.2) is not surprising.

A penalty function can also be added to the schemes (4.1)–(4.2) and (3.50)–(3.51) to get similar improvements presented in Theorem 5.

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