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A SPECTRAL-TAU APPROXIMATION FOR THE STOKES AND NAVIER-STOKES EQUATIONS (*)

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Abstract. — *A new spectral-Tau formulation for Stokes problem is introduced and analyzed. The pressure approximation of the resulting system does not contain any spurious modes. Moreover, the scheme is easy to implement numerically. The spectral convergence of the scheme is proved and is affirmed by numerical results.*

Résumé. — *Une nouvelle formulation de type spectrale-Tau pour les équations de Stokes est introduite et analysée. L'approximation de la pression du nouveau schéma ne contient aucun mode parasite. De plus, le schéma est facile à implémenter numériquement. La convergence spectrale est démontrée et confirmée par des résultats numériques.*

1. INTRODUCTION

The aim of this paper is to study, both theoretically and numerically, a spectral-Tau method for the Stokes problem with Dirichlet boundary conditions.

Let Ω be an open set in R^d (d integer $\geqslant 2$). The Stokes equations in the velocity-pressure formulation are

$$(1.1)\qquad \begin{cases} -\nu\,\Delta u + \nabla p = f, & x \in \Omega \\ \operatorname{div} u = 0, & x \in \Omega \\ u = 0\,|_{\partial\Omega} & \end{cases}$$

where ν is the coefficient of kinematic viscosity, and f an external force field.

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In the recent years, a number of algorithms using spectral methods have been successfully implemented for solving the Stokes and Navier-Stokes equations. Meanwhile, various theoretical results dealing with spectral-Galerkin and spectral-collocation methods have been established (*cf.* [3], [4], [5] and the references therein). However, to the author's knowledge, the spectral-Tau method seems to be less studied, although it is frequently used in practice because of its efficiency in solving Helmholtz type equations (*cf.* [11], [10] for the fast Helmholtz solver by Chebychev-Tau method). Recently, Sacchi Landeriani analyzed in [13] a usual Legendre-Tau formulation which has a large number of spurious modes, consequently, this formulation is difficult to implement numerically. Moreover, the method he used is restricted to two dimensional case and the error estimates he obtained are somewhat unsatisfactory. We propose in this paper a new spectral-Tau formulation for the Stokes problem which does not contain any spurious modes and for which a better error estimate, though might not be optimal, is achieved for arbitrary dimensional case.

A great problem in solving the Stokes problem with Dirichlet boundary condition numerically is the treatment of the pressure. The pressure p in (1.1) can be considered as a Lagrange multiplier which ensures satisfaction of the incompressibility condition « div $u = 0$ », and its calculation is then coupled with that of the velocity. One of the efficient methods for separating the calculation of u from that of p is using the Uzawa algorithm (*cf.* [1]). It is found that our new scheme is well adapted to the Uzawa algorithm.

We analyze the spectral-Tau method for the Helmholtz equations in section 2 where an optimal error estimate is given. In section 3, we introduce and analyze a new spectral-Tau formulation for the Stokes problem. Finally, in section 4, we adapt an Uzawa type algorithm to the new system and the algorithm is implemented to solve the 2-D Stokes problem as well as the 2-D Navier-Stokes problem.

Notations

Let $I = (-1, 1)$, we consider the parrallelepiped $\Omega = I^d$ in R^d. Throughout the paper, we shall use the weighted Sobolev spaces $H^s_\omega(\Omega)$ and $H^s_{0,\omega}(\Omega)$ with the standard weighted norm $\|\cdot\|_{s,\omega}$ where ω is either the Chebychev weight function

$$\omega(x) = \prod_{i=1}^{d} \omega_i(x_i) = \prod_{i=1}^{d} (1 - x_i^2)^{-\frac{1}{2}}$$

or the Legendre weight function

$$\omega(x) \equiv 1 .$$

In the later case, ω will be omitted in the notations. We set

$$X = H^1_{0,\omega}(\Omega)\,, \quad a_\omega(u, v) = (\nabla u, \nabla(v \,.\, \omega))\,, \quad \forall u, v \in X\,.$$

If $\omega \equiv 1$, we derive from the Poincaré inequality that $\|\cdot\|_\omega = a_\omega(\cdot\,,\cdot\,)^{\frac{1}{2}}$ is a norm equivalent to $\|\cdot\|_{1,\omega}$. It is also proved in [6] that the same result is true for then Chebychev weight function. We shall then denote the norm in X (resp. in $L^2(\Omega)$) by $\|\cdot\|_\omega$ (resp. $|\cdot|_\omega$). Throughout the paper, we shall use c and c_i to denote constants which can vary from one equation to another.

The following finite dimensional spaces will be used :

S_N : the set of polynomials such that the order of each variable is less than or equal to N ;

$X_N = \{f \in S_N : f(x)|_{\partial\Omega} = 0\}$.

2. HELMHOLTZ EQUATIONS

We consider in this section the Helmholtz equations with homogeneous boundary condition :

$$\text{(2.1)} \qquad \begin{cases} \alpha u - \Delta u = f & \text{in} \quad \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$$

where $\alpha > 0$ is a constant. Since the term αu does not introduce any difficulty, we study only the case $\alpha = 0$:

$$\text{(2.2)} \qquad \begin{cases} -\Delta u = f & \text{in} \quad \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$$

which is indeed a Poisson equation.

The Tau method applied to the system (2.2) is :

$$\text{(2.3)} \qquad \begin{cases} \text{find } u_N \in X_N \text{ such that} \\ a_\omega(u_N, v) = (f, v)_\omega\,, \quad \forall v \in S_{N-2} \end{cases}$$

where $(f, v)_\omega = (f, v\omega)$ is the scalar product of f and v in $L^2_\omega(\Omega)$.

In the Chebychev case, the system (2.3) can be solved efficiently by the diagonalization method proposed by D. B. Haidvogel and T. Zang (*cf.* [11]).

Let P_{N-2} be the orthogonal projection operator in $L^2_\omega(\Omega)$ onto S_{N-2}. We define :

$$\text{(2.4)} \quad a_{N,\omega}(u, v) = -(\Delta u, P_{N-2}\, v)_\omega\,, \quad (f, v)_{N,\omega} = (f, P_{N-2}\, v \,.\, \omega)\,.$$

By noting that :

$$P_{N-2} X_N = S_{N-2} \tag{2.5}$$

we can rewrite the system (2.3) in a symmetric form :

$$\begin{cases} \text{find } u_N \in X_N \text{ such that} \\ a_{N,\omega}(u_N, v) = (f, v)_{N,\omega}, \quad \forall v \in X_N \end{cases} \tag{2.6}$$

Let us prove first a preliminary lemma.

LEMMA 1 : *The bilinear form $a_{N,\omega}(\cdot, \cdot)$ is continuous and coercive on $X_N \times X_N$, more precisely :*

$$\begin{cases} a_{N,\omega}(u, v) \leqslant \|u\|_\omega \cdot \|v\|_\omega, & \forall u, v \in X_N \\ a_{N,\omega}(v, v) \geqslant cN^{1-d} \|v\|_\omega^2, & \forall v \in X_N. \end{cases}$$

Proof: We shall only prove the two dimensional Chebychev case. The result can be easily established for the Legendre case and extended for any integer $d \geqslant 2$ by similar argument.

We consider first one variable function : $\phi \in X_N(I)$. By definition, we have :

$$-(\phi_{xx}, P_{N-2}\phi)_\omega = -(\phi_{xx}, \phi)_\omega = \|\phi\|_\omega^2 . \tag{2.7}$$

Then $\forall u \in X_N(I^2)$, we can expand it in the Chebychev series :

$$u = \sum_{n,m=0}^{N} u_{nm} T_n(x) T_m(y) \in X_N$$

where $T_n(x)$ is the Chebychev polynomial which possesses the following orthogonal property :

$$\int_I T_n(x) T_m(x) \omega(x) dx = c_n \delta_{nm}, \quad \forall n, m \geqslant 0 \tag{2.8}$$

(where $c_0 = 2$ and $c_n = 1$, $\forall n \geqslant 1$).

We can also write u as :

$$u = \sum_{n=0}^{N} U_n(y) T_n(x), \quad \text{with} \quad U_n(y) = \sum_{m=0}^{N} u_{nm} T_m(y) \in X_N(I) \quad \text{or}$$

$$u = \sum_{m=0}^{N} \tilde{U}_m(x) T_m(y), \quad \text{with} \quad \tilde{U}_m(x) = \sum_{m=0}^{N} u_{nm} T_n(x) \in X_N(I).$$

Hence

$$\Delta u = \sum_{m=0}^{N} \tilde{U}_m^{(2)}(x)\, T_m(y) + \sum_{n=0}^{N} U_n^{(2)}(y)\, T_n(x)$$

where $\tilde{U}_m^{(2)}(x)$ and $U_n^{(2)}(y)$ are respectively the second derivative of $\tilde{U}_m(x)$ and $U_n(y)$.

It follows that :

$$-(\Delta u, \mathscr{P}_{N-2}\, u)_\omega = -\left(\sum_{m=0}^{N} \tilde{U}_m^{(2)}(x)\, T_m(y), \sum_{m=0}^{N-2} \tilde{U}_m(x)\, T_m(y)\right)_\omega - \left(\sum_{n=0}^{N} U_n^{(2)}(y)\, T_n(x), \sum_{n=0}^{N-2} U_n(y)\, T_n(x)\right)_\omega$$

Then by using successively (2.8) and (2.7), we get :

$$(2.9)\quad -(\Delta u, \mathscr{P}_{N-2}\, u)_\omega = = -\sum_{m=0}^{N-2} c_m \left\{ (\tilde{U}_m^{(2)}(x), \tilde{U}_m(x)\, \omega(x)) + (U_m^{(2)}(y), U_m(y)\, \omega(y)) \right\} = \sum_{m=0}^{N-2} c_m \left\{ \|\tilde{U}_m(x)\|_\omega^2 + \|U_m(y)\|_\omega^2 \right\}, \quad \forall u \in W_N .$$

We denote from now on the last sum by $\|\mathscr{K}_{N-2}\, u\|_\omega^2$. The next step is then to majorize $\|u\|_\omega$ in term of $\|\mathscr{K}_{N-2}\, u\|_\omega$.

Since $u|_{\partial\Omega} = 0$, we have :

$$u(\pm 1, y) = \sum_{m=0}^{N} U_m(y)\, T_m(\pm 1) = \sum_{m=0}^{N} U_m(y)(\pm 1)^m = 0$$

which imply :

$$U_N(y) = -\sum_{m=0}^{\frac{N}{2}-1} U_{2m}(y), \quad U_{N-1}(y) = -\sum_{m=0}^{\frac{N}{2}-1} U_{2m-1}(y).$$

Hence

$$(2.10)\qquad \|U_N(y)\|_\omega^2 + \|U_{N-1}(y)\|_\omega^2 \leqslant \frac{N}{2} \sum_{m=0}^{N-2} \|U_m(y)\|_\omega^2 .$$

Similarly

$$(2.11)\qquad \|\tilde{U}_N(x)\|_\omega^2 + \|\tilde{U}_{N-1}(x)\|_\omega^2 \leqslant \frac{N}{2} \sum_{m=0}^{N-2} \|\tilde{U}_m(x)\|_\omega^2 .$$

On the other hand

$$(2.12)\quad \|u\|_{\omega}^{2} = -(\Delta u, \mathscr{P}_{N-2}u)_{\omega} = \sum_{m=0}^{N} \left\{ \left\|\tilde{U}_m(x)\right\|_{\omega}^{2} + \|U_m(y)\|_{\omega}^{2} \right\}.$$

It then follows that

$$(2.13)\qquad \|\mathscr{K}_{N-2}u\|_{\omega}^{2} \geqslant \left(\frac{N}{2}+1\right)^{-1} \|u\|_{\omega}^{2}$$

which also means that

$$-(\Delta u, \mathscr{P}_{N-2}u)_{\omega} \geqslant \left(\frac{N}{2}+1\right)^{-1} \|u\|_{\omega}^{2}.$$

One can readily check that in arbitrary dimensional case the last two inequalities become :

$$(2.13bis)\qquad \|\mathscr{K}_{N-2}u\|_{\omega}^{2} \geqslant \left(\left(\frac{N^{d-1}}{2}\right)+1\right)^{-1} \|u\|_{\omega}^{2}$$

and

$$-(\Delta u, \mathscr{P}u_{N-2}u)_{\omega} \geqslant \left(\frac{N^{d-1}}{2}+1\right)^{-1} \|u\|_{\omega}^{2}.$$

The second inequality of this lemma can be established by using the same argument as before. Namely

$$\begin{aligned}(2.14)\quad -(\Delta u, \mathscr{P}_{N-2}v)_{\omega} &= \\ &= -\sum_{m=0}^{N-2} c_m \left\{ (\tilde{U}_m^{(2)}(x), \tilde{V}_m(x)\,\omega(x)) + (U_m^{(2)}(y), V_m(y)) \right\} \\ &\leqslant \sum_{m=0}^{N-2} c_m \left\{ \left\|\tilde{U}_m(x)\right\|_{\omega} \cdot \left\|\tilde{V}_m(x)\right\|_{\omega} + \|U_m(y)\|_{\omega} \cdot \|V_m(y)\|_{\omega} \right\} \\ &\leqslant \|\mathscr{K}_{N-2}u\|_{\omega}\, \|\mathscr{K}_{N-2}v\|_{\omega} \\ &\leqslant \|u\|_{\omega}\, \|v\|_{\omega}, \quad \forall u, v \in X_N.\end{aligned}$$

The proof is complete. ¶

We can now prove the following theorem.

THEOREM 1 : *There exists an unique solution u_N for the system* (2.6). *Moreover, if the solution u of* (2.1) *belongs to $H_{\omega}^{s}(\Omega)$, we have the following error estimate :*

$$\|u-u_N\|_{\omega} \leqslant cN^{\frac{d+1}{2}-s} \|u\|_{s,\omega}.$$

Proof: The existence and uniqueness of the solution for the system (2.6) follows immediately from Lemma 1 and the classical Lax-Milgram theorem.

By subtracting (2.1) from (2.3), we find :

$$a_{N,\omega}(u_N - u, v) = 0\,, \qquad \forall v \in X_N$$

thus

$$a_{N,\omega}(u_N - \phi, v) = a_{N,\omega}(u - \phi, v)\,, \quad \forall v, \phi \in X_N\,. \tag{2.15}$$

We replace v by $u_N - \phi$ in (2.15), by using (2.9) and (2.14), we find :

$$\begin{aligned} \|\mathscr{K}_{N-2}(u_N - \phi)\|_\omega^2 &= a_{N,\omega}(u_N - \phi, u_N - \phi) \\ &\leqslant \|\mathscr{K}_{N-2}(u - \phi)\|_\omega \, \|\mathscr{K}_{N-2}(u_N - \phi)\|_\omega\,. \end{aligned}$$

We then derive from (2.13*bis*) that

$$\begin{aligned} \|u_N - \phi\|_\omega &\leqslant cN^{\frac{d-1}{2}} \|\mathscr{K}_{N-2}(u - \phi)\|_\omega \\ &\leqslant cN^{\frac{d-1}{2}} \|u - \phi\|_\omega\,, \quad \forall \phi \in X_N\,. \end{aligned}$$

Hence

$$\begin{aligned} \|u_N - u\|_\omega &\leqslant \inf_{\phi \in X_N} \left\{ \|u_N - \phi\|_\omega + \|u - \phi\|_\omega \right\} \\ &\leqslant cN^{\frac{d-1}{2}} \inf_{\phi \in X_N} \|u - \phi\|_\omega \\ &\leqslant cN^{\frac{d-1}{2}} N^{1-s} \|u\|_{s,\omega}\,. \end{aligned}$$

The last inequality follows from an approximation result in [7]. ¶

Remark 1 : We note that the Tau method does not yield the best error estimate among the class of polynomials as the Galerkin and collocation methods do (*cf.* [3], [5]). However, the results in Lemma 1 and Theorem 1 are optimal since the only critical points in the proof are those of (2.10) and (2.11) which are not improvable.

3. STOKES EQUATIONS

We analyze in this section a spectral-Tau approximation for the Stokes problem. In the sequel, we shall use the calligraphic letter $\mathscr{A}$ to denote vector space A^d.

A usual spectral-Tau formulation for this problem is :

$$\text{find } (u_N, p_N) \in \mathscr{X}_N \times M_N \text{ such that :}$$

$$\begin{cases} \nu a_\omega(u_N, v) + (\nabla p_N, v)_\omega = (f, v)_\omega, & \forall v \in \mathscr{S}_{N-2} \\ \operatorname{div} u_N = 0 \end{cases} \tag{3.1}$$

where M_N is to be determined.

As for the continuous problem (1.1), the constant in the pressure space should be filtered out, so it is necessary for the discrete problem to choose a space M_N which does not contain any spurious mode whose discrete gradient vanishes. The compatibility problem between $\mathscr{X}_N$ and M_N is extensively investigated for finite elements methods (*cf.* e.g. [9]) as well as for the spectral methods (*cf.* [3], [5]). At first view, the most natural choice for M_N seems to be the space S_N. But unfortunately, the resulting system is not well posed because S_N contains spurious modes. It is found that the number of such modes in 2-D case is 8 and it seems in 3-D case the number will be increased to be proportional to N (*cf.* [5] for Galerkin and collocation case). The presence of spurious modes introduces a significant difficulty for the theoretical study of the system, it also increases considerably the numerical complexity since a large number of spurious modes should be filtered out.

We propose here a new spectral-Tau formulation for the Stokes problem which enjoys the following properties.

(i) no spurious mode is present ;

(ii) the resulting system can be easily solved by using an Uzawa type algorithm.

The scheme we propose is the following :

$$\text{find } (u_N, p_N) \in \mathscr{X}_N \times M_N \text{ such that :}$$

$$\begin{cases} -\nu a_\omega(u_N, v) + (\nabla p_N, v)_\omega = (f, v)_\omega, & \forall v \in \mathscr{S}_{N-2} \\ P_{N-2} \operatorname{div} u_N = 0 \end{cases} \tag{3.2}$$

where $M_N = \left\{ v \in S_{N-2} : \int_\Omega v \, dx = 0 \right\}$.

We observe first there is no spurious modes in M_N. In fact, it is easy to check that

$$\{ q \in M_N : (\nabla q, v)_\omega = 0, \quad \forall v \in \mathscr{S}_{N-2} \} = \Phi \quad \text{(the empty set)}.$$

The price we pay, compared to the usual formulation (3.1), is that the divergence of solution u_N of (3.2) is not identically zero. We will show late that this particularity does not alter the precision of the scheme.

In the rest of this section, we will only consider the Legendre case. Numerical experiences suggest that similar results should hold for the Chebychev case.

As well as in case of the Helmholtz equations, it is preferable to transform the discrete Stokes system to a symmetric one. Since $P_{N-2} X_N = S_{N-2}$, we can then rewrite (3.2) under the form :

$$\text{find } (u_N, p_N) \in \mathscr{X}_N \times M_N \text{ such that :}$$

$$(3.3) \qquad \begin{cases} \nu a_N(u_N, v) + b(v, p_N) = (f, v)_N, & \forall v \in \mathscr{X}_N \\ b(u_N, q) = 0, \quad \forall q \in M_N \end{cases}$$

where we have set

$$b(v, q) = (\nabla q, v) = -(\operatorname{div} v, q), \quad \forall (v, q) \in \mathscr{X}_N \times M_N .$$

The system (3.3) is now well suited to apply the saddle point theory developed by Brezzi (*cf.*[2]).

We can now prove our main result.

THEOREM 2 : *The system* (3.3) *admits a unique solution* (u_N, p_N). *Moreover, if the solution* (u, p) *of problem* (2.1) *belongs to* $H^s(\Omega) \times H^{s-1}(\Omega)$, *then :*

$$(3.4) \qquad \|u - u_N\| \leq c N^{d-s} (\|u\|_s + \|p\|_{s-1})$$

$$(3.5) \qquad \|p - p_N\| \leq c N^{2+d-s} (\|u\|_s + \|p\|_{s-1}) .$$

Proof: We derive from (2.5) that $\forall q \in M_N$, there exists $q_u \in X_N$ such that $\mathscr{P}_{N-2} q_u = \nabla q$. By using the same technique as in Lemma 1, one can prove :

$$\|q_u\| \leq c N^{\frac{d-1}{2}} \|\nabla q\| .$$

Therefore

$$\forall q \in M_N, \ \sup_{v \in \mathscr{X}_n} \frac{b(v, q)}{\|v\|} \geq \frac{(\nabla q, \nabla q)}{\|q_u\|}$$

$$\geq c N^{\frac{1-d}{2}} \frac{|\nabla q|^2}{\|\nabla q\|} .$$

Then by using the following inverse inequality (*cf.* [7]) :

$$(3.6) \qquad \|u\|_{t,\omega} \leq c N^{2(t-s)} \|u\|_{s,\omega}, \quad \forall u \in S_N, \quad t > s$$

we obtain

$$(3.7)\quad \forall q \in M_N \,, \quad \sup_{v \in \mathscr{X}_N} \frac{b(v,q)}{\|v\|} \geqslant cN^{\frac{-d-3}{2}} \, |\nabla q| \geqslant cN^{\frac{-d-3}{2}} \, |q| \,.$$

We derive also from Lemma 1 that

$$(3.8)\qquad \forall u \in \mathscr{X}_N : \sup_{v \in \mathscr{X}_N} \frac{a_N(u,v)}{\|v\|} \geqslant \frac{a_N(u, \mathscr{P}_{N-2}\, u)}{\|u\|} \geqslant \frac{cN^{1-d} \, \|u\|^2}{\|u\|} = cN^{1-d} \, \|u\|$$

(3.7) and (3.8) are the well known inf-sup conditions for the bilinear forms $a_N(\cdot\,,\cdot\,)$ and $b(\cdot\,,\cdot\,)$. It then follows from Corollary 4.1, Ch. 1 in [9] that the system (3.3) admits an unique solution.

In order to derive an error estimate, we need the following approximation result (*cf.* [7] and [13]) :

$$\inf_{\phi \in S_N} \|p - \phi\|_t \leqslant cN^{t-s} \, \|p\|_s \,, \quad \forall p \in H^s(\Omega) \quad (s > t)$$

$$\inf_{\phi \in \mathscr{V}_N} \|v - \phi\|_t \leqslant cN^{t-s} \, \|v\|_s \,, \quad \forall v \in \mathscr{V} \cap \mathscr{H}^s(\Omega) \quad (s > t)$$

where $\mathscr{V} = \{ f \in \mathscr{X} : \operatorname{div} f = 0 \}$ and $\mathscr{V}_N = \mathscr{V} \cap \mathscr{S}_N$.

Consequently, there exist $\bar{u}_N \in \mathscr{V}_N$ and $\bar{q}_N \in S_{N-2}$ such that :

$$(3.9)\qquad \|u - \bar{u}_N\| \leqslant cN^{1-s} \, \|u\|_s$$

$$(3.10)\qquad \|p - \bar{q}_N\| \leqslant cN^{2-s} \, \|p\|_{s-1} \,.$$

We now subtract (1.1) from (3.3) :

$$(3.11)\qquad a_N(u - u_N, v) + b(v, p - p_N) = 0 \,, \forall v \in \mathscr{X}_N \,.$$

Hence

$$(3.12)\quad a_N(\bar{u}_N - u_N, v) = a_N(\bar{u}_N - u, v) - b(v, p - p_N) \,, \quad \forall v \in \mathscr{X}_N \,.$$

We then replace v by $\bar{u}_N - u_N$ in (3.12) and we majorize the terms on the right hand side as follows :

Using Lemma 1 and (3.9) :

$$a_N(\bar{u}_N - u, \bar{u}_N - u_N) \leqslant \|\bar{u}_N - u_N\| \cdot \|\bar{u}_N - u\| \leqslant cN^{1-s} \, \|\bar{u}_N - u_N\| \,.$$

Since

$$b(\bar{u}_N - u_N, p_N) = 0 \quad \text{and} \quad P_{N-2} \operatorname{div} u_N = 0$$

we find

$$\begin{aligned} b(\bar{u}_N - u_N, p_N - p) = \\ &= -b(\bar{u}_N - u_N, p) \\ &= (\nabla p, \bar{u}_N - u_N) = (p, \operatorname{div}(u_N - \bar{u}_N)) \\ &\leqslant \inf_{\phi \in S_{N-2}} (p - \phi, \operatorname{div}(u_N - \bar{u}_N)) \leqslant cN^{1-s} \|p\|_{s-1} \|\bar{u}_N - u_N\| . \end{aligned}$$

By combining these inequalities into (3.12) and taking into account Lemma 1, we obtain :

$$\begin{aligned} c_1 N^{1-d} \|\bar{u}_N - u_N\|^2 &\leqslant a_N(\bar{u}_N - u_N, \bar{u}_N - u_N) \\ &\leqslant c_2 N^{1-s}(\|u\|_s + \|p\|_{s-1}) \|\bar{u}_N - u_N\| . \end{aligned}$$

Hence

$$\|\bar{u}_N - u_N\| \leqslant c_3 N^{d-s}(\|u\|_s + \|p\|_{s-1})$$

(3.4) then follows from this inequality and (3.9).

By using the inverse inequality (3.6), we also have :

$$\|u_N - u\|_2 \leqslant c_3 N^{2+d-s}(\|u\|_s + \|p\|_{s-1}) . \tag{3.13}$$

Finally, we deduce from (3.11), (3.13) and (3.9) that :

$$\begin{aligned} \|p_N - \bar{q}_N\| &\leqslant c \sup_{v \in \mathcal{S}_{N-2}} \frac{b(v, p_N - \bar{q}_N)}{|v|} \\ &\leqslant c \cdot \sup_{v \in \mathcal{S}_{N-2}} \frac{b(v, p_N - p) + b(v, p - \bar{q}_N)}{|v|} \\ &\leqslant c \cdot \sup_{v \in \mathcal{S}_{N-2}} \frac{a_N(u - u_N, v) + b(v, p - \bar{q}_N)}{|v|} \\ &\leqslant c\left\{ \|u - u_N\|_2 + \|p - \bar{q}_N\| \right\} \\ &\leqslant cN^{2+d-s}(\|u\|_s + \|p\|_{s-1}) . \end{aligned}$$

The proof is complete by combining this inequality and (3.10). ¶

4. NUMERICAL ALGORITHM AND RESULTS

4.1. Uzawa algorithm

We proved in the previous section that the scheme (3.2) converges exponentially to (1.1) provided that the solution of (1.1) is infinitely differentiable. However, the numerical calculation of (3.2) is still very difficult since u_N and the p_N are coupled by the condition « $P_{N-2} \operatorname{div} u_N = 0$ ». One way to overcome this difficulty is to use the so called influence matrix method (*cf.* [12]). It consists of solving a cascade of Helmholtz equations for the velocity as well as for the pressure subjected to an implicit boundary condition « $\operatorname{div} u_N = 0$ ». This method requires a large number of preliminary operations and memories for the construction and storage of the influence matrix. It is then preferable only in the 2-D case and when there is a large number of Stokes-like equations to be calculated. Another strategy is to use an iterative procedure, for example the Uzawa algorithm (*cf.* [1] and [14]), to separate the calculation of u from that of p. For our discrete system (3.2), the Uzawa algorithm takes the following form :

find $(u_N^n, p_N^n) \in \mathscr{X}_N \times M_N$ such that :

$$\nu a_N(u_N^n, v) = (f - \nabla p_N^n, v)\,, \quad \forall v \in \mathscr{S}_{N-2} \tag{4.1a}$$

$$p_N^{n+1} = p_N^n - \rho P_{N-2} \operatorname{div} u_N^n \tag{4.1b}$$

with p_N^0 arbitrary given in M_N.

At each iteration step, only a spectral-Tau system of the Poisson equation (4.1*a*) is to be solved. In the Chebychev case, this can be done by using the efficient diagonalization method.

We establish in the following a convergence result for the Legendre case.

LEMMA 2 : *The solution* (u_N^n, p_N^n) *of scheme* (4.1) *converge to that of* (3.2) *under the condition* $0 < \rho < \nu$. *More precisely, we have :*

$$|p_N^{m+1} - p_N|^2 + 2\,\rho c(\nu - \rho)\,N^{1-d} \sum_{n=0}^{m} \|u_N^n - u_N\|^2 \leq |p_N^0 - p_N|^2\,.$$

Proof: We set

$$q^n = p_N - p_N^n \quad \text{and} \quad e^n = u_N - u_N^n\,.$$

We derive from (3.2) and (4.1) that :

$$\nu a_N(e^n, v) = (-\nabla q^n, v)_N\,, \quad \forall v \in \mathscr{X}_N \tag{4.2a}$$

$$q^{n+1} = q^n - \rho P_{N-2} \operatorname{div} e^n\,. \tag{4.2b}$$

We replace v by e^n in (4.2a) and integrate (4.2b) multiplied by $2\, q^n$, by taking into account (2.12), we get :

$$\nu \|\mathscr{K}_{N-2}\, e^n\|^2 \leqslant (-\nabla q^n, \mathscr{P}_{N-2}\, e^n) \tag{4.3a}$$

$$|q^{n+1}|^2 - |q^n|^2 = |q^{n+1} - q^n|^2 - 2\,\rho\,(P_{N-2} \operatorname{div} e^n, q^n)\,. \tag{4.3b}$$

It follows from (4.2b) that :

$$|q^{n+1} - q^n|^2 = \rho^2 |P_{N-2} \operatorname{div} e^n|^2\,. \tag{4.4}$$

We deduce from integration by parts :

$$(P_{N-2} \operatorname{div} e^n, q^n) = (\operatorname{div} e^n, q^n) = -\ (e^n, \nabla q^n) = -\ (\mathscr{P}_{N-2}\, e^n, \nabla q^n)\,.$$

We now add (4.3b) with (4.3a) multiplied by $2\,\rho$, by using (4.4) and the last equality, we obtain :

$$|q^{n+1}|^2 - |q^n|^2 + 2\,\rho\nu\, \|\mathscr{K}_{N-2}\, e^n\|^2 \leqslant \rho^2 |P_{N-2} \operatorname{div} e^n|^2 \tag{4.5}$$

one can readily check that :

$$|P_{N-2} \operatorname{div} e^n|^2 \leqslant 2\, \|\mathscr{K}_{N-2}\, e^n\|^2\,.$$

By summing (4.5) for $n = 0, \ldots, m$, we arrive to :

$$|q_N^{m+1}|^2 + 2\,\rho\,(\nu - \rho) \sum_{n=0}^{m} \|\mathscr{K}_{N-2}\, e^n\|^2 \leqslant |q^0|^2\,.$$

The proof is complete by using (2.13*bis*). ¶

Remark 1 : For the usual spectral-Tau approximation (3.1), a similar Uzawa algorithm will lead to a very restrictive convergence condition, namely, $\rho < \nu/N$. As a consequence, the resulting scheme converges very slowly, especially when N is large.

4.2. Numerical results

The schemes (4.1) in its Chebychev form are used to solve 2-D steady Stokes equations. In order to test our algorithm, we calculated the problem (1.1) with following exact solution :

$$\begin{cases} \bar{u} = (u, v) = (\cos \pi x \,.\, \cos \pi y, \sin \pi x \,.\, \sin \pi y\,) \\ p = \cos \pi x \,.\, \cos \pi y,\ \nu = 1\,. \end{cases} \tag{4.6}$$

One of the nice properties of the scheme (4.1) is that the convergence speed does not depend on the number of discretization modes as we proved in Lemma 2. We observe in practice that the convergence speeds for different numbers « N » of discretization modes are almost identical. This property is particularly interesting when N is large.

The convergence speed of the scheme (4.1) varies drastically with the value of parameter ρ. In all cases, the condition « $0 < \rho < \nu$ » ensures stability as we proved in Lemma 2. However, the optimal value of ρ usually does not lie in this interval. In fact, for each concrete problem, there exists a constant $\rho_{\max} \in [\nu, 2\nu)$ such that instability phenomena occurs for all $\rho > \rho_{\max}$ and the fastest convergence is achieved for a ρ near $\rho_{\max}$ but not equal to it. Our experiences suggest that the optimal value usual lies in $(1.2\,\nu, 1.6\,\nu)$. We traced in figure 1 the number of iterations, required to get a relative l^2 residue less than « $10\,e-6$ », in function of ρ for our test problem. We also implemented the Uzawa algorithm using a collocation-Chebychev formulation. It is carried out that its convergence speed is much slower then our tau-Chebychev formulation. The high convergence speed of Uzawa algorithm applied to the new tau-Chebychev formulation may be explained by the fact that the last two rows and columns of high frequency modes are filtered out.

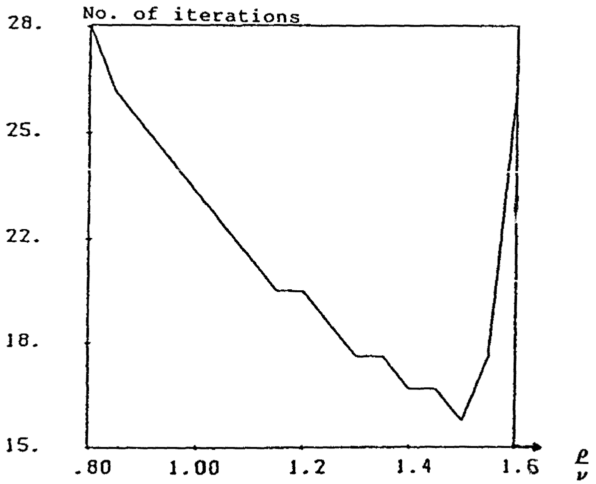

Fig. 1. — Convergence rate of Uzawa algorithm.

The relative l^2 errors at the collocation points with different N are presented in table 1. We observe that the spectral convergence is achieved for both velocity and pressure.

TABLE 1

Relative l^2 errors for the test problem

$\varepsilon(f) = \max_{x \in D} |f_{ex}(x) - f_{ap}(x)|$, where D is the set of collocation points in Ω ;

N, M : number of modes in each direction.

$N = M =$	8	10	16	20
$\varepsilon(u) =$	3.416 *E*-4	1.836 *E*-7	2.686 *E*-11	4.085 *E*-14
$\varepsilon(p) =$	5.398 *E*-3	3.844 *E*-6	7.294 *E*-10	5.242 *E*-13

TABLE 2

Driven cavity

$f \max = \max_{x \in D} |\psi|$, where D is the set of 65×65 equidistant points in $\bar{\Omega}$.

PV ($= f \max$) : value in the center of principal vortex ;
LV (resp. RV) : value in the center of left (resp. right) secondary vortex.

$N = M$	R	PV ($= f$ max)	LV	RV
16	100	0.08369 (0.63, 0.77)	– 2.692 *E*-8 (0.01, 0.13)	– 5.530 *E*-6 (0.97, 0.06)
24	100	0.08366 (0.63, 0.77)	– 1.266 *E*-6 (0.05, 0.05)	– 4.907 *E*-6 (0.97, 0.06)
16	400	0.08569 (0.59, 0.64)	– 6.881 *E*-6 (0.08, 0.05)	– 2.562 *E*-4 (0.91, 0.13)
24	400	0.08573 (0.59, 0.63)	– 2.591 *E*-6 (0.05, 0.06)	– 2.563 *E*-4 (0.92, 0.13)
24	1000	0.08691 (0.53, 0.55)	– 3.249 *E*-4 (0.078, 0.093)	– 1.564 *E*-3 (0.86, 0.19)

For the steady Navier-Stokes equations, we can adapt a similar algorithm :

$$(4.7) \quad \begin{cases} \left(\dfrac{u^n - u^{n-1}}{k}\right)_\omega - \nu(\Delta u^n, v)_\omega = \\ = (f - u^{n-1} \cdot \nabla u^{n-1} - \nabla p^n, v)_\omega, \quad \forall v \in \mathcal{S}_{N-2} \\ u^n = g|_{\partial\Omega} \\ p^{n+1} = p^n - \rho \cdot P_{N-2} \operatorname{div} u^n . \end{cases}$$

The scheme (4.7) is nothing more than the artificial compressibility method if we replace ρ by kc^2 (*cf.* for instance [14]). The stability of this scheme can be determined numerically.

We applied this scheme to the classical driven cavity problem. The boundary condition for the velocity is modified as shown in figure 2 such that the solution will meet the regularity requirement of the spectral methods.

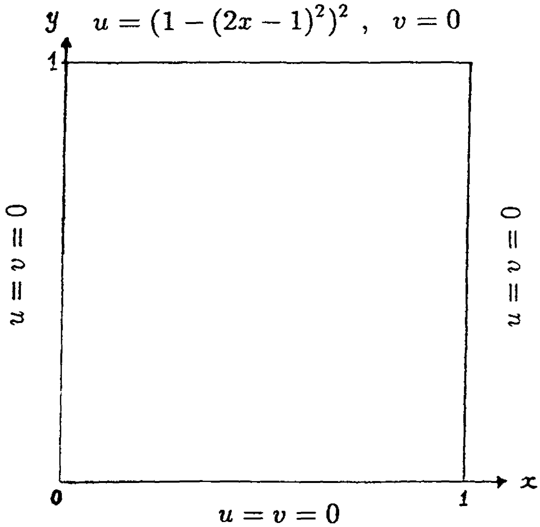

Fig. 2. — Driven cavity.

For Reynold number up to 400, we compared our result with that of U. Ehrenstein (*cf.* [8]) who used an influence matrix method applied to the collocation Chebychev approximation of the Navier-Stokes equations under the streamlinevortex formulation. The results are also compared with that of L. B. Zhang (*cf.* [15]) who used a second order multi-grids method on 128×128 points for Reynold number up to 1 000. The differences between these results obtained by totally different methods are within 0.5 %.

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