

SOME LOCAL-GLOBAL NON-VANISHING RESULTS OF THETA LIFTS FOR SYMPLECTIC-ORTHOGONAL DUAL PAIRS

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ABSTRACT. Following the approach of B. Roberts, we characterize the non-vanishing of the global theta lift for symplectic-orthogonal dual pairs in terms of its local counterpart. In particular, we replace the temperedness assumption present in Robert's work by a certain weaker assumption, and apply our results to small rank similitude groups. Among our applications is a certain instance of Langlands functorial transfer of a (non-generic) cuspidal automorphic representation of $\mathrm{GSp}(4)$ to $\mathrm{GL}(4)$.

1. INTRODUCTION

In this paper, we consider both local and global theta lifting for the symplectic-orthogonal dual pair $(\mathrm{Sp}(2n), \mathrm{O}(V_r))$, where V_r is a symmetric space of an even dimension m and the Witt index r , with emphasis on the non-vanishing problem of global theta lifts for this pair. In particular by following the approach by B. Roberts [R4], we characterize the global non-vanishing in terms of its local counterpart. Our main theorem is

Theorem 1.1. *Let π (resp. σ) be a cuspidal automorphic representation of $\mathrm{Sp}(2n, \mathbb{A})$ (resp. $\mathrm{O}(V_r, \mathbb{A})$) realized in a space V_π (resp. V_σ) of cusp forms such that at each place v , π_v (resp. σ_v) is bounded by some $e_v < 1$. Assume χ is the quadratic character associated with V_r . Then*

- (1) *(From orthogonal to symplectic.) Suppose $n = \frac{1}{2} \dim V_r$. If σ_v has a non-zero theta lift to $\mathrm{Sp}(2n, F_v)$ at all the places v and the (incomplete) standard L -function $L^S(s, \sigma)$ does not vanish at $s = 1$ (a pole is allowed), then the global theta lift $\Theta_n(V_\sigma)$ to $\mathrm{Sp}(2n, \mathbb{A})$ does not vanish, and further if $L^S(s, \sigma)$ has a pole at $s = 1$, then the global theta lift $\Theta_{n-1}(V_\sigma)$ to $\mathrm{Sp}(2n - 2, \mathbb{A})$ does not vanish, provided $n \geq 1$.*
- (2) *(From symplectic to orthogonal.) Suppose $\frac{1}{2} \dim V_r = n + 1$. If π_v has a non-zero theta lift to $\mathrm{O}(V_r, F_v)$ at all the places v and the (incomplete) standard L -function $L^S(s, \pi)$ does not vanish at $s = 1$ (a pole is allowed), then the global theta lift $\Theta_{V_r}(V_\pi)$ to $\mathrm{O}(V_r, \mathbb{A})$ does not vanish, and further if $L^S(s, \pi, \chi)$ has a pole at $s = 1$, then the global theta lift $\Theta_{V_{r-1}}(V_\pi)$ to $\mathrm{O}(V_{r-1}, \mathbb{A})$ does not vanish, provided $r \geq 1$.*

Here we use the notion of ‘‘boundedness’’ of each local representation. To introduce this notion, let us recall that for a classical reductive group G over a (not necessarily non-archimedean) local field, each irreducible admissible representation π is the Langlands quotient of the standard module $\delta_1 \times \cdots \times \delta_t \rtimes \tau := \mathrm{Ind}_P^G \delta_1 \otimes \cdots \otimes \delta_t \otimes \tau$, where each δ_i is an essentially tempered representation of some $\mathrm{GL}(n_i)$ and τ is a tempered representation of a lower rank group of the same type as G . Then for each i , there exists $e(\delta_i) > 0$ so that $\delta_i \otimes | \cdot |^{-e(\delta_i)}$ is tempered, and $e(\delta_1) > \cdots > e(\delta_t) > 0$. Now we make the following definition.

Definition 1.2. *Let G be a classical reductive group over a (not necessarily non-archimedean) local field F . If π is an irreducible admissible representation which is the Langlands quotient of the*

Date: April 19, 2010.

Key words and phrases. automorphic representation, theta correspondence, theta lifting.

standard module $\delta_1 \times \cdots \times \delta_t \rtimes \tau$, then we say that π is bounded by e if $e(\delta_1) \leq e$. If $t = 0$, i.e. π is tempered, we say that π is bounded by e for all $e \geq 0$.

Let us mention that in the above theorem if σ is tempered, then part (1) of the theorem is essentially the main theorem of [R4] by Roberts, although he also assumes some other technical assumptions for the archimedean place. (See also [Tk] for those issues.) Indeed, our proof is a modification of the one by Roberts. What forced him to impose the temperedness assumption on his theorem is that he needed his own local result in [R2] which required the representation be tempered. The main part of this paper is to replace his temperedness assumption in [R2, R4] with the boundedness assumption as above. The key technical point is that the ‘‘bound’’ of the local representation as defined above is closely related to the size of the ‘‘local exponents’’ of the representation. We will show that Roberts’ arguments in [R4] also work under this weaker assumption. We also consider not only the lift from the orthogonal group to the symplectic one, but also from the symplectic group to the orthogonal group as in the theorem. Indeed, the method of Roberts works for the ‘‘symplectic-to-orthogonal’’ case.

Next we apply this theorem, as Roberts did, to global theta lifting for groups of similitudes of small ranks. The first case we consider is the situation of [R5], in which he considered the theta lift from (any form of) $\mathrm{GO}(V)$ with $\dim V = 4$ to $\mathrm{GSp}(4)$. In [R5], he needed the temperedness assumption because the same assumption is present in [R4]. We can replace his temperedness assumption by our boundedness assumption, but for the group $\mathrm{GO}(V)$ with $\dim V = 4$ one can check that, except certain degenerate cases, the boundedness assumption is always satisfied. To state our theorem, let us recall that each cuspidal automorphic representation of $\mathrm{GO}(V)$ is of the form $\sigma = (\tau, \delta)$ where τ is a cuspidal representation of $\mathrm{GSO}(V)$ and δ is what we call an ‘‘extension index’’. Then if the discriminant $d = \mathrm{disc} V = 1$, then $\mathrm{GSO}(V, \mathbb{A})$ is of the form $(D^\times(\mathbb{A}) \times D^\times(\mathbb{A}))/\mathbb{A}^\times$ where D is a (possibly split) quaternion algebra and so τ is identified with a cuspidal representation $\pi_1 \otimes \pi_2$ of $D^\times(\mathbb{A}) \times D^\times(\mathbb{A})$ such that π_1 and π_2 have a same central character. In this case, we write $\tau = \tau(\pi_1, \pi_2)$. If $d \neq 1$, then $\mathrm{GSO}(V, \mathbb{A})$ is closely related with an inner form $\mathrm{GSO}(V_{D,E}, \mathbb{A})$ of $\mathrm{GL}(2, \mathbb{A}_E)$ where $E = F(\sqrt{d})$, and τ is naturally identified with a cuspidal representation π of this inner form whose central character is of the form $\omega \circ N_F^E$, where N_F^E is the norm map from \mathbb{A}_E^\times to \mathbb{A} . In this case, we write $\tau = \tau(\pi, \omega)$. (See section 6 for the detail.) Then we prove

Theorem 1.3. *Let $\sigma = (\tau, \delta)$ be an infinite dimensional cuspidal unitary automorphic representation of $\mathrm{GO}(V, \mathbb{A})$. Then*

- (1) *Assume that τ is NOT of the form $\tau = \tau(\pi_1, \pi_2)$ with one of π_i finite dimensional. Then the global theta lift of σ to $\mathrm{GSp}(4)$ does not vanish if and only if each local constituent σ_v has a non-zero theta lift to $\mathrm{GSp}(4)$.*
- (2) *Assume $\tau = \tau(\pi_1, \pi_2)$ with one of π_i , say π_2 , finite dimensional, and so we can write $\pi_1 = \chi \circ N$ where χ is a Hecke character on \mathbb{A}_F^\times and N is the norm map on $D^\times(\mathbb{A})$. Then the global theta lift of σ to $\mathrm{GSp}(4)$ does not vanish if each local constituent σ_v has a non-zero theta lift to $\mathrm{GSp}(4)$ and the (incomplete) L -function $L^S(s, \pi_1 \otimes \chi)$ does not vanish at $s = \frac{1}{2}$.*

Let us note that part (1) of the above theorem has been already proven by Gan and Ichino in their recent preprint [GI] by an entirely different method. However in this paper we give a proof of this theorem in our method and demonstrate that the project of Roberts in his series of papers [R2, R3, R4, R5] can be essentially completed.

The second case we consider is global theta lifting from $\mathrm{GSp}(4)$ to $\mathrm{GO}(V_D)$ where $V_D = D \oplus \mathbb{H}$ with D possibly split, which implies a certain instance of Langlands functorial lift. Namely we will show

Theorem 1.4. *Let π be a non-generic irreducible cuspidal automorphic representation of $\mathrm{GSp}(4, \mathbb{A})$ which satisfies the following assumptions.*

- (1) The standard degree 5 L -function $L^S(s, \pi)$ does not vanish at $s = 1$.
- (2) There is a (non-complex) place v_0 at which π_{v_0} has a non-zero theta lit to $\mathrm{GO}(V_D)$ for both split and non-split D .
- (3) At each place v , π_v is bounded by $e_v < 1$.

Then there is an automorphic representation Π on $\mathrm{GL}(4, \mathbb{A})$ which is the strong functorial lift of π corresponding to the map

$$\mathrm{GSp}(4) = \mathrm{GSp}(4, \mathbb{C}) \hookrightarrow \mathrm{GL}(4, \mathbb{C}) = \widehat{\mathrm{GL}}(4).$$

Here we have to impose unfortunate local assumptions, especially assumption (3) in the theorem. The issue of how much those assumptions are needed will be discussed in some detail after the proof of this theorem.

As our last application, we prove the following facts, which are well-known if the base field is a totally real number field.

Theorem 1.5. *Let π be a cuspidal representation of $\mathrm{GSp}(4, \mathbb{A})$ over a (not necessarily totally real) number field F . Assume the incomplete standard degree 5 L -function $L^S(s, \pi)$ does not vanish at $s = 1$ (a pole is allowed). Then if π_v is generic for each v , then π is globally generic.*

Corollary 1.6. *The multiplicity one theorem holds for the generic representations for $\mathrm{GSp}(4)$ over a (not necessarily totally real) number field F .*

As we mentioned, those are essentially well-known and indeed the first one is Theorem 8.1 of [KRS] and the second one is the main theorem of [JS]. However in both of their works, they assumed that the base field F is a totally real number field. This is because this assumption is present in [KRS]. However our method implies the same result as [KRS, Theorem 8.1] without the restriction on the base field.

The main structure of this paper is as follows. In Section 2, we set up our notations. In Section 3, we define the notion of the bounds of exponents of irreducible admissible representations and prove certain facts related to this notion, which are necessary for our main purposes. In Section 4, we prove the key fact on the local theta correspondence, which significantly improves the main theorem of [R2]. Then in Section 5, we prove Theorem 1.1, and in Section 6 we prove all the theorems related to $\mathrm{GSp}(4)$, namely Theorems 1.3, 1.4 and 1.5, and Corollary 1.6.

Acknowledgements

The author would like to thank Wee Teck Gan for letting him know about his work with A. Ichino [GI]. Thanks are also due to Freydoon Shahidi for answering several questions about generic representations, and Annegret Paul for her help with the archimedean theta correspondence. Finally, he would like to thank Brooks Roberts for showing his interests in this work. Indeed a part of the remark after the proof of Theorem 1.4 is based on a conversation the author had with B. Roberts.

2. NOTATIONS AND PRELIMINARIES

In this paper, F is a local or global field of char $F = 0$, and if F is a global field we denote the ring of adèles by \mathbb{A} . If E is a quadratic extension of F , then we denote by N_F^E (or simply by N) the norm map, and by $\chi_{E/F}$ the quadratic character obtained by local or global class field theory.

We work with smooth representations instead of K -finite ones. Namely if G is a reductive group over a global field F , then by a (cuspidal) automorphic form we mean a smooth (cuspidal) automorphic form on $G(\mathbb{A}_F)$ in the sense of [Co, Definition 2.3].

For a reductive group G over a local field, we denote by $\text{Irr}(G)$ the class of (equivalence classes of) irreducible admissible representations of $G(F)$. For $\pi \in \text{Irr}(G)$, we denote the space of representation by V_π , though we occasionally identify the space of π with π itself when there is no danger of confusion. Also we denote the contragredient by π^\vee . Now assume G is a classical reductive group and P is a standard parabolic subgroup whose Levi is isomorphic to $\text{GL}(n_1) \times \cdots \times \text{GL}(n_t) \times G'$, where G' is a lower rank group of the same type as G . Let δ_i be an admissible representation of $\text{GL}(n_i)$ and τ an admissible representation of G' . Following Tadic, we write

$$\delta_1 \times \cdots \times \delta_t \rtimes \tau := \text{Ind}_P^G \delta_1 \otimes \cdots \otimes \delta_t \otimes \tau \quad (\text{normalized induction}).$$

Here and elsewhere, induction is always normalized. If δ_i is essentially tempered, we denote by $e(\delta_i)$ the real number so that $\delta_i \otimes |\cdot|^{-e(\delta_i)}$ is tempered. If all the δ_i are essentially tempered with

$$e(\delta_1) > \cdots > e(\delta_t) > 0,$$

and τ is tempered, we call $\delta_1 \times \cdots \times \delta_t \rtimes \tau$ a standard module.

Now for $\pi \in \text{Irr}(G)$ and a parabolic subgroup P , we denote by $R_P(\pi)$ the normalized Jacquet module of π along P , and by $\overline{R}_P(\pi)$ the normalized Jacquet module of π along the opposite parabolic \overline{P} of P . It is well-known that

$$\overline{R}_P(\pi) = R_P(\pi^\vee)^\vee.$$

Assume $G = \text{GL}(n)$. For $\pi \in \text{Irr}(G)$, we denote the central character of π by ω_π . Also we denote by $P_{n_1, \dots, n_t}^{\text{GL}}$ the standard parabolic of $\text{GL}(n)$ whose Levi is $\text{GL}(n_1) \times \cdots \times \text{GL}(n_t)$.

Let V be an even dimensional symmetric space defined over a field F of even dimension m equipped with a symmetric bilinear form. If V is defined over a local or global field F of char $F = 0$, then we denote by $\text{disc } V \in F^\times / F^{\times 2}$ the discriminant of V when V is viewed as a quadratic form. We let $\chi_V : F^\times \rightarrow \{\pm 1\}$ be the quadratic character of V , namely $\chi_V(a) = (a, (-1)^{\frac{m(m-1)}{2}} \text{disc } V)_F$ for $a \in F^\times$, where $(\cdot, \cdot)_F$ is the Hilbert symbol of F . Sometimes we omit V and simply write χ .

Also assume the Witt index of V is r and so $V = V_a \oplus \mathbb{H}^r$ where V_a is anisotropic and \mathbb{H} is the hyperbolic plane. If $\dim V = m$, we write $\dim V_a = m_a$. Now we fix a Witt decomposition

$$V_r = V'_r \oplus V_a \oplus V''_r$$

with bases $\{v_1, \dots, v_r\}$ for V'_r and $\{v'_1, \dots, v'_r\}$ for V''_r satisfying $(v_i, v_j) = (v'_i, v'_j) = 0$ and $(v_i, v'_j) = \delta_{ij}$. Then we denote by P_k (or sometimes Q_k) the parabolic subgroup of $\text{O}(V_r)$ that fixes $\{v_1, \dots, v_k\}$, which is a standard maximal parabolic of $\text{O}(V_k)$. Also if V_r is split, we sometimes write $\text{O}(V_r) = \text{O}(r, r)$. Similarly for W a symplectic space of rank n , we fix a polarization

$$W = W' \oplus W''$$

with fixed symplectic bases $\{e_1, \dots, e_n\}$ for W' and $\{e'_1, \dots, e'_n\}$ for W'' . Then we denote by P_k the parabolic subgroup of $\text{Sp}(W) = \text{Sp}(2n)$ that fixes $\{e_1, \dots, e_k\}$, which is a standard maximal parabolic of $\text{Sp}(2n)$.

We denote the (local or global) Weil representation for $\text{O}(V) \times \text{Sp}(2n)$ by $\omega_{V,n}$ or simply by ω when V and n are clear from the context. If F is an archimedean local field, then the Weil representation is a smooth representation $\omega_{V,n}$ of the group $\text{O}(V) \times \text{Sp}(2n)$ of moderate growth in the sense of [C]. We say that $\sigma \in \text{Irr}(\text{O}(V))$ and $\pi \in \text{Irr}(\text{Sp}(2n))$ correspond, or σ corresponds to π if there is a non-zero homomorphism of Harish-Chandra modules from the underlining Harish-Chandra module of $\omega_{V,n}$ to the underlining Harish-Chandra module of $\sigma \otimes \pi$. It is known that for each non-zero π , if a

non-zero σ corresponds to it, then σ is unique up to infinitesimal equivalence, and hence the canonical completion of the underlining Harish-Chandra module of σ is unique. We denote this canonical completion by $\theta_V(\pi)$. Similarly, we define $\theta_n(\sigma)$. Next assume F is non-archimedean. We say that $\sigma \in \text{Irr}(\text{O}(V))$ and $\pi \in \text{Irr}(\text{Sp}(2n))$ correspond, or σ corresponds to π if there is a non-zero $\text{O}(V) \times \text{Sp}(2n)$ homomorphism from $\omega_{V,n}$ to $\sigma \otimes \pi$, i.e. $\text{Hom}_{\text{O}(V) \times \text{Sp}(2n)}(\omega_{V,n}, \sigma \otimes \pi) \neq 0$. For an irreducible cuspidal automorphic representation σ of $\text{O}(V, \mathbb{A})$, we denote by $\Theta_n(V_\sigma)$ or $\Theta_n(\sigma)$ (when V_σ is clear from the context) the space of the global theta lifts of σ to $\text{Sp}(2n, \mathbb{A})$, i.e. the space generated by the forms of the form $\theta(f, \phi)$ for $f \in V_\sigma$ and $\phi \in \mathcal{S}(V(\mathbb{A}_F)^n)$ which are define by

$$\theta(f; \varphi)(g) = \int_{\text{O}(V, F) \backslash \text{O}(V, \mathbb{A}_F)} \left(\sum_{x \in V(F)^n} \omega(h, g) \varphi(x) \right) f(h) dh$$

for each $g \in \text{Sp}(2n, \mathbb{A})$. Similarly for a cuspidal automorphic representation π of $\text{Sp}(2n, \mathbb{A})$, we denote by $\Theta_V(V_\pi)$ or $\Theta_V(\pi)$ the space of the global theta lifts of π to $\text{O}(V, \mathbb{A})$.

3. BOUNDS OF EXPONENTS

In this section we prove certain facts on exponents of admissible representations, which are necessary for proving our theorems on theta lifting. First recall the following definition we made in Introduction. Namely if $\pi \in \text{Irr}(G)$ with G a classical group is the Langlands quotient of the standard module $\delta_1 \times \cdots \times \delta_t \rtimes \tau$, then we say that π is bounded by $e \in \mathbb{R}$ if $e \geq e(\delta_1)$. If $t = 0$, i.e. π is tempered, we say that π is bounded by all $e \geq 0$. Clearly if π is bounded by some e , then it is bounded by all $e' \geq e$. Note that although this notion applies to archimedean F , throughout this section, we assume that our field F is non-archimedean.

This notion of boundedness can be shown to be closely related to the notion of exponents of π . So let us recall the notion of exponent of a representation π . Let P be a parabolic subgroup of G whose Levi is M and A the split maximal torus whose centralizer is M . Then an exponent ω of π along P is the restriction to A of the central character of a constituent of the (normalized) Jacquet module $\overline{R}_P(\pi)$. For example, if $G = \text{Sp}(2n)$ and $P = P_k$ is the standard maximal parabolic whose Levi is $\text{GL}(k) \times \text{Sp}(2n - 2k)$, then A is the center of $\text{GL}(k)$, and so if $\pi_1 \otimes \pi_2 \in \text{Irr}(\text{GL}(k) \times \text{Sp}(2n - 2k))$ is a non-zero constituent of $\overline{R}_{P_k}(\pi)$, then the central character ω_{π_1} of π_1 is an exponent along P_k . (Note that occasionally in the literature, an exponent of π is defined as the real number s when one writes $|\omega| = |\cdot|^s$, i.e. literally the ‘‘exponent’’ of ω . But in this paper, we simply call the character ω an exponent. We essentially follow [S].) Now in this paper, we only consider an exponent along a maximal parabolic subgroup, and hence by an exponent we always mean an exponent along a maximal parabolic. Then let us make the following definition.

Definition 3.1. *Let π be an admissible representation of $\text{Sp}(2n)$ (resp. of $\text{O}(V_\tau)$). We say that the exponents of π are bounded by $e \geq 0$, if for all exponents ω along the standard maximal parabolic subgroup P_k for all $1 \leq k \leq n$, we have*

$$|\omega(a)| \leq |a|^{ke}$$

for $a \in F^\times$ with $|a| > 1$. Also if $\overline{R}_{P_k}(\pi) = 0$ for all P_k , i.e. π is supercuspidal, we say that the exponents are bounded by any $e \geq 0$.

Let us note that for each admissible representation π of finite length, there are only finitely many exponents along each parabolic, and hence for each such π the exponents are bounded by some non-negative number. Also clearly if the exponents of π are bounded by some e , then they are also bounded by all $e' \geq e$.

For tempered representations, the exponents are bounded by 0. Namely,

Proposition 3.2. (a) *Let π be a non-supercuspidal admissible tempered representation of $\mathrm{Sp}(2n)$ (resp. $\mathrm{O}(V_r)$) over F which is of finite length. Then for all standard maximal parabolic P_k whose Levi is $\mathrm{GL}(k) \times \mathrm{Sp}(2n - 2k)$ (resp. $\mathrm{GL}(k) \times \mathrm{O}(V_{r-k})$) and every exponent ω along P_k , we have*

$$|\omega(a)| \leq 1$$

for $a \in F^\times$ with $|a| > 1$. In particular, the exponents of π are bounded by 0.

(b) *Let σ be a non-supercuspidal admissible representation of $\mathrm{GL}(n)$ over F which is essentially tempered and of finite length. Then for all standard maximal parabolic $P_{k,n-k}^{\mathrm{GL}}$ whose Levi is $\mathrm{GL}(k) \times \mathrm{GL}(n - k)$, let $\sigma_1 \otimes \sigma_2 \in \mathrm{Irr}(\mathrm{GL}(k) \times \mathrm{GL}(n - k))$ be a non-zero constituent of $\overline{R}_{P_{k,n-k}^{\mathrm{GL}}}$. Then*

$$\left| \frac{\omega_{\sigma_1}(a_1)}{\omega_{\sigma_2}(a_2)} \right| \leq \left| \frac{a_1^k}{a_2^{n-k}} \right|^{e(\sigma)}$$

for $a_1, a_2 \in F^\times$ with $\left| \frac{a_1}{a_2} \right| > 1$, where recall that $e(\sigma)$ is the real number such that $\sigma \otimes |\cdot|^{-e(\sigma)}$ is tempered.

Proof. This easily follows from [S, Corollary 2.6]. \square

Note that by combining this proposition with our convention that for a supercuspidal representation the exponents are bounded by any $e \geq 0$, we can simply say that for a tempered representation the exponents are bounded by any $e \geq 0$.

Now the following gives the relation between the bound of π and its exponents, and this is the key proposition for our computation of local theta correspondences.

Proposition 3.3. *Let π be an irreducible admissible representation of $\mathrm{Sp}(2n)$ (resp. $\mathrm{O}(V_r)$) which is a non-zero constituent of the standard module*

$$\delta_1 \times \cdots \times \delta_t \rtimes \tau.$$

Then the exponents of π are bounded by $e(\delta_1)$. In particular, if π is the Langlands quotient and so is bounded by $e(\delta_1)$, then the exponents of π are bounded by $e(\delta_1)$.

To prove the proposition, we need the following lemma.

Lemma 3.4. *Let π be an irreducible admissible representation of $\mathrm{Sp}(2n)$ (resp. $\mathrm{O}(V_r)$) which is a non-zero constituent of*

$$\sigma \rtimes \tau,$$

where σ is an essentially tempered representation of $\mathrm{GL}(l)$ with $e(\sigma) > 0$, and τ is an admissible representation of $\mathrm{Sp}(2n - 2l)$ (resp. $\mathrm{O}(V_{r-l})$) of finite length in which the exponents of every constituent are bounded by $e(\sigma)$. Then the exponents of π are bounded by $e(\sigma)$.

Proof. Let us first treat the case for the symplectic group $\mathrm{Sp}(2n)$. Let P_k be the standard maximal parabolic with the Levi $\mathrm{GL}(k) \times \mathrm{Sp}(2n - 2k)$. Recall $\overline{R}_{P_k}(\pi) = R_{P_k}(\pi^\vee)^\vee$, and let $\pi_1 \otimes \pi_2$ be a constituent of $\overline{R}_{P_k}(\pi)$. We first compute the Jacquet module of

$$\mathrm{Ind}_{P_l}^{\mathrm{Sp}(2n)}(\sigma \otimes \tau)^\vee = \mathrm{Ind}_{P_l}^{\mathrm{Sp}(2n)}(\sigma^\vee \otimes \tau^\vee)$$

along P_k . For this, let $l = l_1 + l_2 + l_3$ be a partition of l and $n - l = m_1 + m_2$ a partition of $n - l$ such that

$$l_3 + l_1 + m_1 = k,$$

where we allow some of l_i and m_i to be zero. By [Ta, Theorem 5.4] each constituent of the Jacquet module of $\text{Ind}_{P_i}^{\text{Sp}(2n)}(\sigma^\vee \otimes \tau^\vee)$ along P_k is a constituent of a representation of the form

$$\text{Ind}_{P_{l_3, l_1, m_1}}^{\text{GL}(k)}(\sigma_3^\vee \otimes \sigma_1 \otimes \tau_1) \otimes \text{Ind}_{P_{l_2}'}^{\text{Sp}(2n-2k)}(\sigma_2 \otimes \tau_2),$$

where $\sigma_1 \otimes \sigma_2 \otimes \sigma_3$ is a constituent of the Jacquet module of σ^\vee along the standard parabolic $P_{l_1, l_2, l_3}^{\text{GL}}$ whose Levi is $\text{GL}(l_1) \times \text{GL}(l_2) \times \text{GL}(l_3)$, and $\tau_1 \otimes \tau_2$ is a constituent of the Jacquet module of τ^\vee along the standard parabolic whose Levi is $\text{GL}(m_1) \times \text{Sp}(2m_2)$. Hence the central character ω_{π_1} of π_1 is

$$\omega_{\pi_1} = \omega_{\sigma_3} \omega_{\sigma_1}^{-1} \omega_{\tau_1}^{-1}.$$

Since σ is essentially tempered and $\sigma_1^\vee \otimes \sigma_2^\vee \otimes \sigma_3^\vee$ is a constituent of $R_{P_{l_1, l_2, l_3}^{\text{GL}}}(\sigma^\vee)^\vee = \overline{R}_{P_{l_1, l_2, l_3}^{\text{GL}}}(\sigma)$, by Proposition 3.2, for $|a| > 1$ we have

$$|\omega_{\sigma_1}^{-1}(a)| \leq |a|^{l_1 e(\sigma)}$$

and

$$|\omega_{\sigma_3}(a)| \leq |a|^{-l_3 e(\sigma)}.$$

Also since the exponents of τ are bounded by $e(\sigma)$ by our assumption and $\tau_1^\vee \otimes \tau_2^\vee$ is a constituent of $R_{P_{m_1}}(\tau^\vee)^\vee = \overline{R}_{P_{m_1}}(\tau)$, we have

$$|\omega_{\tau_1}^{-1}(a)| < |a|^{m_1 e(\sigma)}.$$

Hence for $|a| > 1$, we have

$$|\omega_{\pi_1}(a)| < |a|^{(l_1 - l_3 + m_1) e(\sigma)}.$$

Notice that since $k = l_3 + l_1 + m_1$,

$$(l_1 - l_3 + m_1) e(\sigma) = (k - 2l_3) e(\sigma) \leq k e(\sigma).$$

So the lemma is proven for the symplectic group.

For the orthogonal group, the proof is identical except that we need to use [Ban] instead of [Ta] for the computation of the Jacquet module of induced representations. Note that in [Ban] she treats only the split orthogonal group, but her proof works for the non-split group. Indeed, the proof of the non-split case is even simpler. The detail is left to the reader. \square

Now we are ready to prove Proposition 3.3.

Proof of Proposition 3.3. We will prove it by induction on t , i.e. the number of inducing data. Assume $t = 1$ and so π is a constituent of $\delta_1 \rtimes \tau$. By Proposition 3.2 (a), the exponents of τ are bounded by 0 and so *a fortiori* by $e(\delta_1)$. So by the lemma the exponents of π are bounded by $e(\delta_1)$. Now assume π is a constituent of $\delta_1 \times \cdots \times \delta_{t+1} \rtimes \tau$. Note that

$$\delta_1 \times \cdots \times \delta_{t+1} \rtimes \tau = \delta_1 \rtimes (\delta_2 \times \cdots \times \delta_{t+1} \rtimes \tau),$$

and so by the induction hypothesis the exponents of every constituent of $\delta_2 \times \cdots \times \delta_{t+1} \rtimes \tau$ are bounded by $e(\delta_2)$ and hence by $e(\delta_1)$. The above lemma immediately implies that the exponents of π are bounded by $e(\delta_1)$. \square

There are a couple of corollaries we need to mention.

Corollary 3.5. *If π is a constituent of $\delta_1 \times \cdots \times \delta_t \rtimes \tau$ as in the proposition, then the exponents of π^\vee are also bounded by $e(\delta_1)$.*

Proof. Note that π^\vee is a constituent of $\delta_1^\vee \times \cdots \times \delta_t^\vee \rtimes \tau^\vee$. But $\delta_1^\vee \times \cdots \times \delta_t^\vee \rtimes \tau^\vee = \delta_1 \times \cdots \times \delta_t \rtimes \tau^\vee$ up to semisimplification, and if τ is tempered, then so is τ^\vee . So the lemma immediately follows from the proposition. \square

Notice that the above proposition tells us that one can tell how large the exponents of every irreducible admissible representation are by looking at the Langlands quotient data. Moreover, the Langlands quotient data gives the lowest bound for the exponents. Namely,

Corollary 3.6. *Assume that π is an irreducible admissible representation of $\mathrm{Sp}(2n)$ (resp. $\mathrm{O}(V_r)$), which is the Langlands quotient of $\delta_1 \times \dots \times \delta_t \rtimes \tau$ with $e(\delta_1) > \dots > e(\delta_t) > 0$. Then $e(\delta_1)$ is the smallest number by which the exponents of π are bounded.*

Proof. Since

$$\mathrm{Hom}_G(\delta_1 \times \dots \times \delta_t \rtimes \tau, \pi) \neq 0,$$

where G is $\mathrm{Sp}(2n)$ (resp. $\mathrm{O}(V_r)$), by Frobenius reciprocity,

$$\mathrm{Hom}_M(\delta_1 \otimes (\delta_2 \times \dots \times \delta_t \rtimes \tau), \overline{R}_{P_{n_1}}(\pi)) \neq 0,$$

where M is $GL(n_1) \times \mathrm{Sp}(2(n - n_1))$ (resp. $GL(n_1) \times \mathrm{O}(V_{r-n_1})$). So there exists an exponent ω along this parabolic such that $\omega = \omega_{\delta_1}$, and hence $|\omega(a)| = |a|^{n_1 e(\delta_1)}$ for $a \in F^\times$. Thus if the exponents of π are bounded by e , we must have $e(\delta_1) \leq e$. But by the proposition, we already know that the exponents of π are bounded by $e(\delta_1)$, which completes the proof. \square

4. ON THE LOCAL THETA LIFT FOR ISOMETRY GROUPS

The major object of this section is to improve upon the result of Roberts [R2] regarding the non-archimedean theta correspondence by applying our notion of ‘‘boundedness’’. In particular we will replace his temperedness assumption in [R2] by a weaker assumption of ‘‘bounded by 1’’. Accordingly, except Corollary 4.6, which will be given at the end of this section, we assume that the base field is a non-archimedean local field of characteristic 0.

The first theorem in this section is

Theorem 4.1. *Recall that $\dim V_r = m = m_a + 2r$. Let $\pi \in \mathrm{Ind}(\mathrm{Sp}(2n))$ and $\pi' \in \mathrm{Ind}(\mathrm{Sp}(2n'))$, and also let $\sigma \in \mathrm{Ind}(\mathrm{O}(V_r))$ and $\sigma' \in \mathrm{Ind}(\mathrm{O}(V_{r'}))$.*

(1) *(From orthogonal to symplectic.) Let n and n' be such that $2n' > 2n \geq \dim V_r$. Assume both π and π' correspond to σ under the theta correspondence. If π is bounded by $e \leq 1$, then π' is an irreducible quotient of*

$$\chi |^{n'-m/2} \times \chi |^{n'-1-m/2} \times \dots \times \chi |^{n+1-m/2} \rtimes \pi.$$

In particular if π is the Langlands quotient of

$$\delta_1 \times \dots \times \delta_1 \rtimes \tau$$

such that $e(\delta_1) < 1$, then π' is the Langlands quotient of

$$\chi |^{n'-m/2} \times \chi |^{n'-1-m/2} \times \dots \times \chi |^{n+1-m/2} \times \delta_1 \times \dots \times \delta_1 \rtimes \tau.$$

(2) *(From symplectic to orthogonal.) Let r and r' be such that $m_a + 2r' > m_a + 2r \geq 2n + 2$. Assume both σ and σ' correspond to π under the theta correspondence. If σ is bounded by $e \leq 1$, then σ' is an irreducible quotient of*

$$|^{m_a/2+r'-1-n} \times |^{m_a/2+r'-2-n} \times \dots \times |^{m_a/2+r-n} \rtimes \sigma.$$

In particular if σ is the Langlands quotient of

$$\delta_1 \times \dots \times \delta_1 \rtimes \tau$$

such that $e(\delta_1) < 1$, then σ' is the Langlands quotient of

$$||^{m_a/2+r'-1-n} \times ||^{m_a/2+r'-2-n} \times \dots \times ||^{m_a/2+r-n} \times \delta_1 \times \dots \times \delta_1 \times \tau.$$

Proof. The proof is essentially a modification of the one by Roberts [R2, Theorem 4.4], but we use our “bounded by 1” assumption instead of his temperedness assumption. In what follows, we give a self-contained proof for the “from symplectic to orthogonal” case because this is the case which is not treated by Roberts. The “from orthogonal to symplectic” case is a straightforward modification of Roberts’ proof and left to the reader.

First let

$$\delta = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix} \in \mathrm{GL}(2n).$$

Notice that if $g \in \mathrm{Sp}(2n)$, then $\delta g \delta \in \mathrm{Sp}(2n)$. We define π^δ by $\pi^\delta(g) = \pi(\delta g \delta)$. Then we have

$$\pi^\delta \cong \pi^\vee.$$

(See, for example, [Kd2, Theorem 1.6, Ch. VI]). Similarly, we define $\omega_{V_r, n}^\delta$ by $\omega_{V_r, n}^\delta(h, g) = \omega_{V_r, n}(h, \delta g \delta)$. Then we have

$$\omega_{V_r, n}^\delta \cong \omega_{-V_r, n},$$

where $-V_r$ is the negative of the quadratic form given by V_r . Now since σ and π correspond via theta correspondence, there is a non-zero $\mathrm{O}(V_r) \times \mathrm{Sp}(2n)$ map

$$\omega_{V_r, n} \rightarrow \sigma \otimes \pi.$$

Hence by twisting by δ we have a non-zero $\mathrm{O}(V_r) \times \mathrm{Sp}(2n)$ map

$$\omega_{V_r, n}^\delta \cong \omega_{-V_r, n} \rightarrow \sigma \otimes \pi^\vee.$$

Also since π and σ' correspond via theta correspondence, there is a non-zero $\mathrm{O}(V_{r'}) \times \mathrm{Sp}(2n)$ map

$$\omega_{V_{r'}, n} \rightarrow \sigma' \otimes \pi.$$

Thus we have a non-zero $\mathrm{O}(V_{r'}) \times \mathrm{O}(V_r) \times \mathrm{Sp}(2n)$ map

$$\omega_{V_{r'}, n} \otimes \omega_{-V_r, n} \rightarrow \sigma' \otimes \sigma \otimes \pi \otimes \pi^\vee.$$

Note that there is a canonical embedding $\mathrm{O}(V_{r'}) \times \mathrm{O}(V_r) \hookrightarrow \mathrm{O}(V_{r'} \oplus -V_r)$, where for $V_{r'} \oplus -V_r$ the bilinear form is defined by $\langle v'_1 + v_1, v'_2 + v_2 \rangle = \langle v'_1, v'_2 \rangle_{V_{r'}} - \langle v_1, v_2 \rangle_{V_r}$. (This embedding might as well be called the embedding of the “generalized doubling method”.) Then $\mathrm{O}(V_{r'} \oplus -V_r)$ is the split orthogonal group of rank

$$R = m_a + r + r'.$$

Via this embedding, one can see that

$$\omega_{V_{r'}, n} \otimes \omega_{-V_r, n} \cong \omega_{V_{r'} \oplus -V_r, n} |_{\mathrm{O}(V_{r'}) \times \mathrm{O}(V_r) \times \mathrm{Sp}(2n)}.$$

Now by composing the canonical $\mathrm{Sp}(2n)$ map $\pi \otimes \pi^\vee \rightarrow \mathbf{1}$, we obtain a surjective $\mathrm{O}(V_{r'}) \times \mathrm{O}(V_r)$ map

$$(\omega_{V_{r'} \oplus -V_r, n})_{\mathrm{Sp}(2n)} \rightarrow \sigma' \otimes \sigma.$$

Recall that for the orthogonal group every irreducible admissible representation is selfdual and so $\sigma'^\vee \otimes \sigma^\vee \cong \sigma' \otimes \sigma$. (See [Kd2, Theorem 1.6, Ch. VI].) Hence by taking the contragredient we obtain an injective $\mathrm{O}(V_{r'}) \times \mathrm{O}(V_r)$ map

$$\sigma' \otimes \sigma \hookrightarrow (\omega_{V_{r'} \oplus -V_r, n})_{\mathrm{Sp}(2n)}^\vee \cong (\omega_{-V_{r'} \oplus V_r, n})_{\mathrm{Sp}(2n)}.$$

Let $I_R(s)$ be the degenerate principal series for the split orthogonal group $\mathrm{O}(-V_{r'} \oplus V_r) = \mathrm{O}(R, R)$. By Theorem 5.1 of [Kd2, Ch. II], there exists an injective $\mathrm{O}(R, R)$ map

$$(\omega_{-V_{r'} \oplus V_r, n})_{\mathrm{Sp}(2n)} \hookrightarrow I_R(-s_0),$$

where

$$s_0 = \frac{R-1}{2} - n.$$

This gives an injective $O(-V_{r'}) \times O(V_r) = O(V_{r'}) \times O(V_r)$ map

$$\sigma' \otimes \sigma \hookrightarrow I_R(-s_0),$$

By taking the contragredient, we have a non-zero $O(V_{r'}) \times O(V_r)$ map

$$I_R(s_0) \rightarrow \sigma' \otimes \sigma,$$

The degenerate principal series $I_R(s_0)$ admits a filtration of $O(V_{r'}) \times O(V_r)$ representations

$$0 = I^{r+1} \subset I^r \subset \cdots \subset I^1 \subset I^0 = I_R(s_0)$$

such that

$$I^i/I^{i+1} \cong \text{Ind}_{P_{r'-i} \times P_{r-i}}^{O(V_{r'}) \times O(V_r)} \left(\left| \left| s_0 + \frac{r-i}{2} \right| \right| \otimes \left| \left| s_0 + \frac{r'-i}{2} \right| \right| \otimes \rho_i \right)$$

where ρ_i is the representation of $O(V_i) \times O(V_i)$ on $\mathcal{S}(O(V_i))$ defined by $\rho_i(h, h')\phi(x) = \phi(h^{-1}xh')$. (This can be shown following the proof of Proposition 2.3 [Kd2] which shows the symplectic case.)

So for some $0 \leq i \leq r$, we have

$$\text{Hom}_{O(V_{r'}) \times O(V_r)}(\text{Ind}_{P_{r'-i} \times P_{r-i}}^{O(V_{r'}) \times O(V_r)} \left(\left| \left| s_0 + \frac{r-i}{2} \right| \right| \otimes \left| \left| s_0 + \frac{r'-i}{2} \right| \right| \otimes \rho_i, \sigma' \otimes \sigma \right) \neq 0.$$

By Frobenius reciprocity,

$$\text{Hom}_{M_{r'-i} \times M_{r-i}} \left(\left| \left| s_0 + \frac{r-i}{2} \right| \right| \otimes \left| \left| s_0 + \frac{r'-i}{2} \right| \right| \otimes \rho_i, \overline{R}_{P_{r'-i}}(\sigma') \otimes \overline{R}_{P_{r-i}}(\sigma) \right) \neq 0.$$

So there exist an irreducible subquotient $\sigma_1 \otimes \sigma_2 \in \text{Irr}(\text{GL}(r-i) \times O(V_i))$ of $\overline{R}_{P_{r-i}}(\sigma)$ and an irreducible subquotient $\sigma'_1 \otimes \sigma'_2 \in \text{Irr}(\text{GL}(r-i) \times O(V_i))$ of $\overline{R}_{P_{r'-i}}(\sigma')$ such that

$$\text{Hom}_{M_{r'-i} \times M_{r-i}} \left(\left| \left| s_0 + \frac{r-i}{2} \right| \right| \otimes \left| \left| s_0 + \frac{r'-i}{2} \right| \right| \otimes \rho_i, \sigma'_1 \otimes \sigma'_2 \otimes \sigma_1 \otimes \sigma_2 \right) \neq 0.$$

Hence we have

$$\left| \left| s_0 + \frac{r'-i}{2} \right| \right| = \sigma_1.$$

First suppose $i < r$ and so $r-i \geq 1$. By our assumption σ is bounded by $e \leq 1$ and so for $|a| > 1$, we have

$$|\det(aI_{r-i})|^{s_0 + \frac{r'-i}{2}} = |\omega_{\sigma_1}(a)| \leq |a|^{(r-i)e} \leq |a|^{r-i},$$

which gives

$$(r-i)\left(s_0 + \frac{r'-i}{2}\right) \leq r-i.$$

Since we are assuming $r-i > 0$, we have $s_0 + \frac{r'-i}{2} \leq 1$. Recalling $s_0 = \frac{R-1}{2} - n$ and $R = m_a + r + r'$, we obtain $m_a + r + r' - 1 - 2n + r' - i \leq 2$. Also by our assumption, $2n+2 \leq m_a + 2r$. So $r-i \leq -2(r'-r) + 1$. Now $r'-r > 0$ and so we have $r-i < 1$, which is a contradiction because $r-i \geq 1$.

Hence we have $i = r$ and so

$$\text{Hom}_{O(V_{r'}) \times O(V_r)}(\text{Ind}_{P_{r'-r} \times O(V_r)}^{O(V_{r'}) \times O(V_r)} \left| \left| s_0 \right| \right| \otimes \rho_r, \sigma' \otimes \sigma) \neq 0.$$

By Frobenius reciprocity, this gives

$$\text{Hom}_{P_{r'-r} \times O(V_r)} \left(\left| \left| s_0 \right| \right| \otimes \rho_r, (\sigma'|_{P_{r'-r}} \otimes \delta_{P_{r'-r}}^{1/2}) \otimes \sigma \right) \neq 0,$$

i.e.

$$\text{Hom}_{P_{r'-r} \times O(V_r)} \left(\rho_r, (\sigma'|_{P_{r'-r}} \otimes \left| \left| -s_0 \right| \right| \delta_{P_{r'-r}}^{1/2}) \otimes \sigma^\vee \right) \neq 0.$$

Here we used the fact that for the orthogonal group every irreducible admissible representation is selfdual. Also notice that on ρ_r , the $\text{GL}(r'-r)$ part of the parabolic $P_{r'-r}$ acts trivially and hence it

also acts trivially on $\sigma'|_{P_{r'-r}} \otimes |^{-s_0} \delta_{P_{r'-r}}^{1/2}$. Now let f be a non-trivial element in this Hom space. By the lemma on p. 59 of [MVW], there is a surjective $O(V_r) \times O(V_r)$ map

$$\rho_r \rightarrow \sigma \otimes \sigma^\vee$$

whose kernel is contained in $\ker f$. Hence we have a non-zero $O(V_r) \times O(V_r)$ map

$$\sigma \otimes \sigma^\vee \rightarrow (\sigma'|_{P_{r'-r}} \otimes |^{-s_0} \delta_{P_{r'-r}}^{1/2}) \otimes \sigma^\vee.$$

Since $GL(r' - r)$ acts trivially on $\sigma'|_{P_{r'-r}} \otimes |^{-s_0} \delta_{P_{r'-r}}^{1/2}$, this map can be extended to a $O(V_r) \times P_{r'-r}$ map. Hence

$$\text{Hom}_{P_{r'-r}}(\sigma, \sigma'|_{P_{r'-r}} \otimes |^{-s_0} \delta_{P_{r'-r}}^{1/2}) \neq 0.$$

By Frobenius reciprocity,

$$\text{Hom}_{O(V_{r'})}(\text{Ind}_{P_{r'-r}}^{O(V_{r'})}(|^{s_0} \boxtimes \sigma), \sigma') \neq 0.$$

One can easily see that there is a surjective $GL(r' - r)$ map

$$\text{Ind}_{P_{1, \dots, 1}}^{GL(r'-r)}(|^{m_a/2+r'-1-n} \times |^{m_a/2+r'-2-n} \times \dots \times |^{m_a/2+r-n}) \rightarrow |^{s_0}.$$

(Indeed the one dimensional representation $|^{s_0}$ of $GL(r' - r)$ is the Langlands quotient of this induced representation.) Therefore there is a surjective $O(V_{r'})$ map

$$|^{m_a/2+r'-1-n} \times |^{m_a/2+r'-2-n} \times \dots \times |^{m_a/2+r-n} \rtimes \sigma \rightarrow \sigma',$$

which completes the proof. \square

Remark 4.2. As we mentioned at the beginning of the proof, the first case of the theorem, *i.e.* the orthogonal-to-symplectic case is even a more straightforward modification of [R2, Theorem 4.4]. However in [R2] Roberts assumes that σ is unitarizable, which he calls pre-unitary. He needed this assumption to have a \mathbb{C} -anti-linear map $\sigma \rightarrow \sigma^\vee$. But it is possible to get around this subtle issue in the following way. First notice that $\overline{\omega_{V_r, n}} \cong \omega_{V_r, n}^\delta$, where $\overline{\omega_{V_r, n}}$ is as in the proof of Theorem 4.4. in [R2] and δ is as in our proof above. So since $\sigma^\vee = \sigma$ and $\pi^\vee = \pi^\delta$, we have a non-zero $O(V_r) \times \text{Sp}(2n)$ map

$$\overline{\omega_{V_r, n}} \rightarrow \sigma^\vee \otimes \pi^\vee.$$

Hence there is a non-zero $O(V_r) \times \text{Sp}(2n) \times \text{Sp}(2n')$ map

$$\overline{\omega_{V_r, n}} \otimes \omega_{V_r, n'} \rightarrow \sigma^\vee \otimes \sigma \otimes \pi^\vee \otimes \pi'.$$

Via the inclusion $\text{Sp}(2n) \times \text{Sp}(2n') \hookrightarrow \text{Sp}(2(n+n'))$ given in p. 1116 of [R2], we have

$$\overline{\omega_{V_r, n}} \otimes \omega_{V_r, n'} \cong \omega_{V_r, n+n'}|_{O(V_r) \times (\text{Sp}(2n) \times \text{Sp}(2n'))}.$$

So by composing with the canonical $O(V_r)$ map $\sigma^\vee \otimes \sigma \rightarrow \mathbf{1}$, we obtain a non-zero $O(V_r) \times \text{Sp}(2n) \times \text{Sp}(2n')$ map

$$\omega_{V_r, n+n'} \rightarrow \pi^\vee \otimes \pi'.$$

This way, we can suppress the unitarizability assumption present in [R2]. The reader can easily verify that the rest of the proof by Roberts does not require σ be unitarizable.

Now a natural question to ask is for which class of $\sigma \in \text{Irr}(O(V_r))$ (resp. $\pi \in \text{Irr}(\text{Sp}(2n))$) the corresponding $\pi \in \text{Irr}(\text{Sp}(2n))$ (resp. $\sigma \in \text{Irr}(O(V_r))$) has the desired boundedness property as in the assumption of the theorem. The tempered version of this question has been settled by Roberts [R2, Theorem 4.2] for the orthogonal-to-symplectic case for the range $n \leq \dim V_r/2$. And he has shown that for this range if σ is tempered, then π is also tempered. We will show an analogue of this in terms of our notion of boundedness. Namely, we have

Theorem 4.3. *Suppose that $\pi \in \text{Irr}(\text{Sp}(2n))$ and $\sigma \in \text{Irr}(\text{O}(V_r))$, and π and σ correspond under the theta correspondence. Then*

- (1) *(From orthogonal to symplectic.) Assume $2n \leq \dim V_r$. If σ is bounded by $e \geq 0$, then π is also bounded by e .*
- (2) *(From symplectic to orthogonal.) Assume $\dim V_r \leq 2n + 2$. If π is bounded by $e \geq 0$, then σ is also bounded by e .*

The proof is by now a standard argument using Frobenius reciprocity and the Jacquet module of the Weil representation. Indeed, the proof is again almost identical to the one given by Roberts [R2, p.1113-1114], especially for the orthogonal-to-symplectic case. But since this theorem is quite crucial for our global applications, we give a self-contained proof with our modified assumption for the symplectic-to-orthogonal case. First we need the following well-known result due to Kudla [Kd1].

Lemma 4.4. *Let Q_j be our choice of the maximal parabolic of $\text{O}(V_r)$ whose Levi is $\text{GL}(j) \times \text{O}(V_{r-j})$, and also let $l = \min(n, j)$. For each $0 \leq k \leq l$ define σ_k to be the representation of $\text{GL}(k) \times \text{GL}(k)$ on the space $\mathcal{S}(\text{GL}(k))$ with the action given by $\sigma_k(g, g')\varphi(x) = \varphi(g^{-1}xg')$. Then the Jacquet module $R_{Q_j}(\omega_{V_r, n})$ of the Weil representation has a filtration*

$$0 = F^{l+1} \subset F^l \subset F^{l-1} \subset \cdots \subset F^1 \subset F^0 = R_{Q_j}(\omega_{V_r, n})$$

such that

$$F^k / F^{k+1} \cong \text{Ind}_{Q_{jk} \times P_k}^{(\text{GL}(j) \times \text{O}(V_{r-j})) \times \text{Sp}(2n)} \xi_{jk} \sigma_k \otimes \omega_{V_{r-j}, n-k},$$

where Q_{jk} is the standard parabolic of $\text{GL}(j)$ whose Levi is $\text{GL}(j) \times \text{GL}(j-k)$, P_k is the standard maximal parabolic of $\text{Sp}(2n)$ whose Levi is $\text{GL}(j) \times \text{Sp}(2n-2k)$, and ξ_{jk} is a character on $Q_{jk} \times P_k$ whose restriction on $(\text{GL}(k) \times \text{GL}(j-k)) \times \text{GL}(k) \subset Q_{jk} \times P_k$ is given by

$$\xi_{jk}(h_1, h_2, g) = |\det h_1|^{-\left(\frac{1}{2}m-j+\frac{k-1}{2}\right)} |\det h_2|^{-\left(\frac{1}{2}m-\frac{1}{2}j-n+\frac{k-1}{2}\right)} \chi(\det g) |\det g|^{\frac{1}{2}m-j+\frac{k-1}{2}},$$

where $(h_1, h_2) \in \text{GL}(k) \times \text{GL}(j-k) \subset Q_{jk}$ and $g \in \text{GL}(k) \subset P_k$.

Proof. This is essentially Theorem 2.8 of [Kd1]. However we should mention that his choice of the parabolic for $\text{O}(V_r)$ is different from ours, and this is the reason we have slightly different actions for ξ_{jk} and σ_k . Our convention follows [R2]. Also a detailed proof for the case of similitude groups appears in [GT4]. \square

Using this, we prove our theorem.

Proof of Theorem 4.3. Once again, we only prove (2) of the theorem. So assume $\pi \in \text{Irr}(\text{Sp}(2n))$ is bounded by $e \geq 0$. Also assume $\sigma \in \text{Irr}(\text{O}(V_r))$ corresponding to π is the Langlands quotient of $\delta_1 \times \cdots \times \delta_t \rtimes \tau$ where δ_1 is a representation of $\text{GL}(j)$. Then we will show that $e(\delta_1) \leq e$, which will prove the theorem. First note that σ is a subrepresentation of $\delta_1^\vee \times \cdots \times \delta_t^\vee \rtimes \tau^\vee$. Let

$$\rho = \delta_1^\vee \quad \text{and} \quad \rho' = \delta_2^\vee \times \cdots \times \delta_t^\vee \rtimes \tau^\vee$$

so that σ is a subrepresentation of $\rho \rtimes \rho'$. Since π and σ correspond, we have

$$\text{Hom}_{\text{O}(V_r) \times \text{Sp}(2n)}(\omega_{V_r, n}, (\rho \rtimes \rho') \otimes \pi) \neq 0.$$

By Frobenius reciprocity, we have

$$\text{Hom}_{(\text{GL}(j) \times \text{O}(V_{r-j})) \times \text{Sp}(2n)}(R_{Q_j}(\omega_{V_r, n}), (\rho \otimes \rho') \otimes \pi) \neq 0.$$

So this Hom space is non-zero at some stage of the filtration of $R_{Q_j}(\omega_{V_r, n})$ given in the above lemma. Hence for some $0 \leq k \leq l$, we have

$$\text{Hom}_{(\text{GL}(j) \times \text{O}(V_{r-j})) \times \text{Sp}(2n)}(\text{Ind}_{Q_{jk} \times P_k}^{(\text{GL}(j) \times \text{O}(V_{r-j})) \times \text{Sp}(2n)} \xi_{jk} \sigma_k \otimes \omega_{V_{r-j}, n-k}, (\rho \otimes \rho') \otimes \pi) \neq 0.$$

Hence by Frobenius reciprocity, there is a non-zero $(\mathrm{GL}(k) \times \mathrm{GL}(j-k) \times \mathrm{O}(V_{r-j})) \times (\mathrm{GL}(k) \times \mathrm{Sp}(2n-2k))$ map

$$\xi_{jk} \sigma_k \otimes \omega_{V_{r-j}, n-k} \rightarrow (\overline{R}_{P_{k,j-k}^{\mathrm{GL}}}(\rho) \otimes \rho') \otimes \overline{R}_{P_k}(\pi).$$

So there exist an irreducible constituent $\rho_1 \otimes \rho_2 \in \mathrm{Irr}(\mathrm{GL}(k) \times \mathrm{GL}(j-k))$ of $\overline{R}_{P_{k,j-k}^{\mathrm{GL}}}(\rho)$ and an irreducible constituent of $\pi_1 \otimes \pi_2 \in \mathrm{Irr}(\mathrm{GL}(k) \times \mathrm{Sp}(2n-2k))$ of $\overline{R}_{P_k}(\pi)$ so that there is a non-zero $(\mathrm{GL}(k) \times \mathrm{GL}(j-k) \times \mathrm{O}(V_{r-j})) \times (\mathrm{GL}(k) \times \mathrm{Sp}(2n-2k))$ map

$$\xi_{jk} \sigma_k \otimes \omega_{V_{r-j}, n-k} \rightarrow (\rho_1 \otimes \rho_2 \otimes \rho') \otimes (\pi_1 \otimes \pi_2).$$

We see that

$$\mathrm{Hom}_{\mathrm{GL}(k) \times \mathrm{GL}(k)}(\xi_{jk} \sigma_k, \rho_1 \otimes \pi_1) \neq 0$$

and

$$\mathrm{Hom}_{\mathrm{GL}(j-k)}(\xi_{jk}, \rho_2) \neq 0.$$

Now since the elements of the form $(aI_k, aI_k) \in \mathrm{GL}(k) \times \mathrm{GL}(k)$ acts trivially on σ_k , from the above lemma, one sees that $\xi_{jk}(aI_k, aI_k) = \chi(a)^k$. Hence we must have

$$|\omega_{\rho_1}(a)| |\omega_{\pi_1}(a)| = 1.$$

Also for $aI_{j-k} \in \mathrm{GL}(j-k)$, one sees that $\xi_{jk}(aI_{j-k}) = |\det(aI_{j-k})|^{-\left(\frac{m-j}{2}-n+\frac{k-1}{2}\right)}$. Hence

$$\omega_{\rho_2} = |a|^{-(j-k)\left(\frac{m-j}{2}-n+\frac{k-1}{2}\right)}.$$

Since $\rho_2 \otimes \rho_1$ is a constituent of $\overline{R}_{P_{k,j-k}^{\mathrm{GL}}}(\rho)$, we have $\omega_{\rho}(a) = \omega_{\rho_1}(a) \omega_{\rho_2}(a)$. Hence

$$|\omega_{\pi_1}(a)| = |\omega_{\rho_1}(a)|^{-1} = |\omega_{\rho_2}(a)| |\omega_{\rho}(a)|^{-1}.$$

Recall that $\rho = \delta_1^{\vee}$ and so $|\omega_{\rho}(a)| = -je(\delta_1)$. Therefore we obtain

$$|\omega_{\pi_1}(a)| = |a|^{-(j-k)\left(\frac{m-j}{2}-n+\frac{k-1}{2}\right)+je(\delta_1)}.$$

Now by our assumption π is bounded by e , and so for $|a| > 1$, we have $|\omega_{\pi_1}(a)| \leq |a|^{ke}$, which gives

$$-(j-k)\left(\frac{m-j}{2}-n+\frac{k-1}{2}\right) + je(\delta_1) \leq ke.$$

Now by our assumption $m = \dim V_r \leq 2n+2$. So in the above inequality, we see that

$$-(j-k)\left(\frac{m-j}{2}-n+\frac{k-1}{2}\right) \geq 0.$$

Hence we have $0 \leq -je(\delta_1) + ke$. Assume for the sake of contradiction that $e(\delta_1) > e$. If $k > 0$, we would have $0 < -(j-k)e(\delta_1)$, which is a contradiction because $-(j-k)e(\delta_1) \leq 0$. And if $k = 0$, then we would have $0 \leq -je(\delta_1)$, which is also a contradiction because $j \geq 1$ and so $je(\delta_1) > e \geq 0$. This completes the proof. \square

Remark 4.5. Let us mention that Roberts considers the range $2n > \dim V_r$ for the orthogonal-to-symplectic case, and has computed the first component of the Langlands quotient data of π to some degree of explicitness, when π is a first occurrence but not tempered. (See Theorem 4.2(2) of [R2] for the details.) An analogous result can be shown in our situation. But we will leave this issue to the reader because the theorem will not play any role for our global applications.

By combining the two theorems above, we have the following, which will be the key result for our global applications.

Corollary 4.6. *Assume $\pi \in \mathrm{Ind}(\mathrm{Sp}(2n))$ and $\sigma \in \mathrm{Ind}(\mathrm{O}(V_r))$. Here we assume that F is not necessarily non-archimedean.*

- (1) (From orthogonal to symplectic.) Let $n = \frac{1}{2} \dim V_r$. Also let $\pi' \in \text{Irr}(\text{Sp}(2n'))$ for $n' > n$. Assume both π and π' correspond to σ under the theta correspondence. If σ is bounded by $e < 1$, then π' is the Langlands quotient of

$$|\chi|^{n'-m/2} \times |\chi|^{n'-1-m/2} \times \cdots \times |\chi|^1 \rtimes L(\pi),$$

where $L(\pi)$ is the Langlands quotient data of π .

- (2) (From symplectic to orthogonal.) Let $\frac{1}{2} \dim V_r = n + 1$. Also let $\sigma' \in \text{Irr}(\text{O}(V_{r'}))$ for $r' > r$. Assume both σ and σ' correspond to π under the theta correspondence. If π is bounded by $e < 1$, then σ' is the Langlands quotient of

$$|^{m_a/2+r'-1-n} \times |^{m_a/2+r'-2-n} \times \cdots \times |^1 \rtimes L(\sigma),$$

where $L(\sigma)$ is the Langlands quotient data of σ .

Proof. The non-archimedean case immediately follows from the two theorems in this section. For $F = \mathbb{R}$, this is Theorem 6.2 of [P]. For $F = \mathbb{C}$, this can be read off from Theorem 2.8 of [AB], although the reader has to be careful in that the complex valuation in [AB] is the usual absolute value *i.e.* $|z| = \sqrt{z\bar{z}}$, but we use the standard complex valuation *i.e.* $|z| = z\bar{z}$. \square

Let us emphasize here that this corollary tells us that for the range as in the corollary the theta lift of a representation which is bounded by $e < 1$ is uniquely determined. The proof never used the Howe duality of any sort, and hence the statement holds for the even residual characteristic case.

5. ON THE GLOBAL THETA LIFT FOR ISOMETRY GROUPS

The theorems in the previous section, especially Corollary 4.6, combined with the method of Roberts [R4] allows us to prove our main non-vanishing theorem for global theta lifts, namely Theorem 1.1.

Proof of Theorem 1.1. The proof is a modification of the beautiful argument by Roberts [R4], which in turn has its origin [BS] in the classical context. We reproduce essential points of the proof for the symplectic-to-orthogonal case. So assume $\pi \cong \otimes_v \pi_v$ is an irreducible cuspidal automorphic representation of $\text{Sp}(2n, \mathbb{A})$ which satisfies the assumption of part (2) of the theorem. Also assume that π is realized in a space V_π of cusp forms. First of all, the Euler product of the (incomplete) standard L -function $L^S(s, \pi, \chi)$ twisted by χ converges absolutely for $\text{Re}(s) \geq 2$. This can be seen as follows. Since each π_v is bounded by some $e_v < 1$, each unramified factor of $L^S(s, \pi, \chi)$ is a product of factors of the form

$$(1 - a_v q_v^{-1})^{-1} \quad \text{with} \quad |a_v| \leq q_v^{e_v}.$$

One sees that the Euler product of such factors converges absolutely for $\text{Re}(s) > 1 + e_v$. (See Lemma 2 in p.187 and the first paragraph of p.188 of [M].) Hence it converges absolutely for $\text{Re}(s) \geq 2$, and in particular does not vanish for $\text{Re}(s) \geq 2$.

Now for the sake of contradiction, assume that the global theta lift $\Theta_{V_r}(V_\pi)$ is zero. Then the global theta lift $\Theta_{V_{r'}}(V_\pi)$ must be non-zero cuspidal for some r' with $r < r' \leq 2n$. Let τ be an irreducible constituent of $\Theta_{V_{r'}}(V_\pi)$. By the functoriality of the unramified theta correspondence, the standard L -function $L^S(s, \tau)$ of τ is written as

$$\begin{aligned} L^S(s, \tau) &= L^S(s, \pi, \chi) \zeta^S(s + r' - r) \zeta^S(s + r' - r - 1) \cdots \zeta^S(s + 1) \cdot \\ &\quad \zeta^S(s) \zeta^S(s - 1) \cdots \zeta^S(s - r' + r + 1) \zeta^S(s - r' + r), \end{aligned}$$

where $\zeta^S(s)$ is the (incomplete) Dedekind zeta function. (See [KR2, Corollary 7.1.4].) Now consider the order of the zero at $s = r' - r$ for this L -function. One sees that $\zeta^S(s - r' + r)$ has a zero of order $|S| - 1$ and $\zeta^S(s - r' + r + 1)$ has a simple pole. Also $L^S(s, \pi, \chi)$ does not vanish (a pole is allowed) at $s = r' - r$. (This is true by our assumption on π if $r' - r = 1$, and by the absolute convergence if $r' - r \geq 2$.) So we see that the order of vanishing of $L^S(s, \tau)$ is at most $|S| - 2$.

On the other hand, we can compute the very same L -function $L^S(s, \tau)$ by a totally different method, namely by the doubling method, which gives

$$L^S(s, \tau) = \frac{b^S(s - \frac{1}{2})Z(s - \frac{1}{2}, f, \Phi)}{\prod_{v \in S} Z_v(s - \frac{1}{2}, f_v, \Phi_v)},$$

where $Z(s - \frac{1}{2}, f, \Phi)$ is the doubling integral with $f = \otimes_v f_v$ a matrix coefficient of τ and $\Phi = \otimes_v \Phi_v$ a K -finite standard section, and $b^S(s)$ is a certain normalizing factor. Now the following lemma due to Roberts [R4] is the key technical ingredient.

Lemma 5.1. *Let v be archimedean or non-archimedean. Assume τ_v is the Langlands quotient of $\delta_1 \times \cdots \times \delta_t \rtimes \rho$ with $\delta_1 = |\cdot|^k$ for some $k > 0$. Then f_v and Φ_v can be chosen so that the local zeta integral $Z_v(s - \frac{1}{2}, f_v, \Phi_v)$ has a pole at $s = k$.*

Proof. This is essentially the “main lemma” of [R4]. Although Roberts considers the zeta integral for the symplectic groups, as he mentions in the introduction of [R4] it is straightforward to modify his computations to obtain the same result for the orthogonal groups. Also see [Tk] for a certain subtle issue for the archimedean place. \square

Now we know that τ_v is the local theta lift of π_v^\vee . Then since π_v has a non-zero theta lift to $O(V_r)$, so does π_v^\vee . Also if π_v is bounded by $e_v < 1$, so is π_v^\vee by Corollary 3.5. Hence if τ_v is the Langlands quotient of $\delta_1 \times \cdots \times \delta_t \rtimes \rho$, then $\delta_1 = |\cdot|^{r'-r}$ by Corollary 4.6. So by the above lemma, we can choose f_v and Φ_v so that $Z_v(s - \frac{1}{2}, f_v, \Phi_v)$ has a pole at $s = r' - r$ for all $v \in S$. It is well-known that the normalized zeta integral $b^S(s - \frac{1}{2})Z(s - \frac{1}{2}, f, \Phi)$ has at most a simple pole at $s = r' - r$ (this follows from [KR1, Proposition 1.1.4]), and so we conclude that $L^S(s, \tau)$ has a zero of order at least $|S| - 1$ at $s = r' - r$.

Therefore the first computation of $L^S(s, \tau)$ shows that it has a zero of order at most $|S| - 2$ at $s = r' - r$, and the second computation shows that it has a zero of order at least $|S| - 1$ at the same $s = r' - r$, which is a contradiction. Hence $\Theta_r(V_\pi) \neq 0$.

Now assume further that $L^S(s, \pi)$ has a pole at $s = 1$, but for the sake of contradiction assume that the theta lift $\Theta_{r-1}(V_\pi)$ to $O(V_{r-1}, \mathbb{A})$ vanishes. Then by what we have shown we know that the theta lift $\Theta_r(V_\pi)$ to $O(V_r, \mathbb{A})$ is non-zero cuspidal. Hence if τ is an irreducible constituent, we have

$$L^S(s, \tau) = \zeta^S(s)L^S(s, \pi, \chi).$$

Since $\zeta^S(s)$ has a pole at $s = 1$, if $L^S(s, \pi, \chi)$ has a pole at $s = 1$, then $L^S(s, \tau)$ would have at least a double pole at $s = 1$. But it is known that the poles of the standard L -function of the orthogonal groups are at most simple [KR1, Theorem 2.0.1]. Hence $\Theta_r(\pi) \neq 0$. \square

Remark 5.2. Let us take this opportunity to point out a mistake in the previous work [Tk] by the author. In the non-vanishing theorem [Tk, Theorem 1.1] we claimed that the temperedness assumption for the archimedean place can be removed from the theorem by Roberts [R4]. But this is a mistake. This is due to my misunderstanding on what is meant by “Langlands quotient” in [P] and [AB]. For example, for a standard module $\delta_1 \times \cdots \times \delta_t \rtimes \tau$ with $e(\delta_1) = 1$, consider the induced representation $|\cdot| \times \delta_1 \times \cdots \times \delta_t \rtimes \tau$. This is not a standard module anymore, but it does have a unique irreducible quotient. And in [P] and [AB], this quotient is also called the Langlands quotient. However, this quotient is not given as the image of the desired intertwining operator discussed in

detail in [Tk]. Rather in our sense of “Langlands quotient”, this unique quotient is the Langlands quotient of $\delta'_1 \times \cdots \times \delta_t \rtimes \tau$, where $\delta'_1 = \text{Ind}_{D_{1,n_1}^{\text{GL}}}^{\text{GL}(n_1+1)} | \cdot | \otimes \delta_1$. Hence we cannot apply Roberts’ computation in [R4] to this case. Indeed, we need to exclude cases like this. But otherwise our argument in [Tk] works, and then instead of the boundedness assumption as in the above theorem, it is sufficient to assume that, say for the orthogonal-to-symplectic case, for the archimedean v , if σ_v is the Langlands quotient of $\delta_1 \times \cdots \times \delta_t \rtimes \tau$, then

$$\{e(\delta_1), e(\delta_2), \dots, e(\delta_t)\} \cap \{1, 2, \dots, \frac{1}{2} \dim V_r\} = \emptyset.$$

Similarly, we can also replace the boundedness assumption by an analogous assumption for the symplectic-to-orthogonal case. However, to do this does not seem to have any merit but simply to complicate the statement of the theorem, and so we state our theorem keeping the assumption for the archimedean place same as that of the non-archimedean one.

6. ON THE GLOBAL THETA LIFT FOR SIMILITUDE GROUPS

In this section, we apply our non-vanishing result in the previous section to small rank similitude groups. In particular we consider the theta lift from $\text{GO}(V)$ with $\dim V = 4$ to $\text{GSp}(4)$, and the one from $\text{GSp}(4)$ to $\text{GO}(V_D)$ with $V_D = D \oplus \mathbb{H}$ where D is a (possibly split) quaternion algebra over F and \mathbb{H} is the hyperbolic place. Hence in this section we consider similitude theta lifting. References for the theory of similitude theta lifting are abound by now ([HST, R5, GT1] and citations therein), and so rather than repeating the detail of this theory here, we will refer the reader to those references.

Now first let us note the following lemma, which will be necessary for our applications.

Lemma 6.1. *Let (G, H) be the pair $(\text{GSp}(2n), \text{Sp}(2n))$ (resp. $(\text{GO}(V_r), \text{O}(V_r))$), and $\hat{\pi} \in \text{Irr}(G)$ be an irreducible admissible representation of G which is the Langlands quotient of the standard module $\delta_1 \times \cdots \times \delta_t \rtimes \tau$. Then every constituent π of the restriction $\hat{\pi}|_H$ is bounded by $e(\delta_1)$.*

Proof. Let H' be $\text{Sp}(2(n - n_1 - \cdots - n_t))$ (resp. $\text{O}(V_{r-n_1-\dots-n_t})$). Also let us write $\tau|_{H'} = \tau_1 \oplus \cdots \oplus \tau_k$, where each τ_i is tempered. Then we have $(\delta_1 \times \cdots \times \delta_t \rtimes \tau)|_H = \bigoplus_{i=1}^k \delta_1 \times \cdots \times \delta_t \rtimes \tau_i$. Hence if π is a constituent of $\hat{\pi}|_H$, it is a quotient of some $\delta_1 \times \cdots \times \delta_t \rtimes \tau_i$. So the lemma follows. \square

6.1. The theta lift from $\text{GO}(4)$. We consider the theta lift from $\text{GO}(V)$ with $\dim V = 4$. The main object here is to remove the temperedness assumption present in the work by Roberts [R5]. However we have to mention that such result has been already proven by Gan and Ichino in their recent preprint [GI] by an entirely different method. But we will demonstrate our method, which is a direct generalization of [R5], also works.

We start with classifying representations on $\text{GO}(V)$. The proofs are found in [R5], [HST], [Tk] or citations therein.

The local case: Let us first consider the local case and so F will be a local field and all the groups will be over F . Let $d = \text{disc } V$ be the discriminant of V . The groups $\text{GSO}(V)$ depend on d as follows.

- (1) If $d = 1$, then $\text{GSO}(V) \cong \text{GSO}(D)$, where D is a (possibly split) quaternion algebra over F made into a quadratic form in the usual way. So D is either the unique division quaternion algebra or the space M_2 of 2×2 matrices. Then there is a natural bijection between $\text{Irr}(\text{GSO}(V), \omega)$ and the set of irreducible admissible representations $\pi_1 \otimes \pi_2$ of $D^\times \times D^\times$ such that both π_1 and π_2 have the same central character ω . In this case, we write $\tau = \tau(\pi_1, \pi_2)$.

- (2) If $d \neq 1$, then $\text{GSO}(V) \cong \text{GSO}(V_E)$, where $V_E = \{x \in M_{2 \times 2}(E) \mid {}^c x^t = x\}$ is the space of Hermitian matrices over $E = F(\sqrt{d})$ with the quadratic form given by $-\det$. Then there is a natural bijection between $\text{Irr}(\text{GSO}(V), \omega)$ and $\text{Irr}(\text{GL}(2, E), \omega \circ N_F^E)$. In this case, we write $\tau = \tau(\pi, \omega)$.

Notice that $\text{GO}(V) \cong \text{GSO}(V) \rtimes \{1, t\}$, where we choose t to act on V as the matrix transpose if $V = M_2$ or $V = X_E$ and the quaternion conjugation if $V = D$. For each $\tau \in \text{Irr}(\text{GSO}(V))$, we define τ^c by taking $V_{\tau^c} = V_{\tau}$ and by letting $\tau^c(g)f = \tau(tgt)f$ for all $g \in \text{GSO}(V)$ and $f \in V_{\tau}$.

Then we have

- If $\tau \not\cong \tau^c$, then $\text{Ind}_{\text{GSO}(V)}^{\text{GO}(V)} \tau$ is irreducible, and we denote it by τ^+ .
- If $\tau \cong \tau^c$, then $\text{Ind}_{\text{GSO}(V)}^{\text{GO}(V)} \tau$ is reducible. Indeed, it is the sum of two irreducible representations, and we write $\text{Ind}_{\text{GSO}(V)}^{\text{GO}(V)} \tau \cong \tau^+ \oplus \tau^-$. Here, t acts on τ^{\pm} via a linear operator θ^{\pm} with the property that $(\theta^{\pm})^2 = \text{Id}$ and $\theta^{\pm} \circ g = tgt \circ \theta^{\pm}$ for all $g \in \text{GSO}(V)$.

We can be more explicit about the irreducible components τ^+ and τ^- . First assume $d = 1$. In this case, it is easy to see that, via ρ , t acts on $\text{GL}(2) \times \text{GL}(2)$ or $D^{\times} \times D^{\times}$ by $t \cdot (g_1, g_2) = (g_2, g_1)$, and if $\tau = \tau(\pi_1, \pi_2)$ is such that $\tau \cong \tau^c$, then $\pi_1 \cong \pi_2$. We can choose θ^{\pm} to be such that $\theta^+(x_1 \otimes x_2) = x_2 \otimes x_1$ and $\theta^-(x_1 \otimes x_2) = -x_2 \otimes x_1$ for $x_1 \otimes x_2 \in \pi_1 \otimes \pi_2$. We choose τ^+ and τ^- accordingly. Note that our choice of τ^+ is the *preferred* extension defined in [PS, p.227].

Next assume $d \neq 1$. In this case t acts, via ρ , on $\text{GL}(2, E)$ in such a way that $t \cdot g = {}^c g$ i.e. the Galois conjugation. If $\tau = \tau(\pi, \omega)$ is such that $\tau \cong \tau^c$, then $\pi \cong \pi^c$. Note that π has a unique Whittaker model, namely it is realized as a space of functions $f : \text{GL}(2, E) \rightarrow \mathbb{C}$ such that $f\left(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\right) = \psi_v(\text{tr } a)f(g)$ for all $a \in E$ and $g \in \text{GL}(2, E)$, where ψ_v is a fixed additive character of F . Then we define θ^{\pm} to be the linear operator that acts on this space of Whittaker functions by $f \mapsto \pm f \circ c$, and θ^+ is chosen to be the one that acts as $f \mapsto f \circ c$. We choose τ^+ and τ^- accordingly.

We should note that our choice of τ^+ and τ^- is different from that of Roberts in [R5], but rather we follow [HST]. The reason we make this choice is because it is consistent with Proposition 6.2 below. However as we will show at the end of this subsection (Proposition 6.5), it turns out that those two choices indeed coincide. Also the reader should notice that in the above discussion the fields F and E do not have to be non-archimedean.

The global case: Now we consider the global case, and hence here we assume that F is a global field of char $F = 0$ and the groups are over F . If $d \neq 1$ and $E = F(\sqrt{d})$, then let c be the non-trivial element in $\text{Gal}(E/F)$. For each quaternion algebra D over F , let $B_{D,E} = D \otimes E$. For each $g \in B_{D,E}$, we define ${}^c g^*$ by linearly extending the operation ${}^c(x \otimes a)^* = x^* \otimes {}^c a$ where c is the Galois conjugation and $*$ is the quaternion conjugation. The space $V_{D,E} = \{g \in B_{D,E} : {}^c g^* = g\}$ can be made into a four dimensional quadratic space over F via the reduced norm of the quaternion algebra $B_{D,E}$. Similarly to the local case, we have

- (1) If $d = 1$, then $\text{GO}(V)$ is isomorphic to $\text{GO}(D)$ for some (possibly split) quaternion algebra over F . Then there is a natural bijective correspondence between an irreducible cuspidal automorphic representation τ of $\text{GSO}(V, \mathbb{A}_F)$ whose central character is ω and an irreducible cuspidal automorphic representation $\pi_1 \otimes \pi_2$ of $D^{\times}(\mathbb{A}_F) \times D^{\times}(\mathbb{A}_F)$ such that both π_1 and π_2 have the central character ω . In this case, we write $\tau = \tau(\pi_1, \pi_2)$.
- (2) If $d \neq 1$, then there exists a quaternion algebra D over F such that $\text{GO}(V) \cong \text{GO}(V_{D,E})$. Then there is a natural bijective correspondence between an irreducible cuspidal automorphic

representation τ of $\mathrm{GSO}(V, \mathbb{A}_F) \cong \mathrm{GSO}(V_{D,E}, \mathbb{A}_F)$ whose central character is ω and an irreducible cuspidal automorphic representation π of $B_{D,E}^\times(\mathbb{A}_E)$ whose central character is of the form $\omega \circ N_{\mathbb{A}_E}^E$. In this case, we write $\tau = \tau(\pi, \omega)$.

We need to consider the relation between irreducible cuspidal automorphic representations of the two groups $\mathrm{GSO}(V, \mathbb{A}_F)$ and $\mathrm{GO}(V, \mathbb{A}_F)$. Recall that as an algebraic group, $\mathrm{GO}(V) \cong \mathrm{GSO}(V) \rtimes \{1, t\}$. First define τ^c by taking $V_{\tau^c} = \{f \circ c : f \in V_\tau\}$, where $c : \mathrm{GSO}(V, \mathbb{A}_F) \rightarrow \mathrm{GSO}(V, \mathbb{A}_F)$ is the isomorphism given by conjugation $g \mapsto tgt$, where $t \in \mathrm{GO}(V, F) \setminus \mathrm{GSO}(V, F)$. Then clearly τ^c is an irreducible cuspidal automorphic representation of $\mathrm{GSO}(V, \mathbb{A}_F)$. (Note that as an admissible representation, τ^c is isomorphic to the representation τ' with $V_{\tau'} = V_{\tau^c}$ and the action defined by $\tau'(g)f = \tau(tgt)f$, and so if we write $\tau \cong \otimes \tau_v$, then $\tau^c \cong \otimes \tau_v^c$.) By multiplicity one theorem, $\tau \cong \tau^c$ implies $V_\tau = V_{\tau^c}$ and in this case $f \circ c \in V_\tau$. Also let σ be an irreducible cuspidal automorphic representation of $\mathrm{GO}(V, \mathbb{A}_F)$. Define $V_\sigma^\circ = \{f|_{\mathrm{GSO}(V, \mathbb{A}_F)} : f \in V_\sigma\}$. Then either $V_\sigma^\circ = V_\tau$ for some irreducible cuspidal automorphic representation τ of $\mathrm{GSO}(V, \mathbb{A}_F)$ such that $\tau = \tau^c$, or $V_\sigma^\circ = V_\tau \oplus V_{\tau^c}$ for some irreducible cuspidal automorphic representation τ of $\mathrm{GSO}(V, \mathbb{A}_F)$ such that $\tau \neq \tau^c$. (See [HST, p.381–382].) Then we have

Proposition 6.2. *Define $\hat{\sigma}$ to be the sum of all the irreducible cuspidal automorphic representations of $\mathrm{GO}(V, \mathbb{A}_F)$ lying above τ , i.e. $\hat{\sigma} = \bigoplus_i \sigma_i$ where σ_i runs over all the irreducible cuspidal automorphic representations of $\mathrm{GO}(V, \mathbb{A}_F)$ such that $V_{\sigma_i}^\circ = V_\tau$ if $\tau = \tau^c$, or $V_{\sigma_i}^\circ = V_\tau \oplus V_{\tau^c}$ otherwise. Then*

$$\hat{\sigma} \cong \bigoplus_{\delta} \otimes_v \tau_v^{\delta(v)},$$

where δ runs over all the maps from the set of all places of F to $\{\pm\}$ with the property that $\delta(v) = +$ for almost all places of F , and $\delta(v) = +$ if $\tau_v \not\cong \tau_v^c$, and further if $\tau \cong \tau^c$, then $\prod_v \delta(v) = +$.

Moreover each $\otimes_v \tau_v^{\delta(v)}$ is (isomorphic to) an irreducible cuspidal automorphic representation of $\mathrm{GO}(V, \mathbb{A}_F)$.

Proof. This is Proposition 5.4 of [Tk]. □

This proposition tells us that if τ is an irreducible cuspidal automorphic representation of $\mathrm{GSO}(V, \mathbb{A}_F)$ and δ is a map from the set of all places of F to $\{\pm\}$ having the property described in the above proposition, then there is an irreducible cuspidal automorphic representation $\sigma = (\tau, \delta)$ of $\mathrm{GO}(V, \mathbb{A}_F)$ lying above τ such that $\sigma \cong \otimes_v \tau_v^{\delta(v)}$. We call such a map δ an “extension index” of τ , and (τ, δ) the extension of τ with an extension index δ .

Once we have this classification of representations of $\mathrm{GO}(V)$, the content of Theorem 1.3 can be understood to the reader. To prove Theorem 1.3, we need the following lemma.

Lemma 6.3. *Let σ_v be an infinite dimensional unitary irreducible admissible representation of $\mathrm{GO}(V_v)$ with $\dim \sigma_v > 1$, where $\dim V_v = 4$. Then σ_v is always bounded by some $e_v < 1$. Here v is not necessarily non-archimedean.*

Proof. In this proof, we suppress the subscript v and simply write σ, V , etc. First note that since σ is infinite dimensional, V can not be anisotropic. Assume $d = 1$. Then $\sigma = \tau^\delta$ where $\delta = +$ or $-$ and $\tau = \tau(\pi_1, \pi_2)$. If each π_i is bounded by e_i , one can see that τ and hence σ are bounded by $e_1 + e_2$. This requires a case-by-case classification of the representations of $\mathrm{GL}(2)$ along with the explicit isomorphism

$$\rho : \mathrm{GL}(2) \times \mathrm{GL}(2) / \Delta \mathbb{G}_m \cong \mathrm{GSO}(2, 2).$$

Under this isomorphism, we have the correspondence of, say, Borel subgroups as

$$\rho\left(\begin{bmatrix} a & * \\ 0 & c \end{bmatrix}, \begin{bmatrix} a' & * \\ 0 & c' \end{bmatrix}\right) = \begin{bmatrix} aa' & * & * & * \\ 0 & ac' & * & * \\ 0 & 0 & cc' & * \\ 0 & 0 & 0 & a'c \end{bmatrix}.$$

(For this, see the proof of Lemma 8.1 of [R5].) From this the assertion is obvious if both π_1 and π_2 are the Langlands quotients of representations induced from the Borel subgroup. Indeed, the spherical case is essentially proven in [R5, Lemma 8.1] in detail. The other cases can be similarly checked, but since the computation is fairly elementary, the detail is left to the reader.

Now if both of π_i are infinite dimensional, then since σ is unitary, so are π_i , which implies $e_i < \frac{1}{2}$. If one of π_i , say π_2 , is one dimensional, then π_2 is bounded by $\frac{1}{2}$, and also $e_1 < \frac{1}{2}$. In either way, σ is bounded by $e_1 + e_2 < 1$.

Similarly if $d \neq 1$ and so $\sigma = \tau^\delta$, where $\delta = +$ or $-$ and $\tau = \tau(\pi, \omega)$, then one sees that if π is bounded by e , then σ is bounded by $2e$. (Again see Lemma 8.1 of [R5] for the spherical case. The non-spherical case can be derived by considering the explicit correspondence of the parabolic subgroups of $GL(2, E)$ and $GSO(V)$. The detail is left to the reader.) And the unitarity implies $e < \frac{1}{2}$ and so $2e < 1$. \square

Now we are ready to prove Theorem 1.3

Proof of Theorem 1.3. (1). Assume that each local constituent σ_v has a non-zero theta lift to $GSp(4, F_v)$. Let σ' be an irreducible constituent of $\sigma|_{O(V, \mathbb{A})}$ (restriction of automorphic forms). It is by now well-known that the standard L -function $L^S(s, \sigma')$ of σ' is $L^S(s, \pi_1 \times \pi_2)$ if $d = 1$ and $\tau = \tau(\pi_1, \pi_2)$, or $L^S(s, \pi, \omega^{-1}, Asai)$ if $d \neq 1$ and $\tau = \tau(\pi, \omega)$. In either way, it does not vanish at $s = 1$. Also by the above lemma together with Lemma 6.1, we know that each σ'_v is bounded by some number less than 1. (Note that if V_v is anisotropic, clearly σ'_v is bounded by less than one.) Moreover by [R1, Lemma 4.2] σ'_v has a non-zero theta lift to $Sp(4, F_v)$. So by the main global non-vanishing result of the previous section, the global theta lift of σ' to $Sp(4, \mathbb{A})$ is non-vanishing and hence $\Theta_2(\sigma) \neq 0$. The converse is clear.

(2). Let σ' be as above. It is again well-known that

$$L^S(s, \sigma') = L^S\left(s + \frac{1}{2}, \pi_1 \otimes \chi\right) L^S\left(s - \frac{1}{2}, \pi_1 \otimes \chi\right).$$

By the same reasoning as above, one sees that σ'_v is bounded by some number less than one. Hence just as above we see that $\Theta_2(\sigma) \neq 0$. \square

Remark 6.4. As we mentioned before, part (1) of the theorem has been already proven by Gan and Ichino in their recent preprint [GI] by an entirely different method. For part (2), the same statement is also proven by their method. See ([GT3] also.) We believe that the converse is also true, and indeed in [GT3] we show it for the case where the central character of π_1 is trivial. Also we should mention that for the case where π_1 has the trivial central character is treated by Schmidt [Sch] in great detail in the context of the Saito-Kurokawa lifting.

As the last thing in this subsection, we will prove the following, which essentially shows that our choice of τ^+ and τ^- coincides with that of Roberts in [R5]. Namely we have

Proposition 6.5. *Let F be a (not necessarily non-archimedean) local field of char $F = 0$. Let $\tau \in \text{Irr}(GSO(V, F))$ be such that $\tau^c \cong \tau$, and so if $d = 1$ then $\tau = \tau(\pi, \pi)$, and if $d \neq 1$ then $\tau = \tau(\pi, \omega)$ with $\pi^c = \pi$. Then*

- (1) *If $d = 1$, then only τ^+ has a non-zero theta lift to $GSp(4, F)$.*
- (2) *If $d \neq 1$, then*

- (a) if π is the base change lift of some τ_0 on $\mathrm{GL}(2, F)$ whose central character is ω , both π^+ and π^- have non-zero theta lifts to $\mathrm{GSp}(4, F)$.
- (b) if π is the base change lift of some τ_0 on $\mathrm{GL}(2, F)$ whose central character is $\omega\chi_{E/F}$, then only π^+ has a non-zero theta lift to $\mathrm{GSp}(4, F)$.

Proof. Note that if F is non-archimedean, then part (2) has been proven by Gan and Ichino [GI, Theorem A.11]. If F is archimedean, then for part (2), F is always \mathbb{R} , and the corresponding statement is [HST, Lemma 12].

Hence we only have to consider part (1), and so let us assume $d = 1$. If $F = \mathbb{R}$ and V is split, then the corresponding statement can be read off from the work of Przebinda [Pr]. To be more specific, the restriction of τ^+ to $\mathrm{O}(2) \times \mathrm{O}(2) \subseteq \mathrm{GO}(2, 2)$ contains the representation of the form $((k) \otimes \mathrm{triv}) \oplus (\mathrm{triv} \otimes (k))$ for some k , where (k) is the representation of $\mathrm{O}(2)$ with weight k and triv is the trivial representation of $\mathrm{O}(2)$. Note that $(k) \otimes \mathrm{triv}$ (resp. $\mathrm{triv} \otimes (k)$) is what Przebinda means by $\pi_{k,0}^0$ (resp. $\pi_{0,k}^0$) in (2.1.6) of [Pr]. Namely in the language of Przebinda τ^+ contains the representations with the lowest degree K -types $\pi_{k,0}^0$ and $\pi_{0,k}^0$. (Also note that the restriction of τ^- to $\mathrm{O}(2) \times \mathrm{O}(2)$ contains $((k) \otimes \det) \oplus (\det \otimes (k))$, which is $\pi_{k,0}^1 \oplus \pi_{0,k}^1$.) Hence the restriction $\tau^+|_{\mathrm{O}(2,2)}$ contains a constituent which contains the representation with the lowest degree K -type $\pi_{k,0}^0$. Then (3.2.4) and (3.4.31) of [Pr] imply that this constituent has a nonzero theta lift to $\mathrm{Sp}(4, \mathbb{R})$ and hence τ^+ has a nonzero theta lift to $\mathrm{GSp}(4, \mathbb{R})$. (The author would like to thank A. Paul for checking this. See [P2].) If $F = \mathbb{C}$, this can be similarly read off from [AB, Theorem 2.8].

Now the only remaining cases are when F is non-archimedean and V is split, and F is non-archimedean or real and V is anisotropic. In either case we can write $\tau = \tau(\pi, \pi)$, where π is a representation of possibly split D^\times . Assume F is non-archimedean. Then we need the following lemma which is due to Roberts.

Lemma 6.6. *Let (π, V_π) be an irreducible admissible representation of possibly split D^\times . Then $(\omega_\pi^{-1} \otimes \pi, V_\pi)$ is known to be isomorphic to the contragredient (π^\vee, V_π^\vee) , and let us fix an isomorphism $R : (\omega_\pi^{-1} \otimes \pi, V_\pi) \rightarrow (\pi^\vee, V_\pi^\vee)$. Let $L : V_\pi \otimes V_\pi \rightarrow \mathbf{1}$ be the composition of $1 \otimes R : V_\pi \otimes V_\pi \rightarrow V_\pi \otimes V_\pi^\vee$ with the canonical map $V_\pi \otimes V_\pi^\vee \rightarrow \mathbf{1}$. If $\tau = \tau(\pi, \pi)$, then τ^δ has a non-zero theta lift to $\mathrm{GSp}(4)$ if and only if for all $v, w \in V_\pi$ we have $L(v \otimes w) = \delta L(w \otimes v)$.*

Assume D is split and so π is a representation of $\mathrm{GL}(2)$. If π is finite dimensional and so it is indeed one dimensional, then clearly $\delta = +$ in the above lemma, and hence τ^+ has a non-zero theta lift. So assume π is infinite dimensional. Then V_π can be identified with the space of the Kirillov model. Then the pairing $L : V_\pi \otimes V_\pi \rightarrow \mathbf{1}$ has been explicitly described by Jacquet-Langlands. (See the proof of part (i) of Theorem 2.18 in [JL].) And from the description there, one can easily see that $\delta = +$ in the above lemma. (Let us note that when π is spherical, this has been proven by Roberts [R5, Proposition 4.3].)

Finally assume D is non-split, and so π is a finite dimensional representation of D^\times . Also here we assume that F is not necessarily non-archimedean. If π is one dimensional, then clearly in the above lemma $\delta = +$. So assume that the dimension of π is > 1 . We need to resort to an unfortunate global argument. (We believe that there must be a purely local proof but at this moment we can provide only global one.) First we need the following well-known lemma, which is essentially the Jacquet-Langlands lift via theta lifting.

Lemma 6.7. *Let $V = D$ be a quaternion algebra over a number field \mathbb{F} , and let $\tau = \tau(\pi, \pi)$ be an infinite dimensional cuspidal automorphic representation of $\mathrm{GSO}(D, \mathbb{A})$. Then there is an extension index δ such that $\sigma = (\tau, \delta)$ has a non-zero theta lift to $\mathrm{GSp}(2, \mathbb{A}) = \mathrm{GL}(2, \mathbb{A})$.*

Now let \mathbb{F} be a number field with a place v_0 such that $\mathbb{F}_{v_0} = F$. Pick up another place v_1 . There exists a quaternion algebra D which ramifies exactly at v_0 and v_1 . One can construct a cuspidal

automorphic representation Π of $D^\times(\mathbb{A})$ such that $\Pi_{v_0} = \pi$ and Π_{v_1} is one-dimensional. Note that at all the other v , Π_v is a representation of $\mathrm{GL}(2, \mathbb{F}_v)$. Now by the above lemma, one sees that there is an extension index δ so that $\sigma = (\tau, \delta)$ has a non-zero theta lift to $\mathrm{GSp}(2, \mathbb{A}) = \mathrm{GL}(2, \mathbb{A})$ and hence to $\mathrm{GSp}(4, \mathbb{A})$. So locally at all v , σ_v has a non-zero theta lift to $\mathrm{GSp}(4, \mathbb{F}_v)$, and hence by what we have shown so far in this proof, one knows that $\delta(v) = 1$ for all v other than v_0 . But by Proposition 6.2, one also knows that $\prod_v \delta(v) = 1$ and hence $\delta(v_0) = 1$. This completes the proof. \square

6.2. The theta lift from $\mathrm{GSp}(4)$. Finally, we consider global theta lifts from $\mathrm{GSp}(4, \mathbb{A})$. Especially, we consider lifts to the orthogonal group $\mathrm{GO}(V_D, \mathbb{A})$, where V_D is the 6 dimensional quadratic space given by $V_D = D \oplus \mathbb{H}$, where D is a (possibly split) quaternion algebra and \mathbb{H} is the hyperbolic plane. This pair $(\mathrm{GSp}(4), \mathrm{GO}(V_D))$, both locally and globally, has been studied in great detail by Gan and the author in [GT1] in the context of Shalika period, especially for generic representations of $\mathrm{GSp}(4)$. But here we apply our non-vanishing theorem in the previous section to not necessarily generic ones. To consider this pair, let us first mention that to consider the theta correspondence for the pair $(\mathrm{GSp}(4), \mathrm{GO}(V_D))$, we have to actually consider the pair $(\mathrm{GSp}(4)^+, \mathrm{GO}(V_D))$ with $\mathrm{GSp}(4)^+$ defined by

$$\mathrm{GSp}(4)^+ = \{g \in \mathrm{GSp}(4) : \lambda(g) \in \mathrm{Im}(\lambda_D)\},$$

where $\lambda_D : \mathrm{GO}(V_D) \rightarrow \mathbb{G}_m$ is the similitude character. It can be checked that if v is non-archimedean, then $\mathrm{GSp}(4, F_v)^+ = \mathrm{GSp}(4, F_v)$, and if v is archimedean and D_v is split, then $\mathrm{GSp}(4, F_v)^+ = \mathrm{GSp}(4, F_v)$. But if v is archimedean and D_v is non-split (so necessarily v is real), then $\mathrm{GSp}(4, F_v)^+$ is the identity component of the real Lie group $\mathrm{GSp}(4, \mathbb{R})$. So if we let Σ_D be the set of places v so that D_v is non-split, and $\Sigma_{D, \infty} \subset \Sigma_D$ the subset of archimedean places in Σ_D , then we have

$$\mathrm{GSp}(4, \mathbb{A})^+ = \prod_{v \in \Sigma_{D, \infty}} \mathrm{GSp}(4, F_v)^+ \times \prod'_{v \notin \Sigma_{D, \infty}} \mathrm{GSp}(4, F_v).$$

Note that if π is a cuspidal automorphic representation of $\mathrm{GSp}(4, \mathbb{A})$, then $\pi|_{\mathrm{GSp}(4, \mathbb{A})^+}$ (restriction of forms) is a non-zero cuspidal automorphic representation of $\mathrm{GSp}(4, \mathbb{A})^+$. By the global theta lift of π to $\mathrm{GO}(V_D, \mathbb{A})$, we mean the global theta lift of $\pi|_{\mathrm{GSp}(4, \mathbb{A})^+}$ to $\mathrm{GO}(V_D, \mathbb{A})$. Also locally at $v \in \Sigma_{D, \infty}$, by the local theta lift of π_v , we mean the local theta lift of one of the constituents of $\pi_v|_{\mathrm{GSp}(4, \mathbb{R})^+}$. Indeed, if $\pi_v|_{\mathrm{GSp}(4, \mathbb{R})^+}$ has two constituents, then both of them have non zero theta lifts to $\mathrm{GO}(V_D, \mathbb{R})$ and their lifts are isomorphic. The following easily follows from the non-vanishing theorem of the previous section.

Proposition 6.8. *Let π be an irreducible cuspidal automorphic representation of $\mathrm{GSp}(4, \mathbb{A})$, and D a division quaternion algebra over F . Assume π satisfies the following three assumptions:*

- (1) *The standard degree 5 L -function $L^S(s, \pi)$ does not vanish at $s = 1$.*
- (2) *At each place v , π_v has a non-zero theta lift to $\mathrm{GO}(V_D)$.*
- (3) *At each place v , π_v is bounded by $e_v < 1$.*

Then the global theta lift of π to $\mathrm{GO}(V_D, \mathbb{A})$ does not vanish.

Proof. Let π' be an irreducible constituent of $\pi|_{\mathrm{Sp}(4, \mathbb{A})}$ (restriction of forms). Then the standard L -function $L^S(s, \pi')$ is the same as the standard degree 5 L -function $L^S(s, \pi)$. Also if each π_v has a non-zero local theta lift to $\mathrm{GO}(V_D, F_v)$, then π'_v has a non-zero local theta lift to $\mathrm{O}(V_D, F_v)$. Hence by Theorem 1.1, the global theta lift of π' to $\mathrm{O}(V_D, \mathbb{A})$ does not vanish and so π has a non-zero global theta lift to $\mathrm{GO}(V_D, \mathbb{A})$. \square

This allows us to prove Theorem 1.4, which is an instance of the Langlands functoriality.

Proof of Theorem 1.4. Let us first mention the following fact, which is a known instance of the conservation conjecture on the first occurrence indices of theta lifting. Let π_v is an irreducible admissible representation of $\mathrm{GSp}(4, F_v)$ with v non-archimedean, and let

$$V_m = \mathbb{H}^m \quad \text{and} \quad V_m^\# = D \oplus \mathbb{H}^{m-2},$$

and consider the theta lifts $\theta_m(\pi)$ and $\theta_m^\#(\pi)$ to $\mathrm{GO}(V_m)$ and $\mathrm{GO}(V_m^\#)$, respectively. Set

$$\begin{cases} m(\pi) = \inf\{m : \theta_m(\pi) \neq 0\} \\ m^\#(\pi) = \inf\{m : \theta_m^\#(\pi) \neq 0\}. \end{cases}$$

Then it is known that

$$m(\pi) + m^\#(\pi) = 6.$$

The proof of this fact is given in [GT1, GT2], especially [GT2, Section 8] and [GT2, Section 5].

Now let Σ_π be the set of all (not necessarily) non-archimedean places v so that π_v has a non-zero theta lift to $\mathrm{GO}(V_D, F_v)$ where D is the division quaternion algebra. Note that for almost all v , π_v has a non-zero theta lift to the split $\mathrm{GO}(2, 2)$ by [GT4, Table 1] and hence by the above ‘‘conservation conjecture’’ for $\mathrm{GSp}(4)$, one can see that π_v does not have a non-zero theta lift to $\mathrm{GO}(V_D, F_v)$. So Σ_π is a finite set. Also note that $v_0 \in \Sigma_\pi$. Now if $|\Sigma_\pi|$ is even, one can find a quaternion algebra D over F so that $\Sigma_D = \Sigma_\pi$. If $|\Sigma_\pi|$ is odd, then one can find a quaternion algebra D over F so that $\Sigma_D = \Sigma_\pi \setminus \{v_0\}$. In either case, D is such that at each place v , π_v has a non-zero theta lift to $\mathrm{GO}(V_D, F_v)$. Hence by the previous proposition, we have a non-zero global theta lift to $\mathrm{GO}(V_D, \mathbb{A})$.

Assume that this global theta lift is cuspidal, and denote an irreducible constituent by σ . The restriction $\sigma|_{\mathrm{GSO}(V_D, \mathbb{A})}$ is identified with a cuspidal automorphic representation Π' of $\mathrm{GL}_2(D, \mathbb{A})$. (See [GT1, Introduction].) By [B, BR], there is a cuspidal automorphic representation Π of $\mathrm{GL}(4, \mathbb{A})$, namely the global Jacquet-Langlands transfer of Π' to $\mathrm{GL}(4, \mathbb{A})$. By the explicit computations of theta lifts carried out in [GT2, GT4], one can check that Π is the desired functorial lift of π .

Next assume that the global theta lift of π to $\mathrm{GO}(V_D, \mathbb{A})$ is not cuspidal. Then it has a non-zero theta lift to $\mathrm{GO}(D, \mathbb{A})$, which is cuspidal. Denote an irreducible constituent by σ . The restriction $\sigma|_{\mathrm{GSO}(D, \mathbb{A})}$ is identified with a cuspidal automorphic representation $\pi_1 \otimes \pi_2$ of $D^\times(\mathbb{A}) \times D^\times(\mathbb{A})$. If we denote the Jacquet-Langlands lift of π_i by π_i^{JL} , then the isobaric sum $\pi_1^{\mathrm{JL}} \boxplus \pi_2^{\mathrm{JL}}$ is the desired lift. \square

Of course, if π is generic, the above transfer has been obtained by Asgari-Shahidi [ASh] without any assumption, which can be shown to be the strong lift in [GT2].

Now let us examine how restrictive the assumptions of this theorem are. The L -function condition in (1) is supposed to be satisfied for the non-CAP representations, though at this moment the author does not know if there is any method of showing it. (Of course, if π is generic, this assumption is known to be satisfied by Shahidi.) Assumption (2) seems to be very small. But it should be mentioned that if π corresponds to a Siegel modular form of level 1, unfortunately this assumption is not satisfied. This can be shown as follows: π_v is unramified for all non-archimedean v and hence does not have a non-zero theta lift to $\mathrm{GO}(V_D, F_v)$ for non-split D , and at the real place v , since π_v is non-generic, it cannot have a theta lift to $\mathrm{GO}(3, 3)$.

Let us consider assumption (3). This is seemingly the most problematic and indeed crucial for our method. Let π_v be an irreducible admissible representation of $\mathrm{GSp}(4)$ which is the Langlands quotient of $\delta_1 \times \cdots \times \delta_t \rtimes \tau$. (Of course for $\mathrm{GSp}(4)$, t is at most 2.) Then it is well-known that if π_v is unitary, then $e(\delta_1) \leq 1$. (For this assertion, for the non-archimedean case see the table of representations of $\mathrm{GSp}(4)$ by Roberts and Schmidt [RS, Appendix], and for the archimedean case see [Pr, A.3.7] and citations therein.) Hence the only cases we need to worry about is when $e(\delta_1) = 1$.

Unfortunately, it is known that such a π_v as $e(\delta_1) = 1$ does occur as a constituent of a cuspidal representation of $\mathrm{GSp}(4, \mathbb{A})$. But all the known examples of such cuspidal representations are obtained as theta lifts of characters on some orthogonal groups, which are all CAP representations. (See [So] for the detail of CAP representations.) Accordingly, we believe that such a π_v occurs as a constituent of only CAP representations for the following reason. First, let us mention

Lemma 6.9. *Let π_v be an irreducible admissible representation of $\mathrm{GSp}(4, F_v)$ which is the Langlands quotient of the standard module $\delta_1 \times \cdots \times \delta_t \rtimes \tau$. If π_v is unitary and generic, then π_v is bounded by $e_v < 1$.*

Proof. For the non-archimedean case, one can verify this by looking at the table of representations of $\mathrm{GSp}(4)$ [RS, Appendix]. But in general, one can argue as follows. First of all, as we mentioned above, if π_v is unitary, then $e(\delta_1) \leq 1$. Now if $e(\delta_1) = 1$, the induced representation $\delta_1 \times \cdots \times \delta_t \rtimes \tau$ is reducible. For the non-archimedean case, this can be also seen in the table [RS, Appendix], and for the archimedean case, the reducibility of the standard modules is found in [Pr, p.91]. (Note that in [Pr], the reducibility is shown only for the real case, but the proof there is completely general, which can be applied to the complex case.) Now by the so called the standard modules conjecture [CSH], which is known for $\mathrm{GSp}(4)$, we know that π_v cannot be generic. Note that for the archimedean case, the standard modules conjecture for the linear groups was proven by Vogan [V] over thirty years ago. \square

Now if π is an irreducible cuspidal automorphic representation of $\mathrm{GSp}(4, \mathbb{A})$ which is not CAP, then it is believed that one can find an irreducible cuspidal automorphic representation π' of $\mathrm{GSp}(4, \mathbb{A})$ which is globally generic such that π_v and π'_v are in the same local L -packet at all v . Hence if π_v is bounded by $e_v < 1$, then so does π'_v . Thus if π is not CAP, then each π_v should be bounded by $e_v < 1$.

Finally, this consideration leads us to our proof of Theorem 1.5, which has been already proven by Kudla, Rallis and Soudry [KRS, Theorem 8.1] with an assumption that the base field F is totally real. However here we give an alternative proof of the theorem which does not require F be totally real and hence remove this unfortunate totally real assumption present in [KRS].

Proof of Theorem 1.5. Note that since π_v is generic, by Lemma 6.9 we know that π_v is bounded by $e_v < 1$. Assume v is non-archimedean. By [GT2, Section 5 and 7] one can see that the theta lift of π_v to the anisotropic $\mathrm{GO}(D)$ is zero, and hence has a nonzero theta lift to the split $\mathrm{GO}(3, 3)$. Next assume that v is complex. Then one sees from [AB, Theorem 2.8] that π_v has a non-zero theta lift to $\mathrm{GO}(3, 3)$. (It is not so immediate to see this from this theorem of [AB]. But one can check it case-by-case. Indeed in the notation of [AB], if π_v is the one with the parameters

$$\mu_2 = (a_1, a_2) \quad \text{and} \quad \nu_2 = (b_1, b_2)$$

then it has a non-zero theta lift to $\mathrm{GO}(3, 3)$ whose parameter is

$$\mu_1 = (a_1, a_2, 0), \quad \nu_1 = (b_1, b_2, 0) \quad \text{and} \quad \epsilon = 1.$$

Also let us mention that for complex v , there is no discrepancy between the isometry theta correspondence and the similitude one, because $\mathrm{GSp}(4, \mathbb{C}) = \mathbb{C}^\times \mathrm{Sp}(4, \mathbb{C})$ and similarly for the orthogonal groups.) Finally assume v is real. First, for the sake of contradiction, assume π_v has a non-zero theta lift to the anisotropic $\mathrm{GO}(D) = \mathrm{GO}(4, 0)$. Then any constituent π'_v of the restriction $\pi_v|_{\mathrm{Sp}(4, \mathbb{R})}$ has a non-zero theta lift to either $\mathrm{O}(4, 0)$ or $\mathrm{O}(0, 4)$. But it is well-known that if this happens, then π'_v must be a (limit of) holomorphic discrete series, which is never generic. Hence each constituent π'_v does not have a non-zero theta lift to $\mathrm{O}(4, 0)$ or $\mathrm{O}(0, 4)$. By [P, Corollary 4.16] one sees that π'_v has a non-zero theta lift to $\mathrm{O}(3, 3)$. (Also see the remark right before Theorem 4.8 of [P].) Hence π_v has a non-zero theta lift to $\mathrm{GO}(3, 3)$.

Therefore π satisfies the assumption of Proposition 6.8 with D split, and hence π has a non-zero global theta lift to $\mathrm{GO}(V_D, \mathbb{A})$ with D split. Assume this theta lift $\Theta_3(V_\pi)$ is in the space of cusp forms, and assume σ is an irreducible constituent of $\Theta_3(V_\pi)$. By the main theorem of [GRS] with necessary modifications to the similitude theta lifting, one can see that σ has a non-zero generic cuspidal theta lift $\Theta_2(V_\sigma)$ to $\mathrm{GSp}(4, \mathbb{A})$. Let Π be an irreducible constituent of $\Theta_2(V_\sigma)$. Then $\Pi_v \cong \pi_v$ for all v , and hence $\Pi \cong \pi$, which implies π is globally generic. If $\Theta_3(V_\pi)$ is in the space of cusp forms, by the tower property of theta lifting one sees that π has a non-zero theta lift to $\mathrm{GO}(2, 2)$. By applying the same argument, one can see that π is globally generic. \square

The corollary of the theorem (Corollary 1.6), which is on the multiplicity one theorem for the generic representation of $\mathrm{GSp}(4)$, is now immediate.

Proof of Corollary 1.6. If F is totally real, this theorem has been proven by Jiang and Soudry [JS]. Looking at their proof, one notices that the only reason they need this totally real assumption is that they needed the above theorem (Theorem 1.5), which had been proven only with the totally real assumption by [KRS]. Hence the above theorem removes this assumption from [JS]. \square

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