Nonlinear Integral Equations for the Inverse Problem in Corrosion Detection from Partial Cauchy Data

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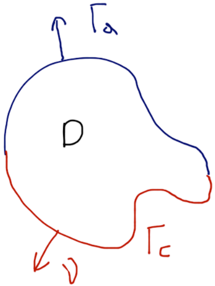
Formulation of the Problem

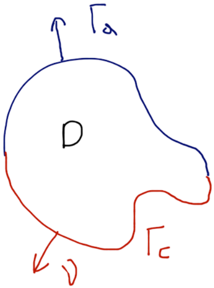


$$\begin{aligned} \Delta u = 0 \quad & \text{in} \quad D \\ u = f \quad & \text{on} \quad \Gamma_a \\ \frac{\partial u}{\partial \nu} + \lambda u = 0 \quad & \text{on} \quad \Gamma_c \end{aligned}$$

We assume that D has Lipshitz boundary ∂D such that $\partial D = \overline{\Gamma_a} \cup \overline{\Gamma_c}$ and $\lambda(x) \geq 0$ is in $L^\infty(\Gamma_c)$.

If $f \in H^{1/2}(\Gamma_a)$ this problem has a unique solution $u \in H^1(D)$

The Inverse Problem



The inverse problem is: given the Dirichlet data $f \in H^{1/2}(\Gamma_a)$ and the (measured) Neumann data

$$g := \frac{\partial u}{\partial \nu} \quad \text{on } \Gamma_a \qquad g \in H^{-1/2}(\Gamma_a)$$

determine the shape of the portion Γ_c of the boundary and the impedance function $\lambda(x)$.

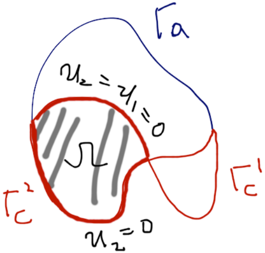
In particular, $\lambda = 0$ corresponds to homogeneous Neumann boundary condition on Γ_c and $\lambda = \infty$ corresponds to homogeneous Dirichlet boundary condition on Γ_c.

Uniqueness of the Inverse Problem

Does one pair of Cauchy data $u|_{\Gamma_a} = f \in H^{1/2}(\Gamma_a)$ and $\left.\frac{\partial u}{\partial \nu}\right|_{\Gamma_a} = g \in H^{-1/2}(\Gamma_a)$ uniquely determine Γ_c?

Consider first the Dirichlet case, i.e. $\lambda = \infty$

Let D_1, D_2 be such that $\partial D_1 = \overline{\Gamma_a} \cup \overline{\Gamma_c^1}$ and $\partial D_2 = \overline{\Gamma_a} \cup \overline{\Gamma_c^2}$

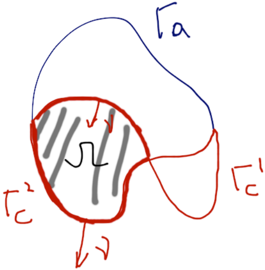


- $\Delta u_i = 0$ in D_i, $i = 1, 2$
- $u_i = 0$ on Γ_c^i, $u_1 = u_2 = f$ and $\partial u_1/\partial \nu = \partial u_2/\partial \nu = g$ on Γ_a.
- Holmgren's theorem $\Longrightarrow u_1 = u_2$ in $D_1 \cap D_2$.
- $\Delta u_2 = 0$ in Ω and $u_2 = 0$ on $\partial\Omega \Longrightarrow u_2 = 0$ and thus $f = 0$.

Uniqueness of the Inverse Problem

This idea does not work in the case of impedance boundary condition.

Indeed by the same reasoning we arrive at the following problem for $w := u_2$ in Ω



$$\Delta w = 0 \qquad \text{in} \quad \Omega$$

$$\frac{\partial w}{\partial \nu} + \lambda_2 w = 0 \qquad \text{on} \quad \partial\Omega_2$$

$$\frac{\partial w}{\partial \nu} - \lambda_1 w = 0 \qquad \text{on} \quad \partial\Omega_1$$

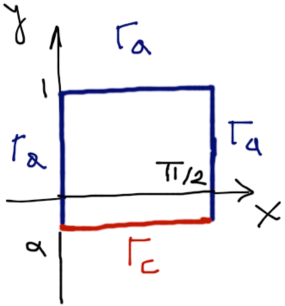
where ν is the normal outward to Ω.

This is not a coercive problem!

Examples of Non-Uniqueness

One pair of Cauchy data does not uniquely determine Γ_c in the case of impedance boundary condition even for known impedance λ.

Example 1: *Cakoni-Kress, Inverse Problems and Imaging (2007)*.



$$D = \left\{ (x, y) \in \mathbb{R}^2 : 0 < x < \pi/2, -\alpha < y < 1 \right\}$$

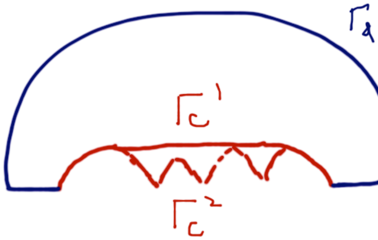
Take $\lambda = 1$ and consider the harmonic function $u(x, y) = (\cos x + \sin x)e^y$. Then

$$\frac{\partial u}{\partial \nu} + u = 0 \qquad \text{on } \Gamma_c.$$

For the data $f := u|_{\Gamma_a}$ and $g := \partial u/\partial \nu|_{\Gamma_a}$ we have infinitely many solutions by changing α.

Examples of Non-Uniqueness

Example 2:



Pagani-Pieroti, Inverse Problems (2009)

- Γ_c^1 consists of two arcs of the form $(x-c)^2 + y^2 = \frac{1}{\lambda^2}$ joined by $y = 1/\lambda$.
- Γ_c^2 consists of arcs of the above form with different c.
- $u(x,y) = y$, $f := y|_{\Gamma_a}$, $g := \partial y/\partial \nu|_{\Gamma_a}$

Examples of non-uniqueness for the case of impedance obstacle surrounded by the measurement surface are given in *Haddar-Kress, J. Inverse Ill-Posed Problems, (2006)* and *Rundell, Inverse Problems, (2008)*.

Uniqueness

Question: What is the optimal measurements that uniquely determine Γ_c?

This was first answered in *Bacchelli, Inverse Problems, (2009)* with improvement in *Pagani-Pieroti, Inverse Problems (2009)*.

Theorem

Assume that Γ_c^i, $i = 1, 2$, are $C^{1,1}$-smooth curves such that $\partial D^i := \Gamma_a \cup \Gamma_c^i$ are $C^{1,1}$-curvilinear polygons and $\lambda^i \in L^\infty(\Gamma_c^i)$. Let $f^1, f^2 \in H^{3/2}(\Gamma_a)$ be such that f^1 and f^2 are linearly independent, and $f^1 > 0$ and u^i, $i = 1, 2$, be the harmonic functions in D^i corresponding to λ^i, f^i. If

$$\frac{\partial u^1}{\partial \nu} = \frac{\partial u^2}{\partial \nu} \qquad \text{on some open arc of } \Gamma_a$$

$$\text{then} \qquad \Gamma_c^1 = \Gamma_c^2 \quad \text{and} \quad \lambda_1 = \lambda_2.$$

Remarks

- The uniqueness result is valid in $\mathbb{R}^2$ or $\mathbb{R}^3$.
- If Γ_c is known then one pair of Cauchy data uniquely determines $\lambda \in L^\infty(\Gamma_c)$. This is a simple consequence of Holmgren's Theorem.
- In the case of Neumann boundary condition (i.e. $\lambda = 0$) one pair of Cauchy data uniquely determines Γ_c. The proof follows the idea of the Dirichlet case with more care to handle irregular $\partial\Omega$ (could have cusps); in $\mathbb{R}^2$ one can use the conjugate harmonic of the solution.
- Logarithmic stability estimates for both Γ_c and λ with two Cauchy data pairs is proven in *Sincich, SIAM J. Math. Anal. (2010)*.

Nonlinear Integral Equation

Cauchy Problem: Given the pair $f \in H^{1/2}(\Gamma_a)$ and $g \in H^{-1/2}(\Gamma_a)$ find $\alpha \in H^{1/2}(\Gamma_c)$ and $\beta \in H^{-1/2}(\Gamma_c)$ such that there exists a harmonic function $u \in H^1(D)$ satisfying

$$u|_{\Gamma_a} = f, \quad \left.\frac{\partial u}{\partial \nu}\right|_{\Gamma_a} = g, \quad u|_{\Gamma_c} = \alpha, \quad \left.\frac{\partial u}{\partial \nu}\right|_{\Gamma_c} = \beta.$$

Let us focus in $\mathbb{R}^2$ and make the ansatz

$$u(x) := (\mathcal{S}\varphi)(x) = \int_{\partial} \Phi(x,y)\varphi(y)\,ds(y), \; x \in D, \; \varphi \in H^{-1/2}(\partial D)$$

where $\Phi(x,y) := 2\pi \ln |x-y|^{-1}$, and for $x \in \partial D$ define

$$(S\varphi)(x) := \int_{\partial D} \Phi(x,y)\varphi(y)\,ds(y)$$

$$(K'\varphi)(x) := \int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(x)}\varphi(y)\,ds(y).$$

Determination of λ

Inverse Impedance Problem: ∂D is known – determine λ from a knowledge of one pair of Cauchy data (f, g) on Γ_a.

This problem is related to completion of Cauchy data.

Theorem

$\alpha := u|_{\Gamma_c}$, $\beta = \left.\frac{\partial u}{\partial \nu}\right|_{\Gamma_a}$ is a solution of the Cauchy if and only if $u := (\mathcal{S}\varphi)(x)$ where $\varphi \in H^{-1/2}(\partial D)$ is a solution of the ill-posed equation

$$A\varphi := \left(\begin{array}{c} S\varphi \\ K'\varphi + \frac{\varphi}{2} \end{array} \right)_{\Gamma_a} = \left(\begin{array}{c} f \\ g \end{array} \right).$$

Determination of λ

We can prove

Theorem

The operator $A : L^2(\partial D) \to L^2(\Gamma_a) \times L^2(\Gamma_a)$ is compact, injective and has dense range.

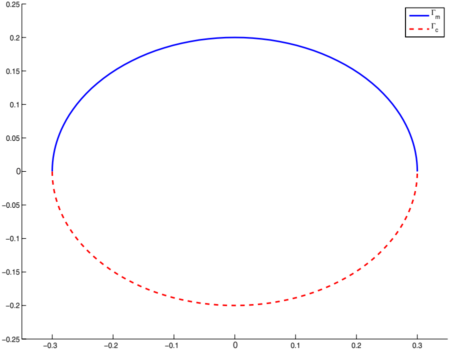
To reconstruct $\lambda(x) \in L^\infty(\Gamma_c)$

- Solve $A\varphi = (f, g)$ for φ using Tikhonov regularization.
- Compute u, α and β.
- Find impedance $\lambda(x)$ as least square solution of

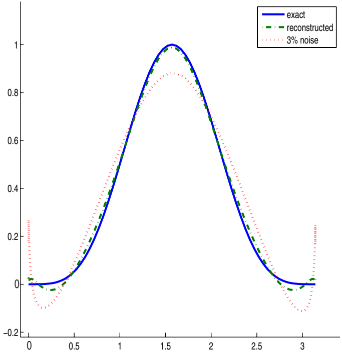
$$\alpha + \lambda\beta = 0$$

Example of Reconstruction of λ

D is the ellipse $z(t) = (0.3\cos t, 0.2\sin t)$, $t \in [0,\ 2\pi]$ and $\lambda(t) = \sin^4 t$, $t \in [\pi,\ 2\pi]$.



(a) Geometry of the boundary



(b) Reconstruction

Nonlinear Integral Equations

Inverse Shape and Impedance Problem: Determine both Γ_c and λ from a knowledge of two pairs of Cauchy data (f, g) on Γ_a.

Theorem

The inverse shape and impedance problem is equivalent to solving

$$S\varphi_i = f_i \qquad \text{on } \Gamma_a$$

$$K'\varphi_i + \frac{\varphi_i}{2} = g_i \qquad \text{on } \Gamma_a$$

and

$$K'\varphi_i + \frac{\varphi_i}{2} + \lambda S\varphi_i = 0 \qquad \text{on } \Gamma_c$$

$i = 1, 2$, *for* Γ_c, φ_1, φ_2 *and* λ.

Remarks

It is possible to obtain a different system of nonlinear integral equations equivalent to the inverse shape and impedance problem by staring with a different ansatz for u. In particular,

$$u(x) := \int_{\partial D} \left(\varphi(y) \frac{\partial \Phi(x,y)}{\partial \nu} - \psi(y) \Phi(x,y) \right) ds(y), \qquad x \in D$$

Here by Green's representation theorem

$$\varphi = u|_{\partial D} \qquad \psi = \left. \frac{\partial u}{\partial \nu} \right|_{\partial D}.$$

Cakoni, Kress and Schuft, Inverse Problems, (2010).

Newton Iterative Method

Assume now that $\partial D := \{z(t) : 0 \leq t \leq 2\pi\}$,
$\Gamma_a := \{z(t) : \pi \leq t \leq 2\pi\}$, $\Gamma_c := \{z(t) : 0 \leq t \leq \pi\}$.

Setting $\psi(t) = |z(t)'|\varphi(z(t))$ we have

$$(\widetilde{S}\psi)(t) = \frac{1}{2\pi}\int_0^{2\pi} \ln \frac{1}{|z(t) - z(\tau)|}\, \psi(\tau) d\tau$$

and

$$(\widetilde{K}'\psi)(t) = -\frac{1}{2\pi|z'(t)|}\int_0^{2\pi} \frac{[z'(t)]^{\perp} \cdot [z(t) - z(\tau)]}{|z(t) - z(\tau)|^2}\, \psi(\tau) d\tau + \frac{\psi(t)}{2|z'(t)|}$$

for $t \in [0, 2\pi]$.

Newton Iterative Method

Then the system of nonlinear integral equations we need to solve reads:

$$\begin{aligned} \widetilde{S}\psi_i &= f_i \quad \text{on } [\pi, 2\pi], \\ \widetilde{K}'\psi_i &= g_i \quad \text{on } [\pi, 2\pi] \end{aligned}$$

and

$$\widetilde{K}'\psi_i + \lambda\widetilde{S}\psi_i = 0 \quad \text{on } [0, \pi]$$

for $i = 1, 2$, where $\lambda = \lambda \circ z$ on $[0, \pi]$, $f_i = f_i \circ z$ and $g_i = g_i \circ z$ on $[\pi, 2\pi]$.

We linearize the system with respect ψ_i, λ and $z_c(t)$, $t \in [0, \pi]$.

Newton Iterative Method

$\psi_i + \chi_i$, $\lambda + \mu$, $z_c + \zeta$ (w.l.o.g. we assume $\zeta = q[z']^\perp$)

$$\begin{aligned} \widetilde{S}(\psi_i, z) + \widetilde{S}(\chi_i, z) + d\widetilde{S}(\psi_i, z; \zeta) &= f_i \quad \text{on } [\pi, 2\pi], \\ \widetilde{K}'(\psi_i, z) + \widetilde{K}'(\chi_i, z) + d\widetilde{K}'(\psi_i, z; \zeta) &= g_i \quad \text{on } [\pi, 2\pi], \end{aligned}$$

and

$$\begin{aligned} &\widetilde{K}'(\psi_i, z) + \widetilde{K}'(\chi_i, z) + d\widetilde{K}'(\psi_i, z; \zeta) \\ &+\lambda \left\{ \widetilde{S}(\psi_i, z) + \widetilde{S}(\chi_i, z) + d\widetilde{S}(\psi_i, z; \zeta) \right\} + \mu \widetilde{S}(\psi_i, z) = 0 \quad \text{on } [0, \pi] \end{aligned}$$

for $i = 1, 2$.

Here, the operators $d\widetilde{K}'$ and $d\widetilde{S}$ denote the Fréchet derivatives with respect to z in direction ζ of the operators $\widetilde{K}'$ and $\widetilde{S}$, respectively.

Local Uniqueness

Theorem

Let $z_c \in C^2[0,\pi]$, $\psi_1, \psi_2 \in L^2[0,2\pi]$, $\lambda \in C[0,\pi]$ be the solutions of the nonlinear system with exact data (f_1, g_1) and (f_2, g_2), where $f_1 > 0$ and f_2 are linearly independent. Assume that $\zeta = q[z']^\perp$, $q \in C^2[0,\pi]$, $\chi_1, \chi_2 \in L^2[0,2\pi]$ and $\mu \in C[0,\pi]$ solve the homogeneous system

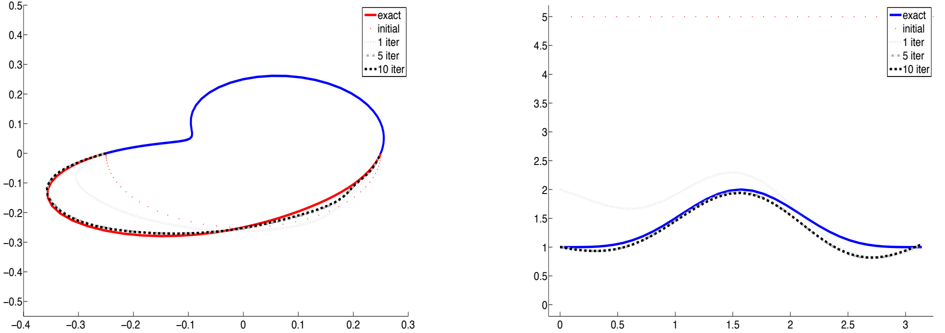
$$
\begin{aligned}
\widetilde{S}(\chi_i, z) + d\widetilde{S}(\psi_i, z; \zeta) &= 0 \quad \text{on } [\pi, 2\pi], \\
\widetilde{K}'(\chi_i, z) + d\widetilde{K}'(\psi_i, z; \zeta) &= 0 \quad \text{on } [\pi, 2\pi] \\
\widetilde{K}'(\chi_i, z) + d\widetilde{K}'(\psi_i, z; \zeta) + \lambda\widetilde{S}(\chi_i, z) & \\
+\lambda d\widetilde{S}(\psi_i, z; \zeta) + \mu\widetilde{S}(\psi_i, z) &= 0 \quad \text{on } [0, \pi].
\end{aligned}
$$

Then $\chi_1 = \chi_2 = 0, \zeta = 0$ and $\mu = 0$.

Newton Iterative Method

1. We make an initial guess for the non-accessible boundary part Γ_c, parameterized by z_c, and for the impedance function λ. Then we find the densities ψ_1 and ψ_2 for the two pairs of Cauchy data (f_1, g_1) and (f_2, g_2) by solving the first two equations of the nonlinear system.
2. Given an approximation for z_c, ψ_1, ψ_2 and λ, the linearized system is solved for ζ, χ_1, χ_2 and μ to obtain the update $z_c + \zeta$ for the parameterization, $\psi_1 + \chi_1, \psi_2 + \chi_2$ for the densities and $\lambda + \mu$ for the impedance.
3. The second step is repeated until a suitable stopping criterion is satisfied.

Example of Reconstructions



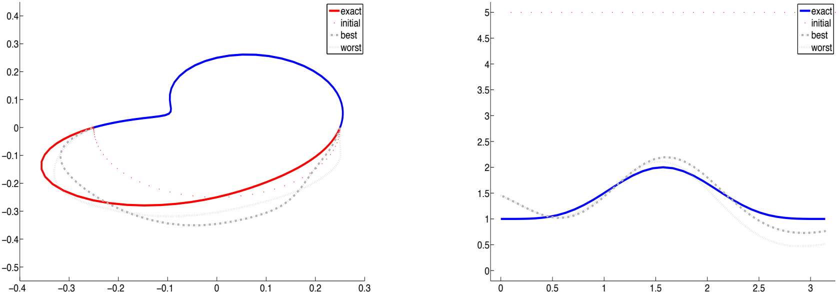
(c) Shape from potential approach

(d) Impedance from potential approach

FIG. 4.2. *Reconstruction of shape* (4.2) *and impedance* (4.1) *with* $\lambda_{\text{initial}} = 5$

$\lambda(t) = \sin^4 t + 1$, $t \in [0, \pi]$

Example of Reconstructions

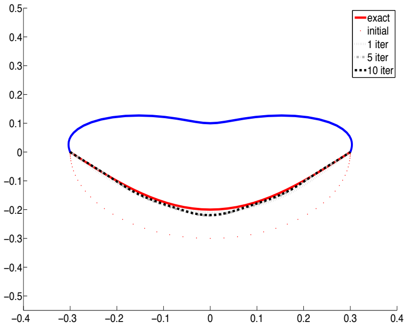


(c) Shape from potential approach

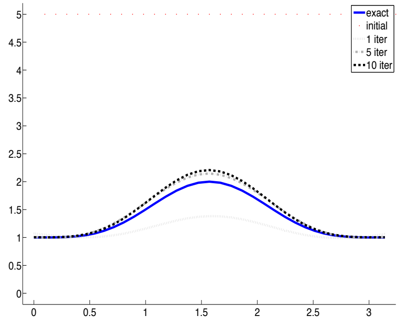
(d) Impedance from potential approach

FIG. 4.5. *Reconstruction of shape* (4.2) *and impedance* (4.1) *with* $\lambda_{\text{initial}} = 5$ *and* 3% *noise*

Example of Reconstructions



(c) Shape from potential approach



(d) Impedance from potential approach

FIG. 4.3. *Reconstruction of shape* (4.4) *and impedance* (4.1) *with* $\lambda_{\text{initial}} = 5$

Literature

This discussion is based on

- F. Cakoni and R. Kress, Integral equations for inverse problems in corrosion detection from partial Cauchy data, *Inverse Probl. Imaging* **1** (2007), no. 2, 229-245.
- V. Bacchelli, Uniqueness for the determination of unknown boundary and impedance with the homogeneous Robin condition, *Inverse Problems* **25** (2009), no. 1, 015004.
- C.D. Pagani and D. Pierotti, Identifiability problems of defects with the Robin condition, *Inverse Problems* **25** (2009), no. 5, 055007.

Literature, cont.

- F. Cakoni, R. Kress and C. Schuft, Integral equations for shape and impedance reconstruction in corrosion detection, *Inverse Problems*, **26** (2010), no. 9.
- F. Cakoni, R. Kress and C. Schuft, Simultaneous reconstruction of shape and impedance in corrosion detection, *Methods Appl. Anal.* **17** (2010), no. 4, 357-377.
- E. Sincich, Stability for the determination of unknown boundary and impedance with a Robin boundary condition, *SIAM J. Math. Anal.* **42** (2010), no. 6, 2922-2943