A Direct D-bar Reconstruction Algorithm for Recovering a Complex Conductivity in 2-D

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The EIT Problem

Current is applied on electrodes on the surface of the body and the resulting voltage is measured. Let $\gamma = \sigma + i\omega\epsilon$. 

Mathematically, this is governed by the inverse admittivity problem: Can the admittivity $\gamma$ be recovered from measurements of the Dirichlet-to-Neuman (DN) map $\Lambda_{\gamma}$?
Medical Applications in 2-D:
- Monitoring ventilation and perfusion in ARDS patients
- Detection of pneumothorax
- Diagnosis of pulmonary edema and pulmonary embolus
Clinical Applications

What does the silent zone represent?
- Pneumothorax?
- Hyperinflation?
- Pulmonary edema?
- Atelectasis?

Figure 1. Computed tomography (CT), ventilation map, and aeration change map obtained at baseline (top) and after injecting 100 mL of air into the pleural space of the left upper quadrant (bottom). The arrow points to the induced pneumothorax (left, bottom).

Figure courtesy of M. Amato from *Real-time detection of pneumothorax using electrical impedance tomography*, Crit Care Med 2008 (Costa et al)
A classic result showed that once differentiable conductivities are uniquely determined by knowledge of the DN map $\Lambda_\sigma$:

**Theorem**

*Let $\Omega \in \mathbb{R}^2$ be a bounded domain with Lipschitz boundary and $\sigma$ be a measurable function bounded away from zero and infinity. If $\sigma_1$ and $\sigma_2$ are two conductivities with $\nabla \sigma_i$ in $L^p(\Omega)$, $p > 2$, and $\Lambda_{\sigma_1} = \Lambda_{\sigma_2}$, then $\sigma_1 = \sigma_2$.***
Assume there exist positive constants $\sigma_0$ and $E$ such that

$$\sigma > \sigma_0 \text{ in } \Omega,$$

$$\|\sigma\|_{W^{2,\infty}(\Omega)} \leq E, \quad \|\epsilon\|_{W^{2,\infty}(\Omega)} \leq E$$

**Theorem**

*Let $\Omega$ be an open bounded domain in $\mathbb{R}^2$ with Lipschitz boundary. Let $\sigma_j$ and $\epsilon_j$ satisfy the conditions above. Then there exists a constant $\omega_0$ such that if $\gamma_j = \sigma_j + i\omega \epsilon_j$ for $j=1,2$ and $\omega < \omega_0$ and if $\Lambda \gamma_1 = \Lambda \gamma_2$, then $\gamma_1 = \gamma_2$.***

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*Francini, Inverse Problems, 2000*
D-bar reconstruction methods capitalize on the direct relationship between the conductivity and CGO solutions to a PDE related to the inverse conductivity problem (possibly through a transformation).

\[ \Lambda_\gamma \rightarrow \text{Scattering transform} \rightarrow \text{CGO solutions} \rightarrow \gamma \]

They are

- Mesh independent
- Trivially parallelizable
The CGO solution depends on an auxiliary variable $k \in \mathbb{C}$.

Typically, a $\bar{\partial}$ equation in $z$ for the CGO solution leads to a direct formula for $\gamma$.

The link between the DN map and the CGO solution is through a nonlinear Fourier transform known as the scattering transform.

A $\bar{\partial}$ equation in the auxiliary variable $k$ for the CGO solution involves the scattering transform and completes the constructive steps.
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To learn more, see our forthcoming book:

Linear and Nonlinear Inverse Problems with Practical Applications, by JM and Samuli Siltanen
In production, SIAM 2012
The Potential Matrix

Define the matrix potential $Q$ by

$$Q = \begin{pmatrix} 0 & -\frac{1}{2} \bar{\partial} \log \gamma \\ -\frac{1}{2} \bar{\partial} \log \gamma & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{\partial \gamma^{1/2}}{\gamma^{1/2}} \\ -\frac{\bar{\partial} \gamma^{1/2}}{\gamma^{1/2}} & 0 \end{pmatrix}$$

and matrices $D$ and $D_k$ by

$$D = \begin{pmatrix} \bar{\partial} & 0 \\ 0 & \partial \end{pmatrix} \quad \quad D_k = \begin{pmatrix} \bar{\partial} & \bar{\partial} - ik \\ \partial + ik & \partial \end{pmatrix}$$

where $\bar{\partial}_z = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$ and $\partial_z = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)$. 
Exponentially Growing Solutions

Given a solution $u \in H^1(\Omega)$ of $\nabla \cdot (\gamma(z)\nabla u(z)) = 0$, the vector

$$
\begin{pmatrix} v \\ w \end{pmatrix} = \gamma^{1/2} \begin{pmatrix} \partial u \\ \bar{\partial} u \end{pmatrix}
$$

solves

$$(D - Q) \begin{pmatrix} v \\ w \end{pmatrix} = 0$$

(1)

For $k = k_1 + ik_2 \in \mathbb{C}$, seek solutions $\psi$ of (??) of the form

$$
\psi(z, k) = M(z, k) \begin{pmatrix} e^{izk} \\ 0 \end{pmatrix}
\begin{pmatrix} 0 \\ e^{-i\bar{z}k} \end{pmatrix}
$$

where $M$ converges to the identity matrix as $|z| \to \infty$. 

Exponentially Growing Solutions

The CGO solutions $M(z, k)$ satisfy

$$(D_k - Q)M = 0$$

Or in integral form

$$M_{11}(z, k) = 1 + \frac{1}{\pi} \int_{\Omega} \frac{Q_{12}(\zeta)M_{21}(\zeta, k)}{z - \zeta} d\zeta$$

$$M_{21}(z, k) = \frac{1}{\pi} \int_{\Omega} \frac{e^{-k(z - \zeta)}Q_{21}(\zeta)M_{11}(\zeta, k)}{\bar{z} - \bar{\zeta}} d\zeta$$

and similarly for $M_{12}$ and $M_{22}$, which are coupled.

Here $e_k(z) = \exp(i(zk + \bar{z}\bar{k}))$. 
Applying FFT’s on a suitable grid of meshsize $h$ to the integral form of the equations

$$M_{11}(z, k) = 1 + h^2 \text{IFFT} \left( \text{FFT} \left( \frac{1}{\pi z} \right) \text{FFT} \left( Q_{12}(z) M_{21}(z, k) \right) \right)$$

$$M_{21}(z, k) = h^2 \text{IFFT} \left( \text{FFT} \left( \frac{e^{-k(z - \zeta)}}{\pi z} \right) \text{FFT} \left( Q_{21}(z) M_{11}(z, k) \right) \right)$$

results in a linear system that can be solved by, eg, GMRES.
Reconstruction of $Q$

Knowledge of the full matrix $M$ results in a direct reconstruction formula for $Q$ and hence $\gamma$.

**Theorem**

*For any $\rho > 0$,*

$$Q(z) = \lim_{k_0 \to \infty} \mu(B_{\rho}(0))^{-1} \int_{k : |k - k_0| < r} D_k M(z, k) \, d\mu(k).$$

This large $k$ limit presents a problem for practical computation.

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0 Theorem 6.2 of Francini, 2000
Reconstruction of $Q$

The following is a direct reconstruction formula for $Q$ and hence $\gamma$ involving a small $k$ limit:

**Theorem**

*Define*

\[
M_+(z, k) \equiv M_{11}(z, k) + e_{-k}(z)M_{12}(z, k)
\]

\[
M_-(z, k) \equiv M_{22}(z, k) + e_k(z)M_{21}(z, k).
\]

*Then*

\[
Q_{12}(z) = \frac{\partial_z M_+(z, 0)}{M_-(z, 0)} \quad Q_{21}(z) = \frac{\partial_z M_-(z, 0)}{M_+(z, 0)}
\]

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Hamilton, 2012
The Scattering Transform

The scattering transform matrix is defined by

\[ S(k) = \frac{i}{\pi} \int_{\mathbb{R}^2} \begin{pmatrix} 0 & e^{-ik \bar{z}} Q_{12}(z) \psi_{22}(z, k) \\ -e^{ik \bar{z}} Q_{21}(z) \psi_{11}(z, k) & 0 \end{pmatrix} dz. \]

The matrix \( M(z, k) \) satisfies the D-bar equation wrt \( k \):

\[ \bar{\partial}_k M(z, k) = M(z, \bar{k}) \begin{pmatrix} e_k(z) & 0 \\ 0 & e_{-k}(z) \end{pmatrix} S(k), \]

Francini, Inverse Problems, 2000
The scattering transform matrix is defined by

\[
S(k) = \frac{i}{\pi} \int_{\mathbb{R}^2} \begin{pmatrix}
0 & e^{-i\bar{k}z}Q_{12}(z)\psi_{22}(z, k) \\
-e^{i\bar{k}z}Q_{21}(z)\psi_{11}(z, k) & 0
\end{pmatrix} dz.
\]

The matrix \(M(z, k)\) satisfies the D-bar equation wrt \(k\):

\[
\bar{\partial}_k M(z, k) = M(z, \bar{k}) \begin{pmatrix}
\psi_{11}(z) & 0 \\
0 & \psi_{22}(z, k)
\end{pmatrix} S(k),
\]
Computation

This results in two coupled systems. The first is

\[ \bar{\partial}_k M_{11}(z, k) = M_{12}(z, \bar{k}) e^{-k(z)} S_{21}(k) \]
\[ \bar{\partial}_k M_{12}(z, k) = M_{11}(z, \bar{k}) e_{\bar{k}}(z) S_{12}(k) \]

or in integral form

\[ 1 = M_{11}(z, k) - \frac{1}{\pi k} \ast (M_{12}(z, \bar{k}) e^{-k(z)} S_{21}(k)) \]
\[ 0 = M_{12}(z, k) - \frac{1}{\pi k} \ast (M_{11}(z, \bar{k}) e_{\bar{k}}(z) S_{12}(k)) \]

This can be discretized and a linear system results. Note that care must be taken with the conjugate with respect to \( k \).
Denote the unit outer normal to $\partial \Omega$ by $\nu = \nu_1 + i\nu_2$ and its conjugate by $\bar{\nu} = \nu_1 - i\nu_2$.

Then

$$S_{12}(k) = \frac{i}{2\pi} \int_{\partial \Omega} e^{-i\bar{k}z} \psi_{12}(z, k) \nu(z) \, ds(z)$$

$$S_{21}(k) = -\frac{i}{2\pi} \int_{\partial \Omega} e^{i\bar{k}\bar{z}} \psi_{21}(z, k) \bar{\nu}(z) \, ds(z).$$
There exist CGO solutions $u_1$ and $u_2$ to the admittivity equation with asymptotic behavior

$$u_1 \sim \frac{e^{ikz}}{ik} \quad \text{and} \quad u_2 \sim \frac{e^{-ik\bar{z}}}{-ik} \quad \text{as} \quad |z|, |k| \to \infty.$$ 

and the following connection to the DN map:

$$u_1(z, k) = \frac{e^{ikz}}{ik} - \int_{\partial \Omega} G_k(z - \zeta) (\Lambda_\gamma - \Lambda_1) u_1(\zeta, k) \, ds(\zeta)$$

$$u_2(z, k) = \frac{e^{-ik\bar{z}}}{-ik} - \int_{\partial \Omega} G_k(z - \zeta) (\Lambda_\gamma - \Lambda_1) u_2(\bar{\zeta}, k) \, ds(\zeta)$$

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0 A. Von Hermann, PhD thesis, Colorado State University, 2010
where $G_k(z)$ is the Faddeev Green’s function

$$G_k(z) = \frac{e^{ikz}}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{e^{iz \cdot \xi}}{\xi(\xi + 2k)} \, d\xi \quad k \in \mathbb{C} \setminus 0.$$ 

These CGO solutions satisfy

$$\begin{pmatrix} \Psi_{11} \\ \Psi_{21} \end{pmatrix} = \gamma^{1/2} \begin{pmatrix} \frac{\partial z u_1}{\partial z} \\ \bar{\frac{\partial z u_1}{\partial z}} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \Psi_{12} \\ \Psi_{22} \end{pmatrix} = \gamma^{1/2} \begin{pmatrix} \frac{\partial z u_2}{\partial z} \\ \bar{\frac{\partial z u_2}{\partial z}} \end{pmatrix},$$

which leads to BIE’s for $\Psi_{12}$ and $\Psi_{21}$...
A Boundary Integral Equation for $\Psi$

Differentiating $u_1$ and $u_2$ leads to BIEs for the CGO solutions $\Psi$:

**Theorem**

The trace of the exponentially growing solutions $\Psi_{12}(z, k)$ and $\Psi_{21}(z, k)$ for $k \in \mathbb{C} \setminus 0$ and $\gamma = 1$ on $\partial\Omega$ can be determined by

$$
\Psi_{12}(z, k) = \int_{\partial\Omega} \frac{e^{i\vec{k}(z-\zeta)}}{4\pi(z-\zeta)} (\Lambda_\gamma - \Lambda_1) u_2(\zeta, k) \, ds(\zeta)
$$

$$
\Psi_{21}(z, k) = \int_{\partial\Omega} \left[ \frac{e^{ik(z-\zeta)}}{4\pi(z-\zeta)} \right] (\Lambda_\gamma - \Lambda_1) u_1(\zeta, k) \, ds(\zeta).
$$

This provides the connection from $\Lambda_\gamma \rightarrow S$. 
Steps of the Method

Given the DN map $\Lambda_\gamma$:

- Compute the traces of the CGO solutions $u_1$ and $u_2$ from the BIE's
- Compute the traces of the CGO solutions $\Psi_{12}$ and $\Psi_{21}$ from knowledge of $u_1$ and $u_2$ on $\partial \Omega$
- Compute the scattering transforms $S_{12}$ and $S_{21}$ from knowledge of $\Psi_{12}$ and $\Psi_{21}$
- Numerically solve the system of $\bar{\partial}_k$ equations for $M$
- Form $M_+$ and $M_-$ and compute $Q_{12}$
- Compute $\gamma$ by solving the $\bar{\partial}$ equation
  \[ \bar{\partial} \log \gamma = -2Q_{21} \]
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$$\bar{\partial} \log \gamma = -2Q_{21}$$
Reconstructions from Simulated Data

Dynamic range: 76% (conductivity) 53% (permittivity)
Reconstructions from Simulated Data

Heart: 1.1 + 0.6 i  Lungs: 0.5 + 0.2 i  Background: 0.8 + 0.4 i

Dynamic range: 61% (no noise)  60%, 53% (noise)
Simulation of fluid in the lung

**Numerical Phantom**

Dynamic range
Reconstruction:
- Conductivity: 80%
- Permittivity: 84%

Difference image:
- Conductivity: 63%
- Permittivity: 67%.
Thank you, Gunther, for all you do, may you have many, many more Happy Birthdays!!