A Direct D-bar Reconstruction Algorithm for Recovering a Complex Conductivity in 2-D

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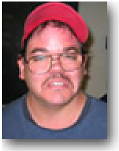
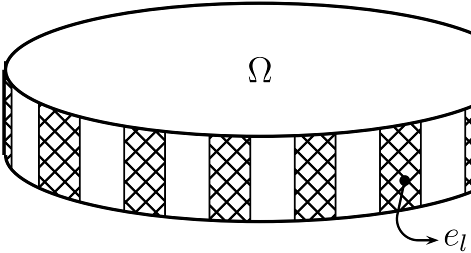


Alan Von Hermann, CSU

The EIT Problem

Current is applied on electrodes on the surface of the body and the resulting voltage is measured. Let $\gamma = \sigma + i\omega\epsilon$.

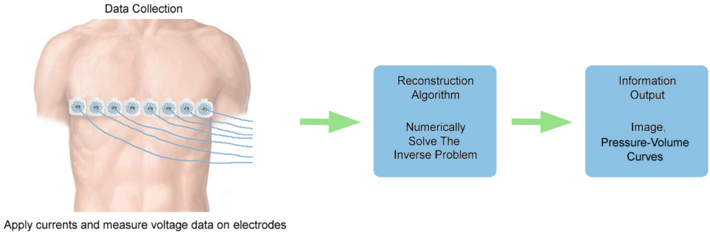
$$\begin{aligned} \nabla \cdot \gamma \nabla u &= 0 \quad \text{in } \Omega \\ u &= f \quad \text{on } \partial\Omega \end{aligned}$$



$$\Lambda_\gamma : f \quad \rightarrow \quad \gamma \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega}$$

Mathematically, this is governed by the inverse admittivity problem: Can the admittivity γ be recovered from measurements of the Dirichlet-to-Neuman (DN) map Λ_γ?

Applications of EIT



Medical Applications in 2-D:

- Monitoring ventilation and perfusion in ARDS patients
- Detection of pneumothorax
- Diagnosis of pulmonary edema and pulmonary embolus

Clinical Applications

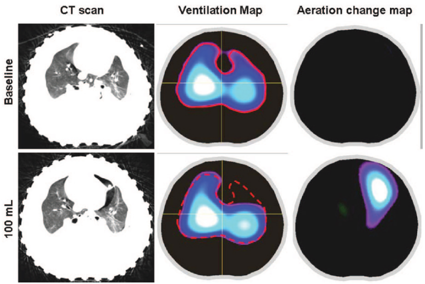


Figure 1. Computed tomography (*CT*), ventilation map, and aeration change map obtained at baseline (*top*) and after injecting 100 mL of air into the pleural space of the left upper quadrant (*bottom*). The arrow points to the induced pneumothorax (*left, bottom*).

What does the silent zone represent?

- Pneumothorax?
- Hyperinflation?
- Pulmonary edema?
- Atelectasis?

[0] Figure courtesy of M. Amato from *Real-time detection of pneumothorax using electrical impedance tomography*, Crit Care Med 2008 (Costa et al)

Global Uniqueness Result: Brown, Uhlmann

A classic result showed that once differentiable conductivities are uniquely determined by knowledge of the DN map Λ_σ:

Theorem

Let $\Omega \in \mathbb{R}^2$ be a bounded domain with Lipschitz boundary and σ be a measurable function bounded away from zero and infinity. If σ_1 and σ_2 are two conductivities with $\nabla\sigma_i$ in $L^p(\Omega)$, $p > 2$, and $\Lambda_{\sigma_1} = \Lambda_{\sigma_2}$, then $\sigma_1 = \sigma_2$.

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[0] Brown and Uhlmann, Comm PDEs, 1997

Global Uniqueness Result of Francini

Assume there exist positive constants σ_0 and E such that

$$\sigma > \sigma_0 \text{ in } \Omega,$$

$$\|\sigma\|_{W^{2,\infty}(\Omega)} \leq E, \quad \|\epsilon\|_{W^{2,\infty}(\Omega)} \leq E$$

Theorem

Let Ω be an open bounded domain in $\mathbf{R}^2$ with Lipschitz boundary. Let σ_j and ϵ_j satisfy the conditions above. Then there exists a constant ω_0 such that if $\gamma_j = \sigma_j + i\omega\epsilon_j$ for j=1,2 and $\omega < \omega_0$ and if $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, then $\gamma_1 = \gamma_2$.

[0] Francini, Inverse Problems, 2000

General Overview: D-bar Methods for EIT

D-bar reconstruction methods capitalize on the direct relationship between the conductivity and CGO solutions to a PDE related to the inverse conductivity problem (possibly through a transformation).

$$\Lambda_\gamma \longrightarrow \text{Scattering transform} \longrightarrow \text{CGO solutions} \longrightarrow \gamma$$

They are

- Mesh independent
- Trivially parallelizable

General Overview: D-bar Methods for EIT

The CGO solution depends on an auxilliary variable $k \in \mathbb{C}$.

Typically, a $\bar{\partial}$ equation in z for the CGO solution leads to a direct formula for γ.

The link between the DN map and the CGO solution is through a nonlinear Fourier transform known as the scattering transform.

A $\bar{\partial}$ equation in the auxilliary variable k for the CGO solution involves the scattering transform and completes the constructive steps.

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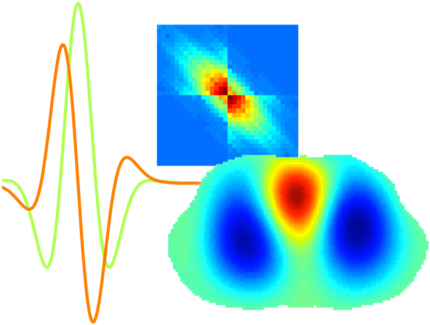
To learn more, see our forthcoming book:

Linear and Nonlinear Inverse Problems with Practical Applications, by JM and Samuli Siltanen
In production, SIAM 2012



The Potential Matrix

Define the matrix potential Q by

$$Q = \begin{pmatrix} 0 & -\frac{1}{2}\partial \log \gamma \\ -\frac{1}{2}\bar{\partial} \log \gamma & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{\partial \gamma^{1/2}}{\gamma^{1/2}} \\ -\frac{\bar{\partial}\gamma^{1/2}}{\gamma^{1/2}} & 0 \end{pmatrix}$$

and matrices D and D_k by

$$D = \begin{pmatrix} \bar{\partial} & 0 \\ 0 & \partial \end{pmatrix} \qquad D_k = \begin{pmatrix} \bar{\partial} & \bar{\partial} - ik \\ \partial + ik & \partial \end{pmatrix}$$

where $\bar{\partial}_z = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$ and $\partial_z = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$.

Exponentially Growing Solutions

Given a solution $u \in H^1(\Omega)$ of $\nabla \cdot (\gamma(z)\nabla u(z)) = 0$, the vector

$$\begin{pmatrix} v \\ w \end{pmatrix} = \gamma^{1/2} \begin{pmatrix} \partial u \\ \bar{\partial} u \end{pmatrix}$$

solves

$$(D - Q) \begin{pmatrix} v \\ w \end{pmatrix} = 0 \tag{1}$$

For $k = k_1 + ik_2 \in \mathbb{C}$, seek solutions ψ of (**??**) of the form

$$\psi(z,k) = M(z,k) \begin{pmatrix} e^{izk} & 0 \\ 0 & e^{-i\bar{z}k} \end{pmatrix}$$

where M converges to the identity matrix as $|z| \to \infty$.

Exponentially Growing Solutions

The CGO solutions $M(z,k)$ satisfy

$$(D_k - Q)M = 0$$

Or in integral form

$$\begin{aligned} M_{11}(z,k) &= 1 + \frac{1}{\pi}\int_{\Omega} \frac{Q_{12}(\zeta)M_{21}(\zeta,k)}{z-\zeta} d\zeta \\ M_{21}(z,k) &= \frac{1}{\pi}\int_{\Omega} \frac{e_{-k}(z-\zeta)Q_{21}(\zeta)M_{11}(\zeta,k)}{\bar{z}-\bar{\zeta}} d\zeta \end{aligned}$$

and similarly for M_{12} and M_{22}, which are coupled.

Here $e_k(z) = exp(i(zk + \bar{z}\bar{k}))$.

Computation of CGO Solutions

Applying FFT's on a suitable grid of meshsize h to the integral form of the equations

$$
\begin{aligned}
M_{11}(z,k) &= 1 + h^2 IFFT(FFT(\frac{1}{\pi z})FFT(Q_{12}(z)M_{21}(z,k))) \\
M_{21}(z,k) &= h^2 IFFT(FFT(\frac{e_{-k}(z-\zeta)}{\pi z})FFT(Q_{21}(z)M_{11}(z,k)))
\end{aligned}
$$

results in a linear system that can be solved by, eg, GMRES.

Reconstruction of Q

Knowledge of the full matrix M results in a direct reconstruction formula for Q and hence γ.

Theorem

For any $\rho > 0$,

$$Q(z) = \lim_{k_0 \to \infty} \mu(B_\rho(0))^{-1} \int_{k:|k-k_0|<r} D_k M(z,k)\, d\mu(k).$$

This large k limit presents a problem for practical computation.

[0] Theorem 6.2 of Francini, 2000

Reconstruction of Q

The following is a direct reconstruction formula for Q and hence γ involving a small k limit:

Theorem

Define

$$M_+(z,k) \equiv M_{11}(z,k) + e_{-k}(z)M_{12}(z,k)$$

$$M_-(z,k) \equiv M_{22}(z,k) + e_k(z)M_{21}(z,k).$$

Then

$$Q_{12}(z) = \frac{\partial_{\bar{z}} M_+(z,0)}{M_-(z,0)} \quad Q_{21}(z) = \frac{\partial_z M_-(z,0)}{M_+(z,0)}$$

[0] Hamilton, 2012

The Scattering Transform

The scattering transform matrix is defined by

$$S(k) = \frac{i}{\pi} \int_{\mathbf{R}^2} \begin{pmatrix} 0 & e^{-i\bar{k}z} Q_{12}(z) \psi_{22}(z,k) \\ -e^{ik\bar{z}} Q_{21}(z) \psi_{11}(z,k) & 0 \end{pmatrix} dz.$$

The matrix $M(z,k)$ satisfies the D-bar equation wrt k:

$$\bar{\partial}_k M(z,k) = M(z,\bar{k}) \begin{pmatrix} e_{\bar{k}}(z) & 0 \\ 0 & e_{-k}(z) \end{pmatrix} S(k),$$

[0] Francini, Inverse Problems, 2000

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Computation

This results in two coupled systems. The first is

$$\begin{aligned}
\bar{\partial}_k M_{11}(z,k) &= M_{12}(z,\bar{k})\, e_{-k}(z)\, S_{21}(k) \\
\bar{\partial}_k M_{12}(z,k) &= M_{11}(z,\bar{k})\, e_{\bar{k}}(z)\, S_{12}(k)
\end{aligned}$$

or in integral form

$$\begin{aligned}
1 &= M_{11}(z,k) - \frac{1}{\pi k} * (M_{12}(z,\bar{k}) e_{-k}(z) S_{21}(k)) \\
0 &= M_{12}(z,k) - \frac{1}{\pi k} * (M_{11}(z,\bar{k}) e_{\bar{k}}(z) S_{12}(k))
\end{aligned}$$

This can be discretized and a linear system results. Note that care must be taken with the conjugate with respect to k.

The Scattering Transform

Denote the unit outer normal to $\partial\Omega$ by $\nu = \nu_1 + i\nu_2$ and its conjugate by $\bar{\nu} = \nu_1 - i\nu_2$.

Then

$$S_{12}(k) = \frac{i}{2\pi} \int_{\partial\Omega} e^{-i\bar{k}z} \; \psi_{12}(z,k) \; \nu(z) \; ds(z)$$

$$S_{21}(k) = -\frac{i}{2\pi} \int_{\partial\Omega} e^{i\bar{k}\bar{z}} \; \psi_{21}(z,k) \; \overline{\nu(z)} \; ds(z).$$

[0] A. Von Hermann, PhD thesis, Colorado State University, 2010

There exist CGO solutions u_1 and u_2 to the admittivity equation with asymptotic behavior

$$u_1 \sim \frac{e^{ikz}}{ik} \quad \text{and} \quad u_2 \sim \frac{e^{-ik\bar{z}}}{-ik} \text{ as } |z|, |k| \to \infty.$$

and the following connection to the DN map:

$$\begin{aligned} u_1(z,k) &= \frac{e^{ikz}}{ik} - \int_{\partial\Omega} G_k(z-\zeta)\,(\Lambda_\gamma - \Lambda_1)\, u_1(\zeta,k)\, ds(\zeta) \\ u_2(z,k) &= \frac{e^{-ik\bar{z}}}{-ik} - \int_{\partial\Omega} G_k(z-\zeta)\,(\Lambda_\gamma - \Lambda_1)\, u_2(\bar{\zeta},k)\, ds(\zeta) \end{aligned}$$

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where $G_k(z)$ is the Faddeev Green's function

$$G_k(z) = \frac{e^{ikz}}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{e^{iz\cdot\xi}}{\xi(\bar{\xi} + 2k)} \, d\xi \quad k \in \mathbb{C} \setminus 0.$$

These CGO solutions satisfy

$$\begin{pmatrix} \Psi_{11} \\ \Psi_{21} \end{pmatrix} = \gamma^{1/2} \begin{pmatrix} \partial_z u_1 \\ \bar{\partial}_z u_1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \Psi_{12} \\ \Psi_{22} \end{pmatrix} = \gamma^{1/2} \begin{pmatrix} \partial_z u_2 \\ \bar{\partial}_z u_2 \end{pmatrix},$$

which leads to BIE's for Ψ_{12} and Ψ_{21}...

A Boundary Integral Equation for Ψ

Differentiating u_1 and u_2 leads to BIEs for the CGO solutions Ψ:

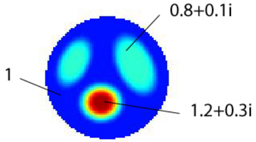
Theorem

The trace of the exponentially growing solutions $\Psi_{12}(z,k)$ and $\Psi_{21}(z,k)$ for $k \in \mathbb{C} \setminus 0$ and $\gamma = 1$ on $\partial\Omega$ can be determined by

$$\begin{aligned} \Psi_{12}(z,k) &= \int_{\partial\Omega} \frac{e^{i\bar{k}(z-\zeta)}}{4\pi(z-\zeta)} (\Lambda_\gamma - \Lambda_1)\, u_2(\zeta,k)\, ds(\zeta) \\ \Psi_{21}(z,k) &= \int_{\partial\Omega} \overline{\left[\frac{e^{ik(z-\zeta)}}{4\pi(z-\zeta)}\right]} (\Lambda_\gamma - \Lambda_1)\, u_1(\zeta,k)\, ds(\zeta). \end{aligned}$$

This provides the connection from $\Lambda_\gamma \to S$.

Steps of the Method

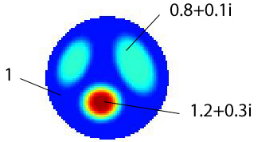


Given the DN map Λ_γ:

- Compute the traces of the CGO solutions u_1 and u_2 from the BIE's
- Compute the traces of the CGO solutions Ψ_{12} and Ψ_{21} from knowledge of u_1 and u_2 on $\partial\Omega$
- Compute the scattering transforms S_{12} and S_{21} from knowledge of Ψ_{12} and Ψ_{21}
- Numerically solve the system of $\bar{\partial}_k$ equations for M
- Form M_+ and M_- and compute Q_{12}
- Compute γ by solving the $\bar{\partial}$ equation

$$\bar{\partial} \log \gamma = -2Q_{21}$$

Steps of the Method



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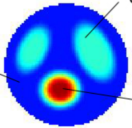

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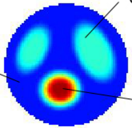

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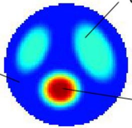

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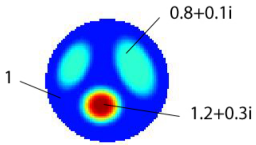

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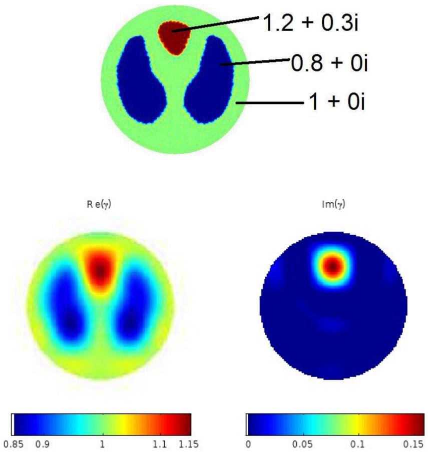
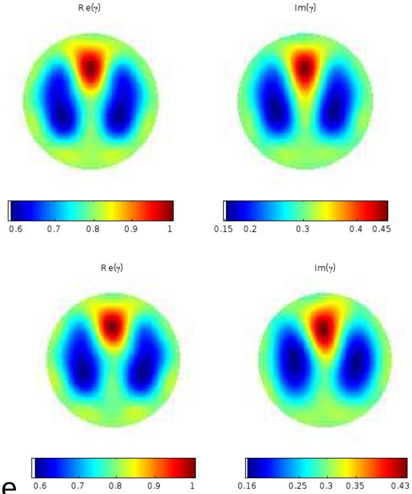
Reconstructions from Simulated Data



Dynamic range: 76% (conductivity) 53% (permittivity)

Reconstructions from Simulated Data

Heart: 1.1+0.6 i   Lungs: 0.5 +0.2 i   Background: 0.8+0.4 i



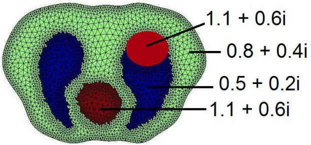
No Noise

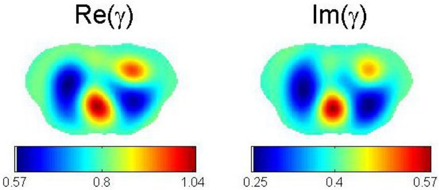
0.01% Noise

Dynamic range: 61% (no noise)   60%, 53% (noise)

Simulation of fluid in the lung

Numerical Phantom





Dynamic range

Reconstruction:

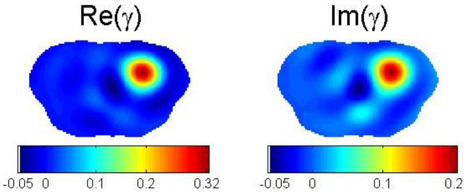
Conductivity: 80%

Permittivity: 84%

Difference image:

Conductivity: 63%

Permittivity: 67%.



Thank you, Gunther, for all you do,
may you have many, many more Happy Birthdays!!