Adiabatic limits and eigenvalues
Gunther Uhlmann’s 60th birthday meeting

Richard Melrose

Department of Mathematics
Massachusetts Institute of Technology

22 June, 2012
Outline

1. Adiabatic limits and operators
   - Adiabatic metrics
   - Adiabatic operators
   - Adiabatic normal operator
   - Invertibility
   - Eigenvalues

2. Harmonic forms and localization
   - Leray-Serre
   - Boundary case
   - Localization
   - Extensions

3. Racetracks
   - Adiabatic structure
   - Boundary conditions
I have not really worked on inverse problems since Gunther and I last collaborated in a project on backscattering. So I thought I would describe some results on adiabatic limits in various settings and finish with some related questions

- Basic structure of an adiabatic problem
- Inversion of operators
- Spectrum of adiabatic operators
- An adiabatic inverse problem
First, let me remind you of a core inverse problem – one that I am would very much like to solve or see solved. This is Kac’s problem.

**Problem**

Do the Dirichlet (and/or Neumann) eigenvalues for a (smooth) bounded strictly convex domain in the plane determine the domain?

- If one drops the smoothness and convexity assumptions then there are counterexamples (but very rigid ones)
- Unfortunately I have nothing new to say about this problem!
- You should talk to Hamid Hezari and Steve Zelditch about their recent work on perturbation of ellipses
The notion of an adiabatic limit in Physics really arose in thermodynamics but the use of the term in differential analysis/geometry follows a paper by Witten.

Witten discusses the adiabatic limit of the eta invariant for a manifold fibred over a circle.

More generally one can think of a fibre bundle

\[ Z \rightarrow M \rightarrow B, \tag{1} \]

For compact manifolds this is the same notion as a submersion, i.e. just a smooth map with surjective differential from each point of the domain.
The inverse image of a small neighbourhood $U$ of each point in $B$ under $\phi : M \to B$ is diffeomorphic to the product $Z \times U$ for a fixed compact manifold $Z$ with smooth transitions between overlaps.

Thus $M$ comes equipped with an exhaustion by disjoint smooth fibres looking like $Z$.

One can give $M$ an ‘adiabatic’ metric, meaning a family of metrics depending on a parameter $\epsilon$ of the form

$$g + \epsilon^{-2} \phi^* h$$

Here $g$ is some metric (maybe only strictly positive on the fibres) on $M$ and $h$ is a metric on the base, $B$.

Thus a fixed tangent vector on $M$ becomes ‘long’ in the base direction as $\epsilon \downarrow 0$. 
Near a point of $M$ there are coordinates $z$ along the fibres and $y$ in the base – these are constant on the fibres.

The vector fields of ‘bounded length’ with respect to an adiabatic metric are then the combinations of $\partial_{z_j}$ and $\epsilon \partial_{y_l}$.

Commutators of these behave sensibly so one can form ‘adiabatic differential operators’ as locally looking like

$$P = \sum_{|\alpha| + |\beta| \leq m} p_{\alpha,\beta}(\epsilon, z, y) \partial_z^\alpha (\epsilon \partial_y)^\beta.$$  \hspace{1cm} (3)

Adiabatic ellipticity means that the polynomial

$$p_m = \sum_{|\alpha| + |\beta| = m} p_{\alpha,\beta}(\epsilon, z, y) \zeta^\alpha \eta^\beta$$  \hspace{1cm} (4)

should have no real zeros.
The symbol here is defined for all $\epsilon \geq 0$

There is an *adiabatic model operator* well defined at $\epsilon = 0$, given locally by

$$A(P) = \sum_{|\alpha| + |\beta| \leq m} p_{\alpha,\beta}(0, z, y) \partial_\alpha^\alpha \zeta^\beta$$  \hspace{1cm} (5)

This is a family of operator on the fibres with conormal parameters from the base

Notice that this is like a partial semiclassical limit with non-commutativity remaining along the fibres.
A relatively easy result is

**Theorem**

If $P$ is an elliptic adiabatic operator and $A(P)$ is invertible (for all values of the parameters) then $P$ is invertible for small $\epsilon > 0$.

This invertibility comes with precise uniformity down to $\epsilon = 0$. 
The Laplacian, $\Delta$, for any adiabatic metric is an example of an elliptic adiabatic family

$$A(\Delta) = \Delta Z_b + |\zeta|^2_b.$$ 

Thus the Theorem above applies to $\Delta - z$ for $z \notin [0, \infty)$

If one thinks about the eigenvalues of Laplacian for an adiabatic metric one can be guided to some extent by the product case for which the eigenvalues are

$$M = Z \times B, \quad g = g_Z + \epsilon^{-2} h_B$$

$$\Delta g = \Delta Z + \epsilon^2 \Delta B$$

$$\lambda(\Delta g) = \lambda_j(Z) + \epsilon^2 \lambda_k(B)$$

(6)
In general of course this does not make sense, since the $\lambda_j(Z_b)$ will be functions on $B$, which parameterizes the fibres, and the $\lambda_k(B)$ do not make sense at all since there is no obvious base operator.

The product case is a reasonable guide provided there is a $\lambda_j$ which is constant – independent of $b \in B$.

Generally there are no such constant eigenvalues.
One such case that Rafe Mazzeo and I looked at some years ago is the Laplacian on forms – for which $A(P)$ is generally not invertible at $\zeta = 0$.

We considered what happens to the Hodge cohomology – the harmonic forms – on $M$ as $\epsilon \downarrow 0$.

The dimension of the harmonic forms of fixed degree is independent of $\epsilon$, being the corresponding Betti number.

**Theorem**

*For any adiabatic metric there is a smooth basis, $u_j(\epsilon)$ of harmonic forms, down to $\epsilon = 0$.***
The limits $u_j(0)$ of these smooth forms consist of harmonic forms on the fibres $Z_b$ which ‘depend in a harmonic way’ on the base variables.

However, not all such forms occur as the limits of truly harmonic forms.

Which harmonic sections occur in the limit can be worked out from the Taylor series in $\epsilon$

This construction implements the Leray-Serre spectral sequence for the cohomology of the total space.
I want to emphasize here that it is very significant that the fibre Laplacians have smoothly varying null space – the harmonic forms on $Z$ (for varying metrics)

Suppose one considers a fibration with fibres which are manifolds with boundary, thus $M$ is also a manifold with boundary

For an adiabatic metric consider the Laplacian on $M$ with Dirichlet boundary conditions

Then the fibre Laplacians are invertible and (a small extension) of the Theorem above shows that $\Delta$ is uniformly invertible down to $\epsilon = 0$
What then happens to the eigenvalues of $\Delta$ as $\epsilon \downarrow 0$?

The lowest fibre eigenvalue for $\lambda_1(Z_b)$ is simple and hence smooth in $b$.

As $\epsilon \downarrow 0$ the lowest eigenvalues of $\Delta$ are close to $l = \inf_{b \in B} \lambda_1(Z_b)$ and concentrate above the point or points in $B$ where this is assumed.

If all the minima are non-degenerate the lowest eigenvalue corresponds to a rescaled harmonic oscillator in the base variable near each of these points and are of the form

$$l + \epsilon^2 t_j + O(\epsilon^3)$$
If we pass from the realm of manifolds with boundary to those with corners there is a natural weakening of the notion of a fibration to a b-fibration.

Picture!
These sorts of considerations can be extended to somewhat more singular settings.

1. The fibration with singular fibres given by a Morse function on a compact manifold – there is an extension of Witten’s theorem on the eta invariant to this case (a question of M. Atiyah)

2. Gluing constructions corresponding to blowing up points in manifolds – for instance the construction of kähler metrics with constant scalar curvature (with M. Singer)

3. The eigenvalues of planar triangles as functions on the moduli space – so corresponding to all collapse modes (with D. Grieser)
Now, let me come back to the planar domain problem I mentioned at the beginning.

Let’s replace the domain by an adiabatic one, or if you like a race track or a wave guide:

\[
\Omega_\epsilon = \{(x, y) \in \mathbb{R}^2; R(\theta) - \epsilon \leq r \leq R(\theta)\}.
\]

Here \(0 < R \in C^\infty(\mathbb{S})\) is a smooth function which is periodic of period \(2\pi\) and \(r, \theta\) are standard polar coordinates. So this domain need not be strictly convex, but is certainly star-shaped around the origin.
To make this look like an adiabatic problem, we can introduce polar coordinates and then rescale, to get coordinates \((t, \theta)\) where

\[
t = \frac{R(\theta) - r}{\epsilon} \in [0, 1].
\]  

Since we have rescaled it, the \(t\) variable, forming the fibre, is being shrunk while the base variable, \(\theta \in S\), is of fixed size.
The adiabatic vector fields can now be seen

\[\partial_r = -\frac{1}{\epsilon} \partial_t, \quad \partial_\theta = \partial_\theta + \frac{R'}{\epsilon} \partial_t.\]  

(9)

Thus the Euclidean Laplacian becomes an elliptic adiabatic operator

\[(\partial_x^2 + \partial_y^2) = \epsilon^{-2} P(\epsilon, t, \theta, \epsilon \partial_t, \partial_\theta)\]  

(10)

We can then ask – what can we recover from knowledge of the eigenvalues for small \(\epsilon\)?
Clearly we need to add boundary conditions.

The Dirichlet condition will mean that

\[ \min \lambda(\Delta_D) > C\epsilon^{-2}, \quad C > 0 \]  

and in particular the family is invertible.

The Neumann condition lead to small eigenvalues.

Perhaps unfortunately the leading terms here are very simple:

\[ \lambda_k(\Delta_D) = ck^2 + \epsilon F(\epsilon, k) \]

where \( c \) is fixed.

Question: Is the problem behind the small eigenvalues for the Neumann problem integrable – are there invariants which can be recovered from them?

In particular of course can one recover \( R \) (up to rotation) from these small eigenvalues?
Instead of scaling the domain one could force the width to be constant in the sense that one could look at the region

\[ \{ z \in \mathbb{R}^2; d(z, C) \leq \epsilon \} \]

where \( C \) is the fixed bounding curve.

What happens to the eigenvalues then?
Best wishes Gunther for many more years and theorems!