Gunther 60

Irvine

Inverse Source Problem

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Forward Source Problem

$$(\Delta + k^2)\, u^+ = f \text{ in } \mathbb{R}^n$$

Theorem $\exists!$ outgoing u^+

For $k \in \mathbb{C}^+$ $\quad \hat{u} = \dfrac{\hat{f}(\xi)}{k^2 - \xi^2}$

For $k \in \mathbb{R}$ $\quad \hat{u} = \lim\limits_{\varepsilon \downarrow 0} \dfrac{\hat{f}(\xi)}{(k+i\varepsilon)^2 - \xi^2}$

Outgoing $\Longleftrightarrow$ Sommerfeld Radiation Condition

$\Longleftrightarrow$ Fourier Transform $(t \Leftrightarrow k)$ in time of solution to the wave equation that vanishes in the past

Far Field = Restricted Fourier Transform

$(\Delta + k^2) u^+ = f$ [compactly supported]

$$\hat{u} = \frac{\hat{f}(\xi)}{(k+i0)^2 - \xi^2}$$

as $r \longrightarrow \infty$

$$u \sim \frac{e^{ikr}}{(kr)^{1/2}} \upsilon^+(\theta) = \frac{e^{ikr}}{(kr)^{1/2}} \hat{f}(k\theta)$$

Rellich's Lemma If u decays faster than $\frac{1}{r^{1/2}}$ at ∞, $u = 0$ outside $\operatorname{ch}(\operatorname{supp} f)$.

Unique Continuation $u \equiv 0$ on $\operatorname{supp}_\infty f$

$\operatorname{supp}_\infty f$ = unbounded connected component of $\mathbb{R}^2 \setminus (\operatorname{supp} f)$

The Inverse Source Problem

$$(\Delta + k^2) u^+ = f$$

Linear but **Non-Unique**

Non-Radiating Sources $f \overset{\mathcal{F}}{\longmapsto} \hat{f}|$ has a big kernel

NR Volume Soures [$F \in L^2$ (compact set)]

$$f \in NR \iff f = (\Delta + k^2)\, \Phi_{oo}$$

Φ_{oo} means H^2_0 (compact set)

Proof

$\Leftarrow$ $\quad ((\Delta + k^2)\Phi_{oo})^\wedge = (-\xi^2 + k^2)\,\hat{\Phi}_{oo} = 0$ on $\{\xi^2 = k^2\}$

$\Rightarrow$ Rellich + unique continuation $\Rightarrow$ $u^+ = \Phi_{oo}$

Free Sources

Corollary Every source supported in Ω has a unique equivalent Ω-free source.

Equivalent sources radiate the same far field $(\hat{f}_1 = \hat{f}_2)$

A free source satisfies $(\Delta+k^2)f = 0$ in Ω

Proof Solve the clamped plate equation

$$(\Delta+k^2)^2 \Phi_{00} = (\Delta+k^2) f$$

NR

$\hat{f} = f - (\Delta+k^2)\Phi_{00}$ is equivalent to f and $(\Delta+k^2)\hat{f} = 0$

Pseudo Inverse The unique free source is also the source with minimal $L^2(\Omega)$ norm

Criticisms of the Ω-free Source

1. we need to start by choosing Ω

2. No matter how small th support of the true source is, the support of the free source is all of Ω

3. But it does have the smallest L^2-norm

The L^2-norm of the Free Source

$$(\Delta+k^2)u^+ = F \qquad\qquad (\Delta+k^2)v^0 = 0$$

$$v^0(x) = \int e^{ik\theta\cdot x}\, v^0(\theta)\, d\theta$$

Standard Far Field Calculation

$$\int_\Omega f v_0 = \int_{\mathbb{R}^n} f v^0 = \int_{\mathbb{R}^n} (\Delta+k^2)u^+ v^0 = \lim_{R\to\infty}\int_{B_R} (\Delta+k^2)u^+ v^0$$

Far Fields

$$= \lim_{R\to\infty}\int_{\partial B_R} \frac{\partial u^+}{\partial \nu} v^0 - u^+\frac{\partial v^0}{\partial \nu} = \int_{S^{n-1}} v^+ v^0\, dS$$

If $\quad v^0 = e^{in\theta}\quad$ then $\quad v^0 = J_n(kr)\, e^{in\theta}$, so

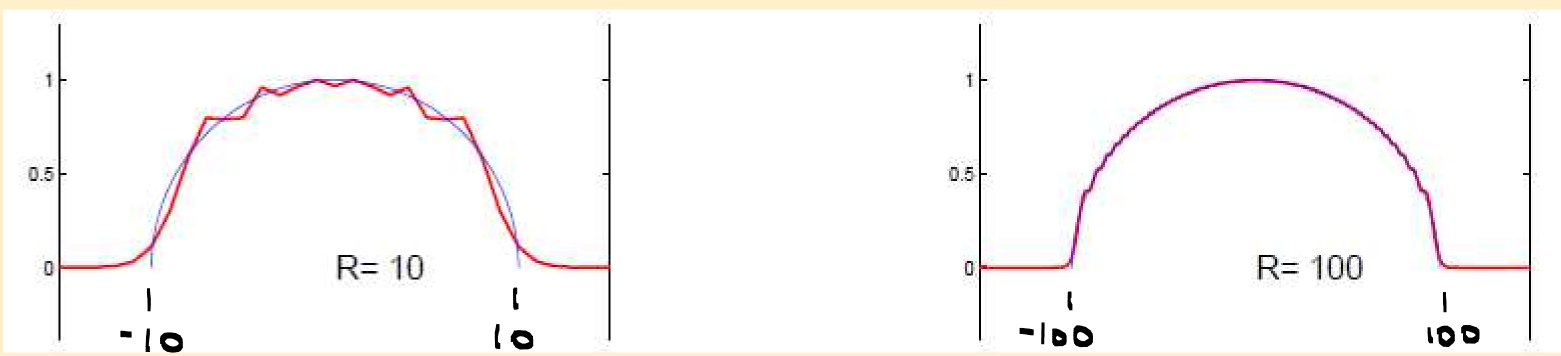
if $\quad v^+ = e^{in\theta}\quad$ then $\displaystyle F_{\text{Free}} = \frac{J_n(kr)\, e^{in\theta}}{\|J_n(kr)\|_{L^2(\Omega)}}\cdot \chi_\Omega$

The Ω-free source (which has minimal L^2 norm)

$$f = \sum f_n e^{in\theta} \frac{J_n(kr)\,\chi_\Omega}{\|J_n(kr)\,\chi_\Omega\|^2} \text{ radiates } \sum f_n e^{in\theta}$$

$$\|f\| \le \sum \frac{|f_n|}{\|J_n(kr)\|_{L^2(\Omega)}}$$

For $\Omega = B_R(0)$, $\|f\|^2_{L^2} = \sum \frac{|f_n|^2}{\|J_n(kR)\|^2_{L^2(B_R(0))}}$



$\|J_n\|^2_{L^2(B_R)}$

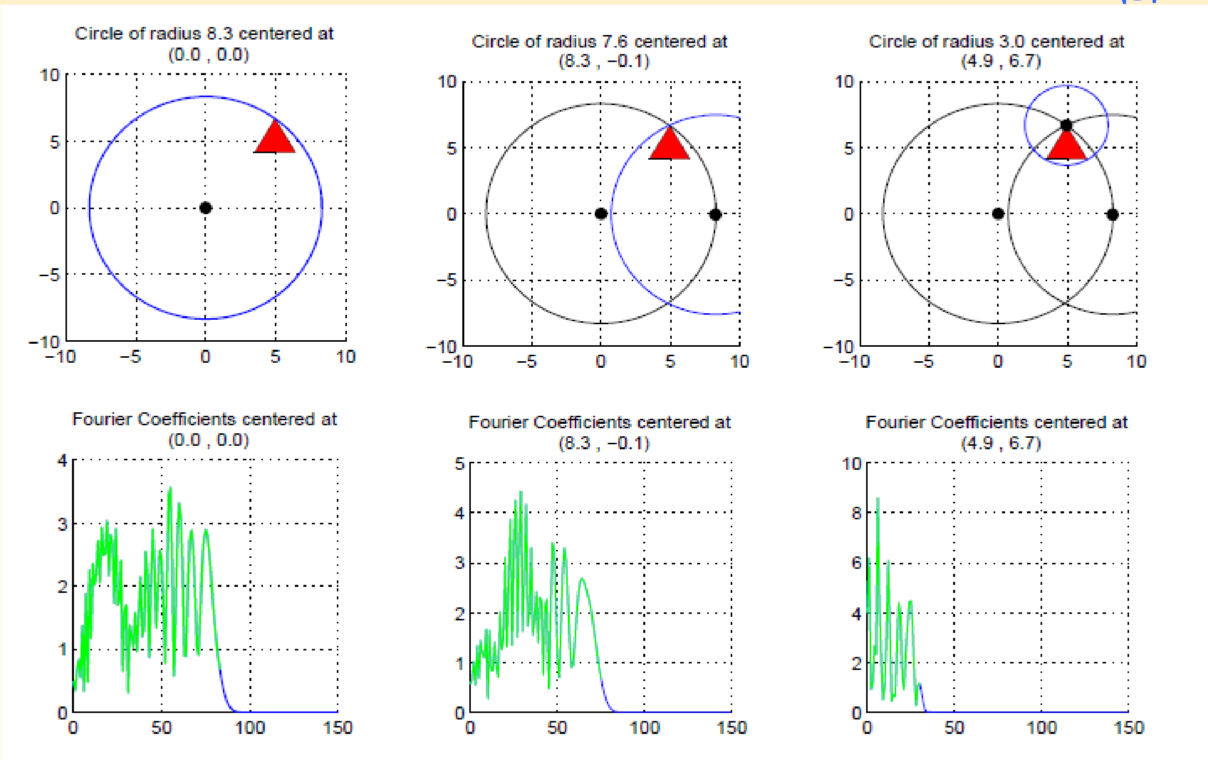
Test for finding supp f

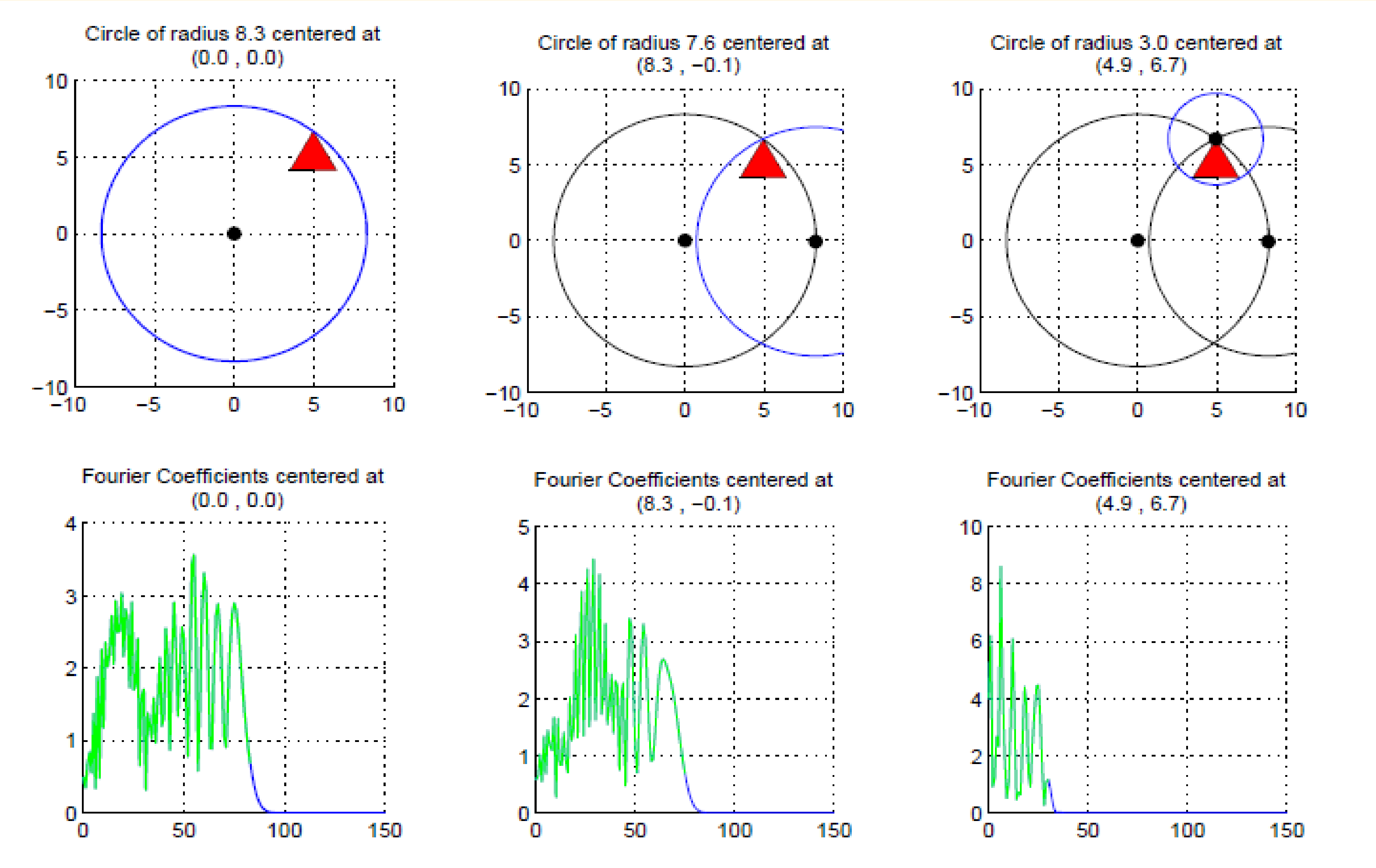
Theorem $\exists f$ supported in $B_R(0) \iff \sum \frac{|f_n|^2}{\|J_n(kR)\|^2_{L^2(B_R(0))}}$ converges

Translation $f^p := f(x-p) \Rightarrow \widehat{f^p}(\xi) = e^{ip\cdot\xi}\hat{f}(\xi)$

$\widehat{f^p}\Big| = \sum f^p_n e^{in\theta}$

Theorem $\exists f$ supported in $B_R(p) \iff \sum \frac{|f^p_n|^2}{\|J_n(kR)\|^2_{L^2(B_R)}}$ converges




Circle of radius 8.3 centered at
(0.0 , 0.0)
Circle of radius 7.6 centered at
(8.3 , −0.1)
Circle of radius 3.0 centered at
(4.9 , 6.7)
Fourier Coefficients centered at
(0.0 , 0.0)
Fourier Coefficients centered at
(8.3 , −0.1)
Fourier Coefficients centered at
(4.9 , 6.7)

Combining the Two Tests

Ω carries α if $\forall \varepsilon > 0$ $\exists f$ such that

① $\operatorname{supp} f \subset N_\varepsilon(\Omega)$

② f radiates α $\left(|\hat{f}| = \alpha \right)$

Question: If Ω_1 and Ω_2 each carry α, does $(\Omega_1 \cap \Omega_2)$ carry α?

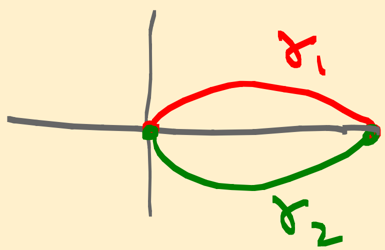
In general, No. If Ω_1 and Ω_2 are convex, yes

Finding Branch Cuts
(An analogy)

$$u(z) = \sqrt{z(z-1)} = z\sqrt{1-\tfrac{1}{z^2}} \sim \underbrace{z - \tfrac{1}{2z} + \dots}_{\text{far Field}} \text{ as } z \to \infty$$

$$(\Delta + k^2)\,u = f$$

$$\bar\partial u = [u]\,dz\Big|_{\gamma_1}$$

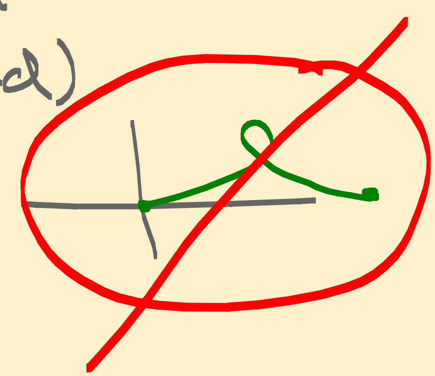


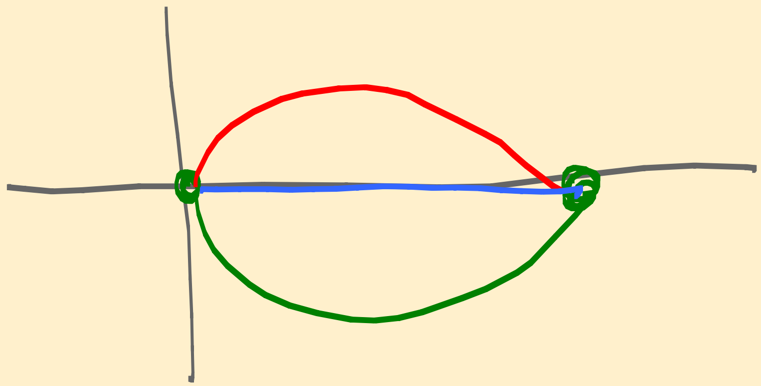
for any γ which joins 0 and 1

① Any curve with the correct endpoints carries the far field.

② Once γ is fixed, f is unique
(as long as $\mathbb{C} \setminus \gamma$ is connected)

③ $\gamma_1 \cap \gamma_2$ doesn't carry the far field





$\exists !$ convex branchcut γ_3

and $\gamma_3 \subset$ convex hull of any other γ

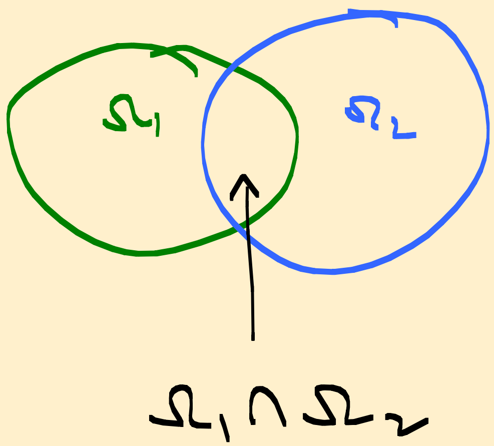
γ_3 = smallest convex set that carries the "far field"

Lemma If Ω_1 carries α and Ω_2 carries α, then $\Omega_1 \cap \Omega_2$ carries α

Proof

$(\Delta+k^2)u_1 = F_1 \qquad (\Delta+k^2)u_2 = F_2$

$u_1 \equiv u_2$



Rellich's Lemma and Unique Continuation guarantee that

$u_1 \equiv u_2$ on $(\mathbb{R}^n\setminus\Omega_1) \cap \mathbb{R}^n\setminus\Omega_2)$

so that

$$v := \begin{cases} u_1 & \text{on } \mathbb{R}^n\setminus\Omega_1 \\ u_2 & \text{on } \mathbb{R}^n\setminus\Omega_2 \\ 0 & \text{on } \Omega_1\cap\Omega_2 \end{cases}$$

is well defined and

$F_3 := (\Delta+k^2)v$ is supported in $N_2(\Omega_1\cap\Omega_2)$

Theorem Every far field [with a compactly supported source] has a unique minimal convex carrier.

Theorem Every far field [with a compactly supported source] has a unique minimal UWSCS carrier.

Union of Well Separated Convex Sets –

distance between convex components > diameter of any component

Proof

Define $\text{c-supp}\,\alpha = \bigcap_{\substack{\Omega \text{ carries } \alpha \\ \Omega \text{ convex}}} \Omega$ It's unique. It's minimal.

Does it carry the far field?

Lemma plus compactness says yes.

Unions of Well Separated Convex Sets [compact]

satisfy

1. Closed under intersection
2. $\mathbb{R}^n \setminus (\Omega_1 \cup \Omega_2)$ connected

which guarantees some lemma.

<u>Moral</u> It makes theoretical sense to look for collections of sources that are small compared to the distance between them

How Small is the c-support of a far field

$f = f(r)$ c-support = point

f = sum of point sources c-support = convex hull
uwscs-support = union of points

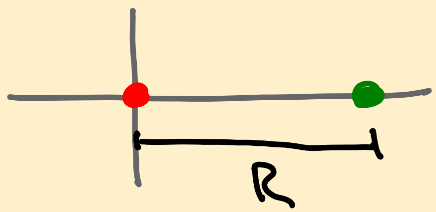
$f = \chi_{Box}$ c-support = box [because of corners]

$= \chi_{Box}\Phi$ smooth c-support = box if Taylor Expansion of Φ at corners starts with harmonic poly

$f = \chi_{Ellipse}$ c-support $\overset{?}{=}$ line connecting foci

Wavelength-Distinguishing well-separated Sources

Point Sources



Far Field $\hat{\delta}_0| = 1$

Far Field $\hat{\delta}_R| = e^{ikR\cos\theta}$

I can distinguish them reliably if the cosine of the angle between them < 1

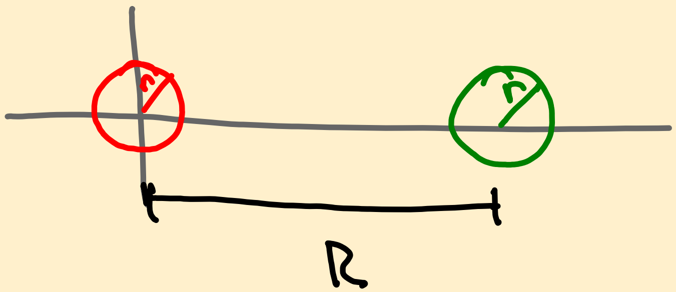
$$\frac{\hat{\delta}_0| \cdot \hat{\delta}_R|}{\|\hat{\delta}_0|\| \, \|\hat{\delta}_R|\|} = \frac{\int_0^{2\pi} 1 \, e^{ikR\cos\theta}}{\sqrt{2\pi}\,\sqrt{2\pi}} = J_0(kR) < 1$$

Since $|J_0(kR)| < \frac{1}{\sqrt{kR}}$, $kR < 1$ suffices.

They need to be more than a wavelength apart

Increasing k increases well-posedness

Distinguishing Bigger Sources



$\alpha(\theta)$ = Far Field carried by B_1

$\beta(\theta)$ = Far Field carried by B_2

Theorem

Conclusion
$$\frac{\int \alpha(\theta)\,\beta(\theta)\,d\theta}{\|\alpha\|\cdot\|\beta\|} \leq \text{const.}\ \frac{kr^2}{R}$$

But $\text{span}\{\alpha\}$ is dense and $\text{Span}\{\beta\}$ is dense

Hypothesis
$$\frac{\|f_\alpha\|^2}{\|\alpha\|^2} < M \qquad \frac{\|f_\beta\|^2}{\|\beta\|^2} < M$$

M = power/sensitivity ratio

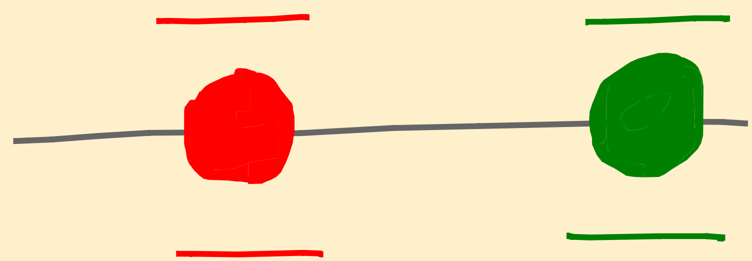
Ratio of transmitted power necessary to generate received power

For $R > 3r$, $\dfrac{\int \alpha(\theta)\,\beta(\theta)\,d\theta}{\|\alpha\|\cdot\|\beta\|} \le \text{const.}\ \dfrac{kr^2}{R}$

and for $\dfrac{kr^2}{R}$ small, it can be that big.

Corollary If you fix the geometry, and increase k, well-posedness get worse

Worst Case



The far fields that are the most similar are the far fields of approximate plane waves perpendicular to the line connecting the centers.

Happy BirthDay

Gunther !!

I wish you

many future collaborators

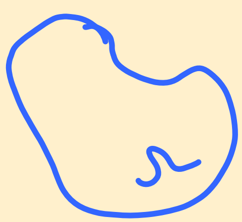
No more tenants !!

Some Examples to Illustrate the Need For Convexity

We expect minimal carriers to be small sets.

<u>Theorem</u> If Ω carries α, then $\partial\Omega$ carries α

Pf//



Replace u^+ by φu^+

$$\varphi = \begin{cases} 0 & \text{inside} \\ 1 & \text{outside } N_\varepsilon(\Omega) \end{cases}$$

and replace f by $(\Delta + k^2)(\varphi u^+)$

Thin Sources (Double and Single Layers)

f is a thin source if ① $f \in H^s(\mathbb{R}^n)$ for $s > -2$

② measure (supp f) $= 0$ + compact

Theorem A thin source is NR $\Longleftrightarrow$ $f = \mathcal{C}_{\partial\Omega} v^0$ [linear span]

v^0 is a free wave $(\Delta + k^2) v^0 = 0$ in Ω and $v^0 \in L^1(\Omega)$

$$\mathcal{C} v^0 \big|_{\partial\Omega} = \frac{\partial v^0}{\partial\nu} \delta_{\partial\Omega} + v^0 \delta'_{\partial\Omega}$$ [Cauchy Data restricted to boundary of Ω]

$$\langle \varphi, \mathcal{C} v^0|_{\partial\Omega} \rangle = \int_\Omega v^0 (\Delta + k^2) \varphi$$ [For Ω any bounded open set + $v^0 \in L^4(\Omega)$]

A thin source is NR $\Longleftrightarrow$ $f = \mathcal{C}_{\partial\Omega} v^0$

Proof $\Longleftarrow$

Start with v^0 free

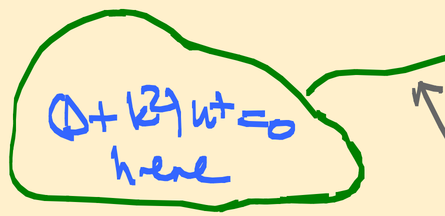
Define $u^+ = \begin{cases} v^0 & \text{inside } \Omega \\ 0 & \text{outside } \Omega \end{cases}$

$$(\Delta + k^2) u^+ = \mathcal{C}_{\partial\Omega} v_0$$

$\Longrightarrow$ $(\Delta + k^2) u^+ = f$

$u^+ \equiv 0$ here

Rellich Lemma
+
Unique Continuation

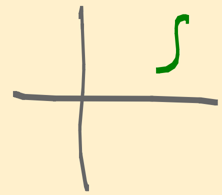


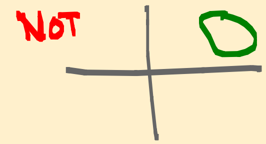
$[u] + \left[\frac{\partial u}{\partial \nu}\right] = 0$
so $f = 0$ here

$[u^+]$ = limit from inside here
$\left[\frac{\partial u^+}{\partial \nu}\right]$ = " " " "

Arcs don't contain boundaries

An arc is a thin source with support γ such that $\mathbb{R}^n \setminus \gamma$ is connected.

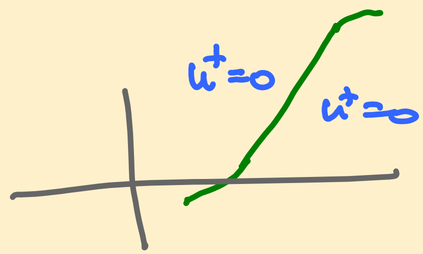




Theorem

A non-radiating thin source supported on an arc is zero.

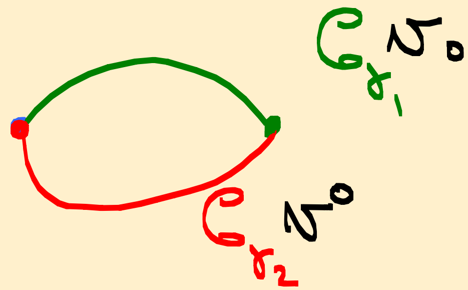
Proof



Corollary Arcs are minimal carriers.

If support of a thin source is an arc, then no subset of the arc carries the same far field.

Example ν^0 Free $\gamma_1 \cup \gamma_2 = \partial\Omega$



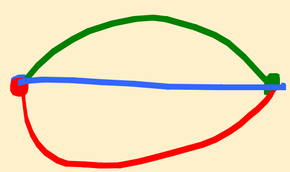
$G_{\gamma_1}\nu^0 + G_{\gamma_2}\nu^0$ is Non-Radiating

$G_{\gamma_1}\nu_0$ and $-G_{\gamma_2}\nu_0$ are equivalent

Both are arcs - so they are minimal

Moral There is no unique minimal carrier

But



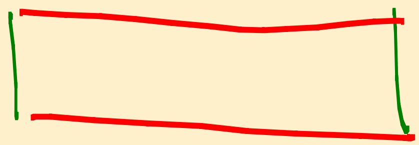
γ_3 is the unique smallest convex set that carries the Far Field

Example

v^0 free (e.g. $e^{ik\theta\cdot x}$ or $J_0(kr)$)

$G_R v_0$ is non-radiating

$R = \gamma_1 \cup \gamma_2$



$G_{\gamma_1} v^0$ and $-G_{\gamma_2} v^0$ radiate the same far field

but only γ_1 is well-separated.