Local Inversions in Ultrasound Modulated Optical Tomography

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Ultrasound Modulated Optical Tomography (Acousto-Optics)

- Acoustic waves are emitted which perturb the optical properties of the medium
- Light propagating through the medium is used to recover the original optical parameters
- Model by G. Bal and J. C. Schotland Phys. Rev. Letters, 2010.

Optical properties perturbed by acoustic waves

$$\sigma_\epsilon = \sigma + \epsilon(2\beta + 1)\cos(k \cdot x + \phi)$$
$$\gamma_\epsilon = \gamma + \epsilon(2\beta - 1)\cos(k \cdot x + \phi)$$

Linearization wrt epsilon and some manipulation yields boundary data

$$\Sigma(k, \phi) = \int_\Omega \left[(2\beta - 1)\gamma(\nabla\phi)^2 + (2\beta + 1)\sigma\phi^2\right]\cos(k \cdot x + \phi)$$

Which is the Fourier transform of some internal data

Mathematical Problem

Given internal data of the form

$$H_{ij}(x) = \gamma \nabla u_i \cdot \nabla u_j + \eta \sigma u_i u_j,$$

where η is a known fixed constant and

$$\begin{aligned} -\nabla \cdot \gamma \nabla u_j + \sigma u_j &= 0 \ \text{ in } \ \Omega \\ u_j &= f_j \ \text{ on } \ \partial\Omega \end{aligned}$$

Find γ and σ

Previous work

- Recovery of γ only, $\sigma = 0$

Capdeboscq, Fehrenback, De Gournay, Kavian (n=2)
Bal, Bonnetier, Monard, Triki (n=3)
Bal, Monard (n>=4)
Kuchment, Kunyansky
Kuchment, Steinhauer- pseudo-differential calculus
Ammari, Capdeboscq, Triki 2012- separation of terms

Assume we have some known background γ_0 and σ_0 .

$$\begin{aligned}
\gamma &= \gamma_0 + \delta\gamma \\
\sigma &= \sigma_0 + \delta\sigma \\
u_j &= u_j^0 + \delta u_j
\end{aligned}$$

Where the background solutions satisfy

$$\begin{aligned}
-\nabla \cdot \gamma_0 \nabla u_j^0 + \sigma_0 u_j^0 &= 0 \;\text{ in }\; \Omega \\
u_j^0 &= f_j \;\text{ on }\; \partial\Omega
\end{aligned}$$

$$L_0 := -\nabla \cdot \gamma_0 \nabla + \sigma_0$$

$$\delta u_j = L_0^{-1}(\nabla \cdot \delta\gamma \nabla u_j^0 - \delta\sigma u_j^0)$$

linearized problem

$$
\begin{aligned}
dH_{ij} &= \delta\gamma \nabla u_i^0 \cdot \nabla u_j^0 + \gamma_0 \nabla \delta u_i \cdot \nabla u_j^0 + \gamma_0 \nabla u_i^0 \cdot \nabla \delta u_j^0 \\
&\quad + \eta \delta\sigma u_i^0 u_j^0 + \eta \sigma_0 \delta u_i u_j^0 + \eta \sigma_0 u_i^0 \delta u_j
\end{aligned}
$$

Really have 3 unknowns here $\delta u_j, \delta\gamma, \delta\sigma$

$$L_0 \delta u_j = \nabla \cdot \delta\gamma \nabla u_j^0 - \delta\sigma u_j^0$$

But they are coupled

One approach: solve for δu_j and substitute back in

Take Laplacian of data

$$\Delta dH_{ij}(\delta\gamma, \delta\sigma) = G_{ij}(\delta\gamma, \delta\sigma) + \Delta T_{ij}(\delta\gamma, \delta\sigma),$$

where T_{ij} is compact, and

$$G_{ij}(\delta\gamma, \delta\sigma) = \nabla u_i^0 \cdot \nabla u_j^0 \Delta\delta\gamma - 2(\nabla u_i^0 \otimes \nabla u_j^0)^s : D^2\delta\gamma + \eta u_i^0 u_j^0 \Delta\delta\sigma$$

Simplest case

- Case where n=2 and $\gamma_0 = 1, \sigma_0 = 0$
- Take $u_0 = 1$ to get

$$dH_{00}(\delta\gamma, \delta\sigma) = \eta\delta\sigma$$

- Eliminate $\delta\sigma$
- Take

$$u_i^0 = x_i$$

then

$$\Delta d\tilde{H}_{11} = (\partial^2_{x_2} - \partial^2_{x_1})$$

$$\Delta d\tilde{H}_{12} = -2\partial_{x_1 x_2}$$

$$\Delta d\tilde{H}_{22} = (\partial^2_{x_1} - \partial^2_{x_2})$$

Separately get hyperbolic, not elliptic
Together elliptic as a redundant system
Hard to invert because redundant

but consider

$$\Gamma^T \Gamma = \sum_{ij} (\Delta d \tilde{H}_{ij})^2$$

$$\Gamma^T \Gamma = 2\partial_{x_1}^4 + 2\partial_{x_2}^4$$

Which is elliptic.

And for $n \geq 3$

$$\sum_{i=1}^{n} \Delta d\tilde{H}_{ii} = (n-2)\Delta$$

Which we can invert

For general σ_0 this doesn't work

So let us consider for general γ_0, σ_0

The highest order part

$$G_{ij}(\delta\gamma, \delta\sigma) = \nabla u_i^0 \cdot \nabla u_j^0 \Delta\delta\gamma - 2(\nabla u_i^0 \otimes \nabla u_j^0)^s : D^2\delta\gamma + \eta u_i^0 u_j^0 \Delta\delta\sigma$$

Define

$$\theta_i = \frac{\nabla u_i^0}{|\nabla u_i^0|}$$

Then we are interested in the system

$$\mathrm{A}_{ij}\delta\gamma + B_{ij}\delta\sigma = F_{ij}\,,$$

where

$$\mathrm{A}_{ij} = \theta_i \cdot \theta_j \Delta - 2(\theta_i \otimes \theta_j)^s : \nabla \otimes \nabla,$$

and

$$\mathrm{B}_{ij} = \eta d_i d_j \Delta, \quad d_i = \frac{u_i}{|\nabla u_i|}.$$

Define the operator

$$\Gamma = \begin{pmatrix} A_{ij} & B_{ij} \end{pmatrix}$$

With a row for each pair (i,j)

We want to show this operator is elliptic so that we can get a parametrix, or Invertibility of the highest order part.

Construction of parametrices for similar problems in

Kuchment and Steinhauer for one coefficient.

Consider the 2x2 system

$$\Gamma^T \Gamma \begin{pmatrix} \delta\gamma \\ \delta\sigma \end{pmatrix} = \Gamma^T F.$$

$$\Gamma^T \Gamma = \begin{pmatrix} \sum_{ij} A_{ij}^T A_{ij} & \sum_{ij} A_{ij}^T B_{ij} \\ \sum_{ij} B_{ij}^T A_{ij} & \sum_{ij} B_{ij}^T B_{ij} \end{pmatrix}.$$

Using symbols, the system is invertible when we always have at least one of the sub-determinants not vanishing

$$\mathrm{Det}\begin{pmatrix} A_{ij} & B_{ij} \\ A_{kl} & B_{kl} \end{pmatrix}$$

These determinants are zero when

$$(\theta_i \cdot \theta_j - 2\theta_i \cdot \hat{\xi}\theta_j \cdot \hat{\xi})d_p d_q = (\theta_p \cdot \theta_q - 2\theta_p \cdot \hat{\xi}\theta_q \cdot \hat{\xi})d_i d_j \qquad \forall (i, j, p, q).$$

$$\theta_i = \frac{\nabla u_i^0}{|\nabla u_i^0|} \qquad \mathrm{d}_i = \frac{u_i^0}{|\nabla u_i^0|}$$

Thanks to Gunther Uhlmann and CGOs

$$
\begin{aligned}
u_\rho &= e^{\rho\cdot x} \\
&= e^{\rho_r\cdot x}(\cos\rho_I\cdot x + i\sin\rho_I\cdot x) \\
\nabla u_\rho &= e^{\rho_r\cdot x}\left[(\rho_r\cos\rho_I\cdot x - \rho_I\sin\rho_I\cdot x) + i(\rho_r\sin\rho_I\cdot x + \rho_I\cos\rho_I\cdot x)\right]
\end{aligned}
$$

Which gives, by taking real and imaginary parts

$$
\theta_1 = \begin{pmatrix} \cos\rho_I\cdot x \\ -\sin\rho_I\cdot x \end{pmatrix} \quad \theta_2 = \begin{pmatrix} \sin\rho_I\cdot x \\ \cos\rho_I\cdot x \end{pmatrix}
$$

$$
d_1 = \frac{\cos\rho_I\cdot x}{|\rho|} \quad d_2 = \frac{\sin\rho_I\cdot x}{|\rho|}
$$

$$
\begin{aligned}
(1-2(\theta_1\cdot\xi)^2)d_2^2 &= (1-2(\theta_2\cdot\xi)^2)d_1^2 \\
-2\theta_1\cdot\xi\theta_2\cdot\xi d_1^2 &= (1-2(\theta_1\cdot\xi)^2)d_1d_2 \\
-2\theta_1\cdot\xi\theta_2\cdot\xi d_2^2 &= (1-2(\theta_2\cdot\xi)^2)d_1d_2
\end{aligned}
$$

Which is

$$
\begin{aligned}
(s^2-c^2)d_2^2 &= (c^2-s^2)d_1^2 \\
-2csd_1^2 &= (s^2-c^2)d_1d_2 \\
-2csd_2^2 &= (c^2-s^2)d_1d_2
\end{aligned}
$$

$$(1) \Rightarrow s^2-c^2=0$$

$$(2) \text{ or } (3) \Rightarrow sc=0$$

- But ellipticity doesn't guarantee injectivity
- Need injectivity for extensions to nonlinear problem

One approach: view as a differential operator with its natural square bilinear form

$$\mathrm{B}\left(\left(\begin{array}{c} v \\ w \end{array}\right), \left(\begin{array}{c} v \\ w \end{array}\right)\right) := \int_\Omega \Gamma\left(\begin{array}{c} v \\ w \end{array}\right) \cdot \Gamma\left(\begin{array}{c} v \\ w \end{array}\right)$$

on

$$\mathrm{H}_0^2(\Omega) \times H_0^2(\Omega)$$

Variational fomulation: find

$$(\delta\gamma, \delta\sigma) \in H_0^2(\Omega) \times H_0^2(\Omega)$$

Such that

$$\mathrm{B}\left(\begin{pmatrix} \delta\gamma \\ \delta\sigma \end{pmatrix}, \begin{pmatrix} v \\ w \end{pmatrix}\right) + L\left(\begin{pmatrix} \delta\gamma \\ \delta\sigma \end{pmatrix}, \begin{pmatrix} v \\ w \end{pmatrix}\right) = \int_\Omega F \cdot \Gamma^T \begin{pmatrix} v \\ w \end{pmatrix}$$

$$\forall (v, w) \in H_0^2(\Omega) \times H_0^2(\Omega)$$

Where L is a lower order operator
(generally nonlocal)

- B is clearly bounded above on $\mathrm{H}_0^2(\Omega) \times H_0^2(\Omega)$
- Know elliptic, can get coercivity bounds explicitly in some cases

Case n=2, constant σ_0, γ_0

- Have the two background solutions

$$u_1^0 = e^{\sqrt{\frac{\sigma_0}{\gamma_0}}x_1}$$

$$u_2^0 = e^{\sqrt{\frac{\sigma_0}{\gamma_0}}x_2},$$

- Which give

$$\theta_i = e_i \text{ and } d_i = \sqrt{\frac{\gamma_0}{\sigma_0}}.$$

$$\Gamma = (\; A_{ij} \;\; B_{ij} \;)$$

Corresponding to (i,j)=(1,1),(1,2),(2,2) where

$$A_{11} = \partial_{yy} - \partial_{xx}$$

$$A_{12} = -2\partial_{xy}$$

$$A_{22} = \partial_{xx} - \partial_{yy}$$

$$B := B_{11} = B_{12} = B_{22} = \eta \frac{\gamma_0}{\sigma_0} \Delta.$$

$$B\left(\left(\begin{array}{c} v \\ w \end{array}\right), \left(\begin{array}{c} v \\ w \end{array}\right)\right) = \int_\Omega 2(v_{xx})^2 + 2(v_{yy})^2 + 3\eta^2 \frac{\gamma_0^2}{\sigma_0^2}(\Delta w)^2 - 2\eta \frac{\gamma_0}{\sigma_0} v_{xy} \Delta w$$

Use Cauchy's inequality

$$|v_{xy}\Delta w| \leq \epsilon v_{xy}^2 + \frac{(\Delta w)^2}{4\epsilon}$$

and integration by parts

$$\int_\Omega v_{xy}^2 = \int_\Omega v_{xx} v_{yy}$$

$$B\left(\begin{pmatrix} v \\ w \end{pmatrix}, \begin{pmatrix} v \\ w \end{pmatrix}\right)$$

$$\geq \int_\Omega \begin{pmatrix} 2 & -|\eta|\frac{\gamma_0}{\sigma_0}\epsilon \\ -|\eta|\frac{\gamma_0}{\sigma_0}\epsilon & 2 \end{pmatrix} \begin{pmatrix} v_{xx} \\ v_{yy} \end{pmatrix} \cdot \begin{pmatrix} v_{xx} \\ v_{yy} \end{pmatrix} + \left(3\eta^2 \frac{\gamma_0^2}{\sigma_0^2} - |\eta| \frac{\gamma_0}{2\epsilon\sigma_0}\right) (\Delta w)^2$$

choose $\epsilon = \dfrac{\sigma_0}{\gamma_0|\eta|}.$ To get

$$\geq \|v_{xx}\|_{L^2}^2 + \|v_{yy}\|_{L^2}^2 + \frac{3}{2}\frac{\gamma_0^2}{\sigma_0^2}\eta^2\|\Delta w\|_{L^2}^2.$$

- If we have injectivity, this means that the linearized solutions

$$\|\hat{\delta\gamma}\|_{H_0^2(\Omega)}, \|\hat{\delta\sigma}\|_{H_0^2(\Omega)} \leq C\|F\|_{L^2(\Omega)}$$

- and we have explicit knowledge of C

- System is elliptic- but don't yet know if injective.
- But since problem is square:

$$\int_\Omega \Gamma \begin{pmatrix} v \\ w \end{pmatrix} \cdot \Gamma \begin{pmatrix} v \\ w \end{pmatrix} = 0 \Rightarrow \Gamma \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Case where domain is small

- If the domain is small, ∇u_i^0 and u_i^0 are close to constants

$$
\begin{aligned}
dH_{ij}(\delta\gamma, \delta\sigma) \quad &= \quad \delta\gamma \nabla u_i^0 \cdot \nabla u_j^0 + \gamma_0 \nabla \delta u_i \cdot \nabla u_j^0 + \gamma_0 \nabla u_i^0 \cdot \nabla \delta u_j \\
&\quad + \eta \delta\sigma u_i^0 u_j^0 + \eta \sigma_0 \delta u_i u_j^0 + \eta \sigma_0 u_i^0 \delta u_j
\end{aligned}
$$

So when we take L_0 of data, lower order terms are differential operators.

$$L_0 = -\nabla \cdot \gamma_0 \nabla + \sigma$$

- if Γ is a differential operator and

$$\Gamma \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

- since $v = \frac{\partial v}{\partial \nu} = w = \frac{\partial w}{\partial \nu} = 0 \text{ on } \partial\Omega$

We can get that $v = w = 0$ from Holmgren’s theorem

Conclusions/Future

- Have ellipticity for linearized system
- Have injectivity with boundary data if the domain is small enough (by variational formulation and Holmgren's theorem)
- So for small domains, can extend to local nonlinear injectivity/inversion
- Still to do: injectivity for more general domains