THE INVERSE PROBLEM FOR THE DIRICHLET-TO-NEUMANN MAP ON LORENTZIAN MANIFOLDS

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ABSTRACT. We consider the Dirichlet-to-Neumann map Λ on a cylinder-like Lorentzian manifold related to the wave equation related to the metric g, a magnetic field A and a potential q. We show that we can recover the jet of g, A, q on the boundary from Λ up to a gauge transformation in a stable way. We also show that Λ recovers the following three invariants in a stable way: the lens relation of g, and the light ray transforms of A and q. Moreover, Λ is an FIO away from the diagonal with a canonical relation given by the lens relation. We present applications for recovery of A and q in a logarithmically stable way in the Minkowski case, and uniqueness with partial data.

1. Introduction and main results

Let (M, g) be a Lorentzian manifold of dimension $1+n$, $n \geq 2$, i.e., g is a metric with signature $(-1, 1, \dots, 1)$. Suppose a part of ∂M is timelike. An example of M is a cylinder-like domain representing a moving and shape changing compact manifold in the x-space (if we have fixed time and space variables) with the requirement that the normal speed of the boundary is less than one, see section 5.

Denote the wave operator by $\Box_g$; in local coordinates $x = (x^0, \dots, x^n)$ it takes the form:

$$\Box_g := \frac{1}{\sqrt{|\det g|}} \partial_j \left(\sqrt{|\det g|} g^{jk} \partial_k \right).$$

Consider the following operator $P = P_{g,A,q}$ which is a first order perturbation of $\Box_g$:

$$P = P_{g,A,q} := \frac{1}{\sqrt{|\det g|}} (\partial_j - iA_j) \sqrt{|\det g|} g^{jk} (\partial_k - iA_k) + q. \tag{1}$$

Here $i = \sqrt{-1}$; A is a smooth 1-form on M; q is a smooth function on M.

The goal of this work is to study the inverse problem of recovery of g, A and q, up to a data preserving gauge transformation, from the outgoing Dirichlet-to-Neumann (DN) Λ map on a timelike boundary associated with the wave equation

$$Pu = 0 \quad \text{in } M. \tag{2}$$

We are motivated by applications in relativity but also in applications to classical wave propagation problems with media moving and/or changing at a speed not negligible compared to the wave speed. We are interested in possible stability results even though some steps in the recovery are inherently unstable. This problem remains widely open. The results we prove are the following. First, we show that one can recover the jet of g, A, q at the boundary (up to a gauge transform) in a Hölder stable way. Next, we show that one can extract the natural geometric invariants of g, A, q from Λ in a Hölder stable way. More precisely, Λ recovers the lens relation $\mathcal{L}$ related to g, in stable way. If we know g, the light ray transform $L_1 A$ of A is recovered stably. If g and A are known, the light ray transform $L_0 q$ of q is recovered stably. The lens relation $\mathcal{L}$ is the canonical relation of the Fourier Integral Operator (FIO) Λ away from the diagonal, and the light ray transforms $L_1 A$ and $L_0 q$ are in fact encoded in the principal and the subprincipal symbol of it. In fact, $\mathcal{L}$ is directly measurable from Λ.

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Since the results we prove are local or semilocal (near a fixed lightlike geodesic); and the proofs are microlocal, we do not formulate a global mixed problem for the wave equation at the beginning but we do consider one in section 5. In fact, existence of solutions of such problems depend on global properties of (M, g), one of them is global hyperbolicity, which are not needed for our weaker formulation and for the proofs. Instead, we define the DN map up to smoothing operators only. In case when one can prove the existence of a global solution, the true DN map would coincide with ours up to a smoothing error, see section 5; and our results are not affected by adding smoothing operators.

This problem has a long history in the stationary Riemannian setting, i.e., when $M = [0, T] \times M_0$, where (M_0, g) is a compact Riemannian manifold with boundary, and the metric is $-\mathrm{d}t^2 + g_{ij}(x)\mathrm{d}x^i\mathrm{d}x^j$. The boundary control method [4] and the Tataru's uniqueness continuation theorem [44, 45] provide uniqueness provided that T is greater than a certain sharp critical value T, as shown by Belishev and Kurylev in [6], see also the survey [5]. Stability however does not follow from such arguments. Stability results for recovering of the metric and lower order terms appeared in [40, 38, 28, 8, 2], with [28] covering the general case. A main assumption in those works is that the metric is simple, i.e., that there are no conjugate points and the boundary is strictly convex (not so essential assumption) and the main technical tool for recovery of the metric is to reduce it to stability for the boundary/lens rigidity problem, see, e.g., [39]. For related results, we refer to [21, 43]. Recently, the progress in treating the local rigidity problem allowed results under the more general foliation condition [41] which allows conjugate points. In any case, some condition is believed to be necessary for stability. It is worth noticing that all inverse (hyperbolic) *scattering* problems for compactly supported perturbations are equivalent to inverse DN map problems.

Recently, there has been increased interest in this problem or in related inverse scattering problems in time-space. Recovery of lower order time-dependent terms for the Minkowski metric has been studied in [36, 33, 32, 47, 34, 1, 7], and for $-\mathrm{d}t^2 + g_{ij}(x)\mathrm{d}x^i\mathrm{d}x^j$ in [23]. In [14], Eskin proved that one can recover g, A, q up to a gauge transformation, assuming existence of a global time variable t and analyticity of all coefficients with respect to it. The proof is based in an adaptation of the boundary control methods and the analyticity is needed so that one can still use the unique continuation results in [45]. Stability does not follow from such arguments. Other inverse problems on Lorentzian manifolds are studied in [24, 25, 27]. The inverse scattering problem of recovery a moving boundary is studied in [10, 42, 15]. The first author showed in [36] that in the case of g Minkowski and $A = 0$, the problem of recovery of q reduces to the inversion of the X-ray transform in time-space over light rays, which was shown there to be injective for functions tempered in time and uniformly compactly supported in space. In [26], it is shown that the linearized metric problem leads to the inversion of a light ray transform of tensor fields. Such light ray transforms are inherently unstable however because they are smoothing on the time-like cone. They require specialized tools for analyzing the singularities near the lightlike cone, not fully developed in the geodesic case, see [16, 17, 18]. The light ray transform has been also studied in [9, 3, 37, 22].

We describe the main results below. Let $x_0 \in \partial M$ and assume that ∂M is timelike near x_0. Then ∂M with the induced metric is a Lorentzian manifold as well and we choose (locally) one of the two time orientations that we call future pointing.

Let $f \in \mathcal{E}'(\partial M)$ be supported near x_0 with $\mathrm{WF}(f)$ close to a fixed timelike $(x_0, \xi^{0\prime}) \in T^*\partial M \setminus 0$. We define the *local outgoing* solution operator $f \mapsto u$, defined up to a smoothing operator, as the operator mapping f to the *outgoing* solution u of

$$Pu \in C^\infty \quad \text{in } M \text{ near } x_0, \qquad u|_{\partial M} = f \mod C^\infty. \tag{3}$$

The term "outgoing" here refers to the following. We chose that microlocal solution (parametrix) for which the singularities of the solution are required to propagate along future pointing bicharacteristics. We refer to section 2.1 for more details. On the other hand, it is "local" because it solves (3) near x_0 only and this keeps the singularities close enough to ∂M without allowing them to hit ∂M again.

Define the associated *local outgoing* Dirichlet-to-Neumann map as

$$\Lambda^{\mathrm{loc}}_{g,A,q} f = \left(\partial_\nu u - i\langle A, \nu\rangle u\right)|_{\partial M}, \tag{4}$$

where ν denotes the unit outer normal vector field to ∂M, and the equality is modulo smoothing operators applied to f. By definition, the $\Lambda^{\rm loc}_{g,A,q}$ is defined near x_0 only, and in fact, in some conic neighborhood of the timelike $(x_0,\xi^{0\prime})$. Since the latter is arbitrary, $\Lambda^{\rm loc}_{g,A,q}$ extends naturally to the whole timelike cone on ∂M but we keep it microlocalized near $(x_0,\xi^{0\prime})$ to emphasize what we can recover given microlocal data only.

As we show in Theorem 3.1, $\Lambda^{\rm loc}_{g,A,q}$ is actually a ΨDO on the timelike cone bundle near x_0. The main result about $\Lambda^{\rm loc}_{g,A,q}$ is Theorem 3.2: a stability estimate about the recovery of the boundary jets of the coefficients.

Let $f\in\mathcal{E}'(\partial M)$ have $\mathrm{WF}(f)$ as above. Let u, as in (3), be the parametrix in a neighborhood of the future pointing null bicharacteristic issued from the unique future pointing lightlike covector $(x_0,\xi^0)\in T^*M\setminus 0$ with orthogonal projection $(x_0,\xi^{0\prime})$. Note that the direction of (x_0,ξ^0) and that of the bicharacteristic might be the same or opposite. Assume that this bicharacteristics hits ∂M again, transversely, at point y_0 in the codirection η^0 and let $\eta^{0\prime}$ be the corresponding orthogonal tangential projection on $T^*_{y_0}\partial M$. Then $(y_0,\eta^{0\prime})$ is timelike, as well. Let $\mathcal{U}$ and $\mathcal{V}$ be two small conic timelike neighborhoods in $T^*\partial M\setminus 0$ of $(x_0,\xi^{0\prime})$ and $(y_0,\eta^{0\prime})$, respectively. If $\mathcal{U}$ is small enough, for every timelike $(x,\xi')\in\mathcal{U}$ close to $(x_0,\xi^{0\prime})$, we can define (y,η') in the same way. This defines the *lens relation*

$$\mathcal{L}:\mathcal{U}\longrightarrow\mathcal{V},\quad \mathcal{L}(x,\xi')=(y,\eta'), \tag{5}$$

see Figure 1. By definition, $\mathcal{L}$ is an even map in the second variable, i.e., $\mathcal{L}(x,-\xi')=(y,-\eta')$. If (x,ξ') is future pointing (i.e., if the associated vector by the metric is such), then $(x,-\xi')$ is past-pointing but we can interpret $(y,-\eta')$ as the end point of the null geodesic with initial point projecting to $(y,-\eta')$ but moving "backward" w.r.t. the parameter over it. This property correlates well with Theorem 4.1 since the wave equation has two wave "speeds" of opposite signs.

The map $\mathcal{L}$ is positively homogeneous of order one in its second variable. Now, for f as above, let u be the outgoing solution to (3) near the bicharacteristic issued from (x_0,ξ^0) all the way to its second contact with ∂M at y_0. At this point, we assume that $(x_0,\xi^{0\prime})$ is not a fixed point for $\mathcal{L}$, which means that the reflected bicharacteristic does not become a periodic one after the first reflection. Since f is smooth near $(y_0,\eta^{0\prime})$ that means no singularity of the solution u at $(y_0,\eta^{0\prime})$, therefore, the singularity reflects at y_0. We extend the solution microlocally over a small segment of the reflected ray before reaching ∂M again, see Proposition 4.1. Then we define the *global DN map* $\Lambda^{\rm gl}_{g,A,q}$ by (4) again but with the r.h.s. localized to V, the projection of $\mathcal{V}$ to the base. In fact, by propagation of singularities, $\Lambda^{\rm gl}_{g,A,q}f$ has a wave front set in $\mathcal{V}$ only and we can cut smoothly outside some neighborhood of y_0. The map $\Lambda^{\rm gl}_{g,A,q}$ is actually just semi-global because it is the DN map restricted to a solution near one geodesic segment connecting boundary points. In Theorem 4.1, we prove that $\Lambda^{\rm gl}_{g,A,q}$ is an FIO associated with the graph of $\mathcal{L}$. In Theorem 4.2, we show that $\Lambda^{\rm gl}_{g,A,q}$ recovers $\mathcal{L}$ in a stable way, which is also a general property of FIOs associated to a local canonical diffeomorphism.

Another fundamental object is the *light ray transform* L which integrates functions or more generally tensor fields along lightlike geodesics. We define L on functions by

$$L_0f(\gamma)=\int f(\gamma(s))\,\mathrm{d}s, \tag{6}$$

and on covector fields of order one by

$$L_1f(\gamma)=\int\langle f(\gamma(s)),\dot\gamma(s)\rangle\,\mathrm{d}s, \tag{7}$$

where $\langle f(\gamma(s)),\dot\gamma(s)\rangle=f_j(\gamma(s))\dot\gamma^j(s)$ in local coordinates and γ runs over a give set of lighlike geodesics, and we always assume that $\operatorname{supp}f$ is such that the integral is taken over a finite interval. In out results below, γ's in L_0 and L_1 are the maximal geodesics through M connecting boundary points. Unlike the Riemannian case, lightlike geodesics do not have a natural speed one parameterization and every rescaling of the parameter along them (even if that rescaling changes from geodesic to geodesic) keeps them being lightlike. The transform L_1 is invariant under reparameterization of the geodesics and can be considered as an integral of $\langle f,\mathrm{d}\gamma\rangle$ over the geodesics. On the other hand, L_0 is not. Despite that freedom, the property

$L_0 f = 0$ does not change. One way to parameterize it is to define it locally near a lightlike geodesic hitting a timelike surface at $s = 0$, in our case, ∂M. Then the orthogonal projection $\dot\gamma'(0)$ of each such γ on $T\partial M$ (the prime stands for projection) determines $\dot\gamma(0)$ and therefore, γ uniquely. To normalize the projections on $T\partial M$, we can choose a timelike covector field Z on $T\partial M$ locally and require $g(\dot\gamma, Z) = \mp 1$ for future/past pointing directions.

In Theorem 4.3, we show that given g, one can recover $L_1 A$ in a Hölder stable way; and if we are given g, A, one can recover $L_0 q$ in a Hölder stable way. Notice that we do not require absence of conjugate points and we do not use Gaussian beams. Instead, we use standard microlocal tools including Egorov's theorem. In section 5, we consider some cases where L_1 and L_0 can be inverted to derive uniqueness results. As we mentioned above, those transforms are unstable. The reason is that they are microlocally smoothing in the spacelike cone, see, e.g., [18, 37, 26]. Therefore, stable recovery of $L_1 A$ and $L_0 q$ does not imply Hölder stable recovery of A_1 (up to a gauge transform) and q but allows for weaker logarithmic estimates using the estimate for recovery of q from $L_0 q$ in the Minkowski case proven in [3], for example. We discuss some of those possible corollaries in section 5. Recovery of g from $\mathcal{L}$ is an open problem with some results about the linearized problems obtained recently in [26].

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2. Preliminaries

2.1. Notation and terminology.

In what follows, we denote by U and V the projections of $\mathcal{U}$ and $\mathcal{V}$ onto the base ∂M. We freely assume that $\mathcal{U}$ and $\mathcal{V}$, and therefore, U and V are small enough to satisfy the needed requirements below.

If ξ is a covector based at a point x on ∂M, we denote by ξ' its orthogonal projection to $T_x^*\partial M$. We routinely denote covectors on $T_x^*\partial M$ by placing primes, like ξ', etc., even if a priori such covector is not a projection of a given one.

Timelike/spacelike/lightlike vectors v are the ones satisfying $g(v,v) < 0$, or $g(v,v) > 0$, or $g(v,v) = 0$, respectively. We identify vectors and covectors by the metric. We choose an orientation in U that we call future pointing (FP). More precisely, we choose some smooth timelike vector Z in U (identified with an open set in the tangent bundle) and we call *future pointing* those timelike vectors v for which $g(v, Z) > 0$. If we have a time variable t, for example, such a choice could be $Z = \partial/\partial t$. In semigeodesic coordinates $(x^0 = t, x)$ near a spacelike hypersurface, see equation (9) after Lemma 2.3, FP $v = (v^0, v')$ means $v^0 > 0$. Notice that for the associated covector $(\tau, \xi) = gv$, we have $\tau < 0$.

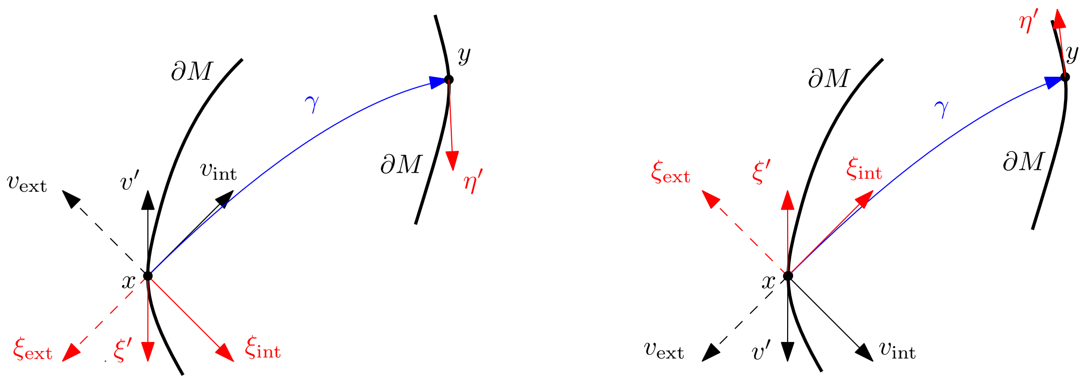

FIGURE 1. A tangent timelike future pointing (FP) vector v' on the left, and a past pointing on the right; and the two lightlike vectors v_{int} and v_{ext} with the same projection, pointing to M and outside M, respectively. The FP geodesic $\gamma = \gamma_{x,\xi'}(s)$ in both cases propagates to the future but on the right, it is determined by negative values of the parameter over it. The corresponding covectors ξ', ξ_{int} and ξ_{ext} are plotted, as well. The lens relation is $\mathcal{L}(x,\xi') = (y, \eta')$.

Given a timelike $(x,\xi')\in\mathcal{U}$, assume first that ξ' is FP. Let ξ be the lightlike covector pointing into M with orthogonal projection ξ', identified with the vector $v=g^{-1}\xi$. The geodesic $\gamma_{x,\xi'}(s)$ issued from (x,v), for $s\ge 0$ will be called the FP geodesic issued from (x,ξ'). In Figure 1 on the left, $v=v_{\text{int}}$ and $\gamma_{x,\xi'}(s)=\gamma$. If (x,ξ') is past pointing, then we choose v to be the lightlike vector projecting to v' pointing to the exterior (v_{ext} in Figure 1 on the right) and take $\gamma_{x,\xi'}(s)$ for $s\le 0$. By propagation of singularities, a boundary singularity (x,ξ') as above would propagate either along the FP geodesics chosen above, or along the past pointing ones (or both) that we did not choose. The choice we made reflects the requirement that singularities should propagate to the future only. We call such microlocal solutions *outgoing*. We borrow that term from scattering theory. In the case of the classical formulation of the Riemannian version of this problem, this is guaranteed by the condition $u=0$ for $t<0$.

2.2. **Gauge Invariance.** There exist some gauge transformations which leave the local and the global versions of the Dirichlet-to-Neumann map $\Lambda_{g,A,q}$ invariant, thus one can only expect to recover the corresponding gauge equivalence class. To simplify the formulations, we assume that the DN map $\Lambda_{g,A,q}$ is well defined globally on M. In our main theorems, we will apply this to the ΨDO part of $\Lambda_{g,A,q}$ first, and then Φ below needs to be identity near a fixed point only. For the semiglobal one, we need Φ to be identity near both ends of the fixed lightlike geodesic only. Since the computations below are purely algebraic, the lemmas remain true for the localized maps with obvious modifications.

We will consider two types of gauge transformations in this part. The first one is a diffeomorphism in M which fixes ∂M.

Lemma 2.1. *Let (M,g) be a Lorentzian manifold with boundary as above, let A be a smooth 1-form and q be a smooth function on M. If $\Phi:M\to M$ is a diffeomorphism with $\Phi|_{\partial M}=\mathrm{Id}$, then*

$$\Lambda_{g,A,q}=\Lambda_{\Phi^*g,\Phi^*A,\Phi^*q}.$$

Here $Id:\partial M\to\partial M$ is the identity map, Φ^*g,Φ^*A,Φ^*q are the pullbacks of g,A,q under Φ, respectively.

Proof. For any $f\in C^\infty(\partial M)$, let u be the solution of $\mathcal{L}_{g,A,q}u=0$ on M with $u|_{\partial M}=f$. Define $v:=\Phi^*u$ as the pull-back of u, then simple calculation in local coordinates shows that $\mathcal{L}_{\Phi^*g,\Phi^*A,\Phi^*q}v=0$ and $v|_{\partial M}=f$. If we write $y=\Phi(x)$ as a local coordinate representation of Φ, then

$$\begin{aligned}\Lambda_{g,A,q}f(y)&=\nu^j(y)\frac{\partial u}{\partial y^j}(y)-i\nu^j(y)A_j(y)u(y)\Big|_{\partial M}\\&=\nu^j(x)\frac{\partial x^l}{\partial y^j}\frac{\partial v}{\partial y^l}-i\frac{\partial x^l}{\partial y^j}\nu^j(x)\frac{\partial y^k}{\partial x^l}A_k(x)v(x)\Big|_{\partial M}\\&=\tilde\nu^j(x)\frac{\partial v}{\partial x^j}(x)-i\tilde\nu^j(x)(\Phi^*A)_j(x)v(x)\Big|_{\partial M}\\&=\Lambda_{\Phi^*g,\Phi^*A,\Phi^*q}f,\end{aligned}$$

where ν and $\tilde\nu$ are the unit normals in the y and the x variables, respectively. The above calculation essentially verifies that $\Lambda_{g,A,q}$ is defined invariantly. Therefore, $\Lambda_{g,A,q}=\Lambda_{\Phi^*g,\Phi^*A,\Phi^*q}$. □

Another type of gauge invariance occurs when one makes a conformal change of the metric g. This type of gauge invariance also occurs when g is a Riemannian metric and $\Lambda_{g,A,q}$ is the corresponding Dirichlet-to-Neumann map for the magnetic Schrödinger equation, see [12, Proposition 8.2].

Lemma 2.2. *Let (M,g) be a Lorentzian manifold with boundary as above, let A be a smooth 1-form and q be a smooth function on M. If φ and ψ are smooth functions such that*

$$\varphi|_{\partial M}=\partial_\nu\varphi|_{\partial M}=0,\quad \psi|_{\partial M}=0,$$

then we have

$$\Lambda_{g,A,q}=\Lambda_{e^{-2\varphi}g,A-d\psi,e^{2\varphi}(q-q_\varphi)}$$

where $q_\varphi:=e^{\frac{n-2}{2}\varphi}\Box_g e^{\frac{2-n}{2}\varphi}$.

Proof. A direct computation in local coordinates shows that

$$e^{\frac{n+2}{2}\varphi}P_{g,A,q}(e^{\frac{2-n}{2}\varphi}u) = P_{e^{-2\varphi}g,A,e^{2\varphi}(q-q_\varphi)}u$$
$$e^{-i\psi}P_{g,A,q}(e^{i\psi}u) = P_{g,A-d\psi,q}u$$

For any $f \in C^\infty(\partial M)$, let u be the solution of $P_{g,A,q}u = 0$ on M with $u|_{\partial M} = f$. Setting $v := e^{\frac{n-2}{2}\varphi}e^{-i\psi}u$, we have

$$\begin{aligned}
P_{e^{-2\varphi}g,A-d\psi,e^{2\varphi}(q-q_\varphi)}v &= P_{e^{-2\varphi}g,A-d\psi,e^{2\varphi}(q-q_\varphi)}(e^{\frac{n-2}{2}\varphi}e^{-i\psi}u)\\
&= e^{\frac{n+2}{2}\varphi}P_{g,A-d\psi,q}(e^{-i\psi}u)\\
&= e^{\frac{n+2}{2}\varphi}e^{-i\psi}P_{g,A,q}u = 0
\end{aligned}$$

Furthermore, notice that $\nu_{e^{-2\varphi}g} = \nu_g$ by the assumption on φ, thus

$$\begin{aligned}
\Lambda_{e^{-2\varphi}g,A-d\psi,e^{2\varphi}(q-q_\varphi)}f &= \nu^j\frac{\partial v}{\partial x^j} - i\nu^j\left(A_j - \frac{\partial\psi}{\partial x^j}\right)v|_{\partial M}\\
&= \nu^j\frac{\partial(e^{\frac{n-2}{2}\varphi}e^{-i\psi}u)}{\partial x^j} - i\nu^j\left(A_j - \frac{\partial\psi}{\partial x^j}\right)(e^{\frac{n-2}{2}\varphi}e^{-i\psi}u)|_{\partial M}\\
&= \nu^j\left(-i\frac{\partial\psi}{\partial x^j}u + \frac{\partial u}{\partial x^j}\right) - i\nu^j i\nu^j\left(A_j - \frac{\partial\psi}{\partial x^j}\right)u|_{\partial M}\\
&= \nu^j\frac{\partial u}{\partial x^j} - i\nu^j A_j u|_{\partial M}\\
&= \Lambda_{g,A,q}f
\end{aligned}$$

which completes the proof. □

2.3. **Gauge equivalent modifications of g, A, q.** It is convenient to work in semi-geodesic normal coordinates on a Lorentzian manifold. These coordinates are the Lorentzian counterparts of the well known Riemannian semigeodesic coordinates for Riemannian manifolds with boundary. We formulate the existence of such coordinate in the following lemma.

Lemma 2.3. *Let S be a timelike hypersurface in M. For every $x_0 \in S$, there exist $\varepsilon > 0$, a neighborhood N of x_0 in M, and a diffeomorphism $\Psi : S \cap N \times [0,T) \to N$ such that*

(i) $\Psi(x', 0) = x'$ for all $x' \in S \cap N$;

(ii) $\Psi(x', x^n) = \gamma_{x'}(x^n)$ where $\gamma_{x'}(x^n)$ is the unit speed geodesic issued from x' normal to S.

Moreover, if $(x^0, \dots, x^{n-1})$ are local boundary coordinates on S, in the coordinate system $(x^0, \dots, x^n)$, the metric tensor g takes the form

$$g = g_{\alpha\beta}dx^\alpha \otimes dx^\beta + dx^n \otimes dx^n, \quad \alpha, \beta \le n-1. \tag{8}$$

Clearly, $g_{\alpha\beta}$ has a Lorentzian signature as well. If M has a boundary, then S can be ∂M and x^n is restricted to $[0, \varepsilon]$. A proof of the lemma can be found in [30] and is based on the fact that the lines $x' = \text{const.}$, $x^n = s$ are unit speed geodesics; therefore the Christoffel symbols Γ^i_{nn} vanish for all i. We will call such coordinates the semi-geodesic normal coordinates. The lemma remains true if S is spacelike with a negative sign in front of $dx^n \otimes dx^n$ in (8) (we replace the index n by 0 below), and this gives us a way to define a time function $t = x^0$ locally, and put the metric in the block form

$$g = -dt^2 + g_{ij}(t,x)dx^i \otimes dx^j, \quad 1 \le i, j \le n \tag{9}$$

with g_{ij} Riemannian.

Now we use the gauge invariance of $\Lambda_{g,A,q}$ to alter g, A, q without changing the DN map. Three types of modifications are made in the following, labeled as **(M1)-(M3)** respectively.

Firstly, given two metrics g and $\tilde g$, one can choose diffeomorphisms as in Lemma 2.1 to obtain common semi-geodesic normal coordinates. In fact, let Ψ and $\tilde\Psi$ be diffeomorphisms like in Lemma 2.3 with respect to g and $\tilde g$ respectively, then $\tilde\Psi \circ \Psi^{-1}$ is a diffeomorphism near ∂M which fixes ∂M. Extend $\tilde\Psi \circ \Psi^{-1}$ as in [29] to be a global diffeomorphism on M. The properties of Ψ and $\tilde{\Psi}$ ensure that the two metrics g and $(\tilde{\Psi}\circ\Psi^{-1})^*\tilde{g}$ have common semi-geodesic normal coordinates near ∂M. Therefore, we may assume

(M1): if (x', x^n) are the semi-geodesic normal coordinates for g, they are also the semi-geodesic normal coordinates for $\tilde{g}$.

Secondly, we employ the conformal gauge invariance to replace $\tilde{g}$ with a gauge equivalent one to obtain some identities which later will help simplify the calculations.

Lemma 2.4. *Let S be either a timelike or a spacelike hyperplane near some point $p_0 \in S$. Given smooth functions $r_2, r_3, \dots$ on S near p_0, there exists a smooth function μ near p_0 with $\mu = 0$, $\partial_\nu\mu = 0$ on S so that if $\hat{\Psi}$ is the diffeomorphism in Lemma 2.3 related to the metric $\hat{g} := e^\mu g$, then*

$$\partial_n^j \det(\hat{\Psi}^*\hat{g}) = r_j, \qquad j = 2, 3, \dots$$

on S near p_0. Here $\partial_n = \frac{\partial}{\partial x^n}$ with $(x^0, \dots, x^n)$ the semi-geodesic normal coordinates for g.

Before giving the proof of the lemma, we remark that $(x^0, \dots, x^n)$ may not be the semi-geodesic normal coordinates for $\hat{g}$.

Proof. The statement of the theorem is invariant under replacing g by Ψ^*g for any local diffeomorphism Φ which preserves the boundary pointwise. Therefore, we may assume that g is replaced by Ψ^*g, i.e., that $x = (x', x^n)$ are semi-geodesic coordinates for g.

Note first that the conformal factor does not change the property of a covector being normal to S but rescales the normal derivative and may change the higher order ones because $\gamma_{x'}$ may change its curvature with respect to the old metric. More precisely, for the vector $e_n = (0, \dots, 0, 1)$ we have $g(e_n, e_n) = \mp 1$ but $\hat{g}(e_n, e_n) = \mp e^\mu$. Therefore, for the corresponding normal derivatives we have $\hat{\partial}_\nu = e^{-\mu/2}\partial_\nu = \partial_n$ on $x^n = 0$. Let $\hat{\gamma}_{x'}(s)$ be the normal geodesic at $x' \in S$ with $\dot{\hat{\gamma}}_{x'}$ consistent with the orientation of S, normalized by $\hat{g}(\dot{\hat{\gamma}}_{x'}(s), \dot{\hat{\gamma}}_{x'}(s)) = \mp 1$. Then for every smooth function f,

$$\partial_n^j \hat{\Psi}^* f(x')|_{x^n=0} = \partial_n^j|_{x^n=0} f(\hat{\gamma}_{x'}(x^n)).$$

For $j = 0, 1$, the results are not affected by the conformal factor and we get

$$\hat{\Psi}^* f(x')|_{x^n=0} = f(x', 0), \quad \partial_n \hat{\Psi}^* f(x')|_{x^n=0} = f_n(x', 0)$$

To compute the higher order normal derivatives, we write

$$\partial_n^2 \hat{\Psi}^* f(x') = f_{ij}\dot{\hat{\gamma}}^i_{x'}\dot{\hat{\gamma}}^j_{x'} + f_i \ddot{\hat{\gamma}}^i_{x'} \quad \text{on } x^n = 0. \tag{10}$$

Under the conformal change of the metric, the Christoffel symbols are transformed by the law

$$\hat{\Gamma}^k_{jk} = \Gamma^k_{ij} + \frac{1}{2}\delta^k_i \partial_j\mu + \frac{1}{2}\delta^k_j\partial_i\mu - g_{ij}\nabla^k\mu.$$

In particular,

$$\hat{\Gamma}^k_{nn} = \Gamma^k_{nn} + \frac{1}{2}\delta^k_n\partial_n\mu + \frac{1}{2}\delta^k_n\partial_n\mu - g_{nn}\nabla^k\mu = \delta^k_n\partial_n\mu - \frac{1}{2}g^{kl}\partial_l\mu. \tag{11}$$

Therefore, $\hat{\Gamma}^k_{nn} = 0$ on $x^n = 0$ and (10) reduces to

$$\partial_n^2 \hat{\Psi}^* f(x') = f_{nn} \quad \text{on } x^n = 0. \tag{12}$$

In a similar way, we may compute $\partial_n^j \hat{\Psi}^* f(x')$ on $x^n = 0$. The result is $\partial_n^j f$ plus normal derivatives of f of order $j-1$ and less with coefficients depending on the normal derivatives of μ up to order $j-1$. For our purposes, the exam expression does not matter.

The metric $\hat{g}$ has the form

$$(\hat{\Psi}^*\hat{g})_{kl} = (\hat{g}_{ij}\circ\hat{\Psi})\frac{\partial\hat{\Psi}^i}{\partial x^k}\frac{\partial\hat{\Psi}^j}{\partial x^l} = (\hat{g}_{\alpha\beta}\circ\hat{\Psi})\frac{\partial\hat{\Psi}^\alpha}{\partial x^k}\frac{\partial\hat{\Psi}^\beta}{\partial x^l} + \frac{\partial\hat{\Psi}^n}{\partial x^k}\frac{\partial\hat{\Psi}^n}{\partial x^l}$$

where the Greek indices range from 0 to $n-1$ (but not n). In particular,

$$\det \hat{\Psi}^*\hat{g} = (\det \mathrm{d}\hat{\Psi})^2 \det(\hat{g}\circ\hat{\Psi}). \tag{13}$$

We need to understand the structure of $\partial_n^k(\det \mathrm{d}\hat{\Psi})|_{x^n=0}$ now. For $k=0$, we have $\mathrm{d}\hat{\Psi}|_{x^n=0} = \mathrm{Id}$. Notice next that

$$\mathrm{d}\hat{\Psi} = (\partial_0\hat{\Psi}, \dots, \partial_{n-1}\hat{\Psi}, \partial_n\hat{\Psi}), \tag{14}$$

where each partial derivative is a vector. Since by (11), $\partial_n^2\hat{\Psi}^i = -\hat{\Gamma}^i_{nn} = 0$ for $x^n = 0$,

$$\partial_n(\det \mathrm{d}\hat{\Psi})|_{x^n=0} = 0.$$

To analyze $k=2$, we notice first that

$$\partial_n^3\hat{\Psi}^i = -\partial_n\hat{\Gamma}^i_{nn} = -\partial_n\left(\delta^i_n\partial_n\mu - \frac{1}{2}g^{il}\partial_l\mu\right) = -\delta^i_n\mu_{nn} + \dots,$$

where the dots represent a term involving lower order ∂_n derivatives of μ. Using this in (14), we get

$$\partial_n^2(\det \mathrm{d}\hat{\Psi})|_{x^n=0} = -\mu_{nn}|_{x^n=0}.$$

Reasoning as above, we see that

$$\partial_n^j(\det \mathrm{d}\hat{\Psi})|_{x^n=0} = -\partial_n^j\mu|_{x^n=0} + \dots, \tag{15}$$

where the dots represent terms involving normal derivatives of μ (possibly differentiated tangentially) up to order $j-1$.

We will analyze the normal derivatives of $\det(\hat{g}\circ\hat{\Psi})$ in (13) now. Since $\det\hat{g} = e^{n\mu}\det g$, we get

$$\begin{aligned}\partial_n \det(\hat{g}\circ\hat{\Psi}) &= \partial_n\left(e^{(n+1)\mu\circ\hat{\Psi}}\det g\circ\hat{\Psi}\right)\\ &= (n+1)\mu_n\det g + \partial_n\det g = \partial_n\det g \quad \text{on } \partial M.\end{aligned} \tag{16}$$

We used the fact that $\mathrm{d}\Psi = \mathrm{Id}$ on ∂M and that $\partial_n\mathrm{d}\Psi = 0$ since $\mathrm{d}\mu = 0$ on ∂M. Therefore, $\partial_n^j\det\hat{g}\circ\hat{\Psi} = \partial_n^j\det g$ on $x^n = 0$ for $j = 0, 1$.

For the highest order derivatives, notice that $\partial_n^j\hat{\Psi}$ involves $\partial_n^{j-1}\mu$ as its highest order normal μ derivative, as the arguments leading to (15) show. Differentiating (16), we therefore get

$$\begin{aligned}\partial_n^j \det(\hat{g}\circ\hat{\Psi}) &= \partial_n^j\left(e^{(n+1)\mu\circ\hat{\Psi}}\det g\circ\hat{\Psi}\right)\\ &= (n+1)(\partial_n^j\mu)\det g + \dots \quad \text{on } \partial M,\end{aligned} \tag{17}$$

where the dots have the same meaning as in (15).

Use (13) in combination with (15) and (17) to get

$$\partial_n^j(\det\hat{\Psi}^*\hat{g})|_{x^n=0} = (n-1)(\partial_n^j\mu)\det g + \dots, \tag{18}$$

To complete the proof of the lemma, we determine the normal derivatives of μ on $x^n = 0$ for $j = 2, \dots$. We get first $\partial_n^2(\det\hat{\Psi}^*\hat{g})|_{x^n=0} = (n-1)\mu_{nn}|_{x^n=0}$, which needs to be equal to r_2; and can be solved for μ_{nn}. Then we can determine the tangential derivatives of the latter. After that, we can solve (17) with $j=3$ for μ_{nnn}, etc. To complete the proof, we use Borel's lemma. □

Let g and $\tilde{g}$ be two metrics satisfying **(M1)** with the two diffeomorphisms Ψ and $\tilde{\Psi}$ respectively as in Lemma 2.3. Applying Lemma 2.4 to $S = \partial M$ and $p = x_0$, we can find a metric $\hat{g} := e^\mu g$ with $\mu = 0$, $\partial_\nu\mu = 0$ on ∂M such that under the semi-geodesic normal coordinates $(x^0, \dots, x^n)$ for g we have

$$\partial_n^j\det(\hat{\Psi}^*\hat{g}) = \partial_n^j\det(\tilde{\Psi}^*\tilde{g}) \qquad j = 2, 3, \dots.$$

on ∂M. Notice that $(x^0, \dots, x^n)$ are also semi-geodesic normal coordinates for $\tilde{g}$ by **(M1)**.

Now consider the metrics $(\hat{\Psi}\circ\tilde{\Psi}^{-1})^*\hat{g}$ and $\tilde{g}$. These metrics have common semi-geodesic normal coordinates (see the argument following Lemma 2.3), which are $(x^0, \dots, x^n)$. In these coordinates the choice of $\hat{g}$ yields

$$\partial_n^j\det(\tilde{\Psi}^*\circ(\hat{\Psi}\circ\tilde{\Psi}^{-1})^*\hat{g}) = \partial_n^j\det(\hat{\Psi}^*\hat{g}) = \partial_n^j\det(\tilde{\Psi}^*\tilde{g}).$$

Thus we may replace g by $(\hat{\Psi}\circ\tilde{\Psi}^{-1})^*\hat{g}$ and change A, q accordingly as in Lemma 2.1 and Lemma 2.2 without affecting $\Lambda_{g,A,q}$. We therefore can assume that g and $\tilde{g}$ satisfy not only **(M1)**, but also

(M2): in the common semi-geodesic normal coordinates (x', x^n),

$$\partial_n^j \det g(x',0) = \partial_n^j \det \tilde{g}(x',0) \qquad j = 2, 3, \dots.$$

Here we have identified the metrics with their coordinate representations under $\tilde{\Psi}$.

Thirdly, we make modifications to the 1-form A. Again the modification does not change the gauge equivalence class of $\Lambda_{g,A,q}$ due to Lemma 2.2.

Lemma 2.5. *Let (M, g) be a Lorentzian manifold with boundary as above, let A be a smooth 1-form and q be a smooth function on M. There exists a smooth functions ψ with $\psi|_{\partial M} = 0$ such that in the semi-geodesic normal coordinates (x', x^n), $B := A - d\psi$ satisfy*

$$\partial_n^j B_n(x',0) = 0 \qquad j = 0, 1, 2, \dots. \tag{19}$$

Proof. We can find a smooth function ψ with

$$\psi(x',0) = 0, \quad \partial_n^{j+1}\psi(x',0) = \partial_n^j A_n(x',0), \quad j = 0, 1, 2, \dots.$$

Extend it in a suitable manner so that $\psi \in C^\infty(M)$ with $\psi|_{\partial M} = 0$. Then $B = A - d\psi$ satisfies (19). ☐

As a result we may further assume

(M3): in the common semi-geodesic normal coordinates (x', x^n) of g and $\tilde{g}$,

$$\partial_n^j A_n(x',0) = \partial_n^j \tilde{A}_n(x',0) = 0 \qquad j = 0, 1, 2, \dots.$$

3. Boundary stability

We choose the semi-geodesic coordinates (x', x^n) near x_0 so that $x_0 = 0$, ∂M locally is given by $x^n = 0$, and the interior of M is given by $x^n > 0$. Let $\xi^{0\prime}$ be a future pointing timelike covector in $T^*_{x_0}\partial M$ at x_0. On Figure 1, the associated vector would look like v' on the left, while the covector $\xi^{0\prime}$ would have the opposite time direction, like the figure on the right. Let $\chi(x', \xi')$ be a smooth cutoff function with small enough support in $\mathcal{U}$ that equals to 1 in a smaller conic timelike neighborhood of $(x_0, \xi^{0\prime})$. Assume also that χ is homogeneous in ξ' of order 0.

For

$$f(x') = e^{i\lambda x'\cdot\xi'}\chi(x',\xi'), \tag{20}$$

and for every $N > 0$, we would like to construct a geometric optics approximation of the outgoing solution u of (3) near x_0 in M of the form

$$u_N(x) := e^{i\lambda\phi(x,\xi')}\sum_{j=0}^{N}\frac{1}{\lambda^j}a_j(x,\xi'). \tag{21}$$

The eikonal and the transport equations below are based on the following identity

$$e^{-i\lambda\phi}Pe^{i\lambda\phi} = -\lambda^2 g^{jk}(\partial_j\phi)(\partial_k\phi) + i\lambda\Box_g\phi + 2i\lambda g^{jk}\partial_j\phi(\partial_k - iA_k) + P.$$

In M near x_0, the phase function $\phi(x, \xi')$ solves the eikonal equation, which in the semi-geodesic coordinates takes the form

$$g^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta\phi + (\partial_n\phi)^2 = 0, \qquad \phi|_{x^n=0} = x'\cdot\xi'. \tag{22}$$

With the extra condition $\partial_\nu\phi|_{\partial M} < 0$, (22) is locally uniquely solvable. Moreover, (22) implies

$$\partial_n\phi(x',0) = \xi_n(x',\xi') > 0 \qquad \text{for any } (x',\xi') \in \mathcal{U}, \tag{23}$$

where

$$\xi_n(x',\xi') := \sqrt{-g^{\alpha\beta}(x',0)\xi_\alpha\xi_\beta}. \tag{24}$$

Notice that the choice of the sign of ξ_n makes ξ a lightlike future-pointing covector, pointing into M. In Figure 1, the associated vector $v = g^{-1}\xi$ looks like v_{int} on the left.

We recall briefly the method of characteristics for solving the eikonal equation. We first determine $\partial\phi$ on $x^n = 0$ to get (23) or the same equation with a negative square root. We choose one of them, and in this case our choice is determined by the requirement that $\partial\phi$ points into M, see Figure 1. Let now $(q_{x',\xi'}(s), p_{x',\xi'}(s))$ be the null bicharacteristic with $q_{x',\xi'}(0) = x'$, $p_{x',\xi'}(0) = (\xi',\xi_n)$. We think of (x',s) as local coordinates and set $\phi(x',s) = x'\cdot\xi'$. More precisely, ϕ is uniquely determined locally by the requirement to be constant along the null bicharacteristics $q_{x',\xi'}$. Moreover,

$$p(s) = \nabla_x\phi(q(s),\xi'). \tag{25}$$

Since by the Hamilton equations, $\dot q^i(s) = g^{ij}p_j(s)$, we get in particular that $g^{ij}\partial_j\phi\partial_i$ is just the derivative $\partial/\partial s$ along the null bicharacteristic.

In M near x_0, the amplitudes a_0 and a_j, $j = 1, 2, \dots$ solve the following transport equations:

$$Ta_0 = 0, \qquad a_0|_{x_n=0} = \chi; \tag{26}$$

$$iTa_j = -Pa_{j-1}, \qquad a_j|_{x_n=0} = 0; \quad j \geq 1. \tag{27}$$

where the operator T is defined as

$$T := 2g^{jk}\partial_j\phi\,(\partial_k - iA_k) + \Box_g\phi. \tag{28}$$

We prefer to express the bicharacteristics through the geodesics

$$\Gamma(s) := (q_{x',\xi'}(s), p_{x',\xi'}(s)) = (\gamma_{x',\xi'}(s), g\dot\gamma_{x',\xi'}(s)).$$

Then along the bicharacteristics, we have

$$T = 2\partial_s - 2\mathrm{i}\langle A, p(s)\rangle + \Box_g\phi = 2\mu\partial_s\mu^{-1}, \tag{29}$$

with the integrating factor μ given by

$$\begin{aligned}\mu(\Gamma(s)) = \exp&\Big\{ -\frac{1}{2}\int_0^s (\Box_g\phi)(\gamma_{x',\xi'}(\sigma))\,\mathrm{d}\sigma\Big\}\\ &\times \exp\Big\{ i\int_0^s \big\langle A(\gamma_{x',\xi'}(\sigma)), \dot\gamma_{x',\xi'}(\sigma)\big\rangle\,\mathrm{d}\sigma\Big\}.\end{aligned} \tag{30}$$

The amplitudes a_j, $j = 0, 1, \dots$ are supported in a neighborhood of the characteristics issued from $x_0 \in \partial M$ in the codirection $\xi(x_0)$. As a result, on some neighborhood of x_0, u_N solves $Pu_N = O(\lambda^{-N})$, $u|_{\partial M} = f$.

Theorem 3.1. *$\Lambda^{\text{loc}}_{g,A,q}$ is an elliptic ΨDO of order 1 in $\mathcal{U}$.*

Proof. Given $f \in \mathcal{E}'(U)$ (not related to (20)) with a wave front set as in the theorem, we are looking for an outgoing solution u of $Pu = 0$ near x_0, $u = f$ on U of the form

$$u(x) = (2\pi)^{-n}\int e^{i\phi(x,\xi')}a(x,\xi')\hat f(\xi')\,\mathrm{d}\xi'. \tag{31}$$

The phase ϕ solves the eikonal equation (22) and therefore coincides with ϕ there. We chose the solution which guarantees a locally outgoing u, which corresponds to the positive square root in (24). We are looking for an amplitude a of the form $a \sim \sum_{j=0}^\infty a_j(x,\xi')$, where a_j is homogeneous in the ξ' variable of degree $-j$. The standard geometric optics construction leads to the transport equations (26), (27). Using the standard Borel lemma argument, we construct a convergent series for a. Then u is the microlocal solution (up to a microlocally smoothing operator applied to f) that we used to define $\Lambda^{\text{loc}}_{g,A,q}$. Then $\Lambda^{\text{loc}}_{g,A,q}f = \partial u/\partial\nu|_U$. Since $\phi = x'\cdot\xi'$ on U, we get that $\Lambda^{\text{loc}}_{g,A,q}$ is a ΨDO with symbol

$$-i\xi_n(x',\xi') - \partial_n a|_{x^n=0}.$$

In particular, for the principal symbol we get

$$\sigma_p\left(\Lambda^{\rm loc}_{g,A,q}\right)(x',\xi') = -i\xi_n = -i\sqrt{-g^{\alpha\beta}(x')\xi_\alpha\xi_\beta}. \tag{32}$$

We proceed in the same way if (x',ξ') is past pointing.

It remains to show that if we use another locally outgoing solution $\tilde{u}$, the resulting $\tilde{\Lambda}^{\rm loc}_{g,A,q}$ would differ by a smoothing operator. This follows by considering $v := u - \tilde{u}$ which is a locally outgoing solution with smooth boundary data, which therefore must be smooth. We omit the details. □

We prove a stable determination result on the boundary next. Let (g, A, q) and $(\tilde{g}, \tilde{A}, \tilde{q})$ be two triples. Denote

$$\delta = \left\|\Lambda^{\rm loc}_{g,A,q} - \Lambda^{\rm loc}_{\tilde{g},\tilde{A},\tilde{q}}\right\|_{H^1(U)\to L^2(U)}, \tag{33}$$

where, as above, $\Lambda^{\rm loc}_{g,A,q}$ and $\Lambda^{\rm loc}_{\tilde{g},\tilde{A},\tilde{q}}$ are the local DN maps associated with (g, A, q) and $(\tilde{g}, \tilde{A}, \tilde{q})$, respectively microlocally restricted to a fixed conic neighborhood $\mathcal{U}$ of a timelike future pointing $(x_0, \xi^{0\prime}) \in T^*U$ with $x_0 \in U \subset \partial M$. As above, we assume that $\xi^{0\prime}$ is future pointing and timelike for both g and $\tilde{g}$, and that $\mathcal{U}$ is small enough so that is included in the future timelike cone on T^*U for both metrics. Therefore, in the theorem below, we need to know the DN map microlocally only near a fixed timelike covector on $T^*\partial M$.

Theorem 3.2. *Let (g, A, q) and $(\tilde{g}, \tilde{A}, \tilde{q})$ be replaced by their gauge equivalent triples satisfying* ***(M1)-(M3)****. Then for any $\mu < 1$ and $m \geq 0$, and some open neighborhood $U_0 \Subset U$ of x_0,*

(1) $\sup_{x\in\overline{U}_0,|\gamma|\le m} |\partial^\gamma(g-\tilde{g})| \le C\delta^{\frac{\mu}{2^m}}$;

(2) $\sup_{x\in\overline{U}_0,|\gamma|\le m} |\partial^\gamma(A-\tilde{A})| \le C\delta^{\frac{\mu}{2^{m+1}}}$;

(3) $\sup_{x\in\overline{U}_0,|\gamma|\le m} |\partial^\gamma(q-\tilde{q})| \le C\delta^{\frac{\mu}{2^{m+2}}}$;

are valid whenever $g, \tilde{g}, A, \tilde{A}, q, \tilde{q}$ are bounded in a certain C^k norm in the semi-geodesic normal coordinates near x_0 with a constant $C > 0$ depending on that bound with $k = k(m, \mu)$.

Proof. We adapt the proofs in [28] and [40] in the Riemannian setting. Let Γ_0 be a small conic neighborhood of $\xi^{0\prime}$. We can assume that $\chi = 1$ on $U_0 \times \Gamma_0$. Let f be as in (20). We restrict (x',ξ') to $U_0\times\Gamma_0$ below. In addition, we normalize ξ' to have unit Euclidean length (in that coordinate system). Since $\partial_\nu = -\partial_n$, the formal Dirichlet-to-Neumann map in the boundary normal coordinates (x', x^n) is given by

$$\begin{aligned}\Lambda^{\rm loc}_{g,A,q} f(x') = &- e^{i\lambda x'\cdot\xi'}\left(i\lambda\partial_n\phi(x',0,\xi') + \sum_{j=0}^N \frac{1}{\lambda^j}\left(\partial_n - iA_n\right)a_j(x',0,\xi)\right)\\ &+ O\left(\lambda^{-N-1}\right).\end{aligned} \tag{34}$$

The expression for $\Lambda_{\tilde{g},\tilde{A},\tilde{q}} f$ is similar, with ϕ and a_j replaced by $\tilde{\phi}$ and $\tilde{a}_j$, respectively.

The representation (34) could be derived from (21) but since u there is an approximate solution only, and we defined $\Lambda^{\rm loc}_{g,A,q}$ microlocally, we need to go back to its definition. To justify (34), notice that by [46, Ch. VIII.7], on the set $\chi = 1$, $e^{-i\lambda x'\cdot\xi'}\Lambda^{\rm loc}_{g,A,q} f$ is equal to the full symbol of $\Lambda^{\rm loc}_{g,A,q}$ with $\lambda = |\xi|$ and ξ in (34) unit.

In the following, C denotes various constants depending only on M, χ in (20), on the choice of $k \gg 1$ and on the a priori bounds of the coefficients of P in C^k. Solving for $\partial_n\phi$ (resp. $\partial_n\tilde{\phi}$) in (34) and taking the difference we obtain

$$\begin{aligned}\partial_n\phi - \partial_n\tilde{\phi} =& \frac{1}{i\lambda}\left(\Lambda^{\rm loc}_{g,A,q} f - \Lambda^{\rm loc}_{\tilde{g},\tilde{A},\tilde{q}} f\right)\\ &+ \frac{1}{i\lambda}\sum_{j=0}^N \frac{1}{\lambda^j}\left[(\partial_n a_j - \partial_n\tilde{a}_j) - i(A_n a_j - \tilde{A}_n\tilde{a}_j)\right] + O\left(\lambda^{-N-1}\right)\end{aligned}$$

in $L^2(U_0)$. Integrating in U_0 yields

$$\left\|\partial_n\phi - \partial_n\tilde{\phi}\right\|_{L^2(U_0)} \le \frac{C}{\lambda}\delta\|f\|_{H^1(U_0)} + \frac{C}{\lambda}. \tag{35}$$

The choice of f in (20) indicates that $\|f\|_{H^1(U_0)} \le C\lambda$. Thus, taking the limit $\lambda \to \infty$ yields

$$\|\xi_n - \tilde{\xi}_n\|_{L^2(U_0)} = \left\|\partial_n\phi - \partial_n\tilde{\phi}\right\|_{L^2(U_0)} \le C\delta. \tag{36}$$

From relation (24) we have

$$\left\|(g^{\alpha\beta} - \tilde{g}^{\alpha\beta})\xi_\alpha\xi_\beta\right\|_{L^2(U_0)} = \|\xi_n^2 - \tilde{\xi}_n^2\|_{L^2(U_0)} = \left\|(\partial_n\phi)^2 - (\partial_n\tilde{\phi})^2\right\|_{L^2(U_0)} \le C\delta. \tag{37}$$

We use the following argument here and in several places below: a quadratic form $h^{\alpha\beta}\xi_\alpha\xi_\beta$ is uniquely determined for ξ' in any fixed in advance open set Γ on the unit sphere. In fact, one can choose $n(n-1)/2$ vectors ξ' in Γ and then the recovery is done by inverting an isomorphism on $\mathbf{R}^{n(n-1)/2}$, and is therefore stable, see [11, Lemma 3.3]. Therefore, (37) implies $\|g - \tilde{g}\|_{L^2(U_0)} \le C\delta$. By interpolation estimates in Sobolev space and Sobolev embedding theorems, we have for any $m \ge 0$ and $\mu < 1$ that

$$\|g - \tilde{g}\|_{C^m(\overline{U}_0)} \le C\delta^\mu \tag{38}$$

provided $k \gg 1$ is sufficiently large.

Second, we show that the first order normal derivatives of g and the 1-form can be stably determined on the boundary. From (34) we have

$$\begin{aligned}(\partial_n - i\tilde{A}_n)\tilde{a}_0 - (\partial_n - iA_n)a_0 = e^{-i\lambda x'\cdot\xi'}\left(\Lambda_{g,A,q}f - \Lambda_{\tilde{g},\tilde{A},\tilde{q}}f\right) + \\ i\lambda(\partial_n\phi - \partial_n\tilde{\phi}) + \sum_{j=1}^{N}\frac{1}{\lambda^j}\left(\partial_n a_j - \partial_n\tilde{a}_j\right) + O\left(\frac{1}{\lambda^{N+1}}\right) \qquad \text{in } L^2(U_0).\end{aligned}$$

Estimate as in (35) to obtain

$$\left\|(\partial_n - iA_n)a_0 - (\partial_n - i\tilde{A}_n)\tilde{a}_0\right\|_{L^2(U_0)} \le C(\delta + \lambda\delta + \frac{1}{\lambda})$$

which holds for all $\lambda > 0$. In particular, we may choose $\lambda = \delta^{-\frac{1}{2}}$ to minimize the right-hand side, then

$$\left\|(\partial_n - iA_n)a_0 - (\partial_n - i\tilde{A}_n)\tilde{a}_0\right\|_{L^2(U_0)} \le C\delta^{\frac{1}{2}}. \tag{39}$$

In order to estimate the difference of first order normal derivatives of the metrics, we consider the transport equation in (26). Since $\chi \equiv 1$ for $x \in U_0$, it follows from the boundary condition in (26) that $\partial_\alpha a_0 = \partial_\alpha\chi = 0$ for $\alpha = 0, \dots, n-1$. Moreover, $g^{nj} = \delta^{nj}$ in the semi-geodesic coordinates, thus the transport equation in (26) becomes

$$2\xi_n\left(\partial_n - iA_n\right)a_0 - 2iA^\alpha\xi_\alpha + \frac{1}{\sqrt{-\det g}}\partial_n\left(\sqrt{-\det g}\,\partial_n\phi\right) + Q(g) = 0, \tag{40}$$

where, as before, Greek indices range from 0 to $n-1$ (but not n). Here $A^\alpha := g^{\alpha\beta}A_\beta$, and $Q(g)$ is defined as follows which is a linear combination of tangential derivatives of g:

$$Q(g) := \frac{1}{\sqrt{-\det g}}\partial_\alpha\left(\sqrt{-\det g}\,g^{\alpha\beta}\right)\xi_\beta.$$

where we have used that $\partial_\beta\phi = \xi_\beta$ in U_0, $\beta = 0, \dots, n-1$. As a consequence of (38),

$$Q(g) - Q(\tilde{g}) = O(\delta^{\frac{1}{2}}). \tag{41}$$

Therefore, combining (39) (40) and (41) we obtain

$$\frac{1}{\sqrt{-\det g}}\partial_n\left(\sqrt{-\det g}\,\partial_n\phi\right) - \frac{1}{\sqrt{-\det\tilde{g}}}\partial_n\left(\sqrt{-\det\tilde{g}}\,\partial_n\tilde{\phi}\right) - 2i(A^\alpha - \tilde{A}^\alpha)\xi_\alpha = O(\delta^{\frac{1}{2}}).$$

Notice that

$$\begin{aligned}\frac{1}{\sqrt{-\det g}}\partial_n\left(\sqrt{-\det g}\partial_n\phi\right) =&\frac{1}{2\det g}\partial_n \det g\partial_n\phi+\partial_n^2\phi\\ =&\frac{\xi_n}{2\det g}\partial_n\det g-\frac{1}{2\xi_n}\partial_n g^{\alpha\beta}\xi_\alpha\xi_\beta\end{aligned}$$

is an even function of ξ'. Here in the computation $\partial_n\phi$ is substituted by ξ_n due to (23) and $\partial_n^2\phi$ is calculated by differentiating the eikonal equation (22). Separating the even and odd parts in ξ' we conclude

$$\left(\frac{\xi_n}{2\det g}\partial_n\det g-\frac{1}{2\xi_n}\partial_n g^{\alpha\beta}\xi_\alpha\xi_\beta\right)-\left(\frac{\tilde\xi_n}{2\det\tilde g}\partial_n\det\tilde g-\frac{1}{2\tilde\xi_n}\partial_n\tilde g^{\alpha\beta}\xi_\alpha\xi_\beta\right)=O(\delta^{\frac12}); \tag{42}$$

$$(A^\alpha-\tilde A^\alpha)\xi_\alpha=O(\delta^{\frac12}). \tag{43}$$

From the odd part (43), varying ξ' locally, we get

$$\|A-\tilde A\|_{L^2(U_0)}\le C\delta^{\frac12}. \tag{44}$$

To deal with the even part, notice (42) states that

$$\frac{\xi_n}{2\det g}\partial_n\det g-\frac{1}{2\xi_n}\partial_n g^{\alpha\beta}\xi_\alpha\xi_\beta$$

is stably determined of order $O(\delta^{\frac12})$. As ξ_n is stably determined on U_0, see (36), their product

$$\begin{aligned}&\frac{\xi_n^2}{2\det g}\partial_n\det g-\frac12\partial_n g^{\alpha\beta}\xi_\alpha\xi_\beta\\ &\qquad=-\frac{1}{2\det g}(\partial_n\det g)g^{\alpha\beta}\xi_\alpha\xi_\beta-\frac12\partial_n g^{\alpha\beta}\xi_\alpha\xi_\beta\\ &\qquad=-\frac12\frac{1}{\det g}\partial_n\left(\det g\cdot g^{\alpha\beta}\right)\xi_\alpha\xi_\beta\end{aligned}$$

is also stably determined. Since $\det g$ is known to be stable and away from zero, it follows that $\partial_n h^{\alpha\beta}$ is stable where $h^{\alpha\beta}:=(\det g)g^{\alpha\beta}$. Hence, the normal derivative of $g=(\det h)^{\frac{1}{1-n}}h$ is also stably determined, that is,

$$\|\partial_n g-\partial_n\tilde g\|_{L^2(U_0)}\le C\delta^{\frac12}. \tag{45}$$

Using interpolation and Sobolev embedding theorems, we obtain from (45) and (44) that for any $m\ge 0$ and $\mu<1$,

$$\|\partial_n g-\partial_n\tilde g\|_{C^m(\overline U_0)}+\|A-\tilde A\|_{C^m(\overline U_0)}\le C\delta^{\frac{\mu}{2}} \tag{46}$$

provided $k\gg 1$ is sufficiently large.

Next we show that the second order normal derivatives of g, the first order normal derivatives of A, and the values of q can be stably determined on the boundary. By (34) up to λ^{-1} we obtain

$$\left\|(\partial_n-iA_n)a_1-(\partial_n-i\tilde A_n)\tilde a_1\right\|_{L^2(U_0)}\le C(\lambda^2\delta+\lambda\delta^{\frac12}+\lambda^{-1}).$$

Choose $\lambda=\delta^{-\frac14}$ to minimize the right-hand side. Then

$$\left\|(\partial_n-iA_n)a_1-(\partial_n-i\tilde A_n)\tilde a_1\right\|_{L^2(U_0)}\le C\delta^{\frac14}. \tag{47}$$

Consider the transport equation (27) for a_1. In the semi-geodesic coordinates this equation takes the form

$$2i\xi_n\left(\partial_n-iA_n\right)a_1=-\partial_n^2a_0+q+O(\delta^{\frac12}), \tag{48}$$

where $O(\delta^{\frac{1}{2}})$ represents the stably determined terms of order $O(\delta^{\frac{1}{2}})$. (In fact, $a_1 = 0$ in these expressions by the boundary condition in (27), but it is left here for the convenience of tracking the corresponding terms.) From the estimates (36) (45) and (48) it follows that

$$(-\partial_n^2 a_0 + \partial_n^2 \tilde{a}_0) + (q - \tilde{q}) = O(\delta^{\frac{1}{4}}). \tag{49}$$

To obtain an expression of $\partial_n^2 a_0$, we differentiate the transport equation in (26) and evaluate it on U_0:

$$\begin{aligned}\partial_n^2 a_0 =& -\frac{1}{4\det g}\partial_n^2 \det g - \frac{1}{2\xi_n}\partial_n^3\phi + \frac{i}{\xi_n} g^{\alpha\beta}\partial_n A_\alpha \xi_\beta + O(\delta^{\frac{1}{2}})\\ =& -\frac{1}{4\det g}\partial_n^2 \det g + \frac{1}{4\xi_n^2}\partial_n^2 g^{\alpha\beta}\xi_\alpha\xi_\beta + \frac{i}{\xi_n} g^{\alpha\beta}\partial_n A_\alpha \xi_\beta + O(\delta^{\frac{1}{2}}),\end{aligned}$$

where the $O(\delta^{\frac{1}{2}})$ terms are estimated by (38) and (46) and we have used that $\partial_n A_n(x', 0) = 0$ in **(M3)**. Inserting this into (49) and separating the even and odd parts in ξ' gives (notice that $\xi_n = \sqrt{-g^{\alpha\beta}\xi_\alpha\xi_\beta}$ is an even function of ξ'.):

$$\begin{aligned}&\left(\frac{1}{4\det g}\partial_n^2 \det g - \frac{1}{4\det \tilde{g}}\partial_n^2 \det \tilde{g} - \frac{1}{4\xi_n^2}\partial_n^2 g^{\alpha\beta}\xi_\alpha\xi_\beta + \frac{1}{4\tilde{\xi}_n^2}\partial_n^2 \tilde{g}^{\alpha\beta}\xi_\alpha\xi_\beta\right)\\ &\qquad + (q - \tilde{q}) = O(\delta^{\frac{1}{4}}).\end{aligned} \tag{50}$$

$$-\frac{i}{\xi_n} g^{\alpha\beta}\partial_n A_\alpha \xi_\beta + \frac{i}{\tilde{\xi}_n}\tilde{g}^{\alpha\beta}\partial_n \tilde{A}_\alpha \xi_\beta = O(\delta^{\frac{1}{4}}). \tag{51}$$

To deal with (51), we multiply the two terms by ξ_n and $\tilde{\xi}_n$ respectively. This is valid since ξ_n is stably determined in (36). By the arguemnt following (37),

$$\left\|\partial_n A_\alpha - \partial_n \tilde{A}_\alpha\right\|_{L^2(U_0)} \le C\delta^{\frac{1}{2}}.$$

To deal with (50), recall the following matrix identity which is valid for any invertible matrix S

$$\partial \log|\det S| = \operatorname{tr}(S^{-1}\partial S).$$

Taking $S = g^{\alpha\beta}$ and applying ∂_n^{j-1} we see that

$$\partial_n^j \log(-\det g^{\alpha\beta}) = \partial_n^{j-1}(g_{\alpha\beta}\partial_n g^{\alpha\beta}), \qquad j = 1, 2, \dots$$

For $j = 2$, it gives

$$g_{\alpha\beta}\partial_n^2 g^{\alpha\beta} = \partial_n^2 \log(-\det g^{\alpha\beta}) - \partial_n g_{\alpha\beta}\partial_n g^{\alpha\beta}.$$

The right-hand side is stably determined by **(M2)** and (45), we thus get on U_0 that

$$g_{\alpha\beta}\partial_n^2 g^{\alpha\beta} - \tilde{g}_{\alpha\beta}\partial_n^2 \tilde{g}^{\alpha\beta} = O(\delta^{\frac{1}{2}}). \tag{52}$$

On the other hand, remember that the two metrics g and $\tilde{g}$ have been modified to satisfy **(M2)**, thus by (37)

$$\frac{1}{4\det g}\partial_n^2 \det g - \frac{1}{4\det \tilde{g}}\partial_n^2 \det \tilde{g} = \left(\frac{1}{4\det g} - \frac{1}{4\det \tilde{g}}\right)\partial_n^2 \det g = O(\delta).$$

This together with (50) gives

$$\left(-\frac{1}{4\xi_n^2}\partial_n^2 g^{\alpha\beta}\xi_\alpha\xi_\beta + \frac{1}{4\tilde{\xi}_n^2}\partial_n^2 \tilde{g}^{\alpha\beta}\xi_\alpha\xi_\beta\right) + (q - \tilde{q}) = O(\delta^{\frac{1}{4}}). \tag{53}$$

Again we multiply the terms without the tilde by ξ_n^2 and those with it by $\tilde{\xi}_n^2$, using (24) we have

$$(\partial_n^2 g^{\alpha\beta} + 4q g^{\alpha\beta} - \partial_n^2 \tilde{g}^{\alpha\beta} - 4\tilde{q}\tilde{g}^{\alpha\beta})\xi_\alpha\xi_\beta = O(\delta^{\frac{1}{4}}).$$

By the arguemnt following (37),

$$(\partial_n^2 g^{\alpha\beta} + 4q g^{\alpha\beta}) - (\partial_n^2 \tilde{g}^{\alpha\beta} + 4\tilde{q}\tilde{g}^{\alpha\beta}) = O(\delta^{\frac{1}{4}}).$$

Multiplying those terms without ˜ by $g_{\alpha\beta}$, those with ˜ by $\tilde{g}_{\alpha\beta}$, then summing up in α, β yields

$$(g_{\alpha\beta}\partial_n^2 g^{\alpha\beta} + 4nq) - (\tilde{g}_{\alpha\beta}\partial_n^2 \tilde{g}^{\alpha\beta} + 4n\tilde{q}) = O(\delta^{\frac{1}{4}}).$$

From (52) we come to the conclusion that

$$\|q - \tilde{q}\|_{L^2(U_0)} \le C\delta^{\frac{1}{4}}.$$

Inserting this into (53) and using the arguemnt following (37),

$$\|\partial_n^2 g^{\alpha\beta} - \partial_n^2 \tilde{g}^{\alpha\beta}\|_{L^2(U_0)} \le C\delta^{\frac{1}{4}}.$$

Putting the estimates on g, A, q together, we have established

$$\|\partial_n^2 g - \partial_n^2 \tilde{g}\|_{L^2(U_0)} + \left\|\partial_n A_\alpha - \partial_n \tilde{A}_\alpha\right\|_{L^2(U_0)} + \|q - \tilde{q}\|_{L^2(U_0)} \le C\delta^{\frac{1}{4}}.$$

As before, interpolation and the Sobolev embedding theorem lead to

$$\|\partial_n^2 g - \partial_n^2 \tilde{g}\|_{C^m(\overline{U}_0)} + \left\|\partial_n A_\alpha - \partial_n \tilde{A}_\alpha\right\|_{C^m(\overline{U}_0)} + \|q - \tilde{q}\|_{C^m(\overline{U}_0)} \le C\delta^{\frac{\mu}{4}}$$

for $m > 0$ and $\mu < 1$. Repeating this type of argument will establish the stability for higher order derivatives of g, A, q on U_0. □

4. Interior Stability

4.1. **The semiglobal microlocal solution.** We construct the semiglobal microlocal solution u sketched in the Introduction in the paragraph following (5) and used to define $\Lambda^{\mathrm{gl}}_{g,A,q}$. We recall the assumptions. We fix a time-like $(x_0, \xi^{0\prime}) \in T^*\partial M \setminus 0$ and a small conic neighborhood $\mathcal{U}$ of it. We choose a local orientation so that $(x_0, \xi^{0\prime})$ is future pointing. Then there is a unique light-like $(x_0, \xi^0) \in T^*M \setminus 0$ which projects orthogonally to $(x_0, \xi^{0\prime})$. Let γ_0 be the zero bicharacteristic issued from $(x_0, \xi^{0\prime})$ extended until hits $T^*\partial M$ again, transversally, by assumption; at some (y_0, η^0) with projection $(y_0, \eta^{0\prime}) = \mathcal{L}(x_0, \xi^{0\prime}) \in T^*\partial M$. Let $\mathcal{V} = \mathcal{L}(\mathcal{U})$ and denote by U and V the projections $\pi(\mathcal{U})$, $\pi(\mathcal{V})$ of $\mathcal{U}$ and $\mathcal{V}$ onto the base, i.e., their "x-parts". Denote by Γ the union of all zero bicharacteristics issued "from $\mathcal{U}$", i.e., from all future pointing (x, ξ) with $x \in \partial M$ which have projections on the boundary in $\mathcal{U}$. Let $\Gamma_0 := \pi(\Gamma) \subset M$ be the projection of Γ onto the base, see Figure 2. We assume below, for convenience, that (M, g) is embedded in a slightly larger manifold.

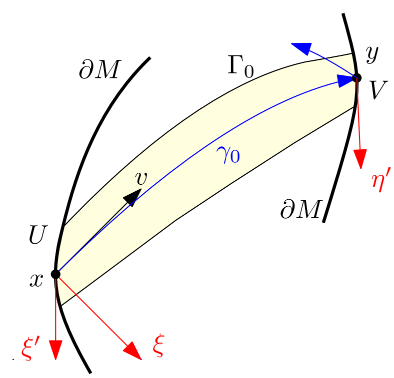

Figure 2. The solution u

Next proposition says that the microlocal solution u used in the Introduction to define $\Lambda^{\mathrm{gl}}_{g,A,q}$ is well defined.

Proposition 4.1. *If $\mathcal{U}$ is small enough, then for every $f \in \mathcal{E}'(\partial M)$ with $\mathrm{WF}(f) \in \mathcal{U}$ there exists a distribution u defined in a neighborhood of $\bar{\Gamma}_0$ so that $Pu \in C^\infty(\Gamma_0)$, $u|_U - f \in C^\infty(U)$ and $u|_V \in C^\infty$. Moreover, u is unique up to a smooth function in Γ_0.*

Proof. We are looking for a solution $u^{\rm inc}$ of the form (31) with f having a wave front set in $\mathcal{U}$. Past pointing codirections can be handled the same way. The solution is the same as in Theorem 3.1 but we are now trying to extend it as far as possible away from ∂M. We know that microlocally, $u^{\rm inc}$ is supported in a small neighborhood of the null bicharacteristic (projecting to a null geodesic on M) issued from (x_0,ξ^0) with ξ^0 future pointing with a projection $\xi^{0\prime}$ on the boundary, i.e., $\xi^0=(\xi^{0\prime},\xi_n(x_0,\xi^0))$, where ξ_n is given by (24), see Figure 1. This follows from the general propagation of singularities theory but in this particular case it can be derived from the fact that T in (28) has its principal part a vector field along such null geodesics; and $\mathrm{WF}(u^{\rm inc})$ can be analyzed directly with the aid of (31).

Such a solution is guaranteed to exist only near some neighborhood of x_0 because the eikonal equation may not be globally solvable. On the other hand, the solution is still a global FIO applied to the boundary data f. Indeed, it can also be viewed as a superposition of a finite number of local FIOs, each one having a representation of the kind (31). We construct $u^{\rm inc}$ first near ∂M, call it u_1. Then we restrict it to a timelike hypersurface S_1 intersecting the null geodesic $\pi(\gamma_0)$ transversely and we chose S_1 so that the geometric optics construction is still valid along $\pi(\gamma_0)$ until it hits S_1, and a bit beyond it. We take the boundary data at S_1, and solve a new similar problem, by taking the outgoing solution (the future pointing cone on S_1 is the one determined by $\mathrm{WF}(u_1|_{S_1})$), etc. By compactness arguments, we can cover the whole null geodesics (the projection of γ_0 to the base) until it hits ∂M again. This construction provides solutions (modulo smooth terms) $u_1,\dots u_k$, each one defined in an open set Γ_k, where $\cup_k\Gamma_k$ covers $\bar\Gamma_0$. Without loss of generality we may assume that the only intersections of the Γ_k's happen among consecutive ones. Then on S_k, near the intersection with $\pi(\gamma_0)$, we have two microlocal solutions: u_k and u_{k+1}. They have the same traces on S_k modulo a smooth function.

Next, in their common domain of definition, u_k and u_{k+1} coincide up to a smooth function. Indeed, the difference v has smooth trace on S_k and it is outgoing. By the last paragraph of the proof of Theorem 3.1, v is near S_k.

We choose a partition of unity $1=\sum_k\chi_k$ near $\bar\Gamma_0$ subordinate to that cover and set $u^{\rm inc}=\sum_k\chi_k u_k$. The latter is a microlocal solution (i.e., a solution up to smooth errors) in a neighborhood of $\bar\Gamma_0$. Indeed, this is not completely obvious only when $\operatorname{supp}\chi_k$ and $\operatorname{supp}\chi_{k+1}$ intersect but then $u_{k+1}=u_k$ modulo C^∞ and therefore, near such a point, $u^{\rm inc}=\chi_k u_k+\chi_{k+1}u_{k+1}=u_k$ modulo C^∞, which is a microlocal solution.

We use this argument several times below. This construction is similar to that in [13] where it is shown that the Cauchy problem on a spacelike surface gives rise to a global FIO. As a result, one gets a microlocal solution $u^{\rm inc}$ in a neighborhood of $\bar\Gamma_0$ (not satisfying the needed boundary conditions on V yet) as a composition of a finite number of FIOs.

We need to reflect $u^{\rm inc}$ at V to satisfy the zero boundary condition. We write the solution u as the sum of the incident wave $u^{\rm inc}$ and the reflected wave $u^{\rm ref}$: $u=u^{\rm inc}+u^{\rm ref}$. The construction of $u^{\rm ref}$ is similar — we start with boundary data $-u^{\rm inc}|_V$ on V and singularities which propagate into M into the future (the past-future orientation near V is determined by declaring the singularities of $u^{\rm inc}$ on $\mathcal{V}$ coming from the past). We refer to (57) below and the construction following it for more details. The solution $u^{\rm ref}$ needs to be extended to a small neighborhood of the geodesics near γ_0 reflected at V until they leave Γ_0. By choosing $\mathcal{U}$ small enough, we guarantee that the reflected geodesics do not hit ∂M again.

Finally, we prove the uniqueness statement. If u_1 and u_2 are two such solutions, then $v:=u_1-u_2$ is smooth on both U and V. A priori, v can be only singular along bicharacteristics close to γ_0 or its reflection from V. By the argument we used above, v must be smooth in Γ_0 with a possible exception some neighborhood of V in M, where $u^{\rm ref}$ might be non-trivial. Near V, v has smooth Cauchy data. An (easier) adaptation of the same argument shows that v has to be smooth near V as well. Indeed, otherwise, for v, extended as zero outside M, we would get that Pv has singularities conormal to V only, and the microlocal propagation of singularities theorem then would yield that v has no singularities near γ_0 or its reflection. □

Having constructed u, then we define $\Lambda^{\rm gl}_{g,A,q}$ as in (4) but with the so constructed u. The uniqueness part of the proposition shows that $\Lambda^{\rm gl}_{g,A,q}$ is defined up to a smoothing operator.

4.2. $\Lambda^{\rm gl}_{g,A,q}$ recovers the lens relation $\mathcal{L}$ in a stable way.

Theorem 4.1. *Under the assumptions in the Introduction, $\Lambda^{\mathrm{gl}}_{g,A,q}$ is an elliptic FIO of order 1 associated with the (canonical) graph of $\mathcal{L}$.*

Note that we excluded lighlike covectors in $\mathrm{WF}(f)$. This excludes bicharacteristics (geodesics) tangent to ∂M carrying singularities of u. This is where the two Lagrangians (one of them being the diagonal) intersect. We also restricted u to the first reflection and shortly after that. Without that, the canonical relations would contain powers of $\mathcal{L}$. The theorem is a direct consequence of the geometric optics construction and propagation of singularities results for the wave equation and can be considered as essentially known.

As a consequence of Theorem 4.1, for every s, $\Lambda^{\mathrm{loc}}_{g,A,q}$ maps $H^s(U)$ into $H^{s-1}(U)$ and $\Lambda^{\mathrm{gl}}_{g,A,q}$ maps $H^s(U)$ into $H^{s-1}(V)$. Fixing $s=1$, one may conclude that the natural norms for those two operators are the $H^1\to L^2$ ones. While both operators are bounded in those norms, their dependence on the metric g is not necessarily continuous if we stay in those norms. For $\Lambda^{\mathrm{loc}}_{g,A,q}$, we will see that the principal symbol (and the whole one, in fact) depends continuously on g; and in fact the whole operator does, as well. On the other hand, while the canonical relation of $\Lambda^{\mathrm{gl}}_{g,A,q}$ depends continuously on g, the operator itself does not. This observation was used in [2], see also [41] for a discussion.

Proof of Theorem 4.1. We will analyze first the map $F : f\mapsto u^{\mathrm{inc}}|_S$, where S is a timelike surface as in the proof of Proposition 4.1, and (31) for $u=u_{\mathrm{inc}}$ is valid all the way to it, and a bit beyond it.

Change the coordinates x so that $S=\{x^n=1\}$. This can be done if S is close enough to ∂M. Then (31) with $x=(x',1)$ is a local representation of the FIO F and its canonical relation is given by (see, e.g., [46, Ch. VIII])

$$(\nabla_{\xi'}\phi|_{x^n=1},\xi')\quad\longmapsto\quad(x',\nabla_{x'}\phi|_{x^n=1}).$$

By (25), with the momentum p projected to $T^*\{x^n=1\}$, we get that this is the lens relation $\mathcal{L}_1$ from $\mathcal{U}\subset T^*\partial M$ to T^*S (instead of the image being on $T^*\partial M$ again).

We can repeat this finitely many steps by choosing S_1, S_2, etc., to get a composition of finitely many canonical relations, starting with $\mathcal{L}_1$, then $\mathcal{L}_2$ maps data on T^*S_1 to T^*S_2, etc. That composition of, say m of them, gives the lens relation from ∂M to S_m. In the final step, we need to take the normal derivative. This shows that the map $f\mapsto\partial_\nu u^{\mathrm{inc}}|_V$ is an FIO of the claimed type.

To prove this for $\Lambda^{\mathrm{gl}}_{g,A,q}$, we need to add $\partial_\nu u^{\mathrm{ref}}|_V$. The latter has an oscillatory representation of the same kind with a different phase, see (57). Its normal derivative on V is the same however and the principal symbol is the same as that of $\partial_\nu u^{\mathrm{ref}}|_V$; see (58) below. This completes the proof. ☐

To prove stable recovery of the lens relation $\mathcal{L}$, we recall that the $H^1\to L^2$ norm of the DN maps is not suitable for measuring how close the canonical relations $\mathcal{L}$ and $\tilde{\mathcal{L}}$ of the FIOs $\Lambda^{\mathrm{gl}}_{g,A,q}$ and $\Lambda^{\mathrm{gl}}_{\tilde g,\tilde A,\tilde q}$ are. Instead, we formulate stability based on measuring propagation of singularities. Given a properly supported ΨDO R on ∂M near (y_0,η^0), with a principal symbol r_0, we consider $\Lambda^*R\Lambda$, where $\Lambda=\Lambda^{\mathrm{gl}}_{g,A,q}$. By the Egorov theorem, this is actually a ΨDO near (x_0,ξ_0) with a principal symbol $(r_0\circ\mathcal{L})\lambda_0$, where λ_0 is the principal symbol of $\Lambda\Lambda^*$ which depends on g. In this way, we do not recover $\mathcal{L}$ directly; instead we recover functions of $\mathcal{L}$ for various choices of r_0, multiplied by λ_0. Choosing a finite number of R's satisfying some non-degeneracy assumption, we can apply the Implicit Function Theorem to recover $\mathcal{L}$ locally. In fact, we choose below the differential operators

$$\{R_j\}=\{1,y^0,\dots,y^{n-1},\partial/\partial y^0,\dots,\partial/\partial y^{n-1}\}. \tag{54}$$

Theorem 4.2. *Let $(y^0,\dots,y^{n-1})$ be local coordinates on ∂M near y_0. Let*

$$\begin{aligned}&\sum_{j=0}^{n-1}\left\|\Lambda^*y^j\Lambda-\tilde\Lambda^*y^j\tilde\Lambda\right\|_{H^2(U)\to L^2(U)}\le\delta,\quad\left\|\Lambda^*\Lambda-\tilde\Lambda^*\tilde\Lambda\right\|_{H^2(U)\to L^2(U)}\le\delta,\\&\sum_{j=0}^{n-1}\left\|\Lambda^*\frac{\partial}{\partial y^j}\Lambda-\tilde\Lambda^*\frac{\partial}{\partial y^j}\tilde\Lambda\right\|_{H^3(U)\to L^2(U)}\le\delta,\end{aligned} \tag{55}$$

with $\Lambda := \Lambda^{\mathrm{gl}}_{g,A,q}$, $\tilde\Lambda := \tilde\Lambda^{\mathrm{gl}}_{\tilde g,\tilde A,\tilde q}$. *Assume that* (g,A,q) *and* $(\tilde g,\tilde A,\tilde q)$ *are* ϵ*–close to a fixed triple* (g_0,A_0,q_0) *in a certain* C^k *norm in the semi-geodesic normal coordinates near* x_0 *and near* y_0. *Then there exist* $k>0$ *and* $\mu\in(0,1)$ *so that*

$$|(\mathcal{L}-\tilde{\mathcal{L}})(x,\xi')| \le C\delta^\mu\sqrt{-g(\xi',\xi')}, \quad \forall (x,\xi')\in\mathcal{U}, \tag{56}$$

if $\mathcal{U}$ *and* $\epsilon>0$ *are small enough.*

A few remarks:

(a) The square root term is just a homogeneity factor.
(b) The cotangent bundle $T^*\partial M$ is not a linear space, therefore the difference $\mathcal{L}-\tilde{\mathcal{L}}$ makes sense in fixed coordinates only.
(c) The norms in (55) are the natural one since the operators we subtract there are ΨDOs of order two and three, respectively.
(d) The norms in (55) are equivalent to studying the quadratic forms $(\Lambda f, R_j\Lambda f)-(\tilde\Lambda f, R_j\tilde\Lambda f)$.
(e) One could reduce the number of the R_j's to $2n-2$; in fact, $R_0=1$ in (55) is not needed, as it follows from Remark 4.1, since we can recover η'/η_n and use the fact $\eta=(\eta',\eta_n)$ is a null covector.

We prove Theorem 4.2 at the end of this section.

4.3. **Stable recovery of the light ray transforms of** A **and** q**.** Let, as in the Introduction, $\xi^0\in T_{x_0}M\setminus 0$ be the future-pointing lightlike co-vector whose projection on $T^*\partial M\setminus 0$ is the timelike co-vector $\xi^{0\prime}$ as in the definition of the semi-global DN map. Let $\gamma_0 := \gamma_{x_0,\xi^{0\prime}}$ be the lightlike geodesic issued from (x_0,ξ^0) which intersects ∂M at another point y_0. Let V be a neighborhood of y_0 containing all endpoints of future pointing geodesics issued from $\bar{\mathcal{U}}$. Choose and fix any parameterization of the lighlike geodesics close to γ_0 by normalizing ξ'. This defines a hypersurface $\mathcal{U}_0$ in $\mathcal{U}$. The theorem below holds if $\mathcal{U}$ is a small enough neighborhood of $(x_0,\xi^{0\prime})$, and therefore $\mathcal{U}_0$ is small enough, as well. Then L_1 and L_0 are well defined on $\mathcal{U}_0$.

Theorem 4.3. *Fix a Lorentzian metric* g, *and* (x_0,ξ^0) *satisfying the assumptions above. Let* (A,q) *and* $(\tilde A,\tilde q)$ *be two pairs of magnetic and electric potentials. Denote* $\delta := \|\Lambda^{\mathrm{gl}}_{g,A,q}-\Lambda^{\mathrm{gl}}_{g,\tilde A,\tilde q}\|_{H^1(U)\to L^2(V)}$. *Then*

(a) for any $\mu<1$ *and* $m\ge 0$, *the following estimates are valid for some integer* N *whenever* $g,A,\tilde A,q,\tilde q$ *are bounded in a certain* C^k *norm*

$$\|L_1(A-\tilde A)-2\pi N\|_{C^m(\bar{\mathcal{U}}_0)} \le C\delta^\mu.$$

(b) Under the a-priori condition $\|A-\tilde A\|_{C^1(M)}\le\delta_1$ *for some* $\delta_1>0$, *for any* $0<\mu<\mu'$ *and* $m\ge 0$, *the following estimate is valid whenever* $g,A,\tilde A,q,\tilde q$ *are bounded in a certain* C^k *norm*

$$\|L_0(q-\tilde q)\|_{C^m(\bar{\mathcal{U}}_0)} \le C(\delta^\mu+\delta_1^\mu).$$

If there are no conjugate points along γ_0, the proof can be done using a geometric optics construction of the kind (21) but with a different phase in (21) all the way along that geodesic and taking the normal derivative in V. Since we do not want to assume the no-conjugate points assumption, we will proceed in a somewhat different way.

The fact that we cannot rule out the case $N\neq 0$ based on those arguments can be considered as a manifestation of the Aharonov-Bohm effect. If $\tilde A$ and A are a priori close, then $N=0$.

We start with a preparation for the proof of the theorem. Consider first the geometric optics parametrix of the kind (31) of the outgoing solution u like in the previous section. We assume that the boundary condition f has a wave front set in the timelike cone on the boundary, and for simplicity, assume that it is in the future pointing one ($\tau<0$ in local coordinates for which $\partial/\partial t$ is future pointing). Assume at this point that the construction is valid in some neighborhood of the maximal γ_0. We microlocalize all calculations below there. All inverses like D^{-1}, etc., below are microlocal parametrices and the equalities between operators are modulo smoothing operators in the corresponding conic microlocal neighborhoods depending on the context.

The construction is the same to that in the previous section, but this time the outgoing solution u is constructed near the bicharacteristic issued from $(x_0,\xi^{0\prime})$ all the way to y_0. Since the solution can reach the other side of the boundary, we need to reflect it at the boundary to satisfy the zero boundary condition. We write the solution u as the sum of the incident wave $u^{\rm inc}$ and the reflected wave $u^{\rm ref}$: $u = u^{\rm inc} + u^{\rm ref}$ where

$$\begin{aligned} u^{\rm inc}(x) &= (2\pi)^{-n} \int e^{{\rm i}\phi(x,\xi')} \left(a_0^{\rm inc} + a_1^{\rm inc} + R^{\rm inc}\right)(x,\xi') \hat{f}(\xi')\, {\rm d}\xi', \\ u^{\rm ref}(x) &= (2\pi)^{-n} \int e^{{\rm i}\phi^{\rm ref}(x,\xi')} \left(a_0^{\rm ref} + a_1^{\rm ref} + R^{\rm ref}\right)(x,\xi') \hat{f}(\xi')\, {\rm d}\xi'. \end{aligned} \tag{57}$$

Here the phase function $\phi^{\rm ref}$ solves the same eikonal equation as ϕ does but satisfies the boundary condition $\phi^{\rm ref}|_V = \phi$. It differs from ϕ by the sign of its (exterior) normal derivative $\partial\phi/\partial\nu = -\partial\phi^{\rm ref}/\partial\nu > 0$ on V. The amplitudes are of order 0 and -1, respectively, and satisfy

$$\begin{aligned} T^{\rm inc} a_0^{\rm inc} &= 0, \qquad & a_0^{\rm inc}|_U &= \chi, \\ T^{\rm ref} a_0^{\rm ref} &= 0, \qquad & a_0^{\rm ref}|_V &= -a_0^{\rm inc}|_V, \\ iT^{\rm inc} a_1^{\rm inc} &= -Pa_0^{\rm inc}, \qquad & a_1^{\rm inc}|_U &= 0, \\ iT^{\rm ref} a_1^{\rm ref} &= -Pa_0^{\rm ref}, \qquad & a_1^{\rm ref}|_V &= -a_1^{\rm inc}|_V, \end{aligned}$$

where $T^{\rm inc}$ and $T^{\rm ref}$ are the transport operators defined in (28), related to the corresponding phase function, and the remainder terms are of order -2.

Replace A and $\tilde{A}$ with their gauge equivalent field satisfying (M3) on V. This does not change their light ray transforms. A direct computation, which can be justified as (34), yields

$$\Lambda^{\rm gl}_{g,A,q} f = (2\pi)^{-n} \int e^{{\rm i}\phi(x,\xi')} \left(2{\rm i}(\partial_\nu\phi)a_0^{\rm inc} + 2{\rm i}(\partial_\nu\phi)a_1^{\rm inc} + \partial_\nu(a_0^{\rm inc} + a_0^{\rm ref}) + a_{-1}\right) \hat{f}(\xi')\, {\rm d}\xi', \tag{58}$$

where a_{-1} is of order -1 and ϕ and the amplitudes are restricted to $x \in V$.

The expression (58) allows us to factorize $\Lambda^{\rm gl}_{g,A,q}$ as $\Lambda^{\rm gl}_{g,A,q} = 2N_0 D$ modulo FIOs of order 0 associated with the same canonical relation, where Df is the trace of $u^{\rm inc}$ on V (a "Dirichlet-to-Dirichlet map") and N_0 is the DN map $\Lambda^{\rm loc}_{g,0,0}$ but localized in V. Note that replacing A and q in N_0 by zeros or not contributes to lower order error terms. Let D_0 be the operator D related to $A = 0$, $q = 0$. Let N_0^{-1} and D_0^{-1} be microlocal parametrices of those operators which are actually parametrices of the local Neumann-to-Dirichlet map and the incoming Dirichlet-to-Dirichlet one from V to U. Then

$$D_0^{-1} N_0^{-1} \Lambda^{\rm gl}_{g,A,q} = 2D_0^{-1} D \mod S^{-1} \tag{59}$$

is a ΨDO of order 0.

In the next lemma, we do not assume that the geometric optics construction is valid along the whole γ_0.

Lemma 4.1. *The operator $D_0^{-1} N_0^{-1} \Lambda^{\rm gl}_{g,A,q}$ is a ΨDO of order zero in $\mathcal{U}$ with principal symbol*

$$2\exp\left\{{\rm i} L_1 A(\gamma_{x',\xi'})\right\}, \tag{60}$$

where $\gamma_{x',\xi'}$ is the future pointing lightlike geodesic issued from x' in direction ξ with projection ξ'.

Proof. By (59), we need to find the principal symbol of $D_0^{-1} D$.

The transport equation for $a_0^{\rm inc}$ is

$$\left[2g^{jk}(\partial_j\phi)(\partial_k - {\rm i}A_k) + \Box_g\phi\right] a_0^{\rm inc} = 0, \qquad a_0^{\rm inc}|_U = 1.$$

As explained right after (25), $g^{jk}(\partial_j\phi)\partial_k$ is the tangent vector field along the null geodesic $\gamma_{x',\xi'}$. Therefore, with $\Gamma(s) := (\gamma_{x',\xi'}(s), g\dot{\gamma}_{x',\xi'}(s))$, as before, on the set $\chi = 1$ we get $a_0^{\rm inc} = \mu$, see (30), i.e.,

$$\begin{aligned} a_0^{\rm inc}(\Gamma(s)) &= \exp\left\{ -\frac{1}{2}\int_0^s (\Box_g\phi)(\Gamma(\sigma))\, {\rm d}\sigma \right\} \\ &\quad \times \exp\left\{ {\rm i}\int_0^s A_k \circ \gamma_{x',\xi'}(\sigma) \dot{\gamma}^k_{x',\xi'}(\sigma)\, {\rm d}\sigma \right\}. \end{aligned} \tag{61}$$

Take $s = s(x, \xi)$ so that $\gamma_{x',\xi'}(s) \in V$ to get

$$(a_0^{\rm inc} \circ \mathcal{L})(x', \xi') = \exp\left\{-\frac{1}{2} L(\Box_g \phi)(x', \xi') + \mathrm{i} L_1 A(x', \xi')\right\},$$

where we use the coordinates (x', ξ') to parameterize the lightlike geodesics locally, and the definition of $L(\Box_g \phi)$ is clear from (61).

To construct a representation for D_0^{-1}, note first that when $A = 0$, the term involving $L_1 A$ is missing above. We look for a parametrix of the incoming solution of $\Box_g u = 0$ with boundary data $u = h$ on V with $\mathrm{WF}(h) \subset \mathcal{V}$ of the form

$$u(x) = (2\pi)^{-n} \int e^{\mathrm{i}\phi(x,\xi')} b(x, \xi') \hat{f}(\xi') \, \mathrm{d}\xi', \tag{62}$$

where ϕ is the same phase as in the first equation in (57) and f (not related to (20)) depending on h as below. The amplitude b solves the transport equation along the same bicharacteristics (with different coefficients since $A = 0$, $q = 0$) with the initial condition:

$$b|_V = a^{\rm inc}|_V,$$

where $a^{\rm inc}$ is the full amplitude in the first equation in (57). Restricted to V, the map $f \to u|_V$ is just Df. Then to satisfy $u = h$ on V, we need to solve $Df = h$, i.e., to take $f = D^{-1}h$ microlocally.

To illustrate the argument below better, suppose that we are solving the ODE

$$y' + ay = 0, \quad y(0) = 1$$

from $t = 0$ to $t = 1$, where $a = a(t)$. Then we solve

$$y_1' + a_1 y_1 = 0, \quad y_1(1) = y(1),$$

where $a_1 = a_1(t)$. A direct calculation yields

$$y(t) = \exp\Big\{ -\int_0^t a(s) \, \mathrm{d}s \Big\}, \quad y_1(t) = \exp\Big\{ -\int_1^t a_1(s) \, \mathrm{d}s \Big\} y(1).$$

In particular,

$$y_1(0) = \exp\Big\{ -\int_0^1 (a_1(s) - a(s)) \, \mathrm{d}s \Big\}.$$

We apply those argument to the transport equation to get

$$b|_U = \exp\big\{\mathrm{i} L_1 A(\gamma_{x',\xi'})\big\}.$$

Then

$$D_0^{-1} Df = (2\pi)^{-n} \int e^{x'\cdot\xi'} \exp\big\{\mathrm{i} L_1 A(\gamma_{x',\xi'})\big\} \hat{f}(\xi') \, \mathrm{d}\xi'.$$

This proves the lemma under the assumption that the geometric optics construction is valid in a neighborhood of γ_0.

To prove the theorem in the general case, we use the partition argument we used in Proposition 4.1. Let $S_1, \dots, S_k$ be small timelike surfaces intersecting γ_0 in increasing order, from U to V so that the geometric optics construction is valid in a neighborhood of each segment of γ_0 cut by two consecutive surfaces of the sequence $\{U, S_1, \dots, S_k, V\}$. This determines Dirichlet-to-Dirichlet maps D_1, from U to S_1; then D_2, from S_1 to S_2, etc., until D_{k+1} from S_k to V. Then $D = D_{k+1} D_k \dots D_1$. Similarly, $D_0 = D_{0,k+1} D_{0,k} \dots D_{0,1}$. Then (59) is still valid and takes the form

$$D_0^{-1} N_0^{-1} \Lambda^{\rm gl}_{g,A,q} = 2 D_{0,1}^{-1} \dots D_{0,k}^{-1} D_{0,k+1}^{-1} D_{k+1} D_k \dots D_1 \mod S^{-1}.$$

By the first part of the proof, $D_{0,k+1}^{-1} D_{k+1}$ is a ΨDO on V with principal symbol $\exp\{\mathrm{i} L_1^{(k+1)} A\}$, where $L_1^{(k+1)}$ is the light ray transform L_1 restricted to geodesics between S_k and V. Then we apply Egorov theorem, see [20, Theorem 25.2.5], to conclude that $D_{0,k}^{-1}\big(D_{0,k+1}^{-1} D_{k+1}\big) D_k$ is a ΨDO with a principal symbol that of $D_{0,k+1}^{-1} D_{k+1}$, pulled back by $\mathcal{L}_{k+1}$, the canonical relation between S_k and V, multiplied by the principal symbol of $D_{0,k}^{-1}D_k$. The result is then (60) without the factor 2 with the integration between S_k (through S_{k+1}) to V. Repeating this argument several times, we complete the proof of the lemma. □

4.4. **Stability of the light ray transform of the magnetic field.**

Proof of Theorem 4.3(a). We have

$$\begin{aligned}(63)\qquad &\big\|D_0^{-1}N_0^{-1}\big(\Lambda^{\rm gl}_{g,A,q}-\Lambda^{\rm gl}_{g,\tilde A,\tilde q}\big)\big\|_{H^1(U)}\\ &\qquad\le C\big\|\Lambda^{\rm gl}_{g,A,q}-\Lambda^{\rm gl}_{g,\tilde A,\tilde q}\big\|_{H^1(U)\to L^2(V)}=C\delta.\end{aligned}$$

Set $R:=D_0^{-1}N_0^{-1}\big(\Lambda^{\rm gl}_{g,A,q}-\Lambda^{\rm gl}_{g,\tilde A,\tilde q}\big)$. By Lemma 4.1, R is a ΨDO in $\mathcal{U}$ of order 0 with principal symbol

$$r_0(x',\xi')=2\exp\left\{\mathrm{i}L_1(\tilde A-A)(\gamma_{x',\xi'})\right\},$$

and we have $\|R\|_{H^1(V)}\le C\delta$, by (63). We need to derive that r_0 is "small" in $\mathcal{U}$, as well. We essentially did that in the proof of Theorem 3.2. Choose f as in (20). By [46, Ch. VIII.7], on the set $\chi=1$, $e^{-i\lambda x'\cdot\xi'}Rf$ is equal to the full symbol of $\Lambda^{\rm loc}_{g,A,q}$ with $\lambda=|\xi|$ and ξ in (34) bounded, say, unit. Therefore,

$$r_0(x',\xi')=e^{-i\lambda x'\cdot\xi'}Rf+O(1/\lambda)\tag{64}$$

in C^k for every k. Since $\|f\|_{L^2}=C$ and $\|f\|_{H^1}\sim\lambda$, (63) yields

$$\|r_0(\cdot,\xi')\|_{H^1(U)}\le C\lambda\delta+C/\lambda,$$

uniformly for ξ in some neighborhood of $\xi^{0\prime}$. With a little more efforts one can remove λ from $C\lambda\delta$ but this is not needed. Take $\lambda=\delta^{-1/2}$ to get

$$\left\|\exp\left\{\mathrm{i}L_1(\tilde A-A)(\gamma_{x',\xi'})\right\}\right\|_{H^1(\bar{\mathcal{U}}')}\le C\delta^{1/2}.$$

Using interpolation estimates, we can replace the H^1 norm by any other one at the expense of lowering the exponent on the right from 1/2 to another positive one, if k in Theorem 4.3 is large enough. Since $|e^{iz}-1|<\varepsilon$ implies $|z-2\pi N|<C\varepsilon$ for some integer N, this proves part (a) of the theorem. □

4.5. **Stability of the light ray transform of the potential.**

Proof of Theorem 4.3(b). First, we will reduce the problem to the case $\tilde A=A$. For $\Lambda^{\rm gl}_{g,\tilde A,\tilde q}-\Lambda^{\rm gl}_{g,A,\tilde q}$, we get a representation as in (58) with a principal symbol with seminorms $O(\delta_1^{\mu'})$, since we can use interpolation estimates to estimate the higher derivatives of $\tilde A-A$. Apply a parametrix $\left(\Lambda^{\rm gl}_{g,A,\tilde q}\right)^{-1}$ to that difference to get a ΨDO Q of order 0 microlocally supported in $\mathcal{U}$. If the geometric optics construction is valid all the way from U to V, we get as in the proof of (a) that $Qf=O(\delta_1^{\mu'})+O(1/\lambda)$ in H^1. This implies the same estimate for $\big\|\big(\Lambda^{\rm gl}_{g,\tilde A,\tilde q}-\Lambda^{\rm gl}_{g,A,\tilde q}\big)f\big\|_{L^2}$. In the general case, we can prove the same estimate as in the proof of (a). We will use this later and for now, we assume $\tilde A=A$.

Lemma 4.2. *The operator $D^{-1}N_0^{-1}\left(\Lambda^{\rm gl}_{g,A,\tilde q}-\Lambda^{\rm gl}_{g,A,q}\right)$ is a ΨDO of order -1 on U with principal symbol*

$$2\left[L_0(\tilde q-q)\right]\circ\gamma_{x',\xi'},\tag{65}$$

where $\gamma_{x',\xi'}$ is the future pointing lightlike geodesic issued from x' in direction ξ with projection ξ'.

Proof. Assume first that the geometric optics construction is valid in a neighborhood of the whole γ_0. In the amplitude

$$-2\mathrm{i}(\partial_\nu\phi)a_0^{\rm inc}-2\mathrm{i}(\partial_\nu\phi)a_1^{\rm inc}+\partial_\nu(a_0^{\rm inc}+a_0^{\rm ref})+a_{-1}$$

in (58), the terms $-2\mathrm{i}(\partial_\nu\phi)a_0^{\rm inc}$ and $\partial_\nu(a_0^{\rm inc}+a_0^{\rm ref})$ do not depend on q, see (61). The other two terms depend on q but they are of different orders. Therefore,

$$\left(\Lambda^{\rm gl}_{g,A,\tilde q}-\Lambda^{\rm gl}_{g,A,q}\right)f=(2\pi)^{-n}\int e^{\mathrm{i}\phi(x,\xi')}\Big(-2\mathrm{i}(\partial_\nu\phi)(\tilde a_1^{\rm inc}-a_1^{\rm inc})+(a_{-1}-\tilde a_{-1})\Big)\hat f(\xi')\,\mathrm{d}\xi'|_V.\tag{66}$$

The order of the FIO above is zero. As in the previous proof, we can represent this as a composition of $2N_0$ with the operator $\tilde D - D$ (the difference of two such Dirichlet-to-Dirichlet maps):

$$\Lambda^{\rm gl}_{g,A,\tilde q} - \Lambda^{\rm gl}_{g,A,q} = 2N_0(\tilde D - D) \tag{67}$$

modulo FIOs of order -2. That operator $\tilde D - D$ is an FIO with a symbol, compare with (66),

$$\sigma(\tilde D - D) = -2\mathrm{i}(\tilde a_1^{\rm inc} - a_1^{\rm inc}) + a_{-2}, \tag{68}$$

with a_{-2} of order -2.

To compute $a_1^{\rm inc}$, recall the transport equation for $a_1^{\rm inc}$

$$\left[2g^{jk}\partial_j\phi(\partial_k - \mathrm{i}A_k) + \Box_g\phi\right] a_1^{\rm inc} = \mathrm{i}Pa_0^{\rm inc}, \qquad a_1^{\rm inc}|_U = 0 \tag{69}$$

where

$$\mathrm{i}Pa_0^{\rm inc} = \mathrm{i}P_{g,A,0}a_0^{\rm inc} + \mathrm{i}qa_0^{\rm inc}.$$

The first term on the right is independent of q. By (29), (30), with $\Gamma(s)$ as in (61), we get

$$\begin{aligned} a_1^{\rm inc}(\Gamma(s)) &= \frac{\mathrm{i}a_0^{\rm inc}}{2}\int_0^s \frac{1}{a_0^{\rm inc}}\left[P_{g,A,0}a_0^{\rm inc} + qa_0^{\rm inc}\right]\circ\Gamma(\sigma)\,\mathrm{d}\sigma \\ &= \frac{\mathrm{i}a_0^{\rm inc}}{2}\int_0^s \left[\frac{1}{a_0^{\rm inc}}P_{g,A,0}a_0^{\rm inc} + q\right]\circ\Gamma(\sigma)\,\mathrm{d}\sigma. \end{aligned} \tag{70}$$

The potential q depends on x only, so $q\circ\Gamma(s) = q\circ\gamma(s)$. In (70), only the last term depends on q and is an integral of q over lightlike geodesics multiplied by an elliptic factor. Note that the integral, as well as $a_1^{\rm inc}$, are homogeneous of order -1 in ξ', as they should be.

We go back to (68) now. Using (70), the terms involving $P_{g,A,0}$ and $P_{g,\tilde A,0}$ cancel below and we get

$$\sigma(\tilde D - D)\circ\mathcal{L} = \mathrm{i}a_0^{\rm inc}L_0(\tilde q - q) + a_{-2}, \tag{71}$$

where a_{-2} is a symbol of order -2, different form the one above.

Similarly to (59), we have

$$D^{-1}N_0^{-1}\left(\Lambda^{\rm gl}_{g,A,\tilde q} - \Lambda^{\rm gl}_{g,A,q}\right) = 2D^{-1}(\tilde D - D) \mod S^{-2}. \tag{72}$$

Therefore, we need to compute the principal symbol of $2D^{-1}(\tilde D - D)$. Let R be a ΨDO in U with principal symbol r_{-1} given by (65). Then, in $\mathcal{U}$, DR is an FIO of the type (62) with $x\in V$ with the same phase function and a principal amplitude b_0 solving $Tb_0 = 0$, $b_0|_U = r_{-1}$. By (29), the solution restricted to $x\in V$ is given by $\mu r_{-1}\circ\mathcal{L}^{-1}|_V$. Recall that $\mu = a_0^{\rm inc}$. By (71), this is $2\sigma(\tilde D - D)$ modulo symbols of order -2. Therefore, $DR = 2(\tilde D - D)$ modulo FIOs of order -2. This proves the lemma under the assumption that the geometric optic construction is valid along the whole γ_0.

In the general case, we repeat the arguments of Lemma 4.1. We represent D and $\tilde D$ as a composition $D = D_{k+1}\dots D_1$, and similarly for $\tilde D$. We will do the first step. Consider $2(D_2D_1)^{-1}(\tilde D_2\tilde D_1 - D_2D_1)$. We have

$$\begin{aligned} &2(D_2D_1)^{-1}(\tilde D_2\tilde D_1 - D_2D_1) \\ &\qquad = 2D_1^{-1}D_2^{-1}\left((\tilde D_2 - D_2)\tilde D_1 + D_2(\tilde D_1 - D_1)\right) \\ &\qquad = D_1^{-1}R_2\tilde D_1 + R_1 = D_1^{-1}R_2D_1 + R_1 \end{aligned}$$

modulo FIOs of order -2, where $R_j = 2D_j^{-1}(\tilde D_j - D_j)$, $j = 1,2$. We apply Egorov's theorem to $D_1^{-1}R_2D_1$ to conclude that it is a ΨDO on U with a principal symbol equal to the sum of two terms as in (65) with L_0 taken over the geodesic segments between U and S_1 first, and S_1 and S_2 second. The sum is equal to (65) with L_0 taken over the union of those segments. Repeating this arguments to include D_2, etc., completes the proof of the lemma. □

We finish the proof of part (b) as we did that for part (a). Set $R = D^{-1}N_0^{-1}\left(\Lambda^{\rm gl}_{g,A,\tilde q} - \Lambda^{\rm gl}_{g,A,q}\right)$. It is a ΨDO of order -1 rather than of order 0 as in (a). The analog of (63) is still true. If, as above, r_{-1} is the principal symbol of R, then by Lemma 4.2,

$$r_{-1}(x',\xi') = -2\mathrm{i}[L_0(\tilde q - q)]\circ\gamma_{x',\xi'} = \lambda e^{-\mathrm{i}\lambda x'\cdot\xi'}Rf + O(1/\lambda)$$

with f as in (20), compare with (64). Then

$$\|r_{-1}(\cdot,\xi')\|_{H^1(U)} \le C\lambda^2\delta + C/\lambda.$$

Choose $\lambda = \delta^{-1/3}$ to get $\|r_{-1}(\cdot,\xi')\|_{H^1(U)} \le C\delta^{1/3}$. This completes the proof of the theorem. □

4.6. **Proof of the stable recovery of the lens relation.**

Proof of Theorem 4.2. We use the notation above. Recall the remark preceding Theorem 4.2 above. The operator $\Lambda^* P\Lambda$ is a ΨDO with a principal symbol $(p_0\circ\mathcal{L})\lambda_0$. Take $p = p_0 = 1$ as in (54) to recover λ_0 first. Knowing the latter, we recover $p_j\circ\mathcal{L}$ for $j = 1,\dots 2n-1$, see (54). That gives us (y,η') in (5) as functions of (x,ξ'). Therefore, we reduce the stability problem to the following: show that the principal symbol of a ΨDO A of order m is determined by $A : H^m \to L^2$ in a stable way which is resolved by the lemma below, see also (35), (36). Note that the lemma is a bit more general than what we need since $\{P_j\}$ are simple multiplication and differentiation operators.

Lemma 4.3. *Let Q be ΨDO in $\mathbf{R}^n$ with kernel supported in $K\times K$, where $K\subset\mathbf{R}^n$ is compact. Let q_m be its principal symbol homogeneous of order m. Then*

$$\|q_m(\cdot,\xi)\|_{L^2} \le C|\xi|^m\|Q\|_{H^m\to L^2}$$

for all $\xi\neq 0$ with $C>0$ depending on K only.

Proof. Take $f = e^{\mathrm{i}x\cdot\xi}\chi(x)$, where $\chi\in C_0^\infty$ equals 1 in a neighborhood K. Then for x in a neighborhood of K, $Qf(x) = e^{\mathrm{i}x\cdot\xi}(q_m(x,\xi) + r(x,\xi))$ with $r\in S^{m-1}$. We have $|\xi|^m/C \le \|f\|_{H^m} \le C|\xi|^m$ for $|\xi|\ge 1$. Therefore, for such ξ,

$$C_1\|Qf\|_{L^2}/\|f\|_{H^m} \ge \|q_m(\cdot,\xi)/|\xi|^m\|_{L^2} - C_2/|\xi|.$$

Take the limit $|\xi|\to\infty$ along radial rays to complete the proof. □

We complete the proof of Theorem 4.2 with the aid of Lemma 4.3. We recover first the L^2-norms w.r.t. x of $\mathcal{L}(x,\xi) - \tilde{\mathcal{L}}(x,\xi)$ uniformly in ξ (in fixed coordinates); we can choose $\mu = 1$ then. Using standard interpolation estimates, we can estimate the L^∞ norm of $\mathcal{L}(x,\xi) - \tilde{\mathcal{L}}(x,\xi)$ with $\mu<1$ in (56), using the a priori bounds on g and $\tilde g$ in some C^k, $k\gg 1$, which imply similar bounds on $\mathcal{L}$ and $\tilde{\mathcal{L}}$. □

Remark 4.1. The symbol λ_0 can be computed. Since we do not use this formula, we will sketch the proof only. Using Green's formula, as in the proof of [38, Prop. 2.1], we can show that $2N_{\mathcal{V}} \cong D^*\Lambda$, where $\cong$ stands for equality modulo lower order terms, and $N_{\mathcal{V}}$ is N above with the subscript $\mathcal{V}$ indicating that it acts microlocally in that set. The same proof implies that Λ^* is the DN map associated with the incoming solution, i.e., the one which starts from $\mathcal{V}$ microlocally and hits $\mathcal{U}$. Therefore, $\Lambda^*\cong 2N_{\mathcal{U}}D^{-1}$, where $N_{\mathcal{U}}$ now acts in $\mathcal{U}$. Those two identities and the Egorov's theorem imply $\lambda_0 = -4(\xi_n\circ\mathcal{L})\xi_n$, where ξ_n is the function defined in (32).

5. Applications and Examples

We start with a partial but still general enough case. We follow [19, §24.1]. Let M be a Lorentzian manifolds with a timelike boundary ∂M. Assume that t is a real valued smooth function on M so that the level surfaces $t=$ const. are compact and spacelike. For every $a<b$, the (compact) "cylinder" $M_{ab} = \{a\le t\le b\}$ (assuming $[a,b]$ is in the range of t) has a boundary consisting of the spacelike surfaces $t^{-1}(a)$, $t^{-1}(b)$ and $\partial M\cap M_{ab}$ which intersect transversely. This is a generalization of $[0,T]\times\Omega$ in the Riemannian case. By [19, Theorem 24.1.1], the following problem is well posed

$$Pu = 0 \text{ in } M,\quad u|_{t<a} = 0,\quad u|_{\partial M} = f$$

with $f \in H^s(\partial M)$, $s \geq 1$, $f = 0$ for $t < a$; with a unique solution $u \in H^s(M)$ vanishing for $t < a$. Moreover, the map $f \mapsto u$ is continuous. Then the Dirichlet-to-Neumann map $\Lambda_{g,A,q}$ defined as in (4), is well defined.

Let $x_0 \in U_0 \Subset U \subset \partial M$ be as in Theorem 3.2. Let χ be a properly supported ΨDO cutoff of order zero localizing near some timelike covector over $x_0 \in U_0$. Since there is a globally defined time function, there are no periodic lightlike geodesics. Then $\chi\Lambda_{g,A,q}\chi$ can be taken as $\Lambda^{\rm loc}_{g,A,q}$ and Theorem 3.2 applies. If we know a priori that $\Lambda_{g,A,q} : H^1_{(0)}(\partial M) \to L^2(\partial M)$ is continuous, where the subscript (0) indicates functions vanishing for $t = 0$, then we can replace $\Lambda^{\rm loc}_{g,A,q}$ by $\Lambda_{g,A,q}$ in (33) and therefore, in Theorem 3.2.

Similarly, with suitable ΨDO cutoffs χ_1 and χ_2, we can take $\Lambda^{\rm gl}_{g,A,q} = \chi_1\Lambda_{g,A,q}\chi_2$, under the assumptions of Theorem 4.3. And again, if we know that $\Lambda_{g,A,q} : H^1_{(0)}(\partial M) \to L^2(\partial M)$ is continuous, we can remove the cutoffs. The results with the cutoffs are actually stronger.

Some special subcases are discussed below. They recover and extend the uniqueness results in [36, 33, 32, 47, 34, 1, 7], and some of the stability results there. Using the results in this paper with the support theorems about the light ray transform in [37, 31], we can get new partial data results.

Example 5.1. Let q be a unknown potential but assume that the metric and the magnetic fields are known. Restrict the DN map to M_{ab} for some $a < b$. Then we can recover $L_0 q$ in a stable way as in Theorem 4.3 over all timelike geodesics intersecting the lateral boundary transversely at their endpoints. If g is real-analytic, then we can apply the results in [37] to recover q in the set covered by those geodesics under an additional foliation condition. Note that in contrast, the results in [14] require A and q to be analytic in time.

Example 5.2. In the example above, assume that g is Minkowski, and $M_{ab} = [0,T] \times \bar\Omega$ for some bounded smooth $\Omega \subset \mathbf{R}^n$. By Theorem 3.2, we can recover $L_1 A$ and $L_0 q$ over all lightlike geodesics (lines) $l_{z,\theta} = \{(t,x) = (s, z + s\theta);\ s \in \mathbf{R}\}$, $(z,\theta) \in \mathbf{R}^n \times S^{n-1}$, not intersecting the top and the bottom of the cylinder. By [37], we can recover q in the set covered by those lines. By [31], we can recover A up to $\mathrm{d}\phi$, $\phi = 0$ on $[0,T] \times \partial\Omega$ in that set as well.

For example, if Ω is the ball $B(0,1) = \{x;\ |x| < 1\}$, the DN map recovers uniquely q and A, up to a gauge transform, in the cylinder $[0,T] \times \bar B(0,1)$ with the upward characteristic cone with base $\{0\} \times B(0,1)$ and the downward with base $\{T\} \times B(0,1)$ removed, see Figure 3. If $T \leq 2$, those two cones intersect; otherwise they do not but the result holds in both cases. This is the possibly reachable region from $[0,T] \times \partial\Omega$, thus the results are sharp since no information about the complement can be obtained by the finite speed of propagation.

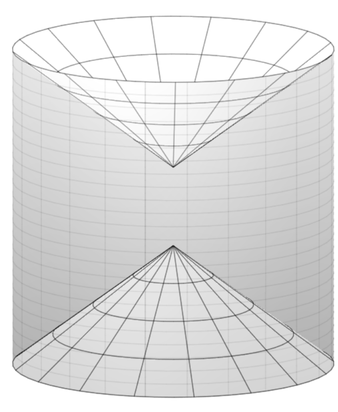

FIGURE 3. The DN map, with g Minkowski, on the lateral boundary of the cylinder determines a potential and a magnetic field up to $\mathrm{d}\phi$ inside the cylinder but outside the two characteristic cones.

This extends further the uniqueness part of the results in [36, 33, 32, 47, 34, 1, 7]. Using the stability estimate in [3] about L_0, and the logarithmic estimate for L_1 in [35], we can use Theorem 4.3 to recover the results in [35]. One important improvement however is that for uniqueness, we do not assume that A and q are known outside $[0,T]$; or that $T=\infty$ because the uniqueness results in [37, 31] do not require the function or the vector field to be compactly supported in time.

Example 5.3. A partial data case of Example 5.2 is the following. Let $\Gamma\subset\partial\Omega$ be relatively open, and assume that $\partial\Omega$ is strictly convex. Assume that we know the DN map for f supported in $[0,T]\times\Gamma$, and we measure Λf there, as well. Then we can recover q (for all $n\ge 2$) and A for $n\ge 3$, up to a gauge transform, in the set covered by the lightlike lines hitting $[0,T]\times\partial\Omega$ in $[0,T]\times\Gamma$ at their both endpoints. When $n=2$, the recovery of A up to a potential $\mathrm{d}\phi$ requires that if we know L_1A for all some lightlike $l_{z,\theta}$, we also know it for $l_{z,-\theta}$, see [31], and this is the reason we restricted n to $n\ge 3$. Those local uniqueness results for the DN maps are new.

Example 5.4. In a recent work [7], an inverse problem for the wave operator

$$P:=\partial_t^2+a(t,x)\partial_t-\Delta+b(t,x)$$

with real valued a, b is studied. The coefficient b causes absorption. We do not restrict A and q to be real valued, so we can take $A=(\frac{\mathrm{i}}{2}a(t,x),0,\dots,0)$, $q=-\frac{\mathrm{i}}{2}\partial_t a(t,x)+b(t,x)$, then P in (1) is the same as the one above. Then Theorem 4.3 proves unique recovery of A, q up to the gauge transform $A\mapsto A-\mathrm{d}\psi$ with $\psi=0$ on $[0,T]\times\partial\Omega$. Since A is restricted to the class of covector fields with spatial components zero, we must have $\psi=\psi(t)$. However, then $\psi=0$ for $x\in\partial\Omega$ implies $\psi\equiv 0$. Therefore, the logarithmic and the double logarithmic stability estimates in [7] for a and for b which are about the DN map can be obtained by Theorem 4.3 combined with the stability estimates in [3, 35]. We can get new uniqueness results however with partial data as in the previous examples. In the Riemannian case studied by Montalto [28] we can allow an absorption term as well to obtain, up to a gauge transform, stable recovery of a Riemannian simple metric in a generic class, a magnetic field, a potential and an absorption term from the DN map.

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