LECTURE NOTES ON GEOMETRIC OPTICS

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1. The wave equation

We want to solve the wave equation

$$(\partial_t^2 - c^2\Delta_g)u = 0, \quad u|_{t=0} = f_1, \quad u_t|_{t=0} = f_2, \tag{1.1}$$

in $\mathbf{R}_t \times \mathbf{R}_x^n$, where g is a Riemannian metric and $c > 0$ is a smooth "speed".

An immediate observation is that the wave operator $\Box = \partial_t^2 - c^2\Delta_g$ is not elliptic (it is hyperbolic). Its principal symbol (multiplied by -1) is

$$p(t, x, \tau, \xi) = \tau^2 - c^2|\xi|_g^2,$$

where $|\xi|_g^2 = g^{ij}(x)\xi_i\xi_j$. The characteristic variety is the conic set $\Sigma = \left\{\tau^2 = c^2|\xi|_g^2\right\}$ (note that it depends on (t, x)), which is a manifold away from the zero section.

A solution exists at least when f_1 and f_2 are test functions. In fact, one can use Stone's theorem to show that a solution exists on the energy space and defines a unitary group.

1.1. **Constant coefficients case.** Assume g Euclidean and c constant.

Let $n = 1$ first. Then it is well known that

$$u = F(x - ct) + G(x - ct)$$

with F and G arbitrary (distributions) represents the general solution. We have two waves moving with speeds c (and velocities c and $-c$, respectively). Having initial conditions, one can easily compute them. If $f_2 = 0$, for example, then $F = G = \frac{1}{2}f_1$. If f_1 is singular, we see that the singularities propagate left and right with speed c. The solution cannot be a ΨDO (applied to f_1).

Let $n \geq 2$. Make the speed c equal to 1. One has explicit formulas then, of course but let us see what happens if we use Fourier analysis. We write

$$(\partial_t^2 + |\xi|^2)\hat{u}(t, \xi) = 0,$$

where $\hat{u}$ is the partial Fourier transform w.r.t. x. This is an ODE, therefore,

$$u(t, \xi) = a_1(\xi)e^{\mathrm{i}t|\xi|} + a_2(\xi)e^{-\mathrm{i}t|\xi|}. \tag{1.2}$$

Using the initial conditions, we get

$$\hat{u}(t, \xi) = \frac{1}{2}e^{\mathrm{i}t|\xi|}\left(\hat{f}_1(\xi) + \frac{1}{|\xi|}\hat{f}_2(\xi)\right) + \frac{1}{2}e^{-\mathrm{i}t|\xi|}\left(\hat{f}_1(\xi) - \frac{1}{|\xi|}\hat{f}_2(\xi)\right). \tag{1.3}$$

This can be also written as

$$\hat{u}(t, \xi) = \cos(t|\xi|)\hat{f}_1(\xi) + \frac{\sin(t|\xi|)}{|\xi|}\hat{f}_2(\xi). \tag{1.4}$$

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Taking the inverse Fourier transform, we get

$$
\begin{aligned}
u(t,x) &= (2\pi)^{-n}\int e^{\mathrm{i}(t|\xi|+x\cdot\xi)}\left(\frac{1}{2}\hat f_1(\xi)+\frac{1}{2|\xi|}\hat f_2(\xi)\right)\mathrm{d}x\\
&\quad+(2\pi)^{-n}\int e^{\mathrm{i}(-t|\xi|+x\cdot\xi)}\left(\frac{1}{2}\hat f_1(\xi)-\frac{1}{2|\xi|}\hat f_2(\xi)\right)\mathrm{d}x.
\end{aligned}\tag{1.5}
$$

This can also be written as

$$
\begin{aligned}
u(t,x) &= (2\pi)^{-n}\int e^{\mathrm{i}(t|\xi|+(x-y)\cdot\xi)}\left(\frac{1}{2}f_1(x)+\frac{1}{2|\xi|}f_2(x)\right)\mathrm{d}x\,\mathrm{d}\xi\\
&\quad+(2\pi)^{-n}\int e^{\mathrm{i}(-t|\xi|+(x-y)\cdot\xi)}\left(\frac{1}{2}f_1(x)-\frac{1}{2|\xi|}f_2(x)\right)\mathrm{d}x\,\mathrm{d}\xi.
\end{aligned}\tag{1.6}
$$

The essential difference with the ΨDOs is the phase functions above: we have

$$\phi_\mp := \pm t|\xi| + x\cdot\xi, \tag{1.7}$$

respectively $\pm t|\xi| + (x-y)\cdot\xi$ instead of $x\cdot\xi$ or $(x-y)\cdot\xi$.

Of course, we also have the classical formulas for the solutions using spherical means, etc.

1.2. **The variable coefficients case.** One can show that $P := -c^2\Delta_g$ is formally self-adjoint w.r.t. the measure $c^{-2}\mathrm{d}\,\mathrm{Vol}$, where $\mathrm{d}\,\mathrm{Vol}(x) = \sqrt{\det g}\,\mathrm{d}x$. Given a domain U, and a function $u(t,x)$, define the energy

$$E_U(t,u) = \int_U \left(|\nabla u|_g^2 + c^{-2}|u_t|^2\right)\mathrm{d}\,\mathrm{Vol},$$

where $|\nabla u|_g^2 = g^{ij}(\partial_i u)(\partial_j u)$. By the finite speed of propagation, the solution with compactly supported Cauchy data will have a compact support for every t expanding at a speed no faster than $\max(c)$. The energy norm for the Cauchy data (f_1,f_2), that we denote by $\|\cdot\|_{\mathcal{H}}$ is then defined by

$$\|(f_1,f_2)\|_{\mathcal{H}}^2 = \int_{\mathbf{R}^n}\left(|\nabla f_1|_g^2 + c^{-2}|f_2|^2\right)\mathrm{d}\,\mathrm{Vol},$$

where $H_D(\mathbf{R}^n)$ This defines the energy space

$$\mathcal{H} = \mathcal{H}(\mathbf{R}^n) = H_D(\mathbf{R}^n)\oplus L^2(\mathbf{R}^n),$$

where $H_D(\mathbf{R}^n)$ is tje completion of $C_0^\infty(\mathbf{R}^n)$ in the norm $\left(\int|\nabla f|_g^2\,\mathrm{d}\,\mathrm{Vol}\right)^{1/2}$. The wave equation then can be written down as the system

$$\mathbf{u}_t = \mathbf{P}\mathbf{u}, \quad \mathbf{P} = \begin{pmatrix}0 & I\\ P & 0\end{pmatrix}, \tag{1.8}$$

where $\mathbf{u} = (u, u_t)$ belongs to the energy space $\mathcal{H}$. The operator $\mathbf{P}$ then extends naturally to a skew-selfadjoint operator ($\mathbf{P}^* = -\mathbf{P}$) on $\mathcal{H}$. In this paper, we will deal with either $U = \mathbf{R}^n$ or $U = \Omega$. This allows us to use the Stone's theorem to conclude that (1.1) is solvable and the solution can be expressed as

$$\mathbf{u}(t,\cdot) = e^{t\mathbf{P}}\mathbf{f}, \tag{1.9}$$

where $\mathbf{f} = (f_1, f_2)$. The group $e^{t\mathbf{P}}$ is unitary (energy preserving) and strongly continuous.

The purpose of this was to show that (1.1) is solvable in the first place, and it also provides energy estimates (the energy norm is preserved). The geometric optics construction aims to get a qualitative information about the solution.

2. The geometric optics ansatz

We start with our microlocal construction now. Motivated by the constant coefficient case, we are looking for a solution of the type (1.5). We cannot expect the phase function to be as there. So we make an educated guess that the solution may have the form

$$u(t,x) = (2\pi)^{-n} \sum_{\sigma=\pm} \int e^{\mathrm{i}\phi_\sigma(t,x,\xi)} \Big(a_{1,\sigma}(t,x,\xi)\hat f_1(\xi) + |\xi|_g^{-1} a_{2,\sigma}(t,x,\xi)\hat f_2(\xi)\Big)\mathrm{d}\xi, \tag{2.1}$$

modulo terms involving smoothing operators of f_1 and f_2. Here the phase functions $\phi_\pm$ are positively homogeneous of order 1 in ξ. Next, we seek the amplitudes in the form

$$a_{j,\sigma} \sim \sum_{k=0}^{\infty} a_{j,\sigma}^{(k)}, \quad \sigma=\pm,\ j=1,2, \tag{2.2}$$

where $a_{j,\sigma}^{(k)}$ is homogeneous in ξ of degree $-k$ for large $|\xi|$. To construct such a solution, we plug (2.1) into (1.1) and try to kill all terms in the expansion in homogeneous (in ξ) terms. The following identity, which we can obtain by a direct calculation, helps with that:

$$e^{-i\phi}\Box e^{i\phi} = -(\partial_t\phi)^2 + c^2 g^{jk}(\partial_j\phi)(\partial_k\phi) + i\Box\phi + 2i\Big((\partial_t\phi)\partial_t - c^2 g^{jk}\partial_j\phi\partial_k\Big) + \Box, \tag{2.3}$$

where $\Box = \partial_t^2 - c^2\Delta_g$ and $e^{\pm i\phi}$ stand for operators of multiplication.

Equating the terms of order 2 yields the *eikonal equation*

$$(\partial_t\phi)^2 - c^2(x)|\nabla_x\phi|_g^2 = 0. \tag{2.4}$$

Write $f_j = (2\pi)^{-n}\int e^{\mathrm{i}x\cdot\xi}\hat f_j(\xi)\,\mathrm{d}\xi$, $j=1,2$, to get the following initial conditions for $\phi_\pm$

$$\phi_\pm|_{t=0} = x\cdot\xi. \tag{2.5}$$

Equate now the order 1 terms in the expansion of $(\partial_t^2 - c^2\Delta)u$ to get that the principal terms of the amplitudes must solve the *transport equation*

$$\big((\partial_t\phi_\pm)\partial_t - c^2 g^{ij}(\partial_i\phi_\pm)\partial_j + C_\pm\big)\, a_{j,\pm}^{(0)} = 0, \tag{2.6}$$

with

$$2C_\pm = (\partial_t^2 - c^2\Delta_g)\phi_\pm. \tag{2.7}$$

Equating terms homogeneous in ξ of lower order we get transport equations for $a_{j,\sigma}^{(k)}$, $j=1,2,\dots$ with the same left-hand side as in (2.6) with a right-hand side determined by $a_{k,\sigma}^{(k-1)}$.

We will solve the eikonal equation first and will return to this construction later. Note that if we have the general differential operator with a principal symbol $p(x,\xi)$, the eikonal equation would be

$$p(x,\nabla\phi) = 0.$$

This is also known as (one of the forms of) the Hamilton-Jacobi equation.

3. Solving the eikonal equation

3.1. **Method of characteristics.** Let $H(x,\xi)$ be a smooth Hamiltonian (so far just a smooth function) of $(x,\xi)\in \mathbf{R}^n\times\mathbf{R}^n$. We want to solve the Hamilton-Jacobi equation (eikonal) with prescribed values on some hypersurface S

$$H(x,\nabla\phi)=0,\quad \phi|_S=\phi_0. \tag{3.1}$$

We work locally near some $x_0\in S$.

There are several versions of that equation which are equivalent to (3.1). The most common form is

$$\partial_t\phi+G(x,\nabla_x\phi)=0. \tag{3.2}$$

This form appears in geometric optics constructions for hyperbolic PDEs. Add one more variable to x by writing $x_0=t$, and also add a zeroth ξ variable, and denote $x=(x_0,x')$, $\xi=(\xi^0,\xi')$. Set $H(x,\xi)=\xi_0+G(x',\xi')$. Then (3.2) reduces to (3.1). In this case, S is assumed to be transversal to ∂_t.

Another version is

$$G(x,\nabla\phi)=1. \tag{3.3}$$

This is the eikonal equation for Hemholtz type of PDEs with a large (or small) parameter. For example, $g^{ij}(x)\phi_{x^i}\phi_{x^j}=1$, with g a Riemannian metric. Write $H=G-1$ to get (3.1).

To solve (3.1), we use the method of characteristics, as in Evans' book. He describes how to solve the more general equation $H(x,\nabla\phi,\phi)=0$. In our case, there are some (small) simplifications.

The idea is to find some curves (characteristics) $x(s)$ so that we can easy compute ϕ restricted to those characteristics: $\phi(x(s))$. Those curves would hit S, where we have an initial value ϕ_0. They would depend on an additional $n-1$ dimensional parameter, for example on the point of intersection with S.

The form of (3.1) suggests that those curves might be the Hamiltonian curves for H, i.e., the solutions of

$$\dot x=H_\xi,\quad \dot\xi=-H_x. \tag{3.4}$$

Then (3.1) would just reflect the fact that H is constant along those curves if we can arrange

$$\xi(t)=\nabla\phi(x(t)) \tag{3.5}$$

(meaning $\nabla\phi$ evaluated at $x=x(t)$). This is just a wild guess, and let us see if we can make it happen. Assume for a moment that (3.1) is locally solvable. Let $x(s)$ be some smooth curve and take (3.5) as a definition of ξ without assuming that it has anything to do with (3.4). Differentiate w.r.t. t to get

$$\dot\xi_i(t)=\phi_{x^ix^j}(x(t))\dot x^j. \tag{3.6}$$

We have unpleasant second derivatives here and to deal with them, differentiate (3.1) to get

$$H_{x^i}+H_{\xi_j}\phi_{x^jx^i}=0,\quad\forall i. \tag{3.7}$$

Comparing the last two equations, we see that it is very convenient to take $\dot x=H_\xi$ to make the expressions involving the second derivatives the same. Once we have done that, the last equation becomes $\dot\xi(t)=-H_x(x(t),\xi(t))$. Those are exactly the Hamiltonian equations (3.4).

So know now that (3.5) holds over every Hamiltonian curve $(x(t),\xi(t))$, for every possible solution ϕ. How to find ϕ? Set $z(t)=\phi(x(t))$. Then

$$\dot z(t)=\nabla\phi\cdot\dot x(t)=\xi(t)\cdot H_\xi(x(t),\xi(t)). \tag{3.8}$$

This an ODE along the characteristics than can be solved using the value of u at the point of intersection of $x(t)$ with S as an initial value.

We can go back now. Solve

$$\dot{z} = \xi \cdot H_\xi \tag{3.9}$$

along the Hamiltonian curves and set $\phi(x(t)) = z(t)$. Define $\xi(t)$ by (3.5). Differentiate to get (3.6) which we can also write as $-H_{x^i} = \dot{\xi}_i = \phi_{x^i x^j} H_{\xi_j}$. Than implies

$$\frac{\mathrm{d}}{\mathrm{d}t} H(x(t), \nabla\phi(x(t))) = 0.$$

If we can arrange (3.1) on S, then we have it along $x(t)$ as well. This brings us to the first thing we should have done: solve (3.1) on S. This is an algebraic equation: we know the tangential gradient $\nabla'\phi = \nabla'\phi_0$ on S because we can differentiate the boundary condition. Then we use (3.1) restricted to S. This equation may have more than one solution smooth near x_0 (or no solutions at all). Assume that it has a smooth solution locally (which means a non-characteristic problem) and fix it. Then $(x, \xi) = (x, \nabla\phi(x))$ is a covector filed with base points on S. Take the Hamiltonian curves with initial points on S and initial conditions given by that field. Solve (3.9) over them with initial conditions as in (3.1). Then this is our solution (defined locally near x_0).

Notice that the Hamiltonian curves preserve the level set of H. Since (3.1) is at the zero level, we need to work with the zero level curves only. Those are called zero bicharacteristics. Note that this solution is defined in the coordinates (x', s), where $x' \in S$ and s is the parameter along the characteristic curves. It is valid as long as $(x', s) \mapsto x$ is a diffeomorphism. In general, that is true in some neighborhood of x_0 only.

Finally, if H is homogeneous of degree m in ξ, then $\xi \cdot H_\xi = mH$.

This gives us the following recipe for solving (3.1) locally near x_0, see also Figure 1.

- Find $\nabla\phi$ on S when S is non-characteristic (for the given ϕ_0). Fix a locally smooth solution, if more than one. We get a covector field $(x', \xi(x'))$, with $x' \in S$ and $\xi(x') = \nabla\phi(x')$.
- Solve the Hamiltonian system (3.4) with initial conditions given by the covector field above. Than defines the bicharacteristics $(x(s), \xi(s))$. They lie on the zero energy level $H = 0$.
- Solve $\dot{z} = \xi \cdot H_\xi$ along the characteristics $x = x(s)$ with initial condition $z(0) = \phi(x', 0)$ in the coordinates (x', s). Then $\phi = z$ in those coordinates.
- Pass to the original coordinates x to get ϕ.

It is useful to remember (3.5). We will use it essentially later.

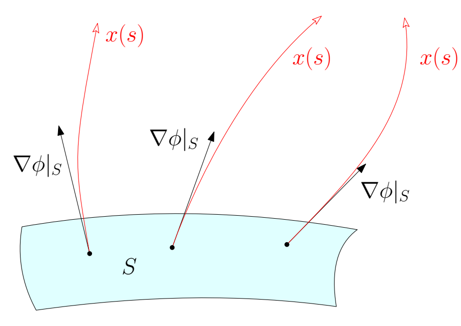

FIGURE 1. Solving the Hamilton-Jacobi equation. We solve $\dot{z} = \xi \cdot H_\xi$ along each characteristic.

3.2. **Special Cases.**

3.2.1. *The Helmholtz equation.* If we are looking for a solution of the Helmholtz equation

$$(-\Delta_g - \lambda^2)u = 0$$

of the form $u = e^{\mathrm{i}\lambda\phi(x)}a(x,\lambda)$ with $a(x,\lambda)$ having an asymptotic expansion in λ, one gets the following eikonal equation

$$g^{ij}\phi_{x^i}\phi_{x^j} = 1, \quad \phi|_S = \phi_0. \tag{3.10}$$

If we have $c^2\Delta$ instead, we have to replace g by $c^{-2}g$. The eikonal equation can be written as $|\nabla\phi|_g^2 = 1$, where $|\cdot|_g$ is the norm of the covector $\mathrm{d}\phi$ at x. The Hamiltonian then is $H(x,\xi) = \frac{1}{2}\left(g^{ij}(x)\xi_i\xi_j - 1\right)$. The factor $1/2$ is just for convenience. We can also say that the Hamiltonian is $H(x,\xi) = \frac{1}{2}g^{ij}(x)\xi_i\xi_j$ and we work on the energy level $H = 1/2$. Then (3.9) takes the form $\dot z = 1$ because $\xi\cdot H_\xi = g^{ij}\xi_i\xi_j = 1$. Then along the characteristics, $\phi = \phi_0 + s$.

The Hamiltonian system take the form

$$\dot x^i = g^{ij}\xi_j, \quad \dot\xi^i = -\frac{1}{2}(\partial_{x^i}g^{kl})\xi_k\xi_l. \tag{3.11}$$

Equation (3.10) implies that those curves lie on the zero energy level of H. They are called zero (null) bicharacteristics of H. It is well known that $x(s) = \gamma(s)$ are then geodesics in the metric g with $\xi_i(s) = g_{ij}\dot\gamma^j(s)$ being the tangent vectors associated to covectors by the metric. They are actually unit geodesics because $H = 0$.

So the solution can be described like this. Solve the eikonal equation at S first, which is equivalent to finding the normal derivative $\nabla_\nu\phi$ on S. It is known that there exist local semigeodesic coordinates in which $S = x^n = 0$, x^n is the signed distance to S (in the metric) in a choosen orientation, and the metric takes the form $g_{\alpha\beta}\mathrm{d}x^\alpha\mathrm{d}x^\beta + (\mathrm{d}x^n)^2$, where α,β take values from 1 to $n-1$, and ∂_{x^n}. The eikonal equation then takes the form

$$g^{\alpha\beta}\phi_{x^\alpha}\phi_{x^\beta} + \phi_{x^n}^2 = 1, \quad \phi|_{x^n=0} = \phi_0(x').$$

On S, we have either two solutions for ϕ_{x^n}, if $g^{\alpha\beta}\phi_{x^\alpha}\phi_{x^\beta}|_S < 1$; no solutions if we have the $>$ inequality, and one if you have equality but in this case, the type may change from point to point. Let us say that we have the first case. Choose one of those solutions in a smooth way near x_0, for example by requiring $\phi_{x^n} > 0$. Then $\nabla\phi$ determines the covector field $\xi = \nabla\phi|_S$, see (3.5) with base points on S. Note that this covector field is unit (in the metric $\{g^{ij}\}$ on $T^*\mathbf{R}^n$) because the eikonal equation says just that! Then we issue bicharacteristics from such (x,ξ). In the (x',s)

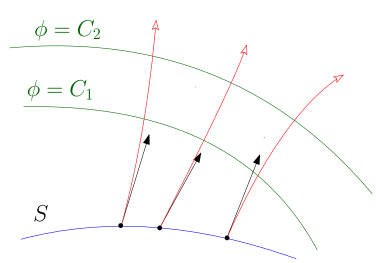

FIGURE 2. Solving $|\nabla\phi|_g^2 = 1$. The characteristics are unit speed geodesics. The solution increases as the arc length (in the metric) along each characteristic. The level hypersurfaces $\{\phi = \text{const.}\}$ are perpendicular (in the metric) to the characteristics. Note that there is another local solution with characteristics going down, like a "distorted mirror image" of the top ones.

coordinates, $\phi = \phi_0(x') + s$. Physically, this corresponds to a wavefront moving away from S in the positive ($x^n > 0$) direction. We can call it outgoing. If we choose the other solution on S, with $\phi_{x^n} < 0$, we get a wave coming to S, we can call it incoming.

Equality (3.5) in this case has a nice geometric meaning. By the Hamiltonian equations (3.11), $\xi = g\dot{x}$ along the bicharacteristics. Then we get

$$\dot{x} = g^{-1}\nabla\phi \tag{3.12}$$

along the bicharacteristics. The expression on the right is the covector field $\nabla\phi$ identified with a vector one by the metric. In Riemannian geometry, this is called $\operatorname{grad}\phi$. We thus get $\dot{x} = \operatorname{grad}\phi$. Therefore, the characteristics $x(t)$ are integral curves of teh gradient field $\operatorname{grad}\phi$ and in particular, they are perpendicular (as always, in the metric) to the level hypersurfaces of ϕ.

A special case is when g is Euclidean. We do not get an explicit solution for general S. When $S = \{x^n = 0\}$, and try to solve

$$|\nabla\phi|^2 = 1, \quad \phi|_{x^n=0} = x \cdot \xi',$$

we get explicit the solutions $\phi_\pm = x \cdot \xi_\pm$, where $\xi_\pm = (\xi', \pm\sqrt{1-|\xi'|^2})$ if $|\xi'| < 1$.

As we saw above, $\phi = x \cdot \omega$ solves the eikonal equation (without worrying about boundary conditions) for every unit ω. The corresponding solutions $u = e^{\mathrm{i}\lambda x\cdot\omega}$ are called harmonic plane waves "propagating" in the direction ω. Another special solution is $\phi = |x|$ for $x \neq 0$. Then we get $u = e^{\mathrm{i}\lambda|x|}a$ and it turns out that $a = 1/|x|$ works as an amplitude, so $u = e^{\mathrm{i}\lambda|x|}/|x|$ is a solution for $x \neq 0$. Also, $u = e^{-\mathrm{i}\lambda|x|}/|x|$ is a solution. In the whole space, up to a constant, they are actually fundamental solutions of the Helmholtz operator. They are called spherical waves. They satisfy a constant boundary condition on any sphere centered at the origin.

3.2.2. *The wave equation.* The eikonal equation for the wave equation

$$(\partial_t^2 - \Delta_g)u = 0$$

is

$$(\partial_t\phi)^2 - g^{ij}\phi_{x^i}\phi_{x^j} = 0, \quad \phi|_S = \phi_0. \tag{3.13}$$

If g is euclidean, it is just $\phi_t^2 - |\nabla_x\phi|^2 = 0$. Even in the metric case, it still looks like that but $|\cdot|$ is the norm of a covector now.

If g is Euclidean, obvious solutions are $\phi_\pm = \mp t|\xi| + x \cdot \xi$, see (1.6). They correspond to solutions $u_\pm = e^{\mathrm{i}\lambda(\mp t|\xi|+x\cdot\xi)}$; or if we combine λ and $|\xi|$, we get $u_\pm = e^{\mathrm{i}(\mp t+x\cdot\omega)}$. Those are the terms we got in (1.5), (1.6). Taking Fourier transform w.r.t. λ gives us solutions again: $w = \delta(\pm t + x \cdot \omega)$ for $|\omega| = 1$ (the factor $|\xi|$ is irrelevant). Both solutions are called plane waves, and the delta type one is propagating in the direction $\pm\omega$.

In the general case, we apply the scheme above. Think of the t variable as another x one, and let τ be its dual one. The Hamiltonian is

$$H(t, x, \tau, \xi) = \frac{1}{2}\left(\tau^2 - |\xi|_g^2\right),$$

where $|\xi|_g^2 = g^{ij}(x)\xi_i\xi_j$. The Hamiltonian system then takes the form

$$\dot{t} = \tau, \quad \dot{x} = -g^{-1}\xi, \quad \dot{\tau} = 0, \quad \dot{\xi} = \nabla_x|\xi|_g^2/2. \tag{3.14}$$

This system can be decoupled. The second and the fourth equation coincide with (3.11); hence the (x, ξ) curves (no t and τ) are the bicharecteristics we had above and $x(s)$ are geodesics. Since τ is constant by the third equation, we also get $t = s\tau$, where s is the parameter along the Hamiltonian curves. Since we work on the energy level $H = 0$, we have $|\tau| = |\xi|_g = \text{const}$. The speed $|\dot{x}|_g$ equals $|\tau| = \text{const}$. If we use t as a parameter along the bicharacteristics, we would get

$$\frac{\mathrm{d}}{\mathrm{d}t}x = \mp g^{-1}\xi/|\xi|_g, \quad \frac{\mathrm{d}}{\mathrm{d}t}\xi/|\xi|_g = \pm(\partial_{x^i}g^{jk})\xi_j\xi_k/|\xi|_g^2, \tag{3.15}$$

where the sign depends on the sign of τ. Those curves are just the unit speed geodesics (after the identification of vectors and covectors) in the phase space.

To solve the eikonal equation now, we solve (3.13) at S first. For this to be possible, we want S to be non-characteristic, i.e., its conormal $\nu = (\nu_t, \nu_x)$ (at the point (t_0, x_0) near which we work) to satisfy $\nu_t^2 \neq |\nu_x|_g^2$. In terms of Lorentzian geometry, that means that ν is non lightlike one for the metric $\mathrm{d}t^2 - g_{ij}\mathrm{d}x^i\mathrm{d}x^j$. If $\nu_t^2 > |\nu_x|_g^2$, the conormal ν is called timelike; if $\nu_t^2 < |\nu_x|_g^2$, it is called spacelike. Then we call S spacelike/timelike, as well. A typical example of a spacelike S is $\{t = 0\}$. A typical example of a timelike one is the cylinder $\{(t, x);\ x \in S_0\}$, where S_0 is a hypersurface in $\mathbf{R}^n_x$.

Notice that the eikonal equation (3.13) says that $\nabla\phi$ must be lightlike. To solve (3.13) at S, assume first that S is spacelike. The problem can then be described in the following way: we are given $\nabla'\phi_0$ (the tangential gradient on S) at a point on S near a fixed one. Is there a lightlike covector which projects to $\nabla'\phi_0$? It is easy to see that there are actually two of them. For example, if $S = \{t = 0\}$, we have $\nabla'\phi_0 = \nabla_x\phi_0$, and we have $\partial_t\phi = \pm|\nabla_x\phi_0|_g$ on S. We may declare one of then future pointing, and the other past pointing. The general case can be reduced to this one by a suitable Lorentzian "rotation" around (t_0, x_0).

If S is a cylinder, for example, we are given $\nabla'\phi_0 = (\partial_t\phi_0, \nabla'_x\phi_0)$, where $\partial'_x\phi_0$ is the tangential derivative on S_0. A lightlike covector with such a projection may not exist however. Indeed, we need to determine $\partial_{x^n}\phi$ from the equation $(\partial_t\phi)^2 = (\nabla'_x\phi)_g^2 + (\partial_{x^n}\phi)^2$ in boundary normal coordinates but that is possible only if $(\partial_t\phi)^2 > (\nabla'_x\phi)_g^2$ (we want to avoid an equality). In other words, $\nabla'\phi_0|_S$ must be timelike (on S)! To say that in a different way, projections of lighlike vectors on S (which is timelike now) must be timelike. The general case (S timelike but not a cylinder can be reduced to this one after some transformation.

After we are done with the boundary determination, we have a covector field $(x, \nabla\phi)$ on S. We have to solve (3.9) now with initial conditions that field. In (3.9), we replace ξ by (τ, ξ). We have a homogeneous Hamiltonian now, and on the zero energy level, the right hand side of (3.9) vanishes! So ϕ is just constant along the characteristics (geodesics) $x(s)$ issued from points on S with initial directions $\dot{x} = g^{-1}\xi$ on S, where ξ is computed in the first step. Note that if $\gamma(s)$ is a geodesic in the x-space with speed $|\dot{\gamma}| = c$, we think of $(\pm ct, \gamma(t))$ as a geodesic in timespace.

The analog of (3.12) now is (by (3.5) and (3.14))

$$(\partial_t\phi, \nabla_x\phi) = (\tau, \xi) = (\dot{t}, -g\dot{x}), \tag{3.16}$$

along the bicharacteristics, and of course, $\dot{t} = \tau = \pm|\xi|_g = \text{const}$. In particular, if g is Euclidean, and $S = \{t = 0\}$, and $\phi|_S = x \cdot \xi$, we first determine $\partial_t\phi|_S = \pm|\xi|$, which gives us $\nabla\phi|_S = (\pm|\xi|, \xi)$. Then ϕ is constant along the lines $(t, x) = (|\xi|s, x_0 \pm s\xi)$ (with tangent vectors $(|\xi|, \pm\xi)$) with an initial condition $\phi_0 = x_0 \cdot \xi$ for $s = 0$. Since $s = t/|\xi|$, $x = x_0 \pm s\xi$, we get

$$\phi_\pm = (x \mp t\xi/|\xi|) \cdot \xi = \mp t|\xi| + x \cdot \xi.$$

Those are the phase functions we considered above.

A special type of solutions of (3.13) is

$$\phi(t, x) = t \mp \psi(x) \tag{3.17}$$

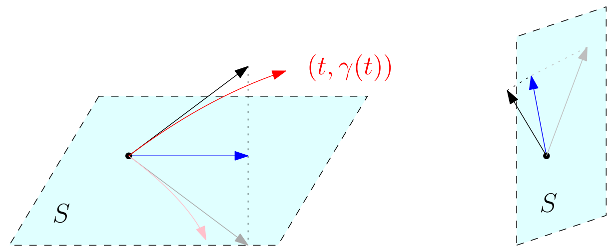

FIGURE 3. Solving $(\partial_t\phi)^2 - |\nabla\phi|_g^2 = 0$. The characteristics are geodesics $(t,\gamma(t))$ in time space. The solution is constant along each characteristic; therefore equal to its value ϕ_0 on S. When S is spacelike (left), there are always two lighlike (co)vectors with a given projections; i.e., $\nabla\phi|_S$ has two locally smooth solutions. When S is timelike (right), there might be two, one or no such solution depending on whether $\partial'\phi|_S$ is timelike/ lightlike/ spacelike.

(possibly multiplied by a constant), where ψ solves (3.10). This observation does not make it easier to solve the problem with boundary data $\phi = \phi_0$ on $S = \{t = 0\}$, for example, because then ψ must be equal to $\mp\phi_0$ but then ϕ_0 must solve (3.10). Therefore, ϕ_0 must belong to some restricted class. When g is Euclidean, there is no problem with $\phi_0 = x\cdot\xi$. We can divide it by $|\xi$ and then the new $\phi_0 = x\cdot\xi/|\xi|$ has a unit gradient. Then $\phi = |\xi|^{-1}(t\mp x\cdot\xi/|\xi|)$ solves (3.13) and those are the solutions we have seen above. But when g is not Eucludean, $\phi_0 = x\cdot\xi/|\xi|$ does not solve (3.10) anymore and the solution of (3.13) cannot be written in the form (3.17) anymore.

3.2.3. *The general Lorentzian case.* Let g be a Lorentzian metric in the "timespace" $\mathbf{R}^{1+n}$ with a signature $(-,+,\dots,+)$. An example is the Minkowski metric $= -\mathrm{d}t^2 + (dx^1)^2 + \dots + (dx^n)^2$, where we think of t as x^0. The corresponding wave equation takes the form

$$|\det g|^{-1/2}\partial_{x^i}|\det g|^{1/2}g^{ij}\partial_{x^j}u = 0.$$

Of course, we can cancel the term on the left but written this way, the operator $\square_g$ applied to u is an invariantly defined hyperbolic operator, formally looking as the Laplace-Beltrami operator in the Riemannian case. Looking for solutions of the kind (2.1), with (t,x) replaced by x there, we arrive at the eikonal equation

$$g^{ij}\phi_{x^i}\phi_{x^j} = 0,$$

which looks like (3.10) but not quite because there is zero on the right. It is actually a generalization of (3.13). Indeed, the wave equation in section 3.2.2 corresponds to the Lorentzian metric

$$-\mathrm{d}t^2 + g_{ij}(x)\mathrm{d}x^i\mathrm{d}x^j,$$

hence it is a special case of the more general Lorentzian case. Without going into details, let us just mention that the analysis is not so different than the one in section 3.2.2. In particular, the characteristics are null geodesics (satisfying $g_{ij}\dot\gamma^i\dot\gamma^j = 0$) and the phase is constant along them.

3.3. **Caustics.** As mentioned above, the eikonal equations are solvable in general locally only. We demonstrate below that indeed, the eikonal equation may not be solvable globally. Those effects are known as caustics.

Let us start with a simple example: solve $|\nabla\phi|^2 = 1$ with boundary data $\phi = 1$ on the unit sphere S. One solution is $\phi = |x|$ (another one is $\phi = 2 - |x|$). We could have started with $\phi = C$ for any fixed constant C, including $C = 0$. The characteristics are easy to compute: they are the radial rays $r = s$ in polar coordinates. Reparameterize them as $r = 1 - s$ so that they start from S and move to the origin. Then $|x| = 1 - s$ would be the level surfaces of ϕ up to $s = 1$, corresponding to $x = 0$. We cannot extend the solution further in a smooth way. Even if S is a part of the unit sphere only; we would have the same problem. The wave fronts develop a caustic at the origin.

Note that nothing wrong happens with the characteristics. They meet at $x = 0$ and keep going. There are no caustics in the phase space. The problem is with the phase function only which is used to represent a particular solution of the wave equation as in (2.1), if we think of ϕ as the spatial component of a time dependent phase function $t - |x|$.

A boundary value problem for the wave equation related to this example would be to solve the constant coefficients wave equation $(\partial_t^2 - \Delta)u = 0$ in the cylinder $\mathbf{R} \times B$, where B is the unit ball with boundary the unit sphere S as above. It is easy to see that $u = \delta(t - |x|)/|x|$ is a local solution for $|t| \ll 1$ and therefore for x close to S. The wave front develops a caustic at the origin at $t = 1$. A distribution valued solution still exists beyond $t = 1$ by the PDE theory, but it is not given by that formula. Physically, we have waves focused at the origin.

Other types of caustics are shown in Figure 4. The curves which are supposed to be the level surfaces of ϕ are either non-smooth (with a vertex) or they self-intersect or they may intersect other such surfaces (which is a contradiction). If a non-smooth curve (hypersurface) is a level set of ϕ, than at the non-smooth points, $\nabla\phi = 0$. This is a contradiction with the eikonal equation.

Note that in all those cases, one can construct a smooth solution having the representation (2.1) with f_1 and f_2 in C_0^∞ up to a time $t = T$ before the caustics are formed; and then solve a Cauchy problem with data at $t = T$. Everything will be smooth, even the extension for $t > T$. On the other hand, (2.1) would not be valid anymore. One can have (f_1, f_2) as a distribution in $\mathcal{E}'$, and again, (2.1) will be valid up to the formation of the caustics but the solution exists globally because the wave equation is solvable for every distribution Cauchy data. At the caustics however, it may get a bit more singular. We will return to this question later because right now, we are talking about the eikonal equation only.

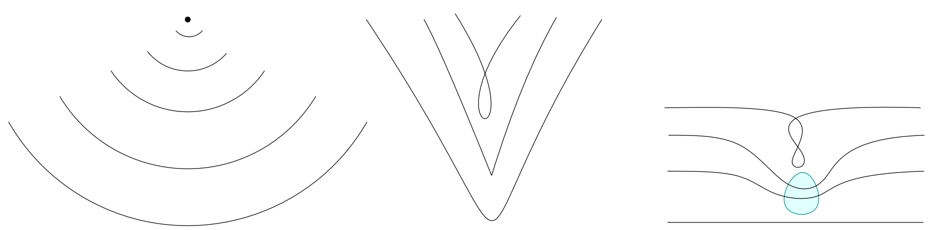

Figure 4. Caustics. Left: a wavefront focusing at a point; Euclidean geometry. Center: cusp that can form even if the geometry is Euclidean. Right: a wave moving upward with a slow region in the center.

4. Solving the transport equations

Let us now solve the transport equations that would allow us to find the amplitudes in (2.1). We are back to the case with c not trivial. Remember that in the previous section, we replaced $c^{-2}g$ by g. Denote

$$L_\pm := (\partial_t\phi_\pm)\partial_t - c^2 g^{ij}(\partial_i\phi_\pm)\partial_j + C_\pm \tag{4.1}$$

$2C_\pm = (\partial_t^2 - c^2\Delta_g)\phi_\pm$. The transport equations for the principal terms $a_{j,\pm}^{(0)}$ take the following form, see (2.6) and the text after it

$$L_\pm a_{j,\pm}^{(0)} = 0, \quad j = 1, 2. \tag{4.2}$$

For the next terms, we get

$$L_\pm a^{(k)}_{j,\pm} = b^{(k)}_{j,\pm}, \quad j = 1, 2, \tag{4.3}$$

where the expressions on the right have been computed in the previous steps, i.e., they depend on the phase functions and on $a^{(l)}_{j,\pm}$ with $l \le k-1$.

The operator $L_\pm$ is a linear first order partial differential operator with real coefficients. Any such operator is of the type $a(x)\cdot\nabla_x$ (here, x is (t,x)) and is a derivative along the characteristic curves of the vector field a solving $\dot{x} = a$. In our case, the vector field is of a gradient type: $(\partial_t\phi, -c^2g^{-1}\nabla_x\phi)$ with $\phi = \phi_\pm$ By (3.16), the latter is $(\dot{t}, \dot{x})$ along the characteristics of the wave operator we used in the previous section, which, a priori, may not be same as those of L but this fact shows that they actually are. But then the integral curves of that field are the wave characteristics themselves by (3.14). Therefore, the transport equations (4.2), (4.3) are just linear ODEs along the characteristics. As we showed above, we can take t as a parameter along those characteristics.

The initial conditions in (1.1) provide initial conditions for the transport equations. We get

$$a_{1,+} + a_{1,-} = 1, \quad a_{2,+} + a_{2,-} = 0 \quad \text{for } t = 0.$$

This is true in particular for the leading terms $a^{(0)}_{1,\pm}$ and $a^{(0)}_{2,\pm}$. Since $\partial_t\phi_\pm = \mp c(x)|\xi|_g$ for $t = 0$, and $u_t = f_2$ for $t = 0$, from the leading order term in the expansion of u_t we get

$$a^{(0)}_{1,+} = a^{(0)}_{1,-}, \quad \mathrm{i}c(a^{(0)}_{2,-} - a^{(0)}_{2,+}) = 1 \quad \text{for } t = 0.$$

Therefore,

$$a^{(0)}_{1,+} = a^{(0)}_{1,-} = \frac{1}{2}, \quad a^{(0)}_{2,+} = -a^{(0)}_{2,-} = \frac{\mathrm{i}}{2c(x)} \quad \text{for } t = 0. \tag{4.4}$$

Note that if $c = 1$, then $\phi_\pm = x\cdot\xi \mp t|\xi|$, and $a_{1,+} = a_{1,-} = 1/2$, $a_{2,+} = -a_{2,-} = \mathrm{i}/2$. Using those initial conditions, we solve the transport equations for $a^{(0)}_{1,\pm}$ and $a^{(0)}_{2,\pm}$. Similarly, we derive initial conditions for the lower order terms in (2.2) and solve the corresponding transport equations. Then we define $a_{j,\sigma}$ by (2.2) as a symbol.

4.1. **Other phase functions.** The parametrix (2.1) consists of terms of the type

$$\begin{aligned} u_0(t,x) &= (2\pi)^{-n}\int e^{\mathrm{i}\phi(t,x,\xi)}a(t,x,\xi)\hat{f}(\xi)\,\mathrm{d}\xi \\ &= (2\pi)^{-n}\iint e^{\mathrm{i}(\phi(t,x,\xi)-x\cdot\xi)}a(t,x,\xi)f(y)\,\mathrm{d}\xi\,\mathrm{d}y. \end{aligned} \tag{4.5}$$

It is not terribly important that we had $x\cdot\xi$ there. Some non-degeneracy conditions would be needed eventually. Such a phase is convenient because we can use the Fourier inversion formula and this simplifies the initial conditions for the transport equations. From Riemannian geometry point of view, that is a strange choice however because it is not invariant under change of variables (which would result in a much worse looking phase). For the purpose of the propagation of the singularities analysis however, it works. We work with an invariant choice in local coordinates but we will get an invariantly formulated result eventually.

Other choices are possible. Let us look for a solution involving terms of the kind

$$u_0(t,x) = (2\pi)^{-n}\iint e^{\mathrm{i}(\phi(t,x,\theta)-\psi(x,\theta))}a(t,x,\theta)f(y)\,\mathrm{d}y\,\mathrm{d}\theta \tag{4.6}$$

with some $\psi(x,\theta)$ positively homogeneous of degree one in θ. We could allow the amplitude a to depend on y, as well. The eikonal equation for ϕ remains the same, (2.4). The transport equations (4.2), (4.3) remain the same with L given by (4.1). At $t=0$, we have

$$u_0(0,x) = (2\pi)^{-n} \iint e^{\mathrm{i}(\phi(0,x,\theta)-\psi(y,\theta))} a(0,x,\theta) f(y)\,\mathrm{d}y\,\mathrm{d}\theta.$$

This imposes new initial conditions for the eikonal equations and the tramsport equations however. One convenient choice of the initial conditions for the former, compare to (2.5), is to take

$$\phi_\pm|_{t=0} = \psi(x,\theta). \tag{4.7}$$

This is not a problem because we have the Jacobi-Hamilton theory for general initial condition.

With that choice of the phase, wee need

5. Justification of the parametrix

Let us see what we have done so far. For every $(x_0,\xi^0)\in T^*\mathbf{R}^n\setminus 0$, we showed that one can solve the eikonal and the transport equations near x_0 and for $\xi=\xi^0$. By continuity, one can do that for ξ in some conic neighborhood of ξ^0.

5.1. The wave equation in $\mathbf{R}^n$ with Cauchy data at $t=0$.

Assume now that $(f_1,f_2)\in\mathcal{E}'(\mathbf{R}^n)$. By compactness, we can use a finite cover of open sets and a pseudo-differential partition of unity associated with it to construct, as in (2.1), a parametric $u_0(t,x)$, defined for $|t|\ll 1$, so that

$$(\partial_t^2 - c^2\Delta_g)u_0 \in C_0^\infty\times C_0^\infty, \quad u_0-u\in C_0^\infty, \quad \partial_t(u_0-u)\in C_0^\infty. \tag{5.1}$$

In fact, one can use a unique representation with phase functions $\phi_\pm$ and amplitudes defined globally in (x,ξ) (with x in a compact set) and for $|t|\ll 1$. Compactness of the support can be gained by finite speed of propagation. This can be expected. We did not really solve (2.1), we only solved it up to smooth errors.

Set $w=u-u_0$. Then w solves

$$(\partial_t^2 - c^2\Delta_g)w \in C_0^\infty\times C_0^\infty, \quad w\in C_0^\infty, \quad \partial_t w\in C_0^\infty. \tag{5.2}$$

We use a priori results for this problem now which say that w must be smooth. One way to see this is to use the system formalism (1.8). Since the initial data $\mathbf{f}$ is smooth, it belongs to the domain of $\mathbf{P}^k$ for every $k=1,2,\dots$. Then so does $\mathbf{u}(t)$, see (1.9), as it follows directly from the theory of unitary groups.

Therefore, $u-u_0\in C_0^\infty$. Thus, our parametrix is really a parametrix — it differs from the actual solution by a smooth function. We actually get more: we constructed an "approximate" solution operator $\mathbf{U}_\mathrm{p}(t)$ for $|t|\ll 1$ which differs from the actual one $\mathbf{U}(t)=e^{t\mathbf{P}}$, see (1.9), by a smoothing operator.

This construction is still not what we really want — to get a microlocal parametrix in a fixed in advance conic set. We will return to this later.

5.2. The wave equation in a domain with boundary with Cauchy data at $t=0$.

Let (f_1,f_2) be supported in a compact set $K\subset\Omega$, where Ω is a bounded domain with a smooth boundary. We want to solve (1.1) with, say, Dirichlet or Neumann boundary conditions. Existence and well posedness of such a solution follows from functional analysis arguments, as well. For that, we need to consider $\mathbf{P}$ in (1.8) with a domain determined by the boundary conditions and to show that we still get a self-adjoint operator.

By finite speed of propagation (the proof is integration by parts over a cone, see Evan's book, for example) for $|t|\ll 1$, the solution is still supported away from $\partial\Omega$ so the boundary conditions play no role (by uniqueness). The parametrix is exactly the same. Note that this does not work for initial conditions supported all the way to $\partial\Omega$. More precisely, the problem is with (f_1, f_2) which do not have the property to be smooth in some neighborhood of $\partial\Omega$.

5.3. **The wave equation with a boundary source.** Consider the mixed problem for the wave equation in Ω

$$\begin{cases} (\partial_t^2 - c^2(x)\Delta_g)u = & 0 \qquad \text{in } \mathbf{R}_t \times \Omega_x, \\ u|_{(0,T)\times\partial\Omega} = & h, \\ u|_{t=0} = & 0, \\ \partial_t u|_{t=0} = & 0. \end{cases} \tag{5.3}$$

We could impose non-zero Cauchy data (f_1, f_2) at $t = 0$ as well but we can always split the problem in two parts: one with non zero (f_1, f_2) and $h = 0$ (which we discussed above) with the other one being (5.3). Then the sum solves the general one. We assume the Dirichlet boundary condition here but the construction works for the Neumann one, as well.

An essential assumption is that h has a support not intersecting $\{t = 0\}$. It would be enough to know that it is smooth in a neighborhood of $t = 0$ and satisfies the compatibility conditions of any order at $t = 0$. In this case, that means $\partial_t^k h = 0$ for $t = 0$, $\forall k$. Then that smooth part would result in a smooth "error".

It is enough to work locally near some $(t_0, x_0) \in \mathbf{R} \times \partial\Omega$. Actually, it is enough to work microlocally, as we will see later. For h supported on near that point, we can pass to local coordinates on $\partial\Omega$ of the type (x', x^n) with $\partial\Omega = \{x^n = 0\}$ and $x^n > 0$ in Ω. It is convenient to assume that x^n is the distance to $\partial\Omega$ in the metric g. One may also take the metric $c^{-2}g$ for this purpose since we want to be able to treat the case g Euclidean and c variable without passing to general metrics, we stick to g here. Then we look for a solution of the form

$$u(t,x) = (2\pi)^{-n} \iint e^{\mathrm{i}\phi(t,x,\tau,\xi)} a(t,x,\tau,\xi)\hat{h}(\tau,\xi)\,\mathrm{d}\tau\,\mathrm{d}\xi, \tag{5.4}$$

where $\hat{h}$ is the (full) Fourier transform of $h(t, x')$ with respect to both variables. It may seem strange why we would have one phase function only and also one boundary condition only. The eikonal equations are obviously the same, and the transport equations look the same was as well but we will have different initial conditions on a different hypersurface for them. Indeed, since we want to use Fourier inversion at x^n, we will request

$$\phi|_{x^n=0} = t\tau + x' \cdot \xi'. \tag{5.5}$$

We will return to the initial conditions for the amplitude later.

5.3.1. *Constant coefficients, flat geometry.* Let g be Euclidean and $c = 1$ again with $\partial\Omega = \{x^n = 0\}$. The phase function, possibly not unique because of the square root, see also (1.7), is

$$\phi = t\tau + x' \cdot \xi' + x^n \xi_n, \quad \xi_n := \text{“}\sqrt{\tau^2 - |\xi'|^2}\text{''}$$

(there is no summation in the last expression). We put quotes because we have to find out which branches of the square roots we need to take. We see below that there are two "physical" choices of a phase, let us denote them by $\phi_\pm$. By the transport equations, the amplitude would be constant, so let us consider an ansatz of the type

$$u_\pm(t,x) = (2\pi)^{-n} \iint e^{\mathrm{i}\phi_\pm}\hat{h}(\tau,\xi)\,\mathrm{d}\tau\,\mathrm{d}\xi'. \tag{5.6}$$

If $\tau^2 - |\xi'|^2 > 0$, i.e., if (τ, ξ') is timelike, we have two smooth square roots: $\pm\sqrt{\tau^2 - |\xi'|^2}$. When $\tau^2 - |\xi'|^2 < 0$ (as spacelike (τ, ξ)), there are two pure imaginary roots. One of them: $-\mathrm{i}\sqrt{|\xi'|^2 - \tau^2}$ would give us an exponentially increasing mode and it could be compensated by $\hat{h}$ only if the latter is a priori exponentially small, which limits the h's we can put there. Even if we can do that, we will get a solution exponentially increasing with x^n. Let us say that we do not want that. So we chose the other branch which generates a term decreasing exponentially with x^n (and with the frequency):

$$e^{\mathrm{i}(t\tau + x'\cdot\xi') - x^n\sqrt{|\xi'|^2 - \tau^2}}, \quad |\tau| < |\xi'|.$$

Such a term is called an evanescent wave. It oscillates in tangential directions and decays exponentially fast with $x^n|(\tau, \xi)|$ if the direction of (τ, ξ) is kept fixed and spacelike. Without going into details, let is mention that the most damaging earthquake waves are of this type (the model is the elastic system then). Evanescent waves are used in super-resolution imaging as well because $|\xi'|$ can be arbitrarily large when the total frequency $\sqrt{\tau^2 + |\xi|^2}$ is fixed, so this gives us one way to beat the diffraction limit. The price we pay is that they live very close to $x^n = 0$.

The final case we need to consider is a neighborhood of the lightlike cone $|\tau| = |\xi'|$. Then the square root, with some fixed branches, loses smoothness but it is still continuous, and there is no problem in this particular case: flat geometry. Things change dramatically if we go to curved boundaries or non-flat metrics/non-constant speeds.

To summarize, we get two "physical" choices of phase functions:

$$\phi_\pm = \begin{cases} t\tau + x'\cdot\xi' \pm x^n\sqrt{\tau^2 - |\xi'|^2} & \text{if } |\tau| \ge |\xi'|\ , \\ t\tau + x'\cdot\xi' + \mathrm{i}x^n\sqrt{|\xi'|^2 - \tau^2} & \text{if } |\tau| < |\xi'|\ . \end{cases}$$

Note that there is no $\pm$ in the spacelike zone. We can also write $\phi_\pm = (t, x)\cdot(\tau, \xi', \xi_n^\pm)$ with

$$\xi_n^\pm = \xi_n^\pm(\tau, \xi') = \begin{cases} \pm\sqrt{\tau^2 - |\xi'|^2} & \text{if } |\tau| \ge |\xi'|\ , \\ \mathrm{i}\sqrt{|\xi'|^2 - \tau^2} & \text{if } |\tau| < |\xi'|\ . \end{cases}$$

With those choices of the phases, we can think of (5.3) as the inverse Fourier transform of something supported on the hyperplane $\xi_n = 0$. That something is $\hat{u}_\pm$, of course. There is a small catch: so far we had $x^n \ge 0$. To talk about Fourier transform, we need a distribution defined everywhere. There is no such problem when we start to microlocalize later but right now, we are trying to get away with the Fourier transform only. If $x^n < 0$, we get exponentially growing terms, as well. To avoid that, we change the sign of $\xi_n^\pm$ in the spacelike zone for $x^n < 0$.

We get

$$\hat{u}_\pm(\tau, \xi) = \hat{h}(\tau, \xi')\delta(\xi_n - \xi_n^\pm(\tau, \xi'))e^{\mathrm{i}x^n\xi_n^\pm(\tau, \xi')}.$$

So the solution operator is a Fourier multiplier (a formal ΨDO) with a singular symbol. This tells us where the wave front set (and in general, the whole Fourier transform). The "symbol" is supported on the characteristic cone $|\tau| = |\xi|$ which is not a surprise. The contribution of $\hat{h}$ restricted to the spacelike cone $|\tau| < |\xi'|$ to $\hat{u}$ is exponentially small. As we will see later, no spacelike singularities of h propagate, so our observation is consistent with that fact. On the other hand, if $\hat{f}$ is supported in a conic neighborhood of some timelike (τ, ξ), this would create a non-negligible content of $\hat{u}$ for (τ, ξ) so that the latter projected to $T^*\{x^n = 0\}$ would be either (τ, ξ', ξ_n^+) or (τ, ξ', ξ_n^-). We do not quite get the localization we want with those simple arguments yet (we will do it later). We will see then that one of those choices means singularities propagating to the future, while the other one means that they propagate to the past.

5.3.2. *Back to the general case.*

6. Towards Microlocalization: A class of Fourier Integral Operators

OK, we have the representation (2.1). What is it good for?

Obviously, the parametrix has the same wave front set as the actual solution. We will compute that wave front set now.

Take one of the terms in (2.1) only and call it u_0 as well:

$$u_0(t,x) = (2\pi)^{-n} \int e^{\mathrm{i}\phi(t,x,\xi)} a(t,x,\xi) \hat{f}(\xi)\,\mathrm{d}\xi. \tag{6.1}$$

To test for wave front set, choose a test function χ non-vanishing at some x_0, muliply by it and take the Fourier transform w.r.t. x

$$\mathcal{F}\chi u_0(t,\cdot) = (2\pi)^{-n} \int e^{\mathrm{i}(\phi(t,x,\xi)-x\cdot\xi)} \chi(x) a(t,x,\xi) \hat{f}(\xi)\,\mathrm{d}\xi\,\mathrm{d}x$$