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WEYL ASYMPTOTICS OF THE TRANSMISSION EIGENVALUES FOR A CONSTANT INDEX OF REFRACTION

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Abstract. We prove Weyl-type asymptotic formulas for the real and the complex internal transmission eigenvalues when the domain is a ball and the index of refraction is constant.

1. **Introduction.** The purpose of this paper is to prove a Weyl-type asymptotic formulas for the counting function of the real Interior Transmission Eigenvalues (ITEs) for a ball and a constant index of refraction. Let $\Omega\subset\mathbb{R}^n$ be an open bounded domain with smooth boundary. Let $m>0$ be a smooth function in $\bar\Omega$. The Interior Transmission problem is given by the following system

$$\begin{cases}(-\Delta-\lambda^2 m)v=0,\\ (-\Delta-\lambda^2)u=0,\\ u|_{\partial\Omega}=v|_{\partial\Omega},\ \partial_\nu u|_{\partial\Omega}=\partial_\nu v|_{\partial\Omega},\end{cases}\tag{1}$$

where ν is the exterior unit normal. Any $\lambda\neq 0$ for which there exist non-zero $u,v\in H^2(\Omega)$ satisfying (1) is called an Interior Transmission Eigenvalue (ITE). In the literature, λ^2 is sometimes replaced by λ. We call the corresponding pair (u,v) an Interior Transmission Eigenpair. The requirement that u and v be both non-zero is unambiguous; if either u or v is identically zero, by unique continuation, the other one also vanishes. A generalization of (1) is obtained by replacing the first equation with a possibly anisotropic Helmholtz-type equation:

$$\begin{cases}-\Delta u(x)-\lambda^2 u(x)=0, & x\in\Omega,\\ -\nabla A(x)\nabla v-\lambda^2 m(x)v=0, & x\in\Omega,\\ u(x)-v(x)=0, & x\in\partial\Omega,\\ \frac{\partial u}{\partial\nu}-\nu\cdot A(x)\nabla v=0, & x\in\partial\Omega.\end{cases}\tag{2}$$

Here $A(x)=\{a_{ij}(x)\}$ is a smooth real symmetric invertible matrix.

ITEs were first studied in 1986 by Kirsch [12], and in 1988 in the context of the inverse scattering problem by Colton and Monk [6]. They were shown to correspond to the frequencies, for which the reconstruction algorithm in inverse scattering based on linear sampling methods breaks down, see e.g. [13], [1] and [3]. When the index of refraction m is radially symmetric, ITEs completely determines m, as shown in [2] and [20].

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In 1989, Colton, Kirsch and Päivärinta [5] showed that the set of real ITEs is at most discrete. The existence of real ITEs was first established for radially symmetric m, see e.g. [6, 20]; the radial symmetry requirement was removed in 2008 by Sylvester and Päivärinta [22], with the assumption that the contrast of the medium $m-1$ is large enough. In 2010, Cakoni et al. [4] showed that the set of real ITEs is infinite and discrete by only requiring that the contrast $m-1$ does not change sign and be bounded away from zero. In 2012, Sylvester [29] showed discreteness under an even weaker assumption that $m \neq 1$ on the boundary. ITEs for isotropic systems were also studied for L^∞ and complex valued m, see e.g. [17] and [25]; and for general elliptic differential operators, see e.g., [9] and [11]. The interior anisotropic transmission problem has been studied as well; for this setting, results on the discreteness and existence of ITEs were established in various papers by Lakshtanov and Vainberg [16, 15, 14, 17]. It should be noted that complex eigenvalues may exist, thus the ITE problem is not self-adjoint. Cakoni et al. [4] also showed the existence of complex TEs under the same assumption on m. For other results on the infiniteness of the set of complex ITEs, see e.g. results in [9], [11] and [19].

There are various recent results about the asymptotic distribution of ITEs, both for the isotropic and for the anisotropic cases. It was shown by Hitrik et al. [10] that almost all ITEs are confined to a parabolic neighborhood of the positive real axis. In 2013, Robbiano [25] gave the following upper bound

$$N(r) = \sharp \left\{ \lambda \in \mathbb{C} : \lambda \text{ is an ITE}, |\lambda| \leq r \right\},$$

$$N(r) \leq C r^{n+2}, \quad r > 1. \tag{3}$$

Here, the ITEs are counted with their geometric multiplicities (see the discussion below and in Section 2). The result was later improved by the same author [24] to an asymptotic formula

$$N(r) = \alpha\, r^n + \mathsf{o}(r^n),$$

where α agrees with the leading constant in (4) below with $A = \text{Id}$. A sharper improvement of (3) was also given by Dimassi and Petkov [8] for the counting function of the complex ITEs (counted with their geometric multiplicities, see the discussion below) in a small sector, namely

$$N(\theta, r) := \sharp \left\{ \lambda \in \mathbb{C} : \lambda \text{ is an ITE}, |\lambda| \leq r, |\arg \lambda| \leq \theta \right\}$$

is of the type

$$N(\theta, r) \leq C r^n \ , \ r \geq r_0(\theta),$$

with an explicit but not optimal constant $C = C(m)$, which is a constant multiple of the integral in (4) below with $A = \text{Id}$. An r^n/C lower estimate on the counting function for real ITEs was obtained by Serov and Sylvester [26] in 2012. There is also a more recent result on the asymptotic by Petkov and Vodev [23] in 2014.

As for the anisotropic case, an asymptotic formula was obtained by Lakshtanov and Vainberg [16] which gives

$$N(r) := \sharp\{\lambda \in \mathbb{C} : \lambda \text{ is an ITE}, |\lambda| \leq r\}$$

has the asymptotic

$$N(r) \sim \frac{\omega_n}{(2\pi)^n} r^n \int_\Omega \left[1 + \frac{m^{n/2}(x)}{\det A(x)} \right] dx, \quad r > 1. \tag{4}$$

Here, $A(x)$ is the matrix given in (2) and ω_n is the volume of the unit ball in $\mathbb{R}^n$. The above result comes with certain assumptions on A, which exclude the case $A = \mathrm{Id}$. These assumptions are needed to guarantee the ellipticity in the semiclassical sense of the problem, when microlocally restricted to the boundary (called there "parameter elliptic"). Later, the above authors [18] also obtained a lower bound of the counting function $N(r)$ of real ITEs.

In all of the above results, the ITEs are counted with a certain notion of multiplicity, which may differ from one work to another. In Section 2, we discuss the different definitions. In our main theorem below, we count the ITEs with their geometric multiplicities, defined as the dimension of the span of the eigenpairs $\{u, v\}$ corresponding to the ITE. Then we define the counting function of the *real* ITEs by

$$N_{\mathbb{R}}^{\mathrm{geom}}(r) = \#\{\lambda;\ 0 < \lambda \le r;\ \lambda \text{ is an ITE}\}.$$

We also define the algebraic multiplicity of an ITE as the order of the ITE as a zero of the determinant $F_{\nu(l)}(\lambda)$ defined in (12). The function $F_{\nu(l)}(\lambda)$ is the determinant of the system which reflects the boundary conditions and is projected to a fixed spherical harmonics eigenspace. We discover the following interesting facts: the algebraic multiplicity is always either one or three (multiplied by $\mu(l)$ if we view $F_\nu(\lambda)$ as a $F_\nu(\lambda)\,\mathrm{Id}_{\mathbb{C}^\mu}$). In the 1D model case, it is one for all ITEs if and only if $\sqrt{m}$ ($\neq 1$) is rational, see Proposition 2.2. In the higher dimensional case, this depends on how $\sqrt{m}$ relates to the zeros of the Bessel functions J_ν, see Section 2.2 and Section 4. On the other hand, the geometric multiplicity is always one in the 1D case, see Section 2.1, and equal to the dimension $\mu(l)$ of the spherical harmonics eigenspace for the corresponding momentum l in the n-dimensional case, see Section 2.2. We refer the reader to Section 2.3 for a more detailed discussion of the multiplicities of the ITEs and for the justification for our choice of the definition.

Our main result (in dimension $n \ge 2$) is as follows.

Theorem 1.1. *Let $0 < m \neq 1$ be constant, and let $\Omega \subset \mathbb{R}^n$, $n \ge 2$ be the unit ball. Then*

$$\begin{aligned} N_{\mathbb{R}}^{\mathrm{geom}}(r) &= |N_1(r) - N_\gamma(r)| + O(r^{n-1}) \\ &= (2\pi)^{-n}\omega_n^2 \left|1 - m^{n/2}\right| r^n + O(r^{n-1}). \end{aligned}$$

Here, ω_n is the volume of the unit ball in $\mathbb{R}^n$. The factor $|1 - m^{n/2}|$ is the same as that which appears in known lower bounds results of real ITEs.

Before proving Theorem (1.1), we study a model problem in dimension 1, also known as scattering on a half-line. For that problem, we give results both for the real ITEs (counted both with geometric and algebraic multiplicities) and for the complex ones (counted with their algebraic multiplicities). The geometric counting function for the real ITEs has the asymptotic type stated in Theorem 1.1; while the algebraic one (for the real ITEs) has, as the leading term, an everywhere discontinuous function of (the constant) m. These results are given in Theorem 3.1 and Theorem 3.2, respectively. The result of Theorem 3.1 is not new and can be found in the existing literature. It can be obtained by taking the contrast to be constant in the Weyl-type asymptotic formula for radially symmetric m, obtained in 1994 by McLaughlin and Polyakov [20]. It is also implicit in the paper of Sylvester [30]. As for the complex ITEs in 1D, Theorem 3.3 says that the asymptotic of the (algebraic) counting function has $1 + m^{1/2}$ as the leading constant (compared to $|1 - m^{1/2}|$ in the real geometric result). This is the same factor that appears in the currently known upper bounds.

The structure of the paper is as follows. In the next section, we give an algebraic characterization of ITEs, both in the 1D and in higher dimension, as zeros of certain functions defined in (7) and (12) respectively. In the same section, we also describe explicitly the eigenspaces for each case. Then we discuss the various notions of multiplicities used in current literature, as well as a justification for ours. In Section 3, we study the model 1D problem, the results of which were briefly summarized in the previous paragraph. Section 4 contains the proof of Theorem 1.1. The last section is devoted to a discussion of the Transmission Eigenvalues and their relation with ITEs.

2. Transmission Eigenvalues, eigenspaces and their multiplicities.

2.1. Analysis of the eigenvalues and eigenspaces — 1D. We analyze here a model problem: scattering on a half-line. In fact, the ITEs we get here are the same as in the 3D case when the angular momentum l is zero, see next section. This is the main reason why we study a half-line instead of the whole line, as in [30]. This setting is only a model problem; more complete results of this type for variable $m(x)$ can be found in [20], [7], and in [19]. Sylvester [30] studied the same problem on the whole line with m constant and got precise results about the distribution of the ITEs in that case.

We set $\gamma = \sqrt{m} > 0$, which is assumed to be constant. The 1D case we study is given by the system

$$\begin{cases} -u'' - \lambda^2 u = 0, \\ -v'' - \lambda^2\gamma^2 v = 0, \\ u(0) = v(0) = 0, \\ u(1) = v(1),\ u'(1) = v'(1) \end{cases}. \tag{5}$$

Any $\lambda \neq 0$ for which a non-trivial pair (u, v) solving that system exists, is an ITE. For the purpose of this definition, γ can be a function.

Then u and v have the form:

$$u = a \sin \lambda x, \quad v = b \sin \gamma\lambda x.$$

The boundary condition at $x = 1$ yields

$$\begin{aligned} u(1) = v(1) \quad &\Longrightarrow \quad a \sin \lambda = b \sin \gamma\lambda, \\ u'(1) = v'(1) \quad &\Longrightarrow \quad a\lambda \cos \lambda = b\lambda\gamma \cos \gamma\lambda, \end{aligned}$$

which is written as, for $\lambda \neq 0$,

$$M(\lambda) \begin{pmatrix} a \\ b \end{pmatrix} := \begin{pmatrix} \sin \lambda & -\sin \gamma\lambda \\ \cos \lambda & -\gamma \cos \lambda\gamma \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0. \tag{6}$$

The above system has a non trivial solution if the determinant of the matrix is zero, which gives us the following condition:

$$F(\lambda) := \gamma \sin \lambda \cos \lambda\gamma - \sin \lambda\gamma \cos \lambda = 0. \tag{7}$$

This implies the following.

Proposition 2.1. *The (possibly complex) ITEs are the zeros of $F(\lambda)$ away from $\lambda = 0$.*

It is easy to compute the following derivatives:

$$
\begin{aligned}
F'(\lambda) &= (1-\gamma^2)\sin\lambda\sin(\gamma\lambda),\\
F''(\lambda) &= (1-\gamma^2)\left(\cos\lambda\sin(\gamma\lambda)+\gamma\sin\lambda\cos(\gamma\lambda)\right),\\
F'''(\lambda) &= (1-\gamma^2)\Big[2\gamma\cos\lambda\cos\gamma\lambda-(1+\gamma^2)\sin\lambda\sin\gamma\lambda\Big].
\end{aligned}
\tag{8}
$$

Proposition 2.2. *Let $\gamma(1-\gamma^2)\neq 0$. Then all (possibly complex) roots of F have multiplicity one or three. If γ is irrational, then $F(\lambda)$ has single roots only. If γ is rational, then it has infinitely many roots of multiplicity three. Moreover, λ_0 is a zero of multiplicity three if and only if $\sin\lambda_0=\sin(\gamma\lambda_0)=0$ and then necessarily λ_0 is real.*

Proof. It is easy to see that $F(\lambda_0)=F'(\lambda_0)=0$ if and only if

$$\sin(\lambda_0)=0,\quad \sin(\gamma\lambda_0)=0.$$

Then for any double root λ_0,

$$F''(\lambda_0)=F'''(\lambda_0)=0,$$

but

$$F'''(\lambda_0)=\pm 2\gamma(1-\gamma^2)\neq 0.$$

Therefore, the multiplicity is either one or three.

Assume now that the multiplicity is three. Then (for $\lambda_0\neq 0$)

$$\begin{cases}F(\lambda_0)=0\\ F'(\lambda_0)=0\end{cases}\Leftrightarrow\begin{cases}\sin\lambda_0=0\\ \sin\gamma\lambda_0=0\end{cases}\Leftrightarrow\begin{cases}\lambda_0=k\pi, & k\in\mathbb{Z}\setminus\{0\}\\ \gamma\lambda_0=l\pi, & l\in\mathbb{Z}\setminus\{0\}\end{cases}$$

$$\Rightarrow \gamma=l/k\in\mathbb{Q}.$$

Therefore, multiplicity three implies that γ is rational. On the other hand, if $\gamma=p/q$, then it follows from above that $kq\pi$, k integer, are all roots of multiplicity three. ☐

Example 2.3. When $\gamma=2$, $F(\lambda)=-2\sin^3\lambda$ which has zeros of multiplicity three only. On the other hand, for $\gamma=3$, $F(\lambda)=-8\cos\lambda\sin^3\lambda$, which has both simple zeros and zeros of multiplicity three.

Then the counting function of the ITEs counted with their algebraic multiplicities (see the discussion below) has to be multiplied by 3 when $\gamma=2$; and by 2, when $\gamma=3$ (modulo $O(1)$ in the latter case).

We next discuss the eigenspaces and its relation to multiplicities. Let γ be irrational and let λ_0 be a simple ITE. Then $(\sin\lambda_0,-\sin\gamma\lambda_0)\neq(0,0)$, and by (6), $a=\sin(\gamma\lambda_0)$, $b=\sin\lambda_0$, modulo a non-zero multiplicative constant. The eigenpair is then given by

$$u=\sin(\gamma\lambda_0)\sin(\lambda_0x),\quad v=\sin\lambda_0\sin(\gamma\lambda_0x).$$

Assume now that λ_0 is a triple zero of F. $\sin\lambda_0=\sin(\gamma\lambda_0)=0$, but the matrix A in (6) still has rank one. Hence, up to a multiplication by a non-zero constant,

$$u=\gamma\cos(\gamma\lambda_0)\sin(\lambda_0x),\quad v=\cos\lambda_0\sin(\gamma\lambda_0x).$$

In this case, although the root is triple, we do not get a space of larger dimension. On the other hand, $A^{-1}(\lambda)=(\lambda-\lambda_0)^{-3}B(\lambda)$, with $B(\lambda_0)\neq 0$ (having rank one), which means the Laurent expansion of $A^{-1}(\lambda)$ at $\lambda=\lambda_0$ has its most singular order -3. When $\gamma=2$, one can compute that the residue is of rank one. Therefore, the candidate for the algebraic multiplicity, 3, is not the rank of the residue (which cannot be larger than 2 anyway), but rather the order of the most singular term. Thus, if the matrix $A(\lambda)$ is taken as our main object, our approach will never get geometric multiplicity three, while the number three is a natural choice for the algebraic multiplicity of λ_0.

Another point of view goes back to the original motivation to study ITE. They are λ's at which we cannot tell $n(x)$ from 1 by looking at Cauchy data on $\partial\Omega$. The Cauchy data for $n = 1$, and $n = \gamma$, respectively, are given by

$$\{C(\sin\lambda, \cos\lambda)\},\ \{C(\sin(\gamma\lambda), \gamma\cos(\gamma\lambda))\}.$$

For $C = 1$, the first vector is unit, and the second one is also unit in another equivalent norm dependent only on γ. The modulus of the determinant $F(\lambda)$ can then be used to measure how close those two 1D spaces are to each other.

The above discussion suggests the following definitions. The geometric multiplicity of an ITE is the dimension of the eigenspace of $A(\lambda)$ (always one). The algebraic one is the multiplicity as a root of $\det A(\lambda)$ (one or three). We will count the ITEs below with their geometric multiplicities (i.e., once). This is consistent with the choice we made in the Introduction.

2.2. **Analysis of the eigenvalues and eigenspaces — Higher dimensions.** Denote by Y_l^m, $l = 0, 1, \dots$, $m = 1, \dots \mu(l)$, an orthonormal set of spherical harmonics on S^{n-1}. They are the eigenfunctions of the Laplacian $\Delta_{S^{n-1}}$ on S^{n-1}. We have

$$-\Delta_{S^{n-1}} Y_l^m = l(l+n-2)Y_l^m, \quad l = 0, 1, \dots;\ m = 1, \dots, \mu(l),$$

where, for each l, the multiplicity of the eigenvalue $l(l+n-2)$ is given by

$$\mu(l) = \frac{2l+n-2}{n-2}\binom{l+n-3}{n-3} = \frac{2l^{n-2}}{(n-2)!}\left(1 + O(l^{-1})\right). \tag{9}$$

The functions

$$j_\nu(\lambda) := \lambda^{1-n/2} J_{l+n/2-1}(\lambda), \quad \nu := l + n/2 - 1 \tag{10}$$

are bounded at $\lambda = 0$; in fact, $J_l(\lambda) \sim c_l\lambda^l$, as $\lambda \to 0$. Any solution u of the Helmholtz equation $(-\Delta - \lambda^2)u = 0$ near 0 has the form

$$u(x) = \sum_{l=0}^{\infty}\sum_{m=1}^{\mu(l)} a_{lm} j_{l+n/2-1}(\lambda r) Y_l^m(\omega), \tag{11}$$

where $x = r\omega$ and $r > 0$, $|\omega| = 1$ are polar coordinates. Similarly, any outgoing solution at ∞ has similar expansion, with J_ν replaced by $H_\nu^{(1)}$. Any solution v of the equation $(-\Delta - \lambda^2\gamma^2)u = 0$ near 0 has the form

$$v(x) = \sum_{l=0}^{\infty}\sum_{m=1}^{\mu(l)} b_{lm} j_{l+n/2-1}(\gamma\lambda r) Y_l^m(\omega).$$

Assume that u and v are in $H^2(\Omega)$, where Ω is the unit ball. Then u and u_r restricted to $r = 1$ are in $H^{3/2}$, and $H^{1/2}$, respectively and the series below converge.

The boundary conditions in (1) imply

$$F_\nu(\lambda) := \gamma j_\nu(\lambda) j_\nu'(\gamma\lambda) - j_\nu(\gamma\lambda) j_\nu'(\lambda) = 0 \tag{12}$$

for some ν, which can be written also as

$$F_\nu(\lambda) = \gamma J_\nu(\lambda) J_\nu'(\gamma\lambda) - J_\nu(\gamma\lambda) J_\nu'(\lambda) = 0.$$

Indeed, the Cauchy data for the unperturbed equation is

$$\left(\sum_{lm} a_{lm} j_{l+n/2-1}(\lambda) Y_l^m(\omega),\ \lambda \sum_{lm} a_{lm} j'_{l+n/2-1}(\lambda) Y_l^m(\omega)\right),$$

and the the Cauchy data for the perturbed equation is

$$\left(\sum_{lm} b_{lm} j_{l+n/2-1}(\gamma\lambda) Y_l^m(\omega),\ \gamma\lambda \sum_{lm} b_{lm} j'_{l+n/2-1}(\gamma\lambda) Y_l^m(\omega)\right).$$

They match if and only if a_{lm} and b_{lm} solve the system

$$M_\nu \begin{pmatrix} a_{lm} \\ b_{lm} \end{pmatrix} := \begin{pmatrix} j_\nu(\lambda) & j_\nu(\gamma\lambda) \\ j'_\nu(\lambda) & \gamma j'_\nu(\gamma\lambda) \end{pmatrix} \begin{pmatrix} a_{lm} \\ b_{lm} \end{pmatrix} = 0. \tag{13}$$

Then $F_\nu = \det M_\nu$. If $F_\nu(\lambda) = 0$ for some ν and λ, then (13) has a nonzero solution for that ν. Next, since J_ν and J'_ν cannot vanish simultaneously, that solution consists of $c_m(a_l, b_l)$ with all coefficients non-zero; note that A_ν depends on l but not on m. We may assume that (a_l, b_l) is a unit vector, generating the 1D null space. Then we get non-zero solutions u and v with that fixed $\nu = l + n/2 - 1$ and any $m = 1, \dots, \mu(l)$ of the form

$$u_\nu(x) = \sum_{m=1}^{\mu(l)} c_m a_l j_{\nu(l)}(\lambda r) Y_l^m(\omega), \quad v_\nu(x) = \sum_{m=1}^{\mu(l)} c_m b_l j_{\nu(l)}(\gamma l)(\gamma\lambda r) Y_l^m(\omega), \tag{14}$$

(recall that $\nu = l + n/2 - 1$). This gives a space of eigenpairs of dimension $\mu(l)$. If the same root λ of F_ν happens to be a root of one or more than one $F_{\nu'}$ with $\nu' \neq \nu$, the corresponding eigenspaces are orthogonal, and the total dimension is the sum of the dimensions. Therefore, each such root contributes $\mu(l)$ ITEs to the counting function. In particular, the algebraic multiplicity of λ, defined as the order of λ as a root of F_ν, plays no role.

Analysis of the zeros of F_ν: The Bessel functions $J_\nu(\lambda)$ solve

$$\lambda^2 J''_\nu + \lambda J'_\nu + (\lambda^2 - \nu^2) J_\nu = 0.$$

Thefefore,

$$J''_\nu = -\lambda^{-1} J'_\nu - (1 - \nu^2\lambda^{-2}) J_\nu$$

$$J''_\nu(\gamma\lambda) = -(\gamma\lambda)^{-1} J'_\nu(\gamma\lambda) - (1 - \nu^2\gamma^{-2}\lambda^{-2}) J_\nu(\gamma\lambda).$$

We drop the subscript ν in F_ν in the computations below. We compute $F'(\lambda)$:

$$\begin{aligned} F'(\lambda) &= \gamma J'_\nu(\lambda) J'_\nu(\gamma\lambda) + \gamma^2 J_\nu(\lambda) J''_\nu(\gamma\lambda) - \gamma J'_\nu(\gamma\lambda) J'_\nu(\lambda) - J_\nu(\gamma\lambda) J''_\nu(\lambda) \\ &= \gamma^2 J_\nu(\lambda) J''_\nu(\gamma\lambda) - J_\nu(\gamma\lambda) J''_\nu(\lambda) \\ &= -\gamma^2 J_\nu(\lambda)(\gamma\lambda)^{-1} J'_\nu(\gamma\lambda) - \gamma^2 J_\nu(\lambda)(1 - \nu^2\gamma^{-2}\lambda^{-2}) J_\nu(\gamma\lambda) \\ &\quad + J_\nu(\gamma\lambda)\lambda^{-1} J'_\nu(\lambda) + (1 - \nu^2\lambda^{-2}) J_\nu(\gamma\lambda) J_\nu(\lambda) \\ &= \lambda^{-1}\Big(-\gamma J_\nu(\lambda) J'_\nu(\gamma\lambda) + J_\nu(\gamma\lambda) J'_\nu(\lambda)\Big) + (1 - \gamma^2) J_\nu(\lambda) J_\nu(\gamma\lambda) \\ &= -\lambda^{-1} F(\lambda) + (1 - \gamma^2) J_\nu(\lambda) J_\nu(\gamma\lambda). \end{aligned}$$

The zeros of $J_\nu(\lambda)$ are at all simple except possibly at $\lambda = 0$. Therefore, $\lambda = \lambda_0 \neq 0$ is a zero of F with multiplicity more than 1 if and only if

$$\begin{cases} F(\lambda_0) = 0 \\ F'(\lambda_0) = 0 \end{cases} \Leftrightarrow \begin{cases} J_\nu(\lambda_0) = 0 \\ J_\nu(\gamma\lambda_0) = 0 \end{cases} . \tag{15}$$

We compute $F''(\lambda)$ now:

$$F''(\lambda) = \lambda^{-2}F(\lambda) - \lambda^{-1}F'(\lambda) + (1-\gamma^2)J'_\nu(\lambda)J_\nu(\gamma\lambda) + (1-\gamma^2)\gamma J_\nu(\lambda)J'_\nu(\gamma\lambda).$$

Hence at $\lambda_0 \neq 0$ satisfying (15), we have $F''(\lambda_0) = 0$.

Next, we compute $F'''(\lambda)$:

$$\begin{aligned} F'''(\lambda) = & -2\lambda^{-3}F(\lambda) + \lambda^{-2}F'(\lambda) + \lambda^{-2}F'(\lambda) - \lambda^{-1}F''(\lambda) \\ & + (1-\gamma^2)\left(J''_\nu(\lambda)J_\nu(\gamma\lambda) + 2\gamma J'_\nu(\lambda)J'_\nu(\gamma\lambda) + \gamma^2 J_\nu(\lambda)J''_\nu(\gamma\lambda)\right). \end{aligned}$$

At $\lambda_0 \neq 0$ satisfying (15), we have $F''(\lambda_0) = 0$ and

$$F'''(\lambda_0) = 2\gamma(1-\gamma^2)J'_\nu(\lambda)J'_\nu(\gamma\lambda).$$

Since J_ν only has simple roots away from $\lambda = 0$, we have the following result:

Proposition 2.4. *For $\gamma(1-\gamma^2) \neq 0$, away from $\lambda = 0$,*

$$F_\nu(\lambda) = \gamma J_\nu(\lambda)J'_\nu(\gamma\lambda) - J_\nu(\gamma\lambda)J'_\nu(\lambda)$$

has roots with possible multiplicities one or three. It is three if and only if the root is a common zero of $J_\nu(\lambda)$ and $J_\nu(\gamma\lambda)$, and in particular, real.

2.3. **Various notions of multiplicity of the ITEs in literature.** As mentioned in the introduction, the notion of a multiplicity of an ITE varies from one work to another. One of the approaches is to view the problem as a spectral one for some non-selfadjoint operator P. Concretely, in [29], [8] and in [25], $P = \mathrm{diag}(-m^{-1}\Delta, -\Delta)$ with boundary conditions as in (1). The ITEs are defined to be the eigenvalues of P. There might be generalized eigenvectors of P, hence the existence of higher dimensional generalized eigenspaces, which are the union of the kernels of $(P-\lambda^2)^k$ for $k = 1, 2, \dots$. Thus, there are two natural notions of multiplicity, one associated with the dimension of the eigenspace, while the other the generalized eigenspace. The latter, also equal to the rank of the residue of the resolvent, is used in [29], [8] and in [25], to define the multiplicity of ITE. This approach is also employed in scattering theory to define the multiplicity of a resonance, after the problem is reduced to a non-selfadjoint one by complex scaling, see, e.g., [27]. With this definition, the multiplicity of a resonance $\lambda_0 \neq 0$ is also the rank of residue of the resolvent at the pole λ_0.

In the analysis of the ITEs, another approach is to write the problem as a non-linear spectral problem for a certain fourth order elliptic differential operator, see, e.g., [22] and study the dimension of corresponding null spaces. With this approach, there is no obvious candidate for generalized eigenvectors.

One can also formulate the ITE problem as one finding the null-space of the difference $\mathrm{DN}_m(\lambda) - \mathrm{DN}(\lambda)$ of the Dirichlet-to-Neumann (DN) maps, see, e.g., [18]; or of the difference of the Neumann-to-Dirichlet maps when $\mathrm{DN}_m(\lambda)$ and $\mathrm{DN}(\lambda)$ have a common pole (which is exactly when λ is not a simple ITE in the case we study). Then the null space consists of functions on the boundary (which can be related to interior solutions, of course). This formulation includes the spectral parameter in the operators in a non-linear way; and the implicit choice of the multiplicity is then the dimension of the null-space, which does not include the generalized eigenvectors mentioned above.

3. Counting the ITEs in the 1D case.

3.1. Counting real ITEs in 1D. Let $N_{\mathbb{R}}^{\text{geom}}(r)$ be the number of the real ITEs not exceeding r, counted once, i.e., with their geometric multiplicities. Let $N_1(r)$ be the number of the Dirichlet eigenvalues λ^2 with $\lambda \in (0, r]$ of $-d^2/dx^2$, and let $N_\gamma(r)$ be related in the same way to $-\gamma^{-2}d^2/dx^2$. Clearly, $N_1(r) = r/\pi + O(1)$, $N_\gamma(r) = \gamma r/\pi + O(1)$.

Theorem 3.1. *Let $0 < \gamma \neq 1$. Then*

$$\begin{aligned} N_{\mathbb{R}}^{\text{geom}}(r) &= |N_1(r) - N_\gamma(r)| + O(1) \\ &= |1-\gamma|\frac{r}{\pi} + O(1). \end{aligned}$$

Proof. Let $0 < \gamma < 1$ first. Assume that γ is irrational. Then $\sin\lambda$ and $\sin(\gamma\lambda)$ do not have common zeros and the ITEs can be characterized as not only as the zeros of F but also as the zeros of the differences $\text{DN} = \text{DN}_\gamma - \text{DN}_1 = F/(\sin(\gamma\lambda)\sin\lambda)$ of the DN maps

$$\text{DN}(\lambda) = \gamma\cot(\gamma\lambda) - \cot\lambda.$$

Since by (8), $\text{DN} = (1-\gamma^2)F/F'$, at the zeros of F, we get $\text{DN}' = 1-\gamma^2 > 0$. This is the crucial observation which makes the counting possible, compared with Proposition 4.1. Therefore, DN has at most one zero on any interval, on which it is continuous. In terms of the branches of the graphs (the graph between two consecutive poles) of $\gamma\cot(\gamma\lambda)$ and $\cot\lambda$, this means if two such branches intersect, the former has a greater slope than the latter (and they are both negative); and therefore, they cannot intersect more than once. On the other hand, comparing their asymptotes at the poles, we see that a branch of $\cot\lambda$ is intersected by a branch of (the slower varying) $\gamma\cot(\gamma\lambda)$ if and only if the interval of definition of the former is contained in the (larger) interval of definition of the latter. Therefore, the number of branches of $\cot\lambda$ which do not contribute (exactly one) zero is equal to the number of intervals $(k\pi, (k+1)\pi)$, $k = 1, 2, \dots$. Each of these types of intervals contains a zero $m\pi/\gamma$, m integer, of $\sin(\gamma\lambda)$, and contains exactly one such zero because $\gamma < 1$.

Let $r > 0$. We apply the arguments above for all intervals between zeros of $\gamma\cot(\gamma\lambda)$ in $[0, r]$. In the partial interval, which contains r, we have $O(1)$ ITEs. Therefore, up to an $O(1)$ error, the number of ITEs is that of zeros of $\sin\lambda$ minus those of $\sin(\gamma\lambda)$.

Next, let $\gamma = p/q$ be rational. In this case, each common pole of $\cot\lambda$ and $\gamma\cot(\gamma\lambda)$ corresponds to common zeros of $\sin\lambda$ and $\sin(\gamma\lambda)$. By the analysis above, any such pole is an ITE. If $\gamma = p/q < 1$ is irreducible, then the zeros of $\sin(\gamma\lambda)$ are $kq\pi/p$, k integer. Those of them which are zeros of $\sin\lambda$ as well are given by $k = mp$, m integer. Consider the interval between two such consecutive zeros, $I := ((m-1)q\pi, mq\pi]$. As above, in I, the branch with domain $I_l := ((l-1)\pi, l\pi)$, $l = (m-1)q+1, \dots mq$, of the faster oscillating $\cot\lambda$ are not intersected by some of those of $\cot(\gamma\lambda)$ if and only if there is a pole of $\cot(\gamma\lambda)$ in $\bar{I}_l$; i.e., if some $kq\pi/p$ belongs to $[l\pi, (l+1)\pi]$, see Figure 1. That pole can be an endpoint of $\bar{I}_l$ only for the most-left and the most-right intervals I_l; and there, there are no intersections of the two graphs. For the rest of the I_l intervals, and there are $q-2$ of them, the pole of $\cot(\gamma\lambda)$ is interior for I_l, and therefore, an interior point for I as well. The number of such $kq\pi/p$ in the interior of I is $p-1$; therefore we have $(q-2)-(p-1)$ ITEs in the interior of I, and thus, $q-p$ ITEs in I, since we include the right endpoint in I but not the left one. Then $N(mq\pi) = m(q-p) = mq(1-\gamma)$. This easily implies $N(r) = r(1-\gamma)/\pi + O(1)$.

In section 4.2 we use a lightly modified counting argument, which we could have applied to this case as well.

When $\gamma > 1$, one can rescale to show $N^\gamma(r/\gamma) = N^{1/\gamma}(r)$, where we temporarily denote by N^γ the counting function $N_{\mathbb{R}}^{\text{geom}}$ of the ITEs related to γ (not to be confused with N_γ). Then we use what we proved above. □

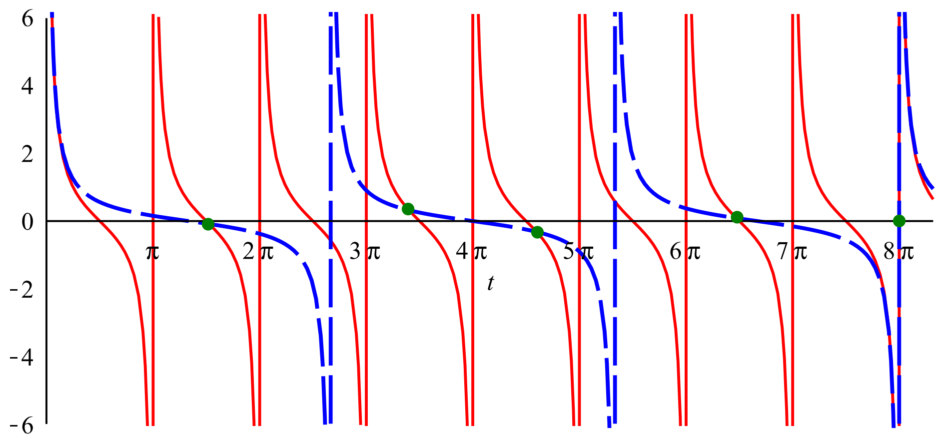

Figure 1. The graphs of $\gamma\cot(\gamma\lambda)$ and $\cot(\lambda)$ on $(0, 8\pi]$ with $\gamma = 3/8$. Here, $p = 3$, $q = 8$. The thick dots represent the ITEs. The ITE 8π has algebraic multiplicity 3, the other three are simple.

Let $N_{\mathbb{R}}^{\text{alg}}(r)$ be the number of the real ITEs not exceeding r counted with their algebraic multiplicities (1 or 3). The asymptotic for this counting function is slightly different.

Theorem 3.2. *Let $\gamma = \sqrt{n}$ be a positive constant. If $\gamma = \sqrt{m} \neq 1$ is irrational, then $N_{\mathbb{R}}^{\text{alg}}(r) = N_{\mathbb{R}}^{\text{geom}}(r)$. If $\gamma = p/q$ is rational and p/q is irreducible, then*

$$N_{\mathbb{R}}^{\text{alg}}(r) = \left(\left| 1 - \frac{p}{q} \right| + \frac{2}{q} \right) r/\pi + O(1).$$

Proof. If γ is irrational, then we just apply Proposition 2.2. Let $0 < \gamma < 1$ be rational. In the counting argument above, we counted each common zero of $\sin(\gamma\lambda)$ and $\sin(\lambda)$ once; and those are exactly the triple roots of F. When we work with the algebraic multiplicities, we should add them two more times. Therefore, in each interval I as in the proof above, we have $q-p+2$ ITEs, instead of $q-p$. As above, we need to divide this by πq to get the leading term. When $\gamma > 1$, we use the rescaling argument above, or direct counting. □

Remark 1. In both cases, $\gamma < 1$ and $\gamma > 1$, we get

$$N_{\mathbb{R}}^{\text{alg}}(r) \leq (1+\gamma)r/\pi + O(1),$$

which is consistent with the theorem below. We have equality if and only if γ or $1/\gamma$ is an integer. We get the same leading term as in (4) which estimates the *complex* ITEs, and is also the upper bound in [8], modulo the multiplicative factor $3\sqrt{3}$.

3.2. **Counting complex ITEs in 1D.** Let $N^{\mathrm{alg}}_{\mathbf{C}}(r)$ be the number of the complex ITEs in $\Re\lambda > 0$ counted with their algebraic multiplicities (1 or 3) of modulus r or less. The fact that they are in a strip parallel to the real line has been proved in [30] and [19]; the latter work provides an explicit bound for $C(\Gamma)$ below for the whole line case.

Theorem 3.3. *Let $\gamma = \sqrt{n}$ be a positive constant, different from 1. Then*

$$N^{\mathrm{alg}}_{\mathbf{C}}(r) = (1+\gamma)\frac{r}{\pi} + o(r).$$

Moreover, all ITEs are symmetric about the real line and included in the strip $|\Im\lambda| \le C(\gamma)$ for some $C(\gamma) > 0$.

Proof. Write

$$4\mathrm{i}F(\lambda) = \gamma(e^{\mathrm{i}\gamma\lambda} + e^{-\mathrm{i}\gamma\lambda})(e^{\mathrm{i}\lambda} - e^{-\mathrm{i}\lambda}) - (e^{\mathrm{i}\gamma\lambda} - e^{-\mathrm{i}\gamma\lambda})(e^{\mathrm{i}\lambda} + e^{-\mathrm{i}\lambda}).$$

When $\lambda = \Re\lambda + \mathrm{i}\Im\lambda$, $\Im \ll 0$, the leading term on the right is $(\gamma-1)e^{\mathrm{i}(\gamma+1)\lambda}$ with modulus $|\gamma-1|e^{(\gamma+1)|\Im\lambda|}$, and we get

$$4|F(\lambda)| \ge |\gamma-1|e^{(\gamma+1)|\Im\lambda|} - C(\gamma)|\gamma-1|e^{|1-\gamma||\Im\lambda|}.$$

Therefore, for $\Im \le -C'(\gamma)$, F cannot vanish. Since $\overline{F(z)} = F(\bar z)$, this covers the $\Im\lambda > 0$ case, as well.

To prove the asymptotic formula, we apply a theorem by Titchmarsh [31] which was generalized to distributions, and successfully used by Zworski [32] to prove a Weyl type of asymptotic of the resonances for the Schrödinger equation in one dimension. If f is a distribution on $\mathbb{R}$ and $[a,b]$ is the smallest closed interval containing $\operatorname{supp}\hat f$, the counting function $N(r)$ of the zeros of f in $\mathbf{C}$ satisfies

$$N(r) = (b-a)r/\pi + o(r).$$

We can write $\hat F(\xi)$ as a linear combination of delta functions supported at $-\gamma-1$, $-\gamma+1$, $\gamma-1$, and $\gamma+1$, all with non-zero coefficients. Therefore, $a = -\gamma-1$, $b = \gamma+1$. We get twice the claimed asymptotic, but this included the zeros in $\Re\lambda < 0$ which are symmetric to those in $\Re\lambda > 0$, because F is odd. This completes the proof. ☐

Remark 2. Combining Theorem 3.2 and Theorem 3.3, we can estimate the asymptotic distribution of the non-real ITEs. The fact that it changes in a singular way when one perturbs γ means that we can view the triple (almost real) ITEs, when γ is rational, as collapsed complex ITEs. In particular, we recover one of the results in [19]: there are infinitely many complex eigenvalues when γ or $1/\gamma$ is not an integer. We also get a linear lower r/C bound on their counting function.

4. **Counting ITEs in higher dimensions.** Denote by $\lambda_{\nu,j}$ the zeros of F_ν defined by (12), with $\nu \in \mathbf{Z} + n/2 - 1$. The discussion above implies the following:

$$N^{\mathrm{geom}}_{\mathbb{R}}(r) = \sum_{\lambda_{\nu,j}<r} \mu(\nu - n/2 + 1). \tag{16}$$

Note that the sum is finite; by (19) below, it contains zeros associated with $\nu = O(r)$ only.

4.1. **Comparing the derivatives at the intersections — Higher dimension.** The zeros of F_ν are also the intersection points, i.e., the zeros of the following equation

$$\gamma J_\nu(\lambda)J'_\nu(\gamma\lambda) = J_\nu(\gamma\lambda)J'_\nu(\lambda). \tag{17}$$

Near every simple zero, we have $J_\nu(\lambda_0), J_\nu(\gamma\lambda_0) \neq 0$. Then we can rewrite (17) near λ_0 as

$$\gamma\frac{J'_\nu(\gamma\lambda)}{J_\nu(\gamma\lambda)} = \frac{J'_\nu(\lambda)}{J_\nu(\lambda)}.$$

We drop the subscripts ν in F_ν the next few lines again. Since

$$P := \gamma\frac{J'_\nu(\gamma\lambda)}{J_\nu(\gamma\lambda)} - \frac{J'_\nu(\lambda)}{J_\nu(\lambda)} = \frac{F}{J_\nu(\lambda)J_\nu(\gamma\lambda)} = (1-\gamma^2)\frac{F}{F' + \lambda^{-1}F},$$

at any simple zero λ_0, we have

$$P'(\lambda_0) = (1-\gamma^2)\frac{F'(F'+\lambda^{-1}F) - F(F'+\lambda^{-1}F)'}{(F'+\lambda^{-1}F)^2} = 1-\gamma^2. \tag{18}$$

So we have proved the following.

Proposition 4.1. *For $0 < \gamma < 1$, at each intersection point (root) λ_0 of $H(\lambda) = \dfrac{J'_\nu(\lambda)}{J_\nu(\lambda)}$ and $G(\lambda) = \gamma H(\gamma\lambda)$, we have*

$$H'(\lambda_0) < G'(\lambda_0).$$

When $\gamma > 1$, we have

$$H'(\lambda_0) > G'(\lambda_0).$$

This proposition is our main counting argument.

4.2. **Proof of Theorem 1.1.** The zeros of F_ν are of two types: (1) points of intersection of the graphs of H and G away from their poles; and (2) common poles of G and H (common zeros of $J_\nu(\lambda)$ and $J_\nu(\gamma\lambda)$). Denote the positive zeroes of J_ν by $j_{\nu,k}$, $k = 1, 2, \dots$ (we suppress the dependence on ν). Let $0 < \gamma < 1$ first. We call below the intervals $(j_{\nu,k}, j_{\nu,k+1}]$ between two consecutive zeros of F_ν "small intervals"; and the intervals $(j_{\nu,k}/\gamma, j_{\nu,k+1}/\gamma]$ between two consecutive zeros of $J_\nu(\gamma\lambda)$ will be called "large intervals". At the endpoints of each small/large interval, the corresponding function H or G, respectively, diverges to ∞ on the left; and to $-\infty$ to the right. If a branch of H intersects a branch of G, this can happen at one point only, by Proposition 4.1. We refer to Figure 2, where $\gamma > 1$.

If a small interval is contained in the interior of large one, then the graph of $J_\nu(\gamma\lambda)$ will intersect that of $J_\nu(\lambda)$, and there is exactly one such point in that small interval. The λ coordinate of that point is an ν-ITE (a zero of F_ν). If $J_\nu(\gamma\lambda)$ and $J_\nu(\lambda)$ have a common pole (vertical asymptote), then that pole is an ν-ITE as well. Those are the two types of ν-ITEs we may have. In the latter case, a small interval is contained in the closure of a large one, and they have a common endpoint. Since we defined all intervals as open on the left and closed on the right; we may attribute an ITE which is a common pole to the small interval to the left of it. Therefore, we established an bijection between the ν-ITEs and the small intervals which are contained entirely in a large one. The small intervals left without an associated ν-ITE are those which have common points with two large ones; i.e., those containing some of the zeros $j_{\nu,k}/\gamma$. Therefore, the number of ν-ITEs not exceeding r is equal

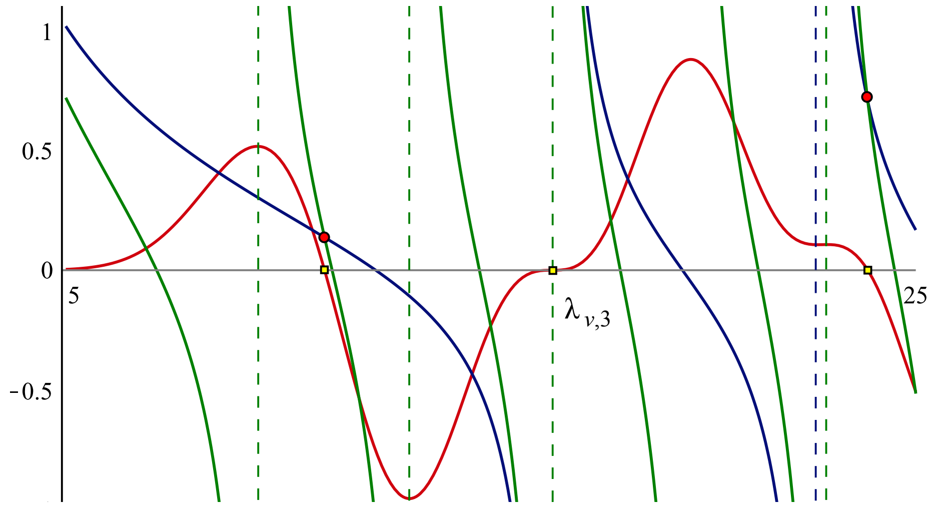

Figure 2. The graphs of F_ν (the smooth curve), H_ν and G_ν for $\lambda \in [5, 25]$ and $\nu = 11/2$ with $\gamma = \lambda_{\nu,3}/\lambda_{\nu,1} \approx 0.57$. The function F_ν has a triple root at $\lambda_{\nu,3} \approx 16.35$ where the two vertical asymptotes coincide. The zeros to the left and right are simple.

to the number of zeros of $J_\nu(\lambda)$ minus that of $J_\nu(\gamma\lambda)$ up to an error 1, depending on the position on r in the small interval to which it belongs. By [21, Ch. 7.6.5],

$$j_{\nu,k} = k\pi + \frac{1}{2}\nu\pi - \frac{1}{4}\pi + O(k^{-1}). \tag{19}$$

Therefore, that error, multiplied by the corresponding multiplicity, see (16), contributes an $O(r^{n-1})$ term to $N(r)$. This proves Theorem 1.1 for $0 < \gamma < 1$. The case $\gamma > 1$ follows by rescaling, as in the 1D case.

4.3. **Another remark about multiplicities.** The geometric multiplicity of each zero λ_0 of $F_\nu(l)$ is $\mu(l)$, if there is only one ν so that $F_\nu(\lambda_0) = 0$; otherwise is a sum of such $\mu(\lambda)$. The ITE λ_0 is multiple (triple) if and only if $J_\nu(\lambda_0) = J_\nu(\gamma\lambda_0) = 0$. We cannot tell whether $\gamma = 1$ or not, if the Cauchy data is $(0, Y_l)$ for any Y_m a linear combination of Y_l^m, $m = 1, \dots, \mu(l)$. However, we can do it for Cauchy data $(Y_l, 0)$, which is the orthogonal complement to that space for a fixed l. Therefore, the algebraic multiplicity $3\mu(\lambda)$ does not play a role here. It only tells us how fast the information about γ, encoded in the Dirichlet data, "degenerates", as $\lambda \to \lambda_0$.

5. **Transmission eigenvalues (TEs).** It is easy to see that in this case, the interior transmission eigenvalues are also transmission eigenvalues (in the whole $\mathbb{R}^n$). Indeed, u_ν in the eigenpair in (14) extends from the unit ball to the whole $\mathbb{R}^n$ in a trivial way, by the same formula. Then the function v_ν, extended as u_ν outside the unit ball as u_ν, is a solution of $(-\Delta - \lambda^2 m)u = 0$, where $m = \gamma^2$ in the unit ball, and $m = 1$ outside. This is a transmission problem. We will use the following facts: u_ν is C^∞, and its the exterior Cauchy data matches the interior one; the interior one is the same as that of v_ν because (u_ν, v_ν) is an eigenpair; the exterior Cauchy data of both functions coincide as well because they are equal outside the unit ball. Therefore, v_ν and its normal derivative do not jump across the unit sphere. The

relative scattering matrix in the spherical harmonic base was computed in [28]. It is a diagonal operator with diagonal entries

$$S_l(\lambda) = -\frac{h_\nu^{(2)'}(\lambda)j_\nu(\gamma\lambda) - \gamma h_\nu^{(2)}(\lambda)j_\nu'(\gamma\lambda)}{h_\nu^{(1)'}(\lambda)j_\nu(\gamma\lambda) - \gamma h_\nu^{(1)}(\lambda)j_\nu'(\gamma\lambda)},$$

where $h_\nu^{(1,2)}(\lambda) = \lambda^{1-n/2}H_{l+n/2-1}^{(1,2)}(\lambda)$.

Then $A_l(\lambda) = S_l(\lambda) - 1$ are the diagonal elements of the scattering amplitude $A(\lambda)$, considered as an operator. A simple calculation yields

$$A_l(\lambda) = \frac{-j_\nu'(\lambda)j_\nu(\gamma\lambda) + \gamma j_\nu(\lambda)j_\nu'(\gamma\lambda)}{h_\nu^{(1)'}(\lambda)j_\nu(\gamma\lambda) - \gamma h_\nu^{(1)}(\lambda)j_\nu'(\gamma\lambda)} = \frac{F_\nu(\lambda)}{h_\nu^{(1)'}(\lambda)j_\nu(\gamma\lambda) - \gamma h_\nu^{(1)}(\lambda)j_\nu'(\gamma\lambda)}.$$

Therefore, A_l has the same zeros as F_ν. The denominator has complex zeros only, at the resonances; which lie in the lower half-plane, see [28]. At each such zero, the eigenspace of $S(\lambda)$ restricted to the spherical harmonics with momentum l, has a kernel coinciding with that space; and its dimension is $\mu(l)$. We can then define the geometric multiplicity of each TE λ as the dimension of the kernel of $A(\lambda)$ (the latter might include more than one but always finitely many l's). Now, $N(\lambda)$ is the number of the real $\lambda \neq 0$, for which $A(\lambda)$ has a non-trivial kernel, counted with their geometric multiplicities. When an ITE λ_0 is not a simple root of F_ν, the algebraic multiplicity shows up if we study $A^{-1}(\lambda)$ — the most singular term in the Laurent expansion is $(\lambda - \lambda_0)^{-3}$. The residue, however, and the whole singular part cannot have rank greater than the dimension $\mu(l)$ of the spherical harmonics, which is the geometric multiplicity.

6. **ITEs are not always TEs.**

We present here a simple example showing that ITEs are not always TEs (the converse is clearly true). Take any solution u of the Helmholtz equation $(-\Delta - \lambda^2)u = 0$ for some $\lambda > 0$ in Ω with the following properties: $u > 0$ in $\bar{\Omega}$, u is C^∞ outside Ω but has no extension as a solution in the whole $\mathbb{R}^n$. Such a solution u, λ and Ω are easy to construct; for example, if $n = 3$, fix $\lambda > 0$ and take $u = \cos(\lambda x)/|x|$ (the real part of the Green's function, up to a constant), and Ω can be any domain in the ball $B(0, \pi/\lambda)$ so that $0 \notin \bar{\Omega}$. Now, take $\phi \in C_0^\infty(\bar{\Omega};\ \mathbb{R})$, and set $v = u + \epsilon\phi$. Then v solves

$$(-\Delta - \lambda^2 m)v = 0 \quad \text{in } \Omega \quad \text{with } m := -\frac{\Delta(u + \epsilon\phi)}{\lambda^2(u + \epsilon\phi)} = \frac{u - \epsilon\lambda^{-2}\Delta\phi}{u + \epsilon\phi}.$$

When $|\epsilon| \ll 1$, m is a well defined positive function in $\bar{\Omega}$ and λ is an ITE, because u and v have the same Cauchy data on $\partial\Omega$. On the other hand, λ is not a TE because u does not extend as a solution in the whole $\mathbb{R}^n$. If $(-\Delta - \lambda^2)\phi \not\equiv 0$, then $m \not\equiv 1$.

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