Inverse Scattering Problems for the Wave Equation with Time Dependent Impurities

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1. Statement of the results

In this paper we present some uniqueness theorems in the inverse scattering for the wave equation with a time dependent potential and for the wave equation in the exterior of a moving obstacle. The complete proofs can be found in author's articles [23-27].

In what follows we assume that the space dimension n is odd, $n \geq 3$. First, consider the wave equation

$$u_{tt} - \Delta u + q(t,x)u = 0, \qquad t \in \mathbb{R},\ x \in \mathbb{R}^n \tag{1}$$

with a potential $q(t,x)$ depending on t. We assume that

(A1) $q \in C^{\infty}$, $|q| \leq C(1+|t|)^N$ with some $C>0$, $N>0$;

(A2) there exists $\varrho>0$ such that $q(t,x) = 0$ for $|x|>\varrho$, all t.

Our aim is to prove that $q(t,x)$ can be determined uniquely by the scattering data corresponding to (1).

Secondly, consider the wave equation in the exterior of a moving obstacle. Let $Q \subset \mathbb{R}^{n+1}$ be an open connected set with a smooth boundary ∂Q. Given $t \in \mathbb{R}$, we denote by $\Omega(t) = \{x \in \mathbb{R}^n;\ (t,x) \in Q\}$ the exterior domain and by $\mathcal{O}(t) = \mathbb{R}^n \setminus \Omega(t)$ the obstacle at the time t. We impose the following conditions.

(B1) There exists $\varrho>0$ such that $\mathcal{O}(t) \subset B_{\varrho} = \{x;\ |x|<\varrho\}$ for each t.

(B2) If $\nu = (\nu_t, \nu_x)$ is the inner unit normal to ∂Q, then $|\nu_t| < |\nu_x|$.

Conditions (B1), (B2) mean that the obstacle remains within a fixed compact set and that the boundary moves with a speed less than 1. Consider the wave equation in Q with Dirichlet boundary conditions

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } Q, \\ u = 0 & \text{on } \partial Q. \end{cases} \tag{2}$$

We are interested in the problem of recovering Q from the scattering data related to (2).

The case of stationary (time independent) potentials and stationary obstacles has been treated by many authors (see for example [12,13,22]). Then the variables can be separated and we can reduce (1) and (2) to stationary elliptic problems. The uniqueness of the inverse scattering problem for (1) with time independent $q(x)$ follows from the Born approximation of the scattering amplitude (see e.g. [22]). In [12] there are two proofs showing that a stationary obstacle can be uniquely recovered from the scattering operator.

In the non-stationary case the situation changes considerably. The separation of variables is no longer available. A time dependent scattering theory for (2) has been developed by Cooper and Strauss (see [3-7] and the references herein), generalizing the Lax-Phillips theory (see also [2,18-20,29]). Equation (1) has been studied in [1,9,16-18,30]. Since the local energy for (1) or (2) may increase [20], the scattering operator S does not exist in general. The main object in the scattering theory for (1), (2) is the *generalized scattering (echo) kernel* $K^{\#}$ introduced by Cooper and Strauss [5]. Below we shall give a brief definition of $K^{\#}$.

Given $f=(f_1,f_2)$ define the energy norms

$$\|f\|_0^2 = \int_{\mathbb{R}^n} (|\nabla f_1|^2 + |f_2|^2) dx, \qquad \|f\|_{(t)}^2 = \int_{\Omega(t)} (|\nabla f_1|^2 + |f_2|^2) dx.$$

Denote by $\mathcal{H}_0$ the completition of $C_0^\infty(\mathbb{R}^n) \times C_0^\infty(\mathbb{R}^n)$ with respect to the norm $\|\cdot\|_0$ and by $\mathcal{H}(t)$ the completition of $C_0^\infty(\Omega(t)) \times C_0^\infty(\Omega(t))$ with respect to the norm $\|\cdot\|_{(t)}$. Denote by $U_q(t,s)$ and $U_Q(t,s)$ the propagators related to (1) and (2) respectively. In other words the solutions to (1) and (2) with initial data $u(s,\cdot) = f_1$, $u_t(s,\cdot) = f_2$ are given by $(u(t,\cdot),u_t(t,\cdot)) = U_q(t,s)f$ and $(u(t,\cdot),u_t(t,\cdot)) = U_Q(t,s)f$, respectively. The operators $U_q(t,s):\mathcal{H}_0 \longrightarrow \mathcal{H}_0$ and $U_Q(t,s)$: $\mathcal{H}(s) \longrightarrow \mathcal{H}(t)$ are bounded ones for fixed t, s but their norms may increase exponentially as $|t-s|\longrightarrow\infty$. Using the finite speed of propagations we can define $U_q(t,s)f$, $U_Q(t,s)f$ for functions f which belong locally to $\mathcal{H}_0$ and $\mathcal{H}(t)$, respectively.

Set $h_n(\xi) = \xi^n/n!$ for $\xi\geq 0$, $h_n(\xi) = 0$ otherwise, $n = 0,1,\ldots$ Thus $h_n' = h_{n-1}$, h_0 is the Heaviside function. Given $(s,\omega) \in \mathbb{R}\times S^{n-1}$ let $\Gamma_q(t,x;s,\omega)$ be the solution of the problem

$$\begin{cases} (\Box + q(t,x))\Gamma_q = 0, \\ \Gamma_q\big|_{t\ll 0} = h_1(t+s-x\cdot\omega), \end{cases}$$

and let $\Gamma_Q(t,x;s,\omega)$ solve the problem

$$\begin{cases} \Box\Gamma_Q = 0 & \text{in } Q, \\ \Gamma_Q = 0 & \text{on } \partial Q, \\ \Gamma_Q \big|_{t \ll 0} = h_1(t+s-x\cdot\omega). \end{cases}$$

Here $\Box = \partial_t^2 - \Delta_x$. Since the pair $(h_1(t+s-x\cdot\omega), h_0(t+s-x\cdot\omega))$ considered as a function of x has locally finite energy, Γ_q and Γ_Q are well defined. The functions $\Gamma_{q,sc} = \Gamma_q - h_1(t+s-x\cdot\omega)$, $\Gamma_{Q,sc} = \Gamma_Q - h_1(t+s-x\cdot\omega)$ possess *asymptotic wave profiles* [5], i.e. the limit

$$\Gamma^{\#}_{sc}(s',\omega';s,\omega) = \lim_{t\to\infty} (t+s')^{(n-1)/2}\partial_t\Gamma_{sc}(t,(t+s')\omega';s,\omega)$$

exists in $L^2_{loc}(\mathbb{R}_{s'}\times S^{n-1}_{\omega'})$, where $\Gamma^{\#}_{sc} = \Gamma^{\#}_{q,sc}$, $\Gamma_{sc} = \Gamma_{q,sc}$ and $\Gamma^{\#}_{sc} = \Gamma^{\#}_{Q,sc}$, $\Gamma_{sc} = \Gamma_{Q,sc}$, respectively.

We define the *generalized scattering kernels* $K^{\#}_q$, $K^{\#}_Q$ by the following relations

$$K^{\#}_q(s',\omega';s,\omega) = (-2\pi)^{(1-n)/2}\partial_s^{(n+1)/2}\Gamma^{\#}_{q,sc}(s',\omega';s,\omega),$$

$$K^{\#}_Q(s',\omega';s,\omega) = (-2\pi)^{(1-n)/2}\partial_s^{(n+1)/2}\Gamma^{\#}_{Q,sc}(s',\omega';s,\omega).$$

$K^{\#}_q$ and $K^{\#}_Q$ are distributions with respect to s' smoothly depending on ω', s, ω. It should be noted that if the scattering operator S exists, then $K^{\#}_q$ (respectively $K^{\#}_Q$) is the Schwartz kernel of $S-Id$ in the Lax-Phillips translation representation.

We are interested in the following inverse problems:

(IP1) Does $K^{\#}_q$ determine q uniquely?

(IP2) Does $K^{\#}_Q$ determine Q uniquely?

As mentioned above if $q(x)$ does not depend on t, then (IP1) has affirmative answer. The first result concerning time dependent q has been obtained by Ferreira and Perla Menzala [9]. They have shown that if $q_1 \geq 0$ and $q_2 \geq 0$ are small in a suitable sense, $q_1 \geq q_2$ for all t, x and $\partial_t q_1$, $\partial_t q_2$ have appropriate decay as $|t| \to \infty$, then the equality of the scattering operators for q_1 and q_2 implies $q_1 = q_2$.

Our first result gives an affirmative answer to (IP1).

Theorem 1. *Let* q_1 *and* q_2 *satisfy* (A1), (A2). *Suppose there exist* $\varepsilon > 0$ *and* $\omega_0 \in S^{n-1}$ *so that* $K^{\#}_{q_1}(s',\omega';s,\omega) = K^{\#}_{q_2}(s',\omega';s,\omega)$ *for* $|\omega-\omega_0| < \varepsilon$, $|\omega'-\omega_0| < \varepsilon$, $|s'-s| < \varepsilon$. *Then* $q_1(t,x) = q_2(t,x)$ *for all* t, x.

The kernel $K_q^{\#}$ will be shown to be smooth off the diagonal $(s',\omega') = (s,\omega)$ and to have a singularity at $(s',\omega') = (s,\omega)$. Theorem 1 states that all the information about q is contained in the behavior of $K_q^{\#}$ near the singularity.

Next we turn our attention to the inverse back-scattering problem for (1), i.e. the problem of recovering $q(t,x)$ from knowledge of $K_q^{\#}$ for $\omega'=-\omega$. This is a still unsolved problem even for stationary q. Recently Eskin and Ralston [8] proved that the map $q(x) \longrightarrow a(k,-\omega,\omega)$ is a local homeomorphism at least for an open dense set of potentials q. Here $a(k,\omega',\omega)$ is the scattering amplitude related to $q(x)$. However the (global) uniqueness problem is still open. In Theorem 2 we give a partial answer to the inverse back-scattering problem.

Let us assume $n=3$ and the following condition

(A1') $q \in C(\mathbb{R}_t; W^{1,\infty}(\mathbb{R}^3_x))$ satisfied instead of (A1). Then we have the following.

Theorem 2. *Let q_i, $i=1,2$ satisfy (A1'), (A2) with $n=3$. Let $q_1(t,x) \geq q_2(t,x)$ for $t_1 \leq t \leq t_2$, all x, where $t_2-t_1 > 4\varrho$. Suppose there exist $\omega_0 \in S^2$, $d > 2\varrho$, such that $K^{\#}_{q_1}(s',-\omega_0;s,\omega_0) = K^{\#}_{q_2}(s',-\omega_0;s,\omega_0)$ for $|s'-s| \leq d$, $t_1+\varrho < -s < t_2+\varrho$. Then $q_1(t,x) = q_2(t,x)$ for $t_1+2\varrho \leq t \leq t_2-2\varrho$ and for all x.*

If q does not depend on t, then $K_q^{\#}$ depends merely on $s'-s$ and if in addition $q(x)$ has no bound states, then the scattering amplitude $a(k,\omega',\omega)$ related to q coincides with the Fourier transform of $K_q^{\#}(s',\omega';0,\omega)$ with respect to s' up to some multiplication factors. This leads to the following.

Corollary 3. *Let $q_i(x) \in L^{\infty}(\mathbb{R}^3)$, q_i have compact support and q_i have no bound states, $i=1,2$. Let a_i be the scattering amplitude related to q_i, $i=1,2$. Suppose $q_1 \geq q_2$ and let for some ω_0 we have $a_1(k,-\omega_0,\omega_0) = a_2(k,-\omega_0,\omega_0)$ for all k. Then $q_1 = q_2$.*

Let $\mu(\omega) = \inf\{x\cdot\omega;\ x \in \operatorname{supp} V\}$ be the support function of $\operatorname{supp} V$. The technique developed for the proof of Theorem 2 enables us to get the following.

Corollary 4. *Let q be a non-negative function in $L^{\infty}(\mathbb{R}^3)$ with compact support and let $a(k,\omega',\omega)$ be the scattering amplitude associated with q. Then $-2\mu(\omega) = \sup_{s'} \operatorname{supp} K_q^{\#}(s',-\omega;0,\omega)$. Therefore, $K_q^{\#}(s',-\omega;0,\omega)$ (respectively $a(k,-\omega,\omega)$) determines uniquely the convex hull of the support of q.*

Next we turn our attention to an inverse boundary value problem for (1). Let D be a bounded domain with smooth boundary ∂D, and let $q(t,x)$ satisfy (A1) and $q=0$ for $x\notin D$. Here the space dimension $n\geq 2$ may be arbitrary. Given $f\in C^{\infty}(\mathbb{R}\times\partial D)$ with $f=0$ for $t\ll 0$, we consider the mixed problem

$$\begin{cases} (\Box+q(t,x))u = 0 & \text{in } \mathbb{R}\times D, \\ u = f & \text{on } \mathbb{R}\times\partial D, \\ u = 0 & \text{for } t\ll 0. \end{cases}$$

We are interested in the following inverse problem:

(IP3) Does the map $\Lambda: f \longrightarrow \partial_N u\big|_{\mathbb{R}\times\partial D}$ determine q uniquely?

Here N is the outer normal to ∂D. A similar problem for time-independent q has been considered in [21]. In this case the results of Nachman, Sylvester and Uhlmann [28,14,15] could also be applied. Although such kind of problems are not scattering problems, there is a close relation between them and the scattering theory. Using the tools developed for the proof of Theorem 1, we can easily prove the following.

Theorem 5. *The map Λ determines q uniquely.*

Now consider (IP2). As mentioned above it is well-known that for stationary bodies the scattering operator S always exists and the answer to (IP2) is affirmative [12]. In the case of moving obstacles it is known that $K_Q^{\#}$ determines uniquely the convex hull of $\mathcal{O}(t)$ for each t [6]. The proof of this result is based on the analysis of the leading singularity of $K_Q^{\#}$ (see also [2,18,19]) This solves (IP2) for convex moving bodies. Nevertheless, nothing is known about (IP2) for moving obstacles of arbitrary geometry. It looks a little surprising that the answer to (IP2) is negative in general. This will be proved in Theorem 7 below. For this reason in order to investigate (IP2) we need to impose some additional restrictions.

Definition. *We say that $Q\in\mathcal{Q}$ if Q satisfies* (B1), (B2) *and the following condition*

(B3) *There exists some $T>0$ such that $\nu_t = 0$ for $|t|>T$.*

In other words the class $\mathcal{Q}$ consists of all obstacles which are stationary in the far past and in the far future. Let $U_0(t)$ be the unitary group related to the Cauchy problem for the wave equation [12]. It is not hard to prove that for $Q \in \mathcal{Q}$ the scattering operator

$$S_Q = \operatorname*{s-lim}_{t\to\infty} U_0(-t)U_Q(t,-t)\chi U_0(-t)$$

exists. Here χ is a multiplication of a smooth function $\chi(x)=1$ for $|x|>3\varrho$, $\chi(x)=0$ for $|x|<2\varrho$ and we assume $\mathcal{H}(t) \subset \mathcal{H}_0$. In the next theorem we show that (IP2) has an affirmative answer if $Q \in \mathcal{Q}$.

Theorem 6. *Let $Q_i \in \mathcal{Q}$, $i=1,2$ and let the scattering operators S_{Q_i}, $i=1,2$, associated with Q_i, coincide. Then $Q_1=Q_2$.*

In contrast to Theorem 1 we note that the proof of Theorem 5 is not based on the analysis of the singularities of $K^{\#}_Q$ but on a specific Holmgren's type theorem (see section 3).

Finally we show that in general the uniqueness of (IP2) fails even for periodically moving obstacles. For this reason we construct in sect. 2 an explicit counter-example.

Theorem 7. *There exists a family of infinitely many distinct periodically moving obstacles satisfying* (B1), (B2) *with the same generalized scattering (echo) kernel.*

2. Outline of the proofs. IP1

First we are going to sketch the proof of Theorem 1. Let $u = \partial_s^2\Gamma_q$ be the solution of the Cauchy problem

$$\begin{cases} (\Box + q(t,x))u = 0, \\ u\big|_{t\ll 0} = \delta(t+s-x\cdot\omega). \end{cases} \tag{3}$$

First we derive the following representation of $K^{\#}_q$:

$$K^{\#}_q(s',\omega';s,\omega) = -2^{-1}(2\pi)^{1-n}\,\partial_{s'}^{(n-1)/2}\partial_s^{(n-3)/2}\iint q(t,x) \times u(t,x;s,\omega)\delta(t+s'-x\cdot\omega')\,dt\,dx, \tag{4}$$

where the integral is to be considered in distribution sense. Secondly, we show that $u(t,x;s,\omega)$ vanishes for $t+s<x\cdot\omega$, u is smooth for $t+s>x\cdot\omega$ and near the plane $t+s=x\cdot\omega$ we have the following singular expansion

$$u(t,x;s,\omega) \sim \delta(t+s-x\cdot\omega) + \sum_{j=0}^{\infty} A_j(t,x,\omega)h_j(t+s-x\cdot\omega), \tag{5}$$

where $A_0(t,x,\omega) = -\frac{1}{2}\int_{-\infty}^{0} q(t+\tau,x+\tau\omega)\,d\tau$, $A_j \in C^\infty$. Therefore $K_q^{\#}(s',\omega';s,\omega)$ is C^∞ off the diagonal. Moreover, all the information about q is contained in the behavior of $K_q^{\#}$ near the singularity $(s',\omega') = (s,\omega)$. More precisely, choosing a unit vector α orthogonal to ω and setting

$$\omega'(\mu) = \omega.\cos\mu + \alpha.\sin\mu,$$

$$s'(\mu) = s + \varkappa.\sin\mu, \qquad (\varkappa,\mu)\in\mathbb{R}^2,$$

we get

$$\lim_{\mu\to 0} |\omega'(\mu)-\omega|M(s'(\mu),\omega'(\mu);s,\omega) = \int_{x\cdot\alpha=\varkappa} q(x\cdot\omega-s,x)\,dS_x,$$

where M is the integral in (4). Now, suppose we know $K_q^{\#}$ for $|\omega-\omega_0|<\varepsilon$, $|\omega'-\omega_0|<\varepsilon$, $|s'-s|<\varepsilon$. Then we can find the integral above for all s, $\varkappa$, $\alpha\perp\omega$, $|\omega-\omega_0|<\varepsilon$. It is easy to deduce that this enables us to determine the integrals

$$\int_{-\infty}^{\infty} q(t+\beta,x+\beta\omega)\,d\beta \tag{6}$$

for all t, x, and $\omega\in S^{n-1}$ such that $|\omega-\omega_0|<\varepsilon$. So we arrive at the following problem of integral geometry: prove that if (6) vanishes for all t, x, $|\omega-\omega_0|<\varepsilon$, then $q = 0$. Note that (6) is the "X"-ray transform of q, but the integral is taken only over characteristic (for $\square$) lines. In case q has compact support in t the solution to the above problem is known [10] and it is based on the analyticity of the Fourier transform $\hat{q}(\tau,\xi)$ of q. In the general case the proof is more complicated and uses the fact that $\hat{q}(\tau,\xi)$ is an analytic function of ξ with values in $\mathcal{S}'(\mathbb{R}_\tau)$.

Next, we are going to sketch the proof of Theorem 2. We shall make some changes in the notations. Denote Γ_q by Γ_q^+ and let Γ_q^- be the solution of (1) with initial data $h_1(-t-s+x\cdot\omega)$ for $t\gg 0$. In other words, we have

$$\begin{cases} (\square + q(t,x))\Gamma_q^+ = 0, \\ \Gamma_q^+\big|_{t\ll 0} = h_1(t+s-x\cdot\omega), \end{cases} \qquad \begin{cases} (\square + q(t,x))\Gamma_q^- = 0, \\ \Gamma_q^-\big|_{t\gg 0} = h_1(-t-s+x\cdot\omega). \end{cases}$$

Let q_1 and q_2 be two potentials satisfying the regularity assumptions of Theorem 2. Since q_i are not necessarily smooth, $K_i^{\#}$ are distributions in s',s depending continuously on ω',ω. We show that the following formula takes place:

$$K^{\#}_{q_1}(s',\omega';s,\omega) - K^{\#}_{q_2}(s',\omega';s,\omega) =$$

$$-\frac{1}{8\pi^2}\partial^3_{s'}\partial^2_{s}\iint\big(q_1(t,x)-q_2(t,x)\big)\Gamma^+_{q_1}(t,x;s,\omega)\Gamma^-_{q_2}(t,x;s',\omega')\,dt\,dx. \tag{7}$$

Since we have $u = \partial^2_s\Gamma^+_q$ in (4) and $\partial^2_{s'}\Gamma^-_{q_2} = \delta(t+s'-x\cdot\omega')$ provided $q_2=0$, the above relation can be considered as a generalization of (4). According to the finite speed of propagation we have $\mathrm{supp}\Gamma^+_{q_1} \subset \{(t,x);\ t+s\geq x\cdot\omega\}$, $\mathrm{supp}\Gamma^-_{q_2} \subset \{(t,x);\ t+s'\leq x\cdot\omega'\}$. Moreover, if t,x,s,s' run over compact sets, we can find a constant $C > 0$ such that

$$|\Gamma^+_{q_1}(t,x;s,\omega) - h_1(t+s-x\cdot\omega)| \leq Ch_2(t+s-x\cdot\omega),$$
$$|\Gamma^-_{q_2}(t,x;s',\omega') - h_1(-t-s'+x\cdot\omega')| \leq Ch_2(-t-s'+x\cdot\omega').$$

As a corollary there exists $\delta > 0$ such that $\Gamma^+_{q_1} > 0$ and $\Gamma^-_{q_2} > 0$ for $0 < t+s-x\cdot\omega < \delta$ and $-\delta < t+s'-x\cdot\omega' < 0$, respectively. Now set

$$A_\mu = \Big\{(t,x);\ x\cdot\omega_0+t_1+\varrho \leq t \leq -x\cdot\omega_0+t_2-\varrho,\ |x|\leq\varrho,\ x\cdot\omega_0 \leq \mu\Big\}.$$

Let $\mu_0 = \sup\{\mu;\ q(t,x)=0 \text{ for } (t,x) \in A_\mu\}$, where $q = q_1-q_2$. If Theorem 2 fails, then $\mu_0<\varrho$. Put $s' = s-2\mu_0-\varepsilon$, where $0 < \varepsilon < \varrho-\mu_0$. Assume in what follows that $t_1+\varrho < -s < t_2-\varrho-2\mu_0-\varepsilon$, $(t,x) \in A_\varrho$. Then t,x,s,s' belong to compact sets and let $\delta>0$ be the corresponding constant. We may assume that $\varepsilon < \delta$. Then by (7) we get

$$\int_{\Omega(s)} q(t,x)\Gamma^+_{q_1}(t,x;s,\omega_0)\Gamma^-_{q_2}(t,x;s-2\mu_0-\varepsilon,-\omega_0)\,dt\,dx = 0. \tag{8}$$

Here the integration is taken over the set

$$\Omega(s) = \Big\{(t,x);\ \mu_0 < x\cdot\omega_0 < \min(t+s,-t-s+2\mu_0+\varepsilon),\ |x|\leq\varrho\Big\}.$$

Since $\varepsilon<\delta$, we get $\Gamma^+_{q_1}>0$, $\Gamma^-_{q_2}>0$ in $\Omega(s)$. Further, by the assumptions of Theorem 2, $q\geq0$ thus each integrand in (8) is nonnegative and moreover $\Gamma^+_{q_1}$ and $\Gamma^-_{q_2}$ are positive in $\Omega(s)$. Therefore $q(t,x) = 0$ for $(t,x) \in \Omega(s)$. Letting s run over the interval $t_1+\varrho < -s < t_2-\varrho-2\mu_0-\varepsilon$, we get $\overline{\bigcup_s\Omega(s)} \cup A_{\mu_0} = A_{\mu_0+\varepsilon}$ hence $q = 0$ on $A_{\mu_0+\varepsilon}$ which contradicts the choice of μ_0.

Corollary 3 follows immediately from Theorem 2. Indeed, if q is stationary, then we have $\Gamma^{\pm}_{q}(t,x;s,\omega) = \Gamma^{\pm}_{q}(t+s,x;0,\omega)$ and $K^{\#}_{q}(s',\omega';s,\omega) = K^{\#}_{q}(s'-s,\omega';0,\omega)$. Moreover, if $-\Delta+q$ has no bound states, then

$$-\frac{k}{2\pi i}a(k,\omega',\omega) = \int e^{-iks'}K^{\#}_{q}(s',\omega';0,\omega)\,ds',$$

where a is the scattering amplitude related to q. Therefore, the knowledge of $a(k,-\omega,\omega)$ enables us to find $K^{\#}_{q}(s',-\omega;0,\omega)$ and to apply Theorem 2.

Remark. Let q be stationary, $-\Delta+q$ has no bound states. Then, after performing the indicated differentiations in (7), setting $s=0$ and making Fourier transform with respect to s', we get the following formula

$$a_1(k,\omega',\omega) - a_2(k,\omega',\omega) = -\frac{1}{4\pi}\int(q_1(x)-q_2(x))\psi_1(k,x,\omega)\psi_2(k,x,-\omega')\,dx, \tag{9}$$

where a_j are the scattering amplitudes related to q_j and ψ_j are the solutions of the problems

$$\begin{cases} (-\Delta-k^2+q_j(x))\psi_j = 0, \\ \psi_j = \exp(ik\omega\cdot x) + O(|x|^{-1}) \quad \text{as } |x|\to\infty, \end{cases}$$

$j=1,2$. If $q_2=0$, then (9) reduces to the well-known representation of the scattering amplitude. We hope that (9) might be useful for solving other inverse problems in the potential scattering.

Let us sketch the proof of Corollary 4. Let q satisfy the assumptions of the corollary and denote by $v(t,x,\omega)$ the solution of the problem

$$(\Box+q)v = 0, \qquad v = h_0(t-x\cdot\omega) \text{ for } t<-\varrho.$$

Recall that h_0 is the Heaviside function. We have the following obvious relations between v, Γ^{+}_{q} and u (see (3)): $u(t,x;s,\omega) = \partial^2_s\Gamma^{+}_{q}(t,x;s,\omega) = \partial_s v(t+s,x,\omega)$. From (4) we deduce

$$K^{\#}_{q}(s',\omega;0,\omega) = -\frac{1}{8\pi^2}\partial^2_{s'}\int q(x)v(-s'-x\cdot\omega,x,\omega)\,dx.$$

Taking into account that $v(t,s,\omega) = h_0(t-x\cdot\omega) + v_{sc}(t,s,\omega)$, where $|v_{sc}| \le Ch_1(t-x\cdot\omega)$, we prove the following

Proposition 8. *Let $q \in L^{\infty}(\mathbb{R}^3)$, q has compact support. Let μ be such that $q(x) = 0$ for $x\cdot\omega \le \mu$. Then $K_q^{\#}(s',-\omega;0,\omega) = 0$ for $s' > -2\mu$ and if q is non-negative near the plane $x\cdot\omega = \mu$ we have*

$$K_q^{\#}(-2\mu-\varepsilon,-\omega;0,\omega) = -\frac{1}{8\pi^2}\partial_\varepsilon^2\Big(\int_{\mu\le x\cdot\omega\le\mu+\varepsilon/2} q(x)\,dx\,(1+O(\varepsilon))\Big) \quad \text{as } \varepsilon \to +0.$$

This proposition shows that if $\mu = \mu(\omega) = \inf\{x\cdot\omega;\ x \in \operatorname{supp} V\}$, then $K_q^{\#}(s',-\omega;0,\omega) \ne 0$ for $s' = -2\mu-\varepsilon$, $\varepsilon>0$ sufficiently small. This proves Corollary 4.

Finally, we give the proof of Theorem 5. Let q_1, q_2 be such that $\Lambda_1=\Lambda_2$. Given $\varphi \in C^{\infty}(\mathbb{R}\times S^{n-1})$, let $\Phi_i = \iint u_i(t,x;s,\omega)\varphi(s,\omega)\,ds\,d\omega \in C^{\infty}$, where u_i is the solution of (3) with $q=q_i$, $i=1,2$. The functions Φ_i solve (1) with $q=q_i$. Denote by w the solution of the problem

$$\begin{cases} (\Box+q_1)w = 0 & \text{in } \mathbb{R}\times D, \\ w = \Phi_2 & \text{on } \mathbb{R}\times\partial D, \\ w = 0 & \text{for } t\ll 0,\ x\in D. \end{cases}$$

Set $\tilde{w} = w$ for $x\in D$, $\tilde{w} = \Phi_2$ for $x \notin D$. We have $w = \Phi_2$ on ∂D and $\Lambda_1=\Lambda_2$ implies $\partial_N w = \partial_N\Phi_2$ on ∂D. Therefore, $\tilde{w}$ solves $(\Box+q_1)\tilde{w} = 0$ in $\mathbb{R}_t\times\mathbb{R}^n_x$. Since $\tilde{w}=\Phi_1$ for $t\ll 0$, we conclude that $\tilde{w} \equiv \Phi_1$ thus $\Phi_1=\Phi_2$ for $x\notin D$, all t. Since φ was arbitrary, we get $(u_1-u_2)(t,x;s,\omega) = 0$ for all t,s,ω and $x\notin D$. Combining this with (5), we get that the integrals (6) for q_1 and q_2 are equal for all t,x,ω, which leads to $q_1=q_2$. Notice that the equality $u_1=u_2$ for $x\notin D$ implies $K_1^{\#} = K_2^{\#}$, which yields another proof of Theorem 5. We note that the requirement $q \in C^{\infty}$ can be relaxed to $q \in C^k$ for some finite k.

Remark. Prof. A.G.Ramm informed the author that he and Prof. J.Sjöstrand have a proof of Theorem 5 different from ours which also leads to the inversion of the light-ray transform (6).

3. Outline of the proofs. IP2

Let us sketch the proof of Theorem 6. Let Q_1 and Q_2 be two distinct domains in Q whose scattering operators S_{Q_i}, $i=1,2$ coincide. Fix some $s \in \mathbb{R}$ and fix $\varphi \in [C_0^{\infty}(\mathbb{R}^n) \times C_0^{\infty}(\mathbb{R}^n)] \cap D_-^{\varrho}$, D_-^{ϱ} being the incoming space of Lax and Phillips [12]. We denote by u_i the solution of (2) related to Q_i having initial data φ for $t=s$. First we prove that $S_{Q_1} = S_{Q_2}$ implies that $u = u_1-u_2$ vanishes for

$|x|>\varrho$ and all $t \in \mathbb{R}$. In the stationary case it is not hard to show that, in fact, u vanishes in the unbounded connected component of $\Omega_1 \cap \Omega_2$ which easily yields $\Omega_1 = \Omega_2$ [12]. Indeed, in this case the function $v(k,x) = \int \exp(-itk)u(t,x)dt$ is a solution to the Helmholtz equation, consequently v is a real-analytic function with respect to x. Clearly, in the case of moving obstacles this argument is no more available. Nevertheless, assuming for definiteness that $\partial Q_2 \cap Q_1 \neq \emptyset$, we aim to prove that there exists $(t_0,x_0) \in \partial Q_2 \cap Q_1$ such that $u(t_0,\cdot)$ vanishes in a neighborhood of x_0 in $\partial\Omega_2(t_0)$. This fact makes possible to obtain a contradiction.

To this end we apply two new ideas. First, we construct a two-parameter family $F_{t,s} : \Omega(s) \longrightarrow \Omega(t)$ of diffeomorphisms related to a domain Q satisfying (B1), (B2). We choose $F_{t,s}$ to be the flow associated with a (time-dependent) vector field $v \in C^{\infty}(\overline{Q};\mathbb{R}^n)$ which is chosen so that $|v|<1$, $v(t,x)=0$ for $|x|>\varrho$ and $(1,v)$ is tangent to ∂Q. In some sense $v(t,x)$ can be considered as the velocity of $x \in \Omega(t)$ at the moment t. Thus we treat all points in the exterior of the obstacle as moving ones. Note that if we introduce the coordinates $(t,y) = (t,F_{t_0,t}(x))$, t_0-fixed, then Q becomes cylindric.

Secondly we formulate and prove a specific Holmgren's type theorem. In order to state it we need some preparation. Using $F_{t,s}$ we can easily show that $\Omega(t)$ is connected for each t. Fix t_0 and $x_0 \in \Omega(t_0)$ and let

$$\gamma = \left\{ x;\ x = x(\sigma),\ 0 \le \sigma \le l \right\} \tag{10}$$

be a smooth curve in $\Omega(t_0)$ which is not self-intersecting, such that $x(0)=x_0$, $x(l)=x_1$ for some $x_1 \in \Omega(t_0)$ and $|x'(\sigma)|=1$. Consider the two-dimensional surface $\Gamma = \bigcup_{t\in\mathbb{R}} \left(t, F_{t,t_0}(\gamma)\right) \subset Q$. Obviously, σ and t are coordinates on Γ. We define vector fields $A^{\pm} \in T\Gamma$ in a following way. Set $A^{\pm} = \frac{\partial}{\partial\sigma} \pm \alpha^{\pm}(t,\sigma)\frac{\partial}{\partial t}$, where $\alpha^{\pm}>0$ is chosen so that for each (t,σ) the vector $A^{\pm}$ considered as a vector in $\mathbb{R}^{n+1}$, is characteristic, i.e. $|A^{\pm}_t|=|A^{\pm}_x|$. Let $\sigma \longrightarrow (t,x) = \left(t_{\pm}(\sigma), F_{t_{\pm}(\sigma),t_0}(x(\sigma))\right)$ be the integral curve of $A^{\pm}$ such that $t_{\pm}(0)=t_0$. Condition (B3) guarantees that $\alpha^{\pm}$ is uniformly bounded, thus the function $t_{\pm}(\sigma)$ is defined for all $\sigma \in [0,l]$. We define a subset X of Γ by the relation

$$X = \left\{(t,x);\ x = F_{t,t_0}(x(\sigma)),\ 0\le\sigma\le l,\ t_-(\sigma)\le t\le t_+(\sigma)\right\}. \tag{11}$$

X depends on t_0, x_0, x_1 and on the choice of γ joining x_0 and x_1.

We have the following Holmgren's type theorem.

Theorem 9. *Let u be a distribution satisfying the wave equation $\Box u = 0$ in an open subset of $\mathbb{R}^{n+1}$ containing X. Suppose that u vanishes in a neighborhood of the curve $\{(t,x);\ t_-(l) \le t \le t_+(l),$ $x = F_{t,t_0}(x(l))\}$. Then* $\operatorname{supp} u \cap X = \emptyset$.

The proof of Theorem 9 is based on the local Holmgren theorem (see e.g. [11], Theorem 8.6.5). In the partial case when $\mathcal{O}(t)$ is stationary and $F_{t,s} = I$, Theorem 9 follows easily from Theorem 1.4 of Ch. IV in [12].

Now, let $Q_i \in \mathcal{Q}$ and $S_{Q_1} = S_{Q_2}$. Let $u = u_1 - u_2$ be the function defined above. As mentioned above, $u = 0$ for $|x|>\varrho$ and all $t \in \mathbb{R}$. Suppose $Q_1 \ne Q_2$. Then we can assume that $Q_1 \cap \partial Q_2 \ne \emptyset$. Let $(t_0, x_0) \in Q_1 \cap \partial Q_2$ and choose some $x_1 \notin B_\varrho$ and a curve $\gamma \subset \Omega_1(t_0)$ of the kind (10), joining x_0 and x_1. Denote by $X \subset Q_1$ the set (11) associated with t_0, x_0, x_1 and the flow $F^1_{t,s}$ related to Q_1. Without loss of generality we can assume that $X \subset \overline{Q}_2$, $X \cap \partial Q_2 = (t_0, x_0)$. Given $\varepsilon > 0$ denote by $X_\varepsilon \subset X$ the set (11) related to t_0, $x(\varepsilon)$, x_1. Since $u=0$ in a neighborhood of $\{(t,x);\ t_-(l) \le t \le t_+(l),\ x=x_1\}$, and $\Box u=0$ in $Q_1 \cap Q_2 \supset X_\varepsilon$, then u vanishes at the point $(t_0, x(\varepsilon))$, in virtue of Theorem 9. Letting $\varepsilon \to 0$, we find $u(t_0, x_0) = 0$. Moreover, we can prove that $u(t_0, x) = 0$ for $x \in V \cap \partial\Omega_2(t_0)$, $V \subset \Omega_1(t_0)$ being a small neighborhood of x_0. Recall that $u_2=0$ on ∂Q_2. Consequently, $u_1(t_0, x) = [U_{Q_1}(t_0, s)\varphi]_1$ vanishes on $V \cap \partial\Omega_2(t_0)$ for all $s \in \mathbb{R}$, $\varphi \in D^\varrho_- \cap C^\infty_0 \times C^\infty_0$. Since such solutions are dense in $\mathcal{H}_1(t_0)$, we conclude that the first component of every $f \in \mathcal{H}_1(t_0)$ vanishes on the surface $V \cap \partial\Omega_2(t_0)$. Thus we obtain a contradiction, which completes the proof of Theorem 6.

Finally, we shall construct explicitly a family of periodically moving obstacles with the same generalized scattering kernel. For the sake of simplicity we assume $n=3$.

It is not hard to see that there exists a function $\psi \in C^\infty_0(\mathbb{R})$ with the properties: $\psi(\sigma) = 1$ for $|\sigma| \le 2$, $\psi(\sigma) = 0$ for $|\sigma| \ge 11/4$, $|\psi| \le 1$, $|\psi'| \le 3/2$. Next set

$$\phi(t,\sigma) = \psi(\sigma)\sin(k(\sigma - t/2)).$$

Here $k \ge 2\pi$ is a large parameter. Consider the hypersurface $\Sigma \subset \mathbb{R}^4$ and the domain $Q_0 \subset \mathbb{R}^4$ given by

$$\Sigma = \left\{(t,x) \in \mathbb{R}^4;\ (x_2 - \phi(t,x_1))^2 + x_3^2 = r^2,\ t \in \mathbb{R},\ |x_1| < 3\right\},$$

$$Q_0 = \left\{(t,x) \in \mathbb{R}^4;\ (x_2-\phi(t,x_1))^2 + x_3^2 < r^2,\ t \in \mathbb{R},\ |x_1| < 3\right\}.$$

Here $r \in (0,1/2)$ is a small parameter. Clearly, Σ is a smooth manifold and $\Sigma \subset \partial Q_0$. We set $\Omega_0(t) = \{x;\ (t,x) \in Q_0\}$. The domain $\Omega_0(t)$ can be considered as a small neighborhood of the curve

$$\gamma_t = \left\{x;\ x_2 = \phi(t,x_1),\ x_3 = 0,\ |x_1| \le 3\right\}.$$

It is not hard to see that Σ is time-like. Let us define a velocity field v associated with Σ. Further we denote by K_α the cube $K_\alpha = \{x \in \mathbb{R}^3;\ |x_i| \le \alpha,\ i = 1,2,3\}$. Pick a function $\chi \in C_0^\infty(\mathbb{R}^3)$, such that $|\chi| \le 1$, $\chi(x) = 1$ for $x \in K_{11/4}$ and $\chi(x) = 0$ for $x \in K_3$. Given $(t,x) \in \mathbb{R}^4$ we set

$$v(t,x) = \tfrac{1}{2}\chi(x)\big(1\ ,\ \psi'(x_1)\sin(k(x_1-t/2))\ ,\ 0\big).$$

Clearly, $v(t,x) = (1/2,0,0)$ for $x \in K_2$ while $v = 0$ for $x \notin K_3$. The properties of ψ imply $|v|<1$. A straightforward calculation shows that $(1,v)$ is tangent to Σ. Thus $v(t,x)$ generates a flow $F_{t,s}:\mathbb{R}^3 \longrightarrow \mathbb{R}^3$ that maps $\Omega_0(s)$ onto $\Omega_0(t)$. Note that $F_{t,s} = Id$ on $\mathbb{R}^3\setminus K_3$.

Let $Q \subset \mathbb{R}^4$ be a domain with smooth boundary such that $\Omega(t) = \{x;\ (t,x) \in Q\}$ satisfies the conditions

C1) $\Omega(t) \cap K_3 = \Omega_0(t)$;

C2) $\Omega(t) \cap (\mathbb{R}^3\setminus K_3)$ is stationary (time-independent);

C3) if $K' = \{\ x;\ -4 \le x_1 \le -3,\ |x_2| \le 1,\ |x_3| \le 1\ \}$, then $\partial\Omega(t) \cap \partial K' = \partial\Omega_0(t) \cap \partial K' = \{\ x;\ x_1 = -3,\ x_2^2 + x_3^2 = r^2\}$;

C4) $\Omega(t) \subset K_5$;

(see Figure 1). Clearly, Q satisfies (B1), (B2) with $\varrho > 5\sqrt{3} = \frac{1}{2}\mathrm{diam}K_5$. Moreover, $\Omega(t+4\pi/k) = \Omega(t)$ thus the motion is periodic with

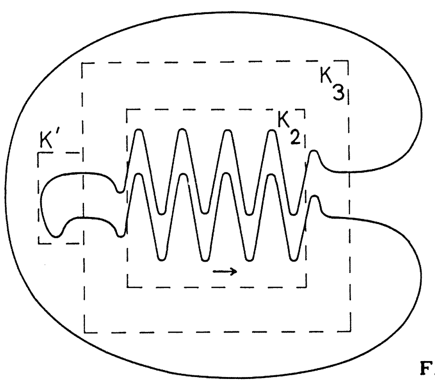

Fig. 1

period $4\pi/k$. Below we denote $\Omega' = \Omega(t) \cap K'$, which does not depend on t by virtue of (C2). We have the following property.

Proposition 10. *There is no piecewise smooth curve* $\{x = x(t),\ a \le t \le b\}$ *such that* $x(a) \notin B_\varrho$, $x(t) \in \overline{\Omega(t)}$ *for all* $t \in [a,b]$, $x(b) \in \Omega'$ *and* $|x'| \le 1$.

In other words, if we travel in the exterior of the obstacle with a speed not greater than 1, starting from $\mathbb{R}^3 \setminus B_\varrho$, we shall never reach Ω'. Using the principle of causality we get the following.

Proposition 11. *For any* $b \ge a$ *and for any* $f \in \mathscr{H}(a)$ *with* $\operatorname{supp} f \cap B_\varrho = \emptyset$ *we have* $U_Q(b,a)f = 0$ *in* Ω'.

Now consider a family $\mathscr{F}$ of domains Q such that each $Q \in \mathscr{F}$ satisfies (C1) -(C4) and $\Omega_1(t) \cap (\mathbb{R}^3 \setminus K') = \Omega_2(t) \cap (\mathbb{R}^3 \setminus K')$ for any $Q_1 \in \mathscr{F}$, $Q_2 \in \mathscr{F}$. Note that we do not impose any restriction on $\Omega(t) \cap K'$, thus the geometry of $\partial\Omega(t) \cap K'$ may be arbitrary, provided that (C3) holds. Hence $\mathscr{F}$ consists of infinitely many distinct domains Q and the corresponding obstacles $\mathcal{O}(t)$ move with the same period $4\pi/k$. Using Proposition 11 we prove that $K^\#_Q$ does not "feel" the shape of $\partial\Omega(t) \cap K'$ therefore all obstacles in $\mathscr{F}$ have the same generalized scattering kernel.

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