Scattering and Inverse Scattering in $\mathbf{R}^n$

Plamen Stefanov

Last updated: March 29, 2018

This is a draft of lecture notes of a graduate course that I was teaching in 2013. Sources are the lecture notes of Melrose [**9**] and Zworski [**26**], Sjöstrand [**16**], Taylor [**25**, Chapter 9], Reed and Simon [**14**], Petkov [**12**], and of course, Hörmander [**4**]. The inverse scattering material comes from many papers, including some of mine.

Contents

CHAPTER I

Introduction

1. Typical examples

We start with the basics of the scattering theory in $\mathbf{R}^n$, $n \geq 2$. We are interested in operators P that are compactly supported perturbations of the Laplacian $-\Delta$, satisfying some additional reasonable "black box" assumptions. The most important examples are potential scattering, obstacle scattering, and scattering by a metric.

In *potential scattering*, P is the Schrödinger operator

$$P = -\Delta + V, \tag{1.1}$$

where the potential $V = V(x) \in L^\infty$ is compactly supported. The perturbation of $-\Delta$ is in a lower order term (-2 degrees lower), which means Euclidean geometry dictated by $-\Delta$.

The PDEs we study are either the wave equation

$$(\partial_t^2 + P)u = 0 \tag{1.2}$$

or the "stationary wave equation"; or the Helmholtz one

$$(P - \lambda^2)u = 0, \tag{1.3}$$

where λ is a spectral parameter, not necessarily real. Equation (1.3) is just a Fourier transformed version of (1.2).

In *obstacle scattering*, we have a compact set $\mathcal{O} \subset \mathbf{R}^n$ with a smooth boundary and some choice of self-adjoint coercive boundary conditions on $\partial\mathcal{O}$, usually Dirichlet or Neumann. We can assume that the exterior $\mathbf{R}^n \setminus \mathcal{O}$ is connected (otherwise there will be connected bounded components disjoint from the unbounded interior, playing no role in the scattering). Then $P = -\Delta$ in $\mathbf{R}^n \setminus \mathcal{O}$ with the corresponding boundary conditions. One can consider a more general second order elliptic self-adjoint operator. The PDEs are still (1.2) or (1.3) but in $\mathbf{R}_t \times (\mathbf{R}^n \setminus \mathcal{O})_x$, and $\mathbf{R}^n \setminus \mathcal{O}$, respectively.

In the scattering by a metric example, $P = -\Delta_g$ is the Laplace-Beltrami operator related to a given Riemannian metric g in $\mathbf{R}^n$ so that g is Euclidean, i.e., $g = \delta_{ij}$, for $|x| > R_0$ with some $R_0 > 0$. One can include lower order terms as well.

Another important example: transparent obstacle...

Scattering Theory studies solutions of time dependent equations like (1.2) or (1.3) for large values of time t and/or large value of the spatial variable x. *Inverse Scattering* tries to recover the operator P (i.e., the potential, or the obstacle, or the metric) from scattering data (from the scattering operator, for example). In some applications, like astronomy or quantum mechanics, inverse scattering is the only (more or less) way to see what is there.

The *time dependent scattering theory* deals with equations of the type (1.2). The *stationary scattering theory* studies equations without a time variable, like (1.2).

2. Abstract (Time-Dependent) Scattering Theory; brief introduction

Reference: [**14**].

The time dependent scattering theory usually compares one unperturbed dynamics to a perturbed one. Let P_0 (say, $P_0 = -\Delta$) and P be self-adjoint operators in a Hilbert space $\mathcal{H}$ as in (1.1), and let the "free dynamics" $U_0(t) = \exp(\mathrm{i}tP_0)$ and the "perturbed one" $U(t) = \exp(\mathrm{i}tP)$ be the corresponding solution unitary groups (Stone's theorem). We want to compare $U(t)$ to $U_0(t)$ for $\pm t \gg 1$.

Take $U(t)f$, where f is the initial condition. We can expect that $U(t)$ would look like a "free solution" for $\pm t \gg 1$. In other words, given f, we would expect that there exist $g_\pm$, viewed as incoming/outgoing "profiles" of $U(t)f$ at $|t| \gg 1$, so that

$$\|U(t)f - U_0(t)g_\pm\| \to 0, \quad \pm t \to \infty. \tag{2.1}$$

This is equivalent to

$$\|g_\pm - U_0(-t)U(t)f\| \to 0, \quad \pm t \to \infty.$$

This is not guaranteed by any means; we expect to be true under "reasonable" assumptions. Then

$$g_\pm = W_\pm(P_0, P)f := \lim_{t\to\pm} U_0(-t)U(t)f.$$

Note that we have a strong limit here. It would be wrong to expect limit in the operator norm. If $W_\pm(P_0, P)$ exist, we have scattering theory for P_0 and P. This is equivalent to the completeness property below. Then

$$S : g_- \longmapsto g_+ \tag{2.2}$$

is the *scattering operator*. It is not well defined yet before we understand the properties of the wave operators. Formally,

$$g_+ = Sg_- = W_+(P_0, P)W_-^{-1}(P_0, P)g_-$$

but this is a correct definition only if the wave operators have the completeness properties below.

Proving existence of $W_\pm(P_0, P)f$ is usually not easy. The following *wave* operators are much easier to show that they exist

$$W_\pm(P, P_0)g := \lim_{t\to\pm} U(-t)U_0(t)g.$$

If they exist (as strong limits, again), we call them *complete* if

$$\operatorname{Ran} W_-(P, P_0) = \operatorname{Ran} W_+(P, P_0) = \mathcal{H}_{\mathrm{ac}}.$$

The latter space corresponds to the absolutely continuous spectrum of P, by the spectral theorem. If the completeness holds (the "$= \mathcal{H}_{\mathrm{ac}}$" condition is not needed for this definition), then the scattering operator is given by

$$Sg_- = W_+^{-1}(P, P_0)W_-(P, P_0)g_-.$$

Now, this is a well defined operator. It is unitary.

The main goal of Scattering Theory is to prove the existence of S, which can be done by showing that the wave operators $W_\pm(P, P_0)$ exists are a complete. It does not stop there however. We want to know as much as we can about S, find a representation, etc.

The wave and the scattering operators were defined by a limit $t \to \pm\infty$. It is not obvious that $|x|$ large are involved there as well. The latter is true but implicit. Typical dynamics $U_0(t)$ have the *local energy decay* property:

$$\|\mathbf{1}_K U_0(t) f\| \to 0, \quad \text{as } |t| \to \infty \tag{2.3}$$

for any compact set K, where $\mathbf{1}_K$ is the multiplication by the characteristic function of K. Then $U_0(t)g$ "goes to infinity" for any g; and if we want (2.1) to be true as well, so would $U(t)f$.

3. A few remarks about the Stationary Scattering Theory

The Stationary Scattering Theory studies systems without an explicit time variable like (1.1) for large values of the spatial variable x. The spectral parameter λ might vary; then we want to know how everything depends on it; including what the analytic or the meromorphic properties of the various objects (the natural version of the scattering operator S, for example) are in the complex plane $\mathbf{C}$. Of particular interests are the *high-frequency asymptotics*, $\lambda \to \infty$.

The frequency $\lambda > 0$ might be fixed. Then we still have scattering theory.

It turns out that for many interesting systems (operators P), the various scattering objects associated with it have certain meromorphic extensions to $\mathbf{C}$. Their poles are the *scattering poles* or the *resonances* (there is a slight difference between those two notions).

Since there is no time involved, the behavior as $|x| \to \infty$ is what determines the scattering properties. We will see below that there is a natural notion of incoming and outgoing term in that asymptotic behavior of the perturbed system; then the scattering matrix $S(\lambda)$ is defined as in (2.2), mapping the incoming term to the outgoing one.

CHAPTER II

Scattering Theory for compactly supported perturbations of the Laplacian in $\mathbf{R}^n$

1. The stationary Scattering Theory for the Laplacian in $\mathbf{R}^n$

In these notes, the unperturbed operator is the Laplace operator $P_0 = -\Delta$, although in some important cases like elasticity, Maxwell's equations, etc., we must make a different choice. The goal of the stationary theory is to understand the large x behavior of the perturbed system (1.3), and to compare it to that of the unperturbed one, corresponding to P_0. Even though that $P_0 = -\Delta$ is a simple enough operator; we need first to understand the behavior of solutions of the Helmholtz equation

$$(-\Delta - \lambda^2)u = 0, \quad |x| \gg 1 \tag{1.1}$$

near infinity.

1.1. The free outgoing resolvent. A fundamental object is the free outgoing (or incoming) resolvent. Take any solution of (1.1), multiply by a smooth cut-off χ equal to one near infinity; and zero in a neighborhood of the compact set where (1.1) may not hold to get

$$(-\Delta - \lambda^2)\chi u = -[\Delta, \chi]u, \quad x \in \mathbf{R}^n. \tag{1.2}$$

The r.h.s. depends on the values of u in some annulus $R_1 \le |x| \le R_2$ only. Alternatively, we may want to solve a boundary value problem in the exterior of some sphere. We therefore naturally get the following inhomogeneous equation

$$(-\Delta - \lambda^2)u = f, \quad x \in \mathbf{R}^n \tag{1.3}$$

where f is compactly supported. Clearly, there is no uniqueness of the solution of (1.3). The homogeneous Helmholtz equation has infinitely many solutions, for example $e^{\mathrm{i}\lambda x\cdot\omega}$, for any unit ω, any linear combination of that; and more generally, any integral w.r.t. ω w.r.t. any reasonable density on the unit sphere. Add any such solution to u in (1.3), and we still get a solution. It turns out that there is unique outgoing/incoming choice however.

We will seek tempered solutions to (1.3). Take a Fourier transform of both sides (assume $f \in C_0^\infty$ for now) to get

$$\left(|\xi|^2 - \lambda^2\right)\hat{u} = \hat{f}. \tag{1.4}$$

Divide to get

$$\hat{u}(\xi) = \frac{\hat{f}(\xi)}{|\xi|^2 - \lambda^2}. \tag{1.5}$$

The problem with this formula is that we are dividing by zero for ξ on the sphere $|\xi| = |\lambda|$, when $\mathbf{R} \ni \lambda \neq 0$ (we assume $\lambda \neq 0$ in what follows). The singularity is bad enough to be not locally integrable. The formula is OK however for $\lambda \in \mathbf{C} \setminus \mathbf{R}$, and represents the resolvent $(-\Delta - \lambda^2)^{-1}$ in functional analysis sense. Note that $-\Delta$ naturally can be considered as a self-adjoint operator on $L^2(\mathbf{R}^n)$ with domain $\{f \in L^2;\ \Delta f \in L^2\}$; and a conjugation with the Fourier transform (scaled to a unitary one) reduces it to a multiplication by $|\xi|^2$. Then non-real λ's are in the resolvent set and "in the Fourier domain", the resolvent is just a multiplication by $(|\xi|^2 - \lambda^2)^{-1}$, i.e.,

$$R_0(\lambda) = \left(-\Delta - \lambda^2\right)^{-1} = \mathcal{F}^{-1}\left(|\xi|^2 - \lambda^2\right)^{-1}\mathcal{F}, \quad \lambda \notin \mathbf{R}. \tag{1.6}$$

The outgoing resolvent $R_0(\lambda)$ is defined as the analytic continuation of the operator (1.5) from $\Im\lambda > 0$ to a small neighborhood (at least) of $\mathbf{R}$ in $\mathbf{C}$. To justify this definition we need to prove that such an extension exists in appropriate spaces. This is certainly not true if $R(\lambda)$ is considered as an operator on $L^2(\mathbf{R}^n)$ because λ^2 is in the spectrum $[0, \infty)$ of $-\Delta$ for real λ. It is possible however of we shrink the domain and expand the range appropriately.

THEOREM 1.1. *Let $n \geq 3$ be odd. Then the operator*

$$R_0(\lambda) = \left(-\Delta - \lambda^2\right)^{-1} : L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$$

defined a priori for $\Im\lambda > 0$ extends to an entire operator-valued function

$$R_0(\lambda) : L^2_{\text{comp}}(\mathbf{R}^n) \to L^2_{\text{loc}}(\mathbf{R}^n)$$

of λ in the whole complex plane $\mathbf{C}$.

Note that the extension is not given by the formula (1.6) for $\Im\lambda < 0$! This would become more explicit below. We could also extend the resolvent from $\Im\lambda < 0$ to $\mathbf{C}$. That would give us the *incoming* resolvent $R_0^-(\lambda)$.

In even dimensions, the extension is to the logarithmic covering of the complex plane.

There are several ways to prove the theorem.

1.1.1. *Time-dependent approach.* See [**26**, Chapter 3]. We go back to the time-dependent wave approach. In odd dimensions $n \geq 3$, we have the strong Huygens' Principle: if $(\partial_t^2 - \Delta)u = 0$ and the support of the Cauchy data $(u, u_t)|_{t=0}$ is in the ball $|x| \leq R$, then $\operatorname{supp} u(t, \cdot)$ is in $t - R < |x| < t + R$. When $t \leq R$, the first inequality carries no information but when $t > R$ it says that the wave has a "back front". In even dimensions, we can only say that $\operatorname{supp} u(t, \cdot)$ is in $|x| < t + R$, which is just finite speed of propagation (speed ≤ 1), actually. The strong Huygens' Principle, valid in odd dimensions, can be interpreted as wave speed equal to one exactly.

The analytic continuation of $R_0(\lambda)$ from $\Im\lambda > 0$ to $\mathbf{C}$ follows directly from the strong Huygens' Principle, as we will show below.

Recall the textbook Kirchoff's formula for the solution of the wave equation in odd dimensions $n \geq 3$:

$$(\partial_t^2 - \Delta)u = 0, \quad (u, u_t)|_{t=0} = (f_1, f_2). \tag{1.7}$$

The solution is given by

$$u = \partial_t U(t) f_1 + U(t) f_2, \tag{1.8}$$

where

$$U(t)\psi(x) = C_n(t^{-1}\partial_t)^{(n-3)/2}t^{n-2}\int_{|\omega|=1}\psi(x+t\omega)\mathrm{d}\omega.$$

The strong Huygens' Principle follow directly now from this formula.

The following formula relates the time-dependent theory and the stationary one:

$$R_0(\lambda)f = \int_0^\infty e^{\mathrm{i}\lambda t}U(t)f\,\mathrm{d}t, \quad f \in L^2, \quad \Im\lambda > 0. \tag{1.9}$$

Indeed, $|e^{\mathrm{i}\lambda t}| = e^{-t\Im\lambda}$ decays exponentially with t. It enough to show that $U(t)f$ grows no faster than a polynomial in L^2 to show that the integral is absolutely convergent. This follows from the representation

$$U(t) = \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}$$

(which in turn follows from using the Fourier transform to solve the wave equation). Since $|\sin(t|\xi|)|/|\xi| \le |t|$ for all t, ξ (easy to prove), we have $\|U(t)\| \le C|t|$. Apply $-\Delta-\lambda^2$ to the r.h.s. of (1.9) now; call it g. Use the fact that $U(t)f$ solves the wave equation with initial conditions $(0,f)$ and integrate by parts. The latter is justified by the fact that all integrals which arise are absolutely convergent. We will get $(-\Delta-\lambda^2)g = f$. For $\Im\lambda > 0$, this has unique solution for g; namely $g = R_0(\lambda)f$. This proves (1.9).

Now, it is enough to show that for any $\chi \in C_0^\infty$, the "cut-off resolvent" $\chi R_0(\lambda)\chi$ has entire continuation from $\Im\lambda > 0$ to $\mathbf{C}$. This follows immediately from (1.9) and the strong Huygens' Principle. Indeed,

$$\chi R_0(\lambda)\chi f = \int_0^\infty e^{\mathrm{i}\lambda t}\chi U(t)\chi f\,\mathrm{d}t, \quad \Im\lambda > 0.$$

If $\operatorname{supp}\chi \subset B(0,R)$ for some R (and such an R always exists), then $U(t)\chi f = 0$ for $|x| \le t-R$. Therefore, $\chi U(t)\chi f = 0$ for $t > 2R$, $\forall x$. The integral above is taken over the finite interval $t \in [0,2R]$ only. Then it clearly extends to an entire function of λ.

Note that there is no clear choice which continuation to call incoming and which outgoing. We could have started from $\Im\lambda < 0$ (which is our incoming choice now) as the physical half-plane. Then we have the formula (1.9) again but with i replaced by $-$i. Then we show that there is a homomorphic extension to $\mathbf{C}$. We can easily declare this to be our outgoing choice; and in fact, this is done in many papers. This discussion reveals something else. In (1.9), t appears as the dual variable to λ because of the choice of the sign of the phase in $e^{\mathrm{i}\lambda t}$. In other words, the Helmholtz equation (1.3) is the inverse Fourier transform of the wave equation (1.2). If we change that sign, we are thinking as λ as the dual variable to t, i.e., the Helmholtz equation (1.3) is now the Fourier transform of the wave equation (1.2). In either case, one of the equations is transformed into the other but with two different (adjoint to each other) transforms. Which way to go is a matter of taste.

1.1.2. *Explicit formula for $R_0(\lambda)$ in 3D.* Another way to prove the analytic extension is just to compute the Schwartz kernel $R_0(x-y,\lambda)$ (which is clearly a convolution, see below) of $R_0(\lambda)$. By (1.6),

$$R_0(x,\lambda) = c_n\int\frac{e^{\mathrm{i}x\cdot\xi}}{|\xi|^2-\lambda^2}\mathrm{d}\xi, \quad \Im\lambda > 0.$$

The integral is a Fourier transform, and not absolutely convergent. When $n = 3$ at least, one could pass to polar coordinates, modify the contour of integration in the integral w.r.t. the radial variable to get the formula in Proposition 1.2 below. One can do the following instead. The kernel $R_0(x, \lambda)f$ solves

$$(-\Delta - \lambda^2)R_0 = \delta, \tag{1.10}$$

and is clearly radially symmetric. Passing to polar coordinates, we get

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{n-1}{r}\frac{\mathrm{d}}{\mathrm{d}r} + \lambda^2\right) R_0 = 0, \quad r > 0.$$

Since the polar change of variables is singular at $r = 0$, there is no simple way of including the delta on the right at this point. The equation we got is of Bessel type and can be reduced to the actual Bessel equation by the substitution $R_0 = r^{1-n/2}\psi(r)$, see also [**24**, Ch.6.3]. Let us consider the case $n = 3$ only. Then

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r} + \lambda^2\right) R_0 = 0, \quad r > 0,$$

and the substitution $R = \phi(r)/r$ (different form the one above!) reduces this to the equation

$$\psi'' + \lambda^2\psi = 0, \quad r > 0$$

That substitution is in any PDE textbook as a way to solve the 3D wave equation. Now, $\psi = c_1 e^{\mathrm{i}\lambda r} + c_2 e^{-\mathrm{i}\lambda r}$; therefore,

$$R_0 = c_1\frac{e^{\mathrm{i}\lambda r}}{r} + c_2\frac{e^{-\mathrm{i}\lambda r}}{r}, \quad r = |x| > 0.$$

Remember, we need a kernel that is a tempered function of λ for $\Im\lambda > 0$, since by the spectral theorem, $\|R_0(\lambda)\| \leq |\Im\lambda^2|^{-1}$. This eliminates the $e^{-\mathrm{i}\lambda r}$ term, so we get $R_0 = Ce^{\mathrm{i}\lambda r}/r$, $r > 0$, $n = 3$. Recall that we do not really know that R_0 solves (1.10). We derived this by assuming that it does.

Now, we go back to Cartesian coordinates and look for a constant C (which may a priori depend on λ!) so that

$$(-\Delta - \lambda^2)C\frac{e^{\mathrm{i}\lambda r}}{r} = \delta \tag{1.11}$$

(and we have an intelligent guess that such a constant must exist). If $\lambda = 0$, then the right constant is $C = (4\pi)^{-1}$ since it is well known that $(4\pi r)^{-1}$ is a fundamental solution of $-\Delta$. The expression we integrate is locally integrable; so it is a distribution. Use the Chain rule, valid for distributions multiplied by a smooth function as well:

$$(\Delta + \lambda^2)\frac{e^{\mathrm{i}\lambda r}}{r} = \frac{(\Delta + \lambda^2)e^{\mathrm{i}\lambda r}}{r} + 2(\partial_x e^{\mathrm{i}\lambda r}) \cdot (\partial_x \frac{1}{r}) + e^{\mathrm{i}\lambda r}\Delta\frac{1}{r}$$

The first derivatives of r and $1/r$ can be computed in the classical way and the results are locally integrable, therefore distributions as well (every distribution is differentiable but the point here is how to compute the derivatives). Since $\Delta(4\pi r)^{-1} = -\delta$, we get

$$(\Delta + \lambda^2)\frac{e^{\mathrm{i}\lambda r}}{r} = \frac{(-\lambda^2 + (2/r)(\mathrm{i}\lambda) + \lambda^2)e^{\mathrm{i}\lambda r}}{r} + 2\mathrm{i}\lambda e^{\mathrm{i}\lambda r}\frac{x}{r} \cdot \left(-\frac{x}{r^2}\right) + e^{\mathrm{i}\lambda r}(-4\pi\delta).$$

Therefore,

$$(\Delta + \lambda^2)\frac{e^{\mathrm{i}\lambda r}}{r} = -4\pi\delta,$$

so $C = (4\pi)^{-1}$,

So we proved the following.

PROPOSITION 1.2. *For $n = 3$,*

$$[R_0(\lambda)f](x) = \int \frac{e^{\mathrm{i}\lambda|x-y|}}{4\pi|x-y|} f(y)\,\mathrm{d}y.$$

The analytic extension from $\Im\lambda > 0$ to $\mathbf{C}$ follows immediately from this formula. The Limiting Absorption Principle below gives another way to prove the existence of an analytic extension.

1.2. The outgoing Green's function. The kernel of $R_0(\lambda)$ is called the outgoing Green's function $G_0(x, y, \lambda)$ of the free Laplacian. Clearly, $G_0(x, y, \lambda) = R_0(x - y, \lambda)$. As we showed above,

$$G_0(x, y, \lambda) = \frac{e^{\mathrm{i}\lambda|x-y|}}{4\pi|x-y|}, \quad n = 3. \tag{1.12}$$

In all dimensions, including even ones, G_0 is expressed in terms of some Hankel function:

$$G_0(x, y, \lambda) = \frac{1}{2\mathrm{i}}(2\pi)^{1-n/2}\lambda^{n-2}(\lambda r)^{1-n/2}H^{(1)}_{n/2-1}(\lambda r), \quad r = |x - y|, \tag{1.13}$$

see [**15**].

1.3. Resolvent estimates.

THEOREM 1.3. *For any $\lambda > 0$ and any $\chi \in C_0^\infty$,*

$$\|\chi R_0(\lambda)\chi\|_{L^2\to L^2} \le \frac{C}{\lambda}.$$

PROOF. We will prove it for n odd only, following [**26**].

Let $U(t)$ be as above. By (1.9), if R is such that $\operatorname{supp}\chi \subset B(0, R)$,

$$\begin{aligned}\lambda\chi U(t)\chi f &= \chi\int_0^{2R} \lambda e^{\mathrm{i}\lambda t}U(t)\chi f\,\mathrm{d}t = \chi\int_0^{2R}\frac{1}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}t}e^{\mathrm{i}\lambda t}U(t)\chi f\,\mathrm{d}t\\ &= \mathrm{i}\chi^2 + \mathrm{i}\chi\int_0^{2R} e^{\mathrm{i}\lambda t}U'(t)\chi f\,\mathrm{d}t.\end{aligned}$$

We have $U'(t) = -\mathrm{i}\cos(t\sqrt{-\Delta}) = O(1)$ as an operator in L^2. Therefore, the right-hand side above is uniformly bounded in λ, which proves the theorem. □

It is worth noticing that

$$\|\chi R_0(\lambda)\chi\|_{L^2\to H^s} \le \frac{C}{\lambda^{1-s}}, \quad s = 0, 1, 2.$$

The proof can be found in [**26**]. The idea is, roughly speaking that to estimate the H^2 norm, for example, we need to apply $-\Delta$ and estimate the result. Then we write $-\Delta = (-\Delta - \lambda^2) + \lambda^2$.

1.4. The Limiting Absorption Principle. The microlocal view of the resolvent $R_0(\lambda)$ is the following. We have a Fourier multiplier with a symbol $(|\xi|^2 - z)^{-1}$, $z = \lambda^2$, see (1.6), where $z = \lambda^2$. When $\Re\lambda > 0$ and $0 \leq \Im\lambda \ll 1$, we have $\Re z \approx \Re\lambda^2$, $\Im z = 2\Re\lambda\Im\lambda$, so we still have $\Re z > 0$, $0 < \Im z \ll 1$. The zeros of $|\xi|^2 - z$ then are simple ($f(\xi) = 0$ implies $\mathrm{d}f(\xi) \neq 0$). So the question is; how do we divide, in the Fourier domain, by functions with simple zeros? A good reference for that is [**4**, XIV.14.2].

If $f(x)$ is a function with a simple zero at, say, zero, then in some local coordinates wit would look as $f = t$, where t is one of the variables. A known fact in theory of distributions is that the limits

$$(t \pm \mathrm{i}0)^{-1}$$

exist in $\mathcal{D}'(\mathbf{R})$. Moreover,

$$\frac{1}{t \pm \mathrm{i}0} = \mathrm{pv}\frac{1}{t} \mp \mathrm{i}\pi\delta(t). \tag{1.14}$$

In particular,

$$\frac{1}{t + \mathrm{i}0} - \frac{1}{t - \mathrm{i}0} = -2\pi\mathrm{i}\delta(t).$$

Now, this is also true if we set $t = p(\xi)$, $\xi \in \mathbf{R}^n$ under the assumption $\mathrm{d}p(0) \neq 0$:

$$\frac{1}{p(\xi) + \mathrm{i}0} - \frac{1}{p(\xi) - \mathrm{i}0} = -2\pi\mathrm{i}\delta(p(\xi)).$$

Recall that $\delta(p(\xi)) = |\mathrm{d}p(\xi)|^{-1}\delta_{p^{-1}(0)}(\xi)$. Apply this to $p(\xi) = |\xi|^2 - \tau$, $\tau > 0$, to get

$$\frac{1}{|\xi|^2 - \tau + \mathrm{i}0} - \frac{1}{|\xi|^2 - \tau - \mathrm{i}0} = -\pi\mathrm{i}|\xi|^{-1}\delta_{|\xi|^2=\tau}(\xi).$$

Set $\tau = \lambda^2$, $\lambda > 0$, to get

$$\frac{1}{|\xi|^2 - (\lambda + \mathrm{i}0)^2} - \frac{1}{|\xi|^2 + (\lambda + \mathrm{i}0)^2} = \pi\mathrm{i}\lambda^{-1}\delta_{|\xi|=\lambda}(\xi).$$

By (1.6),

$$\begin{aligned}
[R_0(\lambda + \mathrm{i}0)f - R_0(\lambda - \mathrm{i}0)f](x) &= \pi\mathrm{i}\lambda^{-1}(2\pi)^{-n}\int_{|\xi|=\lambda} e^{\mathrm{i}x\cdot\xi}\hat{f}(\xi)\,\mathrm{d}S(\xi) \\
&= \pi\mathrm{i}\lambda^{-1}(2\pi)^{-n}\lambda^{n-1}\int_{|\theta|=1} e^{\mathrm{i}\lambda x\cdot\theta}\hat{f}(\lambda\theta)\,\mathrm{d}\theta \\
&= \frac{\mathrm{i}}{2}(2\pi)^{-n+1}\lambda^{n-2}\int_{|\theta|=1} e^{\mathrm{i}\lambda x\cdot\theta}\hat{f}(\lambda\theta)\,\mathrm{d}\theta \\
&= \frac{\mathrm{i}}{2}(2\pi)^{-n+1}\lambda^{n-2}\iint_{|\theta|=1} e^{\mathrm{i}\lambda(x-y)\cdot\theta}f(y)\,\mathrm{d}\theta\,\mathrm{d}y.
\end{aligned}$$

Recall that we call $R_0(\lambda + \mathrm{i}0)$ just $R_0(\lambda)$. Then $R_0(\lambda + \mathrm{i}0)$ is the incoming resolvent, which is easily seen to be $R_0^*(\lambda)$ for $\lambda \in \mathbf{R}$, and also $R_0(-\lambda)$.

We proved the following limiting absorption principle (Stone's theorem in fact):

THEOREM 1.4. *For $\lambda \in \mathbf{R}$, $n \geq 2$,*

$$R_0(\lambda)f - R_0(-\lambda)f = \frac{\mathrm{i}}{2}(2\pi)^{-n+1}\lambda^{n-2} \iint_{|\theta|=1} e^{\mathrm{i}\lambda(x-y)\cdot\theta} f(y)\,\mathrm{d}\theta\,\mathrm{d}y. \tag{1.15}$$

We proved the theorem for $\lambda > 0$. The proof for all real λ follows by analytic continuation once we establish that the latter exists near the real axis (true in even dimensions as well but we have to remove the origin then). We can continue analytically anywhere when the right-hand side is analytic, of course.

The operator on the right can be interpreted as a spectral projection, see [**9**]. The theorem in particular shows that the difference between the outgoing continuation (from $\Im\lambda > 0$) and the incoming one (from $\Im\lambda < 0$) to the real axis is a non-trivial operator.

REMARK 1.1. The kernel in the integral above is just the Fourier transform of the delta function on the unit sphere evaluated at $\xi = -\lambda(x-y)$. This can be computed — that delta function is a radial distribution, and then Fourier transform reduces to the Hankel transform of order $n - 1/2$

$$F(\rho) = (2\pi)^{n/2}\rho^{n/2-1}\int_0^\infty f(r)J_{n/2-1}(\rho r)r^{n/2}\,\mathrm{d}r,$$

where $F(\rho)$, $\rho = |\xi|$, is the Fourier transform of the radial function $f(r)$. When $f = \delta(r-1)$, we get

$$F(\rho) = (2\pi)^{n/2}\rho^{n/2-1}J_{n/2-1}(\rho),$$

see `http://math.unc.edu/Faculty/met/bessel.pdf`. Then the integral in (1.15) is just $F(\lambda|x-y|)$ for $\lambda > 0$, compare it with (1.13). When $n = 3$, since $\rho^{-1/2}J_{1/2}(\rho) = \sqrt{2/\pi}\sin\rho/\rho$, we get $(n = 3)$

$$R_0(\lambda)f - R_0(-\lambda)f = \frac{\mathrm{i}}{2\pi}\int \frac{\sin(\lambda|x-y|)}{|x-y|}f(y)\,\mathrm{d}y.$$

We could have obtained the same result by just subtracting $R_0(\lambda)$ in Proposition 1.2 and its incoming version $R_0(-\lambda)$.

1.5. Asymptotics.

THEOREM 1.5. *For $n \geq 2$ and any $f \in \mathcal{E}'(\mathbf{R}^n)$, we have*
(a)

$$[R_0(\lambda)f](r\theta) = \frac{e^{\mathrm{i}\lambda r}}{r^{(n-1)/2}}h(r,\theta), \tag{1.16}$$

$$h(r,\theta) \sim \sum_{j=0}^\infty r^{-j}h_j(\theta), \quad \textit{as } r \to \infty,$$

and

$$h_0(\theta) = \frac{1}{4\pi\mathrm{i}}\left(\frac{\lambda}{2\pi}\right)^{\frac{1}{2}(n-3)} e^{\frac{1}{4}\pi i(n-1)}\hat{f}(\lambda\theta).$$

In particular, if $n = 3$, then

$$[R_0(\lambda)f](r\theta) = \frac{e^{\mathrm{i}\lambda r}}{4\pi r}\left(\hat{f}(\lambda\theta) + O\left(\frac{1}{r}\right)\right), \quad \textit{as } r \to \infty.$$

(b) Next, $u = R_0(\lambda)f$ satisfies the following ***Sommerfeld radiation condition***

$$(\partial_r - \mathrm{i}\lambda)\, u(r\theta) = o\left(r^{-(n-1)/2}\right), \quad r \to \infty, \tag{1.17}$$

uniformly in $\theta \in S^{n-1}$.

PROOF. I will prove the $n = 3$ case only. For the general case, see e.g., [**26**].
For $n = 3$, write

$$|r\theta - y| = r - \theta \cdot y + O(1/r), \quad |r\theta - y|^{-1} = r^{-1}(1 + O(r^{-1})), \quad r \to \infty$$

and use Proposition 1.2 with $x = r\theta$.

The proof of (b) follows directly by applying $(\partial_r - \mathrm{i}\lambda)$ to the kernel. Notice that we actually get $O\left(r^{-(n+1)/2}\right)$. □

1.6. Outgoing and incoming solutions.

DEFINITION 1.6. Given $\lambda \in \mathbf{C}$, we say that the function u is λ-outgoing (or simply, outgoing, if λ is understood from the context), if there exists $a > 0$ and $f \in \mathcal{E}'$ such that $u|_{|x|>a} = R_0(\lambda)f|_{|x|>a}$.

When n is even, one has to assume that λ belongs to the logarithmic covering of $\mathbf{C}$, actually. If we stay close to $\mathbf{R}$, then there is no problem even then.

Clearly, any outgoing u solves (1.1). By elliptic regularity, it is real analytic for $|x| \gg 1$. One defines incoming solution in a similar way. Note that outgoing and incoming solutions satisfy

$$u = O(1/r^{(n-1)/2}), \quad r = |x| \to \infty. \tag{1.18}$$

There are many solutions to the Helmholtz equation for $|x| \gg 1$ which do not satisfy such an estimate. For example, $e^{\mathrm{i}\lambda\rho\cdot x}$, for $\rho \in C^n$ is a solution whenever $\rho_1^2 + \cdots + \rho_n^2 = 1$, and it is exponentially increasing for $\rho \notin \mathbf{R}^n$; take for example $\rho = (t, \mathrm{i}\sqrt{t^2-1}, 0, \ldots, 0)$, $t > 1$.

We show below that the Sommerfeld radiation condition (1.17) guarantees the outgoing properties of solutions to the Helmholtz equation. This condition is clearly satisfied by the first term in (1.16) (as well as for the whole expansion, as we proved above). The incoming condition is the same with $\mathrm{i}\lambda$ replaced by $-\mathrm{i}\lambda$. Another way to think about it is the following. Since t is the dual variable to λ, ∂_t corresponds to $-\mathrm{i}\lambda$. The Sommerfeld condition then say that, up to lower order terms, $(\partial_t + \partial_r)v = 0$, where v is the Fourier transform of u w.r.t. λ. Then $v = f(r - t)$ modulo lower order terms. This is a wave going to infinity, as $t \to \infty$, so it deserves to be called outgoing. The incoming condition would be $f(r + t)$.

THEOREM 1.7. *Let $\lambda > 0$. The function u is λ-outgoing if and only if u solves the Helmholtz equation for $|x| \gg 1$ and satisfies the Sommerfeld outgoing condition* (1.17). *Moreover,*

(a) For any compact set $K \subset \mathbf{R}^n$ with smooth boundary and connected complement and for any $h \in H^{3/2}(\partial K)$, the problem

$$(-\Delta - \lambda^2)u = 0 \quad \text{in } \mathbf{R}^n \setminus K, \quad u|_{\partial K} = h \tag{1.19}$$

has unique outgoing solution.

(b) For any $f \in L^2_{\mathrm{comp}}(\mathbf{R}^n)$*, the problem*

$$(-\Delta - \lambda^2)u = f \quad \text{in } \mathbf{R}^n \tag{1.20}$$

has unique outgoing solution given by $u = R_0(\lambda)f$*.*

We will start with the following uniqueness statement, see also (a). We follow [**25**, Ch. 9] here.

PROPOSITION 1.8. *Let* $\lambda > 0$*. If*

$$(-\Delta - \lambda^2)u = 0 \quad \text{in } \mathbf{R}^n \setminus K, \quad u|_{\partial K} = 0 \tag{1.21}$$

and u *satisfies* (1.17)*, then* $u = 0$*.*

PROOF. Let $S_R = \{|x| = R\}$. For $R \gg 1$, $S_R \subset \mathbf{R}^n \setminus K$, and

$$\int_{S_r} |u_r - \mathrm{i}\lambda u|^2 \mathrm{d}S = \int_{S_R} \left(|u_r|^2 + \lambda^2 |u|^2\right) \mathrm{d}S - \mathrm{i}\lambda \int_{S_R} (u\bar{u}_r - \bar{u}u_r)\, \mathrm{d}S. \tag{1.22}$$

The integral on the right vanishes because by Green's formula

$$\int_{S_R} (u\bar{u}_r - \bar{u}u_r)\, \mathrm{d}S = \int_{\partial K} \left(u\frac{\partial \bar{u}}{\partial \nu} - \bar{u}\frac{\partial u}{\partial \nu} \right) \mathrm{d}S = 0.$$

On the other hand, by (1.17),

$$\int_{S_R} |u_r - \mathrm{i}\lambda u|^2 \mathrm{d}S \longrightarrow 0, \quad \text{as } R \to \infty$$

because in polar coordinates, $\mathrm{d}S = R^{n-1}\mathrm{d}S_0$, where $\mathrm{d}S_0$ is the natural measure on the unit sphere. Then (1.22) implies

$$\int_{S_R} |u|^2 \, \mathrm{d}S \longrightarrow 0, \quad \text{as } R \to \infty. \tag{1.23}$$

The proof is completed by the following lemma. □

LEMMA 1.9. *If* u *solves the Helmholtz equation* $(-\Delta - \lambda^2)u = 0$ *for* $|x| > R_0 > 0$ *and* (1.23) *holds, then* $u = 0$ *for* $|x| > R_0$*.*

PROOF. Any such solution u has the representation (1.12). By Parseval's identity,

$$\left| a_{lm}(\lambda) h_l^{(1)}(\lambda r) + b_{lm}(\lambda) C_2 h_l^{(2)}(\lambda r) \right|^2 = o(r^{(n-1)/2}), \quad \text{as } r \to \infty.$$

It is easy to see, using (1.9) in Chapter V, that no non-trivial linear combination satisfies this, thus $a_{lm} = b_{lm} = 0$. □

PROOF OF THEOREM 1.7. We start with (b). Clearly, $R_0(\lambda)f$ is an outgoing solution, by definition, satisfies the Sommerfeld radiation condition (1.17), by Theorem 1.5; therefore it is unique by what we just proved. This in particular proves the "only if" part of the first statement of the theorem.

The proof of the existence part of (b) will be given in Theorem 3.1, see also [**25**], pp.147–151. We first show there that the problem is uniquely solvable for $\Im\lambda > 0$ by reducing it to a problem for the inhomogeneous equation (1.20) outside K with a compactly supported f and zero Dirichlet data on ∂K. Then we can just apply the resolvent $(-\Delta_D - \lambda^2)^{-1}$, where

Δ_D is the Dirichlet realization of the Laplacian. Then we show that the limiting absorption principle applies, and we can take the limit $\Im\lambda \to 0+$.

Finally, we need to prove the "if" statement. Let u solve the Helmholtz equation for $|x| \gg 1$ and satisfy (1.17). Then it solves (1.2) as well. Then by (b), $u = -R_0(\lambda)[\Delta, \chi]u$. Therefore, u is outgoing. □

1.7. Far field pattern. Our definition of incoming/outgoing solutions, combined with Theorem 1.5, implies the following.

COROLLARY 1.10. *Every outgoing (incoming) solution u satisfies the following asymptotic*

$$u(x) = \frac{e^{\pm i\lambda|x|}}{|x|^{(n-1)/2}} h(x/|x|) + O\left(\frac{1}{|x|^{(n+1)/2}}\right), \quad \text{as } |x| \to \infty \tag{1.24}$$

with some $h \in C^\infty(S^{n-1})$.

The function h is called the applied literature the *far field pattern* of u. Equation (1.2) and Theorem 1.5 imply the following

$$f(\theta) = c_n(\lambda)\mathcal{F}([\Delta, \chi]u)(\lambda\theta), \quad c_n(\lambda) = -\frac{1}{4\pi i}\left(\frac{\lambda}{2\pi}\right)^{\frac{1}{2}(n-3)} e^{\frac{1}{4}\pi i(n-1)}, \tag{1.25}$$

where $\chi \in C_0^\infty$ is such that u solves the Helmholtz equation on $\operatorname{supp}\chi$. This shows that h is actually a real analytic function of θ.

THEOREM 1.11 (The Rellich Iniqueness Theorem). *Let u be an outgoing (incoming) solution with zero far field pattern. Then* $\operatorname{supp} u$ *is compact.*

PROOF. It follows directly from Lemma 1.9. In this case, it is even simpler because $b_{lm} = 0$. □

1.8. The absolute scattering matrix. We follow [**9**] here. For any unit θ, the function $e^{i\lambda\theta\cdot x}$ solves the Helmholtz equation. Those functions are viewed as *harmonic plane waves* "moving" in the direction θ. Their Fourier transform in λ, up to a constant factor, is $\delta(t - x\cdot\omega)$, known as *plane waves.*

Clearly,

$$\int_{S^{n-1}} e^{i\lambda\theta\cdot x} g(\theta)\, d\theta \tag{1.26}$$

is also a solution of the Helmholtz equation in $\mathbf{R}^n$ for any $g \in \mathcal{D}'(S^{n-1})$ (the integral is understood in distributional sense). In fact, all tempered solutions u of the Helmholtz equation in $\mathbf{R}^n$ are of that type for $\lambda > 0$. Indeed, any such u has a Fourier transform $\hat{u}$ and the latter solves $(|\xi|^2 - \lambda^2)\hat{u}(\xi) = 0$. Therefore, $\hat{u}$ is supported on λS^{n-1}. Since $|\xi|^2 - \lambda^2$ has simple zeros on the sphere λS^{n-1}, $\hat{u}$ is a delta-type of distribution on it.

LEMMA 1.12. *For $\lambda > 0$,*

$$\int_{S^{n-1}} e^{\mathrm{i}\lambda\theta\cdot x} g(\theta)\,\mathrm{d}\theta \sim e^{\frac{1}{4}\pi(n-1)\mathrm{i}}(2\pi)^{(n-1)/2}\frac{e^{-\mathrm{i}\lambda|x|}}{(\lambda|x|)^{(n-1)/2}}\sum_{j\geq 0}|x|^{-j}h_j^-(x/|x|) + e^{-\frac{1}{4}\pi(n-1)\mathrm{i}}(2\pi)^{(n-1)/2}\frac{e^{\mathrm{i}\lambda|x|}}{(\lambda|x|)^{(n-1)/2}}\sum_{j\geq 0}|x|^{-j}h_j^+(x/|x|),$$

as $|x| \to \infty$, where, in particular, $h_0^\pm(\omega) = g(\pm\omega)$, and $h_j^\pm$ for $j \geq 1$ are given by linear combinations of certain derivatives of $g(\pm\theta)$.

The proof is based on the stationary phase method. The phase $\lambda r\theta\cdot\omega$ (we set $x = r\omega$) has stationary points at $\theta = \pm\omega$.

Note that in particular,

$$\int_{S^{n-1}} e^{\mathrm{i}\lambda\theta\cdot x} g(\theta)\,\mathrm{d}\theta = (2\pi\mathrm{i})^{(n-1)/2}\Big(\frac{e^{-\mathrm{i}\lambda|x|}}{(\lambda|x|)^{(n-1)/2}}g\Big(-\frac{x}{|x|}\Big) + \frac{e^{\mathrm{i}\lambda|x|}}{(\lambda|x|)^{(n-1)/2}}(-\mathrm{i})^{n-1}g\Big(\frac{x}{|x|}\Big)\Big) + O\Big(\frac{1}{|x|^{(n+1)/2}}\Big). \tag{1.27}$$

As in the time dependent case, we want to define the scattering operator as the operator mapping the incoming part of any solution to its the outgoing one. Here we do not have a perturbed dynamics yet but the question is still meaningful one. In the classical Newtonian case, where the dynamics is given by $(x, v) \mapsto (x+tv, v)$, the incoming and the outgoing parts can be both associated with the same point in the phase space: (x, v). Then the scattering operator S would be identity, $S = \mathrm{I}$. This has a lot to do with the way we parametrize the incoming and the outgoing points and directions. We could also think of $-v$ as an incoming direction — where we have to look at to see the particle coming. Then $S = -\mathrm{I}$.

Relation (1.27) shows that the incoming-to-outgoing maps would be

$$g(\omega) \quad\longmapsto\quad (-i)^{n-1}g(-\omega)$$

as the following lemma shows.

LEMMA 1.13. *For each $\lambda > 0$ and for each $h \in C^\infty(S^{n-1})$, there is a unique solution u of the Helmholtz equation in $\mathbf{R}^n$ so that*

$$u(r\theta) = \frac{e^{-\mathrm{i}\lambda r}}{r^{(n-1)/2}}h(\theta) + \frac{e^{\mathrm{i}\lambda r}}{r^{(n-1)/2}}\tilde{h}(\theta) + O\Big(\frac{1}{r^{(n+1)/2}}\Big) \tag{1.28}$$

and necessarily,

$$\tilde{h}(\theta) = (-\mathrm{i})^{n-1}h(-\theta). \tag{1.29}$$

The map (1.29) is the *absolute scattering matrix* (the term "matrix" comes from physics). We would expect to be identity but it is not. On the other hand, it is a unitary isomorphism in $L^2(S^{n-1})$. Once and tweak a bit the definition of what is considered incoming and outgoing patterns (see (1.28)) to make it exactly identity.

PROOF. The existence follows from (1.27): u is given by the l.h.s. of (1.27) with g being a multiple of $h(-\theta)$. To prove the uniqueness, let u_1 and u_2 be two such solutions. Then $v = u_1 - u_2$ would be an outgoing solution in the whole space $\mathbf{R}^n$. Then both (a) and (b) of Theorem 1.1 hold. Take $x \neq 0$ then, to deduce $2c_{lm} = a_{lm} = b_{lm}$, by (1.8). On the other hand, $b_{lm} = 0$ since u is outgoing. Then $b_{lm} = 0$ as well. □

2. Potential Scattering

The first scattering problem we consider is the stationary scattering theory for the "stationary Schrödinger equation"

$$\big(-\Delta + V(x) - \lambda^2\big)u = 0 \quad \text{in } \mathbf{R}^n. \tag{2.1}$$

Here, the potential V will be assumed to be real-valued and in $L^\infty_{\rm comp}$ but sometimes we may assume higher regularity, like $C_0^\infty(\mathbf{R}^n)$. The compactness of the support is not so essential as long as V is *short-range*

$$|V(x)| \le C(1+|x|)^{-1-\varepsilon}, \quad \varepsilon > 0. \tag{2.2}$$

This condition can be relaxed a bit but when $\varepsilon = 0$, or more generally, for $-1 < \varepsilon < 0$, the potential is called *long-range* and the whole theory needs to redone.

Equation (2.1) is related to the wave equation

$$(\partial_t^2 - \Delta + V)u = 0 \tag{2.3}$$

through the Fourier transform $\mathcal{F}_{\lambda\to t}$; and to the Schrödinger equation

$$(-\mathrm{i}\partial_t - \Delta + V)u = 0 \tag{2.4}$$

through the Fourier transform $\mathcal{F}_{\lambda^2\to t}$. The links to the time-dependent scattering theory in each case will be discussed later.

2.1. The perturbed resolvent $R(\lambda)$. Note first that $P = -\Delta + V$ is a self-adjoint (with the same domain as $-\Delta$). It is actually a relatively compact perturbation of the latter and as such, the essential spectrum of P is the same as that of $-\Delta$, namely, $[0,\infty)$. The whole spectrum is in $[-C_0,\infty)$, where $-C_0 \le 0$ is such that $-C_0 \le V$. Then the true resolvent

$$R(\lambda) = (-\Delta + V - \lambda^2)^{-1}$$

exists for λ away from the spectrum. If V can take negative values, it is possible, in principle, that P would have negative eigenvalues $-\mu_j$. Then $\Im\lambda > 0$ is not sufficient for λ to be in the resolvent set since at the points $\lambda_j = \mathrm{i}\sqrt{\mu_j}$, the resolvent would have a pole. For $\Im\lambda > \sqrt{C_0}$ this cannot happen however.

Sometimes, complex values potentials are of interest as well. Then P is not self-adjoint anymore but $(P-\lambda^2)^{-1}$ can be shown to exist for $\Im\lambda \gg 0$ as well, as the first part of the theorem below asserts.

THEOREM 2.1. *Let $V \in L^\infty_{\rm comp}(\mathbf{R}^n;\mathbf{C})$, and let $n \ge 3$ be odd. Then*

(a) the resolvent $R(\lambda)$ is a meromorphic family of operators in $\Im\lambda > 0$ (and also, in $\Im\lambda < 0$).

(b) $R(\lambda)$ extends as a meromorphic operator-valued function

$$R(\lambda) : L^2_{\rm comp}(\mathbf{R}^n) \to L^2_{\rm loc}(\mathbf{R}^n)$$

from $\Im\lambda > 0$ to the whole complex plane $\mathbf{C}$.

(c) The following estimate holds: for any $\chi \in C_0^\infty$,

$$\|\chi R(\lambda)\chi\| \leq \frac{C}{\lambda}, \quad \lambda > 0.$$

Note that when V is real-valued, (a) is trivial, and $R(\lambda)$ is actually without poles there, as we already mentioned.

PROOF. First we show that $R(\lambda)$ exists for $\Im\lambda \gg 1$. If course, for real-valued potentials, this is trivial. If there were a resolvent for some λ, then it would satisfy the resolvent identity

$$R(\lambda) - R_0(\lambda) = -R(\lambda)VR_0(\lambda). \tag{2.5}$$

This, and some of its versions, is known as the *Lipmann-Schwinger equation*. Hence,

$$R(\lambda)(\mathrm{I} + VR_0(\lambda)) = R_0(\lambda). \tag{2.6}$$

If we can invert $\mathrm{I} + VR_0(\lambda)$, we are done, and we can write

$$R(\lambda) = R_0(\lambda)(\mathrm{I} + VR_0(\lambda))^{-1}. \tag{2.7}$$

One can check directly that this is a left and right inverse. On the other hand,

$$\|R_0(\lambda)\| \leq 1/\mathrm{dist}(\Im\lambda^2, \mathbf{R}_+) \leq 1/|\Im\lambda|^2,$$

therefore, for $\Im\lambda \gg 1$, $\mathrm{I} + VR_0(\lambda)$ is invertible, indeed. For $\Im\lambda > 0$, $VR_0(\lambda)$ is compact, and the analytic Fredholm theorem completes the proof of (a).

To prove (b), we would like to apply the analytic Fredholm theorem again. Note first that we cannot apply it directly to (2.7) because we have proved existence of analytic extension of $VR_0(\lambda)$ only as an operator from $L^2_{\mathrm{comp}}(\mathbf{R}^n)$ to L^2 but not from L^2 to itself (the latter is not true). Instead, we will do the following. Choose $\chi \in C_0^\infty$ so that $\chi V = V$. Then by (2.5),

$$R(\lambda)\chi - R_0(\lambda)\chi = -R(\lambda)\chi VR_0(\lambda)\chi.$$

Therefore,

$$R(\lambda)\chi\,(\mathrm{I} + VR_0(\lambda)\chi) = R_0(\lambda)\chi.$$

For $\Im\lambda \gg 1$, the operator $\mathrm{I} + VR_0(\lambda)\chi$ is invertible by the arguments above. Then

$$R(\lambda)\chi = R_0(\lambda)\chi\,(\mathrm{I} + VR_0(\lambda)\chi)^{-1} : L^2(\mathbf{R}^n) \longmapsto L^2_{\mathrm{loc}}(\mathbf{R}^n). \tag{2.8}$$

We are ready to apply the analytic Fredholm theorem now. The operator $VR_0(\lambda)\chi$ is compact, depends analytically on $\lambda \in \mathbf{C}$, and $\mathrm{I} + VR_0(\lambda)\chi$ is invertible for $\Im\lambda \gg 1$. Therefore, $\chi R(\lambda)\chi$ extends to a meromorphic family.

Part (c) follows from (2.8) and Theorem 1.3 directly because $\|(\mathrm{I} + VR_0(\lambda)\chi)^{-1}\| \leq 2$ for $\lambda \gg 1$. ☐

REMARK 2.1. Note that we can get (2.8) by expanding (2.7) in Neumann series for $\Im\lambda \gg 0$ and applying χ to the right. Indeed, for $\Im\lambda \gg 0$,

$$R(\lambda) = R_0(\lambda) + R_0(\lambda)VR_0(\lambda) + R_0(\lambda)VR_0(\lambda)VR_0(\lambda) + \dots. \tag{2.9}$$

Therefore,

$$\begin{aligned} R(\lambda)\chi &= R_0(\lambda)\chi - R_0(\lambda)\chi V R_0(\lambda)\chi + R_0(\lambda)\chi V R_0(\lambda)\chi V R_0(\lambda)\chi + \dots \\ &= R_0(\lambda)\chi\,(\mathrm{I} - V R_0(\lambda)\chi + V R_0(\lambda)\chi V R_0(\lambda)\chi + \dots) \\ &= R_0(\lambda)\chi\,(\mathrm{I} + V R_0(\lambda)\chi)^{-1} \end{aligned}$$

for $\Im\lambda \gg 0$, and by meromorphic continuation, is true everywhere. We also note for future reference that the same arguments imply the following

$$\chi R(\lambda) = (\mathrm{I} + \chi R_0(\lambda) V)^{-1}\,\chi R_0(\lambda). \tag{2.10}$$

Definition 2.2. The poles of $R(\lambda)$ are called (scattering) *resonances*. The multiplicity $m(\lambda)$ of each resonance is given by

$$m(\lambda) = \operatorname{rank} \oint_{|\zeta-\lambda|=\varepsilon} R(\zeta)\,\mathrm{d}\zeta,$$

where the circle $|\zeta - \lambda| = \varepsilon > 0$ is so small that λ is the only pole inside it.

When V is real-valued, the only resonances in $\Im\lambda > 0$ are the ones corresponding to negative eigenvalues, $\lambda_j = \mathrm{i}\sqrt{\mu_j}$. Also, there are no resonances on $\mathbf{R} \setminus 0$, see [**26**].

Theorem 2.3. *Let $L^\infty_{\mathrm{comp}}(\mathbf{R}^n)$ be real valued. Then*
(a) There are no non-trivial outgoing solutions of (2.1)*.*
(b) There are no non-zero real scattering poles.

Proof. For (a), we follow the proof of Proposition 1.8. Let u be an outgoing solutions of (2.1). We start with (1.22). The integral in the r.h.s. there vanishes as well; to see that, we integrate $\bar{u}(\Delta + V - \lambda^2)u = 0$ over the ball $B(0, R)$. Then we arrive at (1.23). Lemma 1.9 then implies that $u = 0$ for $|x| \gg 1$. By unique continuation, $u = 0$ everywhere.

To prove (b), it is enough to show that $\mathrm{I} + \chi R_0(\lambda)V$, see (2.10), is invertible. Since the operator $V R_0(\lambda)\chi$ is compact, it is enough to show that $\mathrm{I} + \chi R_0(\lambda)V$ has a trivial kernel.

Let

$$(\mathrm{I} + \chi R_0(\lambda)V)\psi = 0, \quad \psi \in L^2. \tag{2.11}$$

Then the outgoing function $v := -R_0(\lambda)V\psi$ solves $(-\Delta - \lambda^2)v + V\psi = 0$. By (2.11), $\psi = \chi v$, therefore,

$$(-\Delta + V - \lambda^2)v = 0.$$

By (a), $v = 0$. □

2.2. The "naive" scattering theory. The scattering amplitude. We start with an intuitive view of scattering theory which is dominant in (some of) the physical and in (most of) the applied math literature.

The "harmonic plane wave" solutions of the Helmholtz equation $e^{\mathrm{i}\lambda x\cdot\theta}$, $|\theta| = 1$, are viewed as elementary waves "propagating" in the direction θ. They are neither incoming nor outgoing. Even though there is no time in the stationary theory, we are thinking about (2.1) as an inverse (by the choice of our convention) Fourier transformed version of the wave equation (2.3). Then $(2\pi)^{-n}\mathcal{F}_{\lambda\to t} e^{\mathrm{i}\lambda x\cdot\theta} = \delta(t - x\cdot\theta)$; and the later is a "plane wave" moving in the direction θ. It is an analog of the classical particles $x = t\theta$ moving in the same direction.

So we illuminate the system with the harmonic plane wave $e^{\mathrm{i}\lambda x\cdot\theta}$, let it interact with the potential, and look for the difference $u_{\mathrm{sc}} = u - e^{\mathrm{i}\lambda x\cdot\theta}$, where u is the solution; as a sign of how the potential has interacted with that illumination. This intuitive view of the scattering process suggests that we should require u_{sc} to be outgoing. If $V = 0$, $u_{\mathrm{sc}} = 0$. The large $|x|$ limit (the far-field pattern) of u_{sc} is our scattering data, called the scattering amplitude a. The latter is a function of the incoming direction θ, the frequency/energy λ and the outgoing direction $\omega = x/|x|$, i.e., $a = a(\lambda, \theta, \omega)$. The scattering amplitude is basically the kernel of the scattering operator S minus identity, as we will see later.

The discussion above suggests the following. We should look for a solution $u(x, \theta, \lambda)$ of (2.1) of the form

$$u = e^{\mathrm{i}\lambda x\cdot\theta} + u_{\mathrm{sc}}, \tag{2.12}$$

with u_{sc} outgoing. Then u_{sc} is the outgoing solution of

$$(-\Delta + V - \lambda^2)u_{\mathrm{sc}} = -Ve^{\mathrm{i}\lambda x\cdot\theta}. \tag{2.13}$$

Then

$$u_{\mathrm{sc}} = -R(\lambda)\big(Ve^{\mathrm{i}\lambda x\cdot\theta}\big), \tag{2.14}$$

where, somewhat incorrectly, $R(\lambda)\big(Ve^{\mathrm{i}\lambda x\cdot\theta}\big)$ stands for $R(\lambda)$ applied to the function $x \to V(x)e^{\mathrm{i}\lambda x\cdot\theta}$. Indeed, by (2.8), for any compactly supported f, $R(\lambda)$ is outgoing.

Another equation for u_{sc} can be obtained by writing

$$(-\Delta - \lambda^2)u_{\mathrm{sc}} = -Vu.$$

Since u_{sc} is outgoing, we get

$$u_{\mathrm{sc}} = -R_0(\lambda)Vu, \tag{2.15}$$

therefore,

$$u(x, \theta, \lambda) = e^{\mathrm{i}\lambda x\cdot\theta} - \int G_0(x, y, \lambda)V(y)u(y, \theta, \lambda)\,\mathrm{d}y, \tag{2.16}$$

which is the classical Lippmann-Schwinger equation in quantum mechanics. Recall that if $n = 3$, the kernel G_0 of R_0 takes the simple form (1.12).

Since u_{sc} is outgoing, it has a far field pattern, see (1.7). Therefore, there exists a function $a(\theta, \omega, \lambda)$, where $\omega = x/|x|$, so that

$$u(x, \theta, \lambda) = e^{\mathrm{i}\lambda x\cdot\theta} + \frac{e^{\mathrm{i}\lambda|x|}}{|x|^{(n-1)/2}}a(x/|x|, \theta, \lambda) + O\left(\frac{1}{|x|^{(n+1)/2}}\right), \quad \text{as } |x| \to \infty. \tag{2.17}$$

DEFINITION 2.4. The function $a(\omega, \theta, \lambda)$, where $(\omega, \theta, \lambda) \in S^{n-1} \times S^{n-1} \times \mathbf{R}_+$ is called the scattering amplitude associated with V.

In other words, the scattering amplitude is what controls the first non-trivial term in the expansion of u as $|x| \to \infty$, up to the obligatory factor $|x|^{-(n-1)/2}$.

To obtain a representation of a, we take the far-field pattern of the integral term in (2.16). By Theorem 1.5,

$$a(\omega, \theta, \lambda) = -\frac{1}{4\pi\mathrm{i}}\left(\frac{\lambda}{2\pi}\right)^{\frac{1}{2}(n-3)} e^{\frac{1}{4}\pi\mathrm{i}(n-1)} \int e^{-\mathrm{i}\lambda\omega\cdot y}V(y)u(y, \theta, \lambda)\,\mathrm{d}y. \tag{2.18}$$

Of course, the interesting part is the integral there, and very often, the scattering amplitude is just defined as

$$a_0(\omega,\theta,\lambda) = \int e^{-\mathrm{i}\lambda\omega\cdot x} V(x) u(x,\theta,\lambda)\,\mathrm{d}x. \tag{2.19}$$

We will call a_0 the reduced scattering amplitude. The representation (2.19), together with the formulas (2.12) and (2.14) allow us to express the reduced scattering amplitude a_0 directly trough the resolvent $R(\lambda)$:

$$a_0(\omega,\theta,\lambda) = \int e^{-\mathrm{i}\lambda\omega\cdot x} V(x) \left((\mathrm{I} - R(\lambda)V)e^{\mathrm{i}\lambda x\cdot\theta}\right)\,\mathrm{d}x. \tag{2.20}$$

We therefore get the following.

Proposition 2.5. *Let $n \geq 3$ be odd. Then the scattering amplitude $a(\omega,\theta,\lambda)$ extends to a meromorphic functions of $\lambda \in \mathbf{C}$ with possible poles at the scattering resonances, with values in the real-analytic function of (θ,ω) .*

In particular, away from the scattering poles, a is analytic w.r.t. all of its variables. Actually, one can take θ and ω in the complex manifold $\xi^2 = 1$. Note that this representation does not prove that a must have a pole at each scattering pole — in principle, the singular part of $R(\lambda)$ may cancel by the integration. In fact, this cannot happen in $\Im\lambda < 0$ but this will be established later.

2.3. The relative and the absolute scattering matrix. The "naive" scattering theory does not go further — if we can determine the scattering effect of the potential for every illumination $\theta \in S^{n-1}$, we have full scattering data, intuitively. We now relate the scattering amplitude to the scattering matrix, defined similarly to section 1.8.

By (1.27) and (2.17), for any $g \in C^\infty(S^{n-1})$,

$$\begin{aligned}
&\int_{S^{n-1}} u(x,\theta,\lambda) g(\theta)\,\mathrm{d}\theta \\
&\quad = (2\pi\mathrm{i})^{(n-1)/2}\Bigg(\frac{e^{-\mathrm{i}\lambda|x|}}{(\lambda|x|)^{(n-1)/2}} g\Big(-\frac{x}{|x|}\Big) \\
&\quad\quad + \frac{e^{\mathrm{i}\lambda|x|}}{(\lambda|x|)^{(n-1)/2}}\Big[(-\mathrm{i})^{n-1} g\Big(\frac{x}{|x|}\Big) + \lambda^{(n-1)/2}\int_{S^{n-1}} a(x/|x|,\theta,\lambda) g(\theta)\,\mathrm{d}\theta\Big]\Bigg) \\
&\quad\quad + O\bigg(\frac{1}{|x|^{(n+1)/2}}\bigg), \quad \text{as } |x| \to \infty.
\end{aligned} \tag{2.21}$$

Therefore, the l.h.s. above is a sum of an outgoing and an incoming functions; with corresponding far-field patterns $g(-\omega)$ and $(-\mathrm{i})^{n-1} g(\omega) + \lambda^{(n-1)/2} a(\omega,\theta,\lambda)$. This motivates the following.

Definition 2.6.
(a) The *absolute scattering matrix* is the map

$$S_{\mathrm{abs}}(\lambda) : g(\omega) \longmapsto (-\mathrm{i})^{n-1} g(-\omega) + \lambda^{(n-1)/2}\int_{S^{n-1}} a(-\omega,\theta,\lambda) g(\theta)\,\mathrm{d}\theta.$$

(b) The *(relative) scattering matrix* is the map

$$S(\lambda) : g(\omega) \longmapsto g(\omega) + \mathrm{i}^{n-1}\lambda^{(n-1)/2} \int_{S^{n-1}} a(\omega, \theta, \lambda) g(\theta)\, \mathrm{d}\theta.$$

Note that the scattering matrix is the inverse of the absolute one for $V = 0$ composed with $S_{\mathrm{abs}}(\lambda)$. In particular, if $V = 0$, then $S(\lambda) = \mathrm{I}$ which restores the harmony.

THEOREM 2.7. *For any $\lambda > 0$ and any $h \in C^\infty(S^{n-1})$, there is unique solution v of* (2.1) *so that*

$$v(r\omega) = \frac{e^{-\mathrm{i}\lambda r}}{r^{(n-1)/2}} h(\omega) + \frac{e^{\mathrm{i}\lambda r}}{r^{(n-1)/2}} \tilde{h}(\omega, \lambda) + O\left(\frac{1}{r^{(n+1)/2}}\right), \tag{2.22}$$

where, necessarily,

$$\tilde{h}(\cdot, \lambda) = S_{\mathrm{abs}}(\lambda) h.$$

PROOF. The solution v is given by the integral in (2.21). The uniqueness follows from Theorem 2.3(a). □

This theorem and its proof is an analog of the completeness property of the wave operators.

We list some properties of the scattering matrix.

THEOREM 2.8. *Let $n \geq 3$ be odd and let $V \in L^\infty_{\mathrm{comp}}(\mathbf{R}^n)$ be complex-valued. Then $S(\lambda)$ is meromorphic with poles of finite rank and satisfies*

$$S^{-1}(\lambda) = S(-\lambda), \quad \lambda \in \mathbf{C}.$$

There are finitely many poles in $\Im\lambda \geq 0$; and for any pole in $\Im\lambda > 0$, λ^2 is in the spectrum of $-\Delta + V$.

If V is real valued, then

$$S^{-1}(\lambda) = S(\bar{\lambda})^*, \quad \lambda \in \mathbf{C}.$$

In particular, $S(\lambda)$ is unitary on $\mathbf{R}$ and analytic on $\mathbf{R}$.

DEFINITION 2.9. The poles of the scattering matrix $S(\lambda)$ are called *scattering poles*.

The scattering poles and the resonances coincide with a possible exception of finitely many on the imaginary axis. This will be shown later (hopefully).

3. Obstacle Scattering

Let $\mathcal{O}$ be a bounded compact (non-empty) set with smooth boundary and a connected exterior $\Omega = \mathbf{R}^n \setminus \mathcal{O}$. We study the Helmholtz equation in Ω with Dirichlet boundary conditions (Neumann boundary conditions are treated similarly)

$$(-\Delta - \lambda^2)u = 0 \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0. \tag{3.1}$$

If we send a harmonic plane wave $e^{\mathrm{i}\lambda\theta\cdot x}$, it will reflect from the obstacle, producing a response $u = e^{\mathrm{i}\lambda\theta\cdot x} + u_{\mathrm{sc}}$ as before. We think of u_{sc} as the scattered wave, which reflected off $\partial\Omega$ and propagates to infinity.

3.1. The resolvent. Let Δ_D be the natural self-adjoint realization of Δ_D corresponding to the Dirichlet boundary condition. In particular, the domain of Δ_D is given by

$$\mathcal{D}(\Delta_D) = H^2(\Omega) \cap H_0^1(\Omega).$$

The spectrum of $-\Delta_D$ coincides with $[0, \infty)$. Then the resolvent

$$R(\lambda) = (-\Delta_D - \lambda^2)^{-1}$$

exists for $\Im\lambda > 0$. We will show below that for n odd, it extends meromorhically to $\mathbf{C}$. The proof of this will given in the more general black-box setting later but we will present the classical approach below which has its own merits. We refer also to [**25**, IX.7].

Theorem 3.1. *For $n \geq 3$ odd, the resolvent $R(\lambda) : L^2_{\mathrm{comp}}(\Omega) \to L^2_{\mathrm{loc}}(\Omega)$ extends meromorphically from $\Im\lambda > 0$ to $\mathbf{C}$ with poles in $\Im\lambda < 0$.*

As before, the resonances are defined as the poles of $R(\lambda)$.

Instead of studying the problem

$$(-\Delta - \lambda^2)u = h, \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0, \tag{3.2}$$

we will study the homogeneous Helmholtz equation in Ω with prescribed data on $\partial\Omega$:

$$(-\Delta - \lambda^2)u = 0 \quad \text{in } \Omega, \quad u|_{\partial\Omega} = f. \tag{3.3}$$

Clearly, each one can easily be converted into the other.

We define the usual single and double layer potentials by

$$\mathrm{SL}(f)(x) = \int_{\partial\Omega} G_0(x,y,\lambda)f(y)\,\mathrm{d}S_y, \quad \mathrm{DL}(f)(x) = \int_{\partial\Omega} \partial_{\nu_y} G_0(x,y,\lambda)f(y)\,\mathrm{d}S_y.$$

Here, ν is the outer unit normal and we recall the notation $G_0(x,y,\lambda)$ for the kernel of $R_0(\lambda)$. Each of those integrals is a solution of the Helmholtz equation away from $\partial\Omega$ for any $f \in \mathcal{D}'(\partial\Omega)$. Formally restricted to $x \in \partial\Omega$, $\partial_{\nu_y} G$ has a singularity of the kind $\mathrm{dist}(x,y)^{2-n}$, which is integrable on $\partial\Omega$. On the other hand, $\partial_y G_0(x,y,\lambda)$ a singularity of the kind $\mathrm{dist}(x,y)^{1-n}$ which is not integrable on $\partial\Omega$, but $\partial_{\nu_y} G_0(x,y,\lambda)$ has a weaker one, same as $G_0(x,y,\lambda)$. Therefore, the following operators are well-defined on $\partial\Omega$, and are in fact compact ones on $L^2(\partial\Omega)$:

$$Kf(x) = \int_{\partial\Omega} G_0(x,y,\lambda)f(y)\,\mathrm{d}S_y, \quad Nf(x) = \int_{\partial\Omega} \partial_{\nu_y} G_0(x,y,\lambda)f(y)\,\mathrm{d}S_y, \quad x \in \partial\Omega.$$

If we let subscripts $\pm$ denote limits to $\partial\Omega$ from the interior, and, respectively, from the exterior of $\partial\Omega$, it is known that

$$\mathrm{SL}_+ = \mathrm{SL}_- = K, \quad \mathrm{DL}_\pm = \pm\frac{1}{2}\mathrm{I} + N. \tag{3.4}$$

We are looking for solutions of (3.3) of the form

$$u = \mathrm{DL}(g)$$

(or $u = \mathrm{SL}(h)$ or any linear combination of both). Then g has to solve

$$f = \frac{1}{2}g + Ng \quad \Longrightarrow \quad (\mathrm{I} + 2N)g = 2f.$$

If we can invert the Fredholm operator $\mathrm{I} + 2N$, we would get

$$u = 2\mathrm{DL}(\lambda)(\mathrm{I} + 2N(\lambda))^{-1}. \tag{3.5}$$

We indicated above the fact that both operators depend on λ and in fact, they are entire functions of $\lambda \in \mathbf{C}$.

Lemma 3.2. $(\mathrm{I} + 2N(\lambda))^{-1}$ *is a meromorphic operator-valued function of* $\lambda \in C$ *with poles in* $\Im\lambda \leq 0$. *The real poles are at* λ *such that* λ^2 *is a Neumann eigenvalue of* $-\Delta$ *in* $\mathcal{O}$.

Proof. Since $N(\lambda)$ is compact, $\mathrm{I} + 2N(\lambda)$ is invertible if and only if the equation $(\mathrm{I} + 2N(\lambda))g = 0$ has the trivial solution $g = 0$. Let g be a solution to that equation, and consider $u = \mathrm{DL}(\lambda)g$ defined away from $\partial\Omega$. It solves the Helmholtz equation there. In Ω, it satisfies the zero boundary condition. For $\lambda \neq 0$, it is outgoing, and by Theorem 1.7(a), $u = 0$ in Ω. For $\lambda = 0$ this follows by the maximum principle. For $\Im\lambda > 0$, λ^2 is in the resolvent set of $-\Delta_D$, therefore $u = 0$ then as well. Then $\partial_\nu u_+ = 0$ on $\partial\Omega$.

It is well known that the interior and the exterior normal derivatives of a double layer potential coincide. Hence, for $\Im\lambda \geq 0$, $\partial_\nu u_-$ on $\partial\Omega$ as well. Therefore, $u|_\mathcal{O}$ is a Neumann eigenfunction of $-\Delta$ with eigenvalue λ^2.

We therefore proved that $(\mathrm{I}+2N(\lambda))^{-1}$ exists for $\Im\lambda \geq 0$ with the exception of λ for which λ^2 is an interior Neumann eigenvalue. By the analytic Fredholm theorem, $(\mathrm{I} + 2N(\lambda))^{-1}$ extends to a meromorphic function on $\mathbf{C}$. □

Proof of Theorem 3.1. Now, assume that λ_0 is an interior Neumann eigenvalue of $-\Delta$. Then $(\mathrm{I} + 2N(\lambda))^{-1}$ has a pole at $\lambda = \lambda_0$, and near λ_0, we have the Laurent expansion

$$(\mathrm{I} + 2N(\lambda))^{-1} = A_0(\lambda) + \sum_{j=1}^{N} \frac{1}{(\lambda - \lambda_0)^j} A_j, \tag{3.6}$$

with $A_0(\lambda)$ holomorphic there, and $N < \infty$ (the latter follows from the analytic Fredholm theorem, part of the definition of a meromorphic function). Since

$$N(\lambda) - N(\lambda_0) = (\lambda - \lambda_0)\int_0^1 N'(\lambda_0 + s(\lambda - \lambda_0))\,\mathrm{d}s,$$

we have $N(\lambda) = N(\lambda_0) + (\lambda - \lambda_0)N_1(\lambda)$. Apply $\mathrm{I} + 2N(\lambda) = \mathrm{I} + 2N(\lambda_0) + 2(\lambda - \lambda_0)N_1(\lambda)$ to (3.6). On the left, we get identity; on the right, the most singular term would be

$$\frac{1}{(\lambda - \lambda_0)^N}(\mathrm{I} + 2N(\lambda_0))A_N.$$

Than term must vanish, therefore,

$$(\mathrm{I} + 2N(\lambda_0))A_N = 0.$$

Choose $f \in L^2(\partial\Omega)$ and let $g = A_N f$. Then we are in the situation above, and we get $u := \mathrm{DL}(\lambda_0)g = 0$ in Ω. If the r.h.s. of (3.5) had a pole at $\lambda = \lambda_0$, its most singular part would be

$$\frac{1}{(\lambda - \lambda_0)^N}\mathrm{DL}(\lambda_0)A_N : L^2(\partial\Omega) \to L^2_{\mathrm{loc}}(\Omega),$$

and we just established that this operator vanishes. Therefore, even though $(\mathrm{I} + 2N(\lambda))^{-1}$ does have poles for λ square root of an interior Neumann eigenvalue, the operator $\mathrm{DL}(\lambda)(\mathrm{I}+2N(\lambda))^{-1}$ does not — they are canceled by $\mathrm{DL}(\lambda)$. □

3.1.1. *The "naive" obstacle scattering theory and the scattering amplitude.* We are looking again for a solution $u(x, \theta, \lambda)$ of (3.1) of the form

$$u = e^{\mathrm{i}\lambda x\cdot\theta} + u_{\mathrm{sc}}, \tag{3.7}$$

with u_{sc} outgoing. Then u_{sc} is the outgoing solution of

$$(-\Delta - \lambda^2)u_{\mathrm{sc}} = 0, \quad u_{\mathrm{sc}}|_{\partial\Omega} = -e^{\mathrm{i}\lambda x\cdot\theta}. \tag{3.8}$$

This solution is well defined for λ away from the resonances as we just showed. Being outgoing, it has a far field pattern; therefore u satisfies (2.17) and we can define the scattering amplitude as in Definition 2.4.

Recall the notation $G_0(x, y, \lambda)$ for the kernel of $R_0(\lambda)$. Applying Green's theorem, we get

$$u_{\mathrm{sc}}(x) = \int_{\partial\Omega}\big((\partial_{\nu_y}G_0)(x, y, \lambda)u_{\mathrm{sc}}(y) - G_0(x, y, \lambda)\partial_\nu u_{\mathrm{sc}}(y)\big)\,\mathrm{d}S_y, \quad x \in \Omega,$$

where we suppressed the dependence of u_{sc} on θ and λ. Here ν is the outer unit normal to $\mathcal{O}$ (and inner for Ω). We take the asymptotic $|x| \to \infty$ using Theorem 1.5. Its proof (which we did for $n = 3$ only) easily implies an asymptotic expansion for $\partial_{\nu_y}G_0(x, y, \lambda)$ as well. We get

$$a(\omega, \theta, \lambda) = \frac{1}{4\pi}\left(\frac{\mathrm{i}\lambda}{2\pi}\right)^{\frac{1}{2}(n-3)}\int_{\partial\Omega}\left(u_{\mathrm{sc}}(y)\partial_{\nu_y}e^{-\mathrm{i}\lambda y\cdot\omega} - e^{-\mathrm{i}\lambda y\cdot\omega}\partial_\nu u_{\mathrm{sc}}(y)\right)\,\mathrm{d}S_y.$$

We proved the following.

THEOREM 3.3. *We have*

$$a(\omega, \theta, \lambda) = \frac{-1}{4\pi}\left(\frac{\mathrm{i}\lambda}{2\pi}\right)^{\frac{1}{2}(n-3)}\int_{\partial\Omega}\left(\mathrm{i}\lambda\omega\cdot\nu(y)e^{-\mathrm{i}\lambda y\cdot(\omega-\theta)} + e^{-\mathrm{i}\lambda y\cdot\omega}\partial_\nu u_{sc}(y, \theta, \lambda)\right)\,\mathrm{d}S_y.$$

We see again that $a(\omega, \theta, \lambda)$ extends to a meromorphic function of λ with possible poles at the resonances, with values in analytic functions of (θ, ω).

3.1.2. *The outgoing DN map.* Let u be the outgoing solution to (3.3). Set

$$N(\lambda)f = \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega}.$$

It is a well defined operator from $H^s(\partial\Omega)$ to $H^{s-1}(\partial\Omega)$ for $s > 1$ (at least, very often $s = 1/2$ actually), meromorphically depending on λ, with possible poles at the resonances. With its aid, the formula above can be written as

$$a(\omega,\theta,\lambda) = \frac{-1}{4\pi}\left(\frac{\mathrm{i}\lambda}{2\pi}\right)^{\frac{1}{2}(n-3)} \int_{\partial\Omega} \left(\mathrm{i}\lambda\omega\cdot\nu(y)e^{-\mathrm{i}\lambda y\cdot(\omega-\theta)} + e^{-\mathrm{i}\lambda y\cdot\omega}N(\lambda)e^{\mathrm{i}\lambda\bullet\cdot\theta}\right) \mathrm{d}S_y.$$

4. Scattering by Metrics

Let $g_{ij}(x)$ be a Riemannian metric equal to the Euclidean one for large $|x|$, say, $|x| > R_0$ for some $R_0 > 0$. The Laplace-Beltrami operator is given by

$$\Delta_g := \frac{1}{\sqrt{\det g}} \partial_{x^i} g^{ij} \sqrt{\det g} \partial_{x^j}.$$

Then we consider the equation $(-\Delta_g - \lambda^2)u = 0$. We could consider the more general second order elliptic operator

$$P = -c^{-2}\Delta_{g,A} := c^2 \frac{1}{\sqrt{\det g}} \left(\partial_{x^i} - A_j\right) g^{ij} \sqrt{\det g} \left(\partial_{x^j} - A_j\right) + q,$$

where the "magnetic field" $A(x) = (A_1(x), \dots, A_n(x))$ and the "electric field" $q(x)$ vanish outside the ball $B(0, R)$. We also assume that the "acoustic speed" $c(x)$ equals 1 outside that ball.

The reason to include c in the above definition is to have a direct way to treat both the operator $c^2\Delta$ and Δ_g. Clearly, modulo first order terms, P coincides with $-\Delta_{c^{-2}g}$, so the metric which determines the high-frequency properties of P is $c^{-2}g$.

Since P is written in (almost) divergence form, when all coefficients are real (which we assume), P is self-adjoint on the space $L^2(\mathbf{R}^n;\ c^{-2}\sqrt{\det g}\, \mathrm{d}x)$. Its essential spectrum is again $[0, \infty)$. One can define the resolvent

$$R(\lambda) = (P - \lambda^2)^{-1}, \quad \Im\lambda > 0.$$

The meromorphic extension properties will be proved in the next section. Note that even though P is a compactly supported perturbation of $-\Delta$, the difference $P - (\Delta)$ is not of lower order. Many properties are much closer to those of obstacle scattering than to the potential one.

5. Black Box Scattering

We present here the black box scattering framework introduced by Sjöstrand and Zworski [**17**]. It turns out that many (but not all!) properties like the existence of a meromorphic extension, some basic properties of the scattering matrix and amplitude, and even some properties of the resonances are the same for potential, obstacle scattering, and for general second order self-adjoint operators. This suggests that there might be a uniform approach which combines them all. Such an approach is the black box scattering framework.

At the beginning, we follow [**16**].

Let $\mathcal{H}$ and $\mathcal{H}_{R_0}$ be two complex Hilbert spaces so that

$$\mathcal{H} = \mathcal{H}_{R_0} \oplus L^2(\mathbf{R}^n \setminus B(0,R_0)),$$

where $R_0 > 0$. In potential scattering, $\mathcal{H}_{R_0} = L^2(B(0,R_0))$, in obstacle one, $\mathcal{H}_{R_0} = L^2(B(0,R_0)\setminus\mathcal{O})$, where $\mathcal{O} \subset B(0,R_0)$; and in the metric case, $\mathcal{H}_{R_0} = L^2(\mathbf{R}^n;\ c^{-2}\sqrt{\det g}\,\mathrm{d}x)$. Other examples are a compact manifolds with boundary (and possible "holes" and "handles") glued to the exterior of $B(0,R_0)$.

We denote by $\mathbf{1}_{B(0,R_0)}$ (or $u|_{B(0,R_0)}$), and by $\mathbf{1}_{\mathbf{R}^n\setminus B(0,R_0)}$ (or $u|_{\mathbf{R}^n\setminus B(0,R_0)}$) the projections to the first or the second component in the orthogonal decomposition above, respectively. Note that the former notation can be misleading: $u|_{B(0,R_0)})$ is the projection to $\mathcal{H}_{R_0}$ which may not be a space of functions defined on $B(0,R_0)$. Multiplication by $C_0^\infty(\mathbf{R}^n)$ functions χ is well-defined when χ is constant near $\overline{B(0,R_0)}$. We set

$$\mathcal{H}_{\mathrm{comp}} = \{u \in \mathcal{H};\ u|_{\mathbf{R}^n\setminus B(0,R_0)} \text{ has bounded support}\}$$
$$\mathcal{H}_{\mathrm{loc}} = \mathcal{H}_{R_0} \oplus L^2_{\mathrm{loc}}(\mathbf{R}^n \setminus B(0,R_0))$$

We define P now as a self-adjoint operator on $\mathcal{H}$ with domain $\mathcal{D}(P)$. We want first P to be equal to $-\Delta$ outside Ω:

$$Pu|_{\mathbf{R}^n\setminus B(0,R_0)} = -\Delta|_{\mathbf{R}^n\setminus B(0,R_0)}. \tag{5.1}$$

This needs to be complemented by the assumption that if $u \in \mathcal{D}(P)$, then its restriction to $\mathbf{R}^n \setminus B(0,R_0)$ is in H^2; and that any $u \in H^2(\mathbf{R}^n \setminus B(0,R_0)$ vanishing near $B(0,R_0)$ is in $\mathcal{D}(P)$ as well.

The second essential property replaces the requirement that P is elliptic (the latter does not actually make sense in this abstract setting):

$$\mathbf{1}_{B(0,R_0)}(P-\mathrm{i})^{-1} \quad \text{is compact on } \mathcal{H}. \tag{5.2}$$

Note that this condition is fulfilled in potential and in obstacle scattering.

Theorem 5.1.
(a) P can have only discrete spectrum in $(-\infty,0)$.
(b) $(P-\lambda^2)^{-1} : \mathcal{H}_{\mathrm{comp}} \mapsto \mathcal{H}_{\mathrm{loc}}$ has a meromorphic extension from $\{\Im\lambda > 0;\ \lambda^2 \notin \mathrm{spec}(P) \cap (-\infty,0)\}$ to $\mathbf{C}$, when $n \geq 3$ is odd; and to the logarithmic covering of $\mathbf{C}$ when n is even.

Proof. The first step is to show that (5.2) implies the following more general property:

$$\mathbf{1}_{B(0,R)}(P-z)^{-1} \quad \text{is compact on } \mathcal{H} \text{ for any } R \geq R_0 \text{ and any } z \in \rho(P), \tag{5.3}$$

where $\rho(P)$ is the resolvent set of P.

Indeed, by the resolvent identity,

$$\mathbf{1}_{B(0,R_0)}(P-z)^{-1} = \mathbf{1}_{B(0,R_0)}(P-z_0)^{-1} - \mathbf{1}_{B(0,R_0)}(P-z_0)^{-1}(z_0-z)(P-z)^{-1} \tag{5.4}$$

for any z_0 and z in the resolvent set. Take $z_0 = \mathrm{i}$ to conclude that $\mathbf{1}_{B(0,R_0)}(P-z)^{-1}$ is compact. Also, $(P-z)^{-1}\mathbf{1}_{B(0,R_0)}$ is compact, being adjoint to a compact operator.

Next, to show that we can replace R_0 with any $R > R_0$, notice that

$$\mathbf{1}_{B(0,R)\setminus B(0,R_0)}(P-z)^{-1}$$

is compact because $\mathbf{1}_{B(0,R)} : H^2(\mathbf{R}^n \setminus B(0,R_0)) \to L^2(\mathbf{R}^n \setminus B(0,R_0))$ is compact.

Next step is to construct an "approximate" resolvent. We use Sjöstrand's notation here: for any functions χ_1, χ_2, we say that $\chi_1 \prec \chi_2$ is $\chi_2 = 1$ near $\operatorname{supp}\chi_1$ (with $\chi_1 \in C_0^\infty$). Choose χ_0, χ_1, χ_2 in C_0^∞ so that $\mathbf{1}_{B(0,R)} \prec \chi_0 \prec \chi_1 \prec \chi_2$ (the first relation has a natural meaning). Then we set

$$Q(\lambda) = (1-\chi_0)R_0(\lambda)(1-\chi_1) + \chi_2 R(\lambda_0)\chi_1, \tag{5.5}$$

where λ_0 with $\lambda_0^2 \in \rho(P)$ will be chosen later, and $\Im\lambda > 0$. Roughly speaking, $Q(\lambda)$ is like $R_0(\lambda)$ outside the black box, and equal to $R(\lambda_0)$ (with λ_0 fixed) near the black box. We would like the latter to be $R(\lambda)$ but we have not proved its meromorphic continuation yet — actually, we are trying to prove it now. The compactness of the difference, with a cutoff applied to the left, see (5.4), will give us a compact operator error only.

Apply $P - \lambda^2$ to the first operator on the r.h.s. of (5.5):

$$(P-\lambda^2)(1-\chi_0)R_0(\lambda)(1-\chi_1) = 1-\chi_1 + [\Delta,\chi_0]R_0(\lambda)(1-\chi_1) = 1-\chi_1 + K_0(\lambda),$$

where $K_0(\lambda)$ is compact. Next,

$$\begin{aligned}(P-\lambda^2)\chi_2 R(\lambda_0)\chi_1 &= \chi_2(P-\lambda^2)R(\lambda_0)\chi_1 - [\Delta,\chi_2]R(\lambda_0)\chi_1\\ &= \chi_1 + \chi_2(\lambda_0^2-\lambda^2)R(\lambda_0)\chi_1 - [\Delta,\chi_2]R(\lambda_0)\chi_1\\ &= \chi_1 + K_1(\lambda,\lambda_0),\end{aligned}$$

where $K_2(\lambda,\lambda_0)$ is compact as well.

So we get

$$(P-\lambda^2)Q = \mathrm{I} + K(\lambda,\lambda_0), \quad K := K_0(\lambda) + K_1(\lambda,\lambda_0).$$

Clearly, K depends analytically on λ in $\Im\lambda > 0$. We only need to show that $\mathrm{I} + K(\lambda,\lambda_0)$ is invertible for some λ, with suitably chosen (but fixed) λ_0. Then we can apply the analytic Fredholm theorem.

We chose $\lambda_0 = e^{\mathrm{i}\pi/4}\mu$, $\mu > 0$; then $\lambda_0^2 = \mathrm{i}\mu^2$. Then we claim that

$$K(\lambda_0,\lambda_0) = O(|\lambda_0|^{-1}) : \mathcal{H} \to \mathcal{H}.$$

By the spectral theorem,

$$R(\lambda_0) = O(|\lambda_0|^{-2}) : \mathcal{H} \to \mathcal{H}, \quad R(\lambda_0) = O(1) : \mathcal{H} \to \mathcal{D}(P).$$

The most difficult term to handle is $[\Delta,\chi_2]R(\lambda_0)\chi_1$, see the definition of $K_1(\lambda)$. Note that $1-\chi_0 = 1$ on the support of the coefficients of $[\Delta,\chi_2]$. Then we have the same estimates for $(1-\chi_0)R(\lambda_0)$ and $\mathcal{H}$, $\mathcal{D}(P)$ replaced by L^2 and H^2, respectively. By interpolation (or by applying the spectral theorem again), $(1-\chi_0)R(\lambda_0) = O(1/|\lambda_0|) : \mathcal{H} \to H^2$. We treat in the same way the other terms in $K(\lambda_0, \lambda_0)$ to get

$$K(\lambda_0, \lambda_0) = O(1/|\lambda_0|) : \mathcal{H} \to \mathcal{H}.$$

We choose and fix λ_0 so that that norm is less than 1/2. Then $(\mathrm{I} + K(\lambda, \lambda_0))^{-1}$ exists for $\lambda = \lambda_0$. By the analytic Fredholm theorem, it extends to a meromorphic family in $\Im\lambda > 0$ (with the singular part in the Laurent expansion operators of finite rank). Note that at this point we cannot (and should not be able to) make the same conclusion for λ in $\Im\lambda < 0$ because the first term in the definition of $Q(\lambda)$ is analytic only in $\Im\lambda > 0$.

We now write (with the notation $K(\lambda) = K(\lambda, \lambda_0)$)

$$(P-\lambda^2)^{-1} = Q(\lambda)(\mathrm{I} + K(\lambda))^{-1}, \quad \Im\lambda > 0.$$

So far, we did not prove many new meromorphic properties of $(P-\lambda^2)^{-1}$ but we did prove some new ones. We know that $(P-\lambda^2)^{-1}$ is an analytic function in $\Im\lambda > 0$ with possible singularities at λon $i\mathbf{R}$ so that λ^2 is in the (negative) spectrum of P. Now, we see that the negative part of the spectrum of P must be discrete, which proves (a).

To prove the meromorphic extension to $\Im\lambda \leq 0$, as in (b), it is enough to show that $R(\lambda)\chi : \mathcal{H} \to \mathcal{H}_{\mathrm{loc}}$ has this property, for any $\chi \in C_0^\infty$. If we expand $(\mathrm{I} + K(\lambda))^{-1}$ in Neumann series, for λ close to λ_0, we get

$$(\mathrm{I} + K)^{-1} = \mathrm{I} - K + K^2 - \dots.$$

A quick inspection of the definition of K reveals that it is a sum of terms all having a compact cut-off on the left; more precisely, their range is always in H_{comp}; with support in $\operatorname{supp}\chi_2$. One of them however, $[\Delta, \chi_0]R_0(\lambda)(1-\chi_1)$, does not have such a cut-off on the right. Since we always apply it to functions supported in $\operatorname{supp}\chi_2$, except for the K term, we can just replace K with $K(1-\chi)$, where $\chi_2 \prec \chi$, and this would affect the K term only but not K^j, $j \geq 2$. Therefore,

$$(\mathrm{I} + K)^{-1} = (\mathrm{I} + K\chi)^{-1}(1 - K(1-\chi)).$$

This is true for λ near λ_0 but also in $\Im\lambda > 0$ away from the poles, by analytic continuation. We can apply the analytic Fredholm theorem again (note that $K\chi$ now is an analytic function of λ in $\mathbf{C}$, if n is odd, and in the logarithmic cover of $\mathbf{C}$, if n is even), to conclude that $R(\lambda)$ has the claimed meromorphic extension (in the sense of (b)) given by

$$R(\lambda) = Q(\lambda)(\mathrm{I} + K(\lambda)\chi)^{-1}(\mathrm{I} - K(\lambda)(1-\chi)).$$

□

As before, the poles of $R(\lambda)$ are called resonances. Some authors exclude those in $\Im\lambda > 0$ which come from the spectrum.

PROPOSITION 5.2.

(a) For any $f \in \mathcal{H}_{\mathrm{comp}}$ and any λ not a resonance, the function $u = R(\lambda)f$ is λ-outgoing. Moreover, if χ is a smooth cut-off function such that $\chi = 1$ for $|x| > a$, and $\chi = 0$ in a neighborhood of $B(0, R_0)$ and $\operatorname{supp} f$, *then we have $R(\lambda)f|_{|x|>a} = -R_0(\lambda)[\Delta, \chi]R(\lambda)f|_{|x|>a}$.*

(b) Assume that $u \in \mathcal{D}_{\mathrm{loc}}(P)$, $(P-\lambda^2)u = f \in \mathcal{H}_{comp}$, λ is not a resonance, and u is λ-outgoing. Then $u = R(\lambda)f$.

(c) Assume that $u \in \mathcal{D}_{\rm loc}(P)$, $(P-\lambda^2)u = 0$ for $0 \neq \lambda \in \mathbf{R}$, and u is λ-outgoing. Then u has compact support.

Proof. We will use the observation (1.2). Let $\mathbf{1}_{B(0,R_0)} \prec 1-\chi \in C_0^\infty$. For $\Im\lambda > 0$,

$$(-\Delta-\lambda^2)\chi = -[\Delta,\chi] + \chi(-\Delta-\lambda^2) = -[\Delta,\chi] + \chi(P-\lambda^2).$$

Apply $R_0(\lambda)$ on the left and $R(\lambda)$ on the right to get

$$\chi R(\lambda) = -R_0(\lambda)[\Delta,\chi]R(\lambda) + R_0(\lambda)\chi. \tag{5.6}$$

This can be extended meromorphically to the "non-physical sheet" considering both sides above as operators from $\mathcal{H}_{\rm comp}$ to $\mathcal{H}_{\rm loc}$ This proves (a). In particular, this show that for any $f \in \mathcal{E}'$, $R(\lambda)f$ is outgoing.

For the proof of (b), see [**16, 20**].

Consider (c). As in (1.22), we get

$$\int_{S_r} |u_r - \mathrm{i}\lambda u|^2 \mathrm{d}S = \int_{S_R} \left(|u_r|^2 + \lambda^2|u|^2\right) \mathrm{d}S - \mathrm{i}\lambda \int_{S_R} (u\bar{u}_r - \bar{u}u_r)\, \mathrm{d}S.$$

The integral on the right vanishes because by Green's formula

$$\int_{S_R} (u\bar{u}_r - \bar{u}u_r)\, \mathrm{d}S = (\mathbf{1}_{B(0,R)}(P-\lambda^2)u, u) - (\mathbf{1}_{B(0,R)}u, (P-\lambda^2)u), \quad R > R_0,$$

which is easy to justify in the black-box framework. Then we show in the same way that $u = 0$ for $|x| > R_0$. □

Theorem 5.3. *Let $\lambda \in \mathbf{R}\setminus\{0\}$ be a resonance. Then λ^2 is an eigenvalue of P and there exists a compactly supported eigenfunction corresponding to λ^2.*

Proof. We will actually prove something more. Let λ_0 be such a resonance. Then $R(\lambda_0)$ has a non-trivial Laurent expansion of the kind (3.6). Let A_N be as there. In the same way, we show that $(P-\lambda_0^2)A_N = 0$. Since $A_N \neq 0$, there exists $f \in \mathcal{H}_{\rm comp}$ so that $u := A_N f \neq 0$. Then $(P-\lambda_0^2)u = 0$. Such functions are called resonant states.

By (5.6), u is outgoing. By Proposition 5.2(c), it has compact support. Then u, which a priori is in $\mathcal{H}_{\rm loc}$ only, is actually in $\mathcal{H}$ and is therefore an eigenfunction. □

In most typical situations, P has no positive eigenvalues; then there are no real non-zero resonances.

5.1. The scattering amplitude. Similarly to what we did before, we can build the "naive" scattering theory in the general black-box setting.

Let $\mathbf{1}_{B(0,R_0)} \prec 1-\chi_1 \in C_0^\infty$. For any $\theta \in S^{n-1}$, $\lambda > 0$, we are looking for a solution $u(x,\theta,\lambda) \in \mathcal{D}_{\rm loc}(P)$ of

$$(P-\lambda^2)u = 0$$

so that

$$u = \chi_1 e^{\mathrm{i}\lambda\theta\cdot x} + u_{\rm sc} \tag{5.7}$$

with $u_{\rm sc}$ outgoing. Then $u_{\rm sc}$ must solve

$$(P-\lambda^2)u_{\rm sc} = [\Delta,\chi_1]e^{\mathrm{i}\lambda\theta\cdot x}.$$

By Proposition 5.2(b) above,

$$u_{\rm sc} = R(\lambda)[\Delta, \chi_1]e^{{\rm i}\lambda\theta\cdot x}.$$

if $\lambda > 0$ is not a resonances. Choose $\chi_2 \in C^\infty$ so that $\chi_2(x) = 0$ for $|x| \gg 1$ and $\chi = 0$ on $\mathrm{supp}(1-\chi_1)$. Then by Proposition 5.2(a),

$$\chi_2 u_{\rm sc} = -R_0(\lambda)[\Delta, \chi_2]R(\lambda)[\Delta, \chi_1]e^{{\rm i}\lambda\theta\cdot x}.$$

Take the asymptotic $x = r\omega$, $r \to \infty$. By Theorem 1.5,

$$\begin{aligned} u_{\rm sc}(r\omega, \theta, \lambda) = &-\frac{e^{\frac{1}{4}\pi i(n-1)}}{4\pi{\rm i}r^{(n-1)/2}}\left(\frac{\lambda}{2\pi}\right)^{\frac{1}{2}(n-3)} \\ &\times \int e^{-{\rm i}\lambda\omega\cdot x}R_0(\lambda)[\Delta, \chi_2]R(\lambda)[\Delta, \chi_1]e^{{\rm i}\lambda\theta\cdot\bullet}{\rm d}x + O\left(\frac{1}{r^{(n+1)/2}}\right). \end{aligned}$$

We define the scattering amplitude a in the same way as in (2.17). Then

$$a(\omega, \theta, \lambda) = -\frac{e^{\frac{1}{4}\pi i(n-1)}}{4\pi{\rm i}}\left(\frac{\lambda}{2\pi}\right)^{\frac{1}{2}(n-3)}\int e^{-{\rm i}\lambda\omega\cdot x}[\Delta, \chi_2]R(\lambda)[\Delta, \chi_1]e^{{\rm i}\lambda\theta\cdot\bullet}{\rm d}x. \tag{5.8}$$

This formula allows us to compute the scattering amplitude a knowing the resolvent $R(\lambda)$. In particular, we get that, as before, $\lambda \mapsto a(\omega, \theta, \lambda)$ extends meromorphically from the physical sheet to the non-physical one with values analytic functions of (ω, θ) and possible poles at the resonances.

Set

$$(E_\pm f)(\omega) = \int e^{\pm{\rm i}\lambda\omega\cdot x}f(x)\,{\rm d}x = \hat{f}(\mp\lambda\omega), \quad E_\pm : \mathcal{S}'(\mathbf{R}^n) \to C^\infty(S^{n-1}), \tag{5.9}$$

The transpose operators are given by

$$(E'_\pm\phi)(x) = \int_{S^{n-1}} e^{\pm{\rm i}\lambda\omega\cdot x}\phi(\omega)\,{\rm d}\omega, \quad E_\pm : \mathcal{D}'(S^{n-1}) \to \mathcal{S}(\mathbf{R}^n). \tag{5.10}$$

Let us now view the scattering amplitude $a(\omega, \theta, \lambda)$ as a kernel of an operator $A(\lambda)$: $C^\infty(S^{n-1}) \to C^\infty(S^{n-1})$. Then

$$A(\lambda) = -\frac{e^{\frac{1}{4}\pi i(n-1)}}{4\pi{\rm i}}\left(\frac{\lambda}{2\pi}\right)^{\frac{1}{2}(n-3)}E_-(\lambda)[\Delta, \chi_2]R(\lambda)[\Delta, \chi_1]E'_+(\lambda). \tag{5.11}$$

This formula appeared for the first time in [**13**], to my knowledge. I am following [**20**] here.

An even closer look at that formula reveals that

$$A(\lambda) = c(\lambda)E_-(\lambda)[\Delta, \chi_2]R(\lambda)\left(E_+(\lambda)[\Delta, \chi_1]\right)'.$$

This shows that we should probably try to understand the operator $E_-(\lambda)[\Delta, \chi_2]$ better. Take the Schwartz kernel $G(x, y, \lambda)$ of $R(\lambda)$. Then the kernel of $E_-(\lambda)[\Delta, \chi_2]R(\lambda)$ is just the far field pattern of $G(x, y, \lambda)$ w.r.t. the x variable. The kernel of $E_-(\lambda)[\Delta, \chi_2]R(\lambda)\left(E_+(\lambda)[\Delta, \chi_1]\right)'$ is then the far field pattern of the result w.r.t. the y variable; and that is the scattering amplitude, up to the factor $c(\lambda)$. We will make this more precise later, when we discuss inverse scattering in the black-box setting.

5.1.1. *Asymptotics of $R(\lambda)f$.* Similarly to Theorem 1.5, we can derive asymptotics for $R(\lambda)f$ with f compactly supported.

THEOREM 5.4. *For any $f \in \mathcal{E}'(\mathbf{R}^n)$,*

$$[R(\lambda)f](r\theta) = \frac{1}{4\pi \mathrm{i}} \left(\frac{\lambda}{2\pi}\right)^{\frac{1}{2}(n-3)} e^{\frac{1}{4}\pi i(n-1)} \frac{e^{\mathrm{i}\lambda r}}{r^{(n-1)/2}} \int \bar{u}(y, -\theta, -\lambda) f(y)\, \mathrm{d}y.$$

PROOF. By Proposition 5.2, one needs to know the asymptotic of $R_0(\lambda)g$ for g compactly supported only. Then Theorem 1.5 implies (f can be a distribution there)

$$\begin{aligned}[R(\lambda)f](r\theta) = &- \frac{1}{4\pi \mathrm{i}} \left(\frac{\lambda}{2\pi}\right)^{\frac{1}{2}(n-3)} e^{\frac{1}{4}\pi i(n-1)} \\ &\times \frac{e^{\mathrm{i}\lambda r}}{r^{(n-1)/2}} \int e^{-\mathrm{i}\lambda\theta\cdot y} \left([\Delta, \chi] R(\lambda) f\right)(y)\, \mathrm{d}y + O\left(r^{-\frac{n-3}{2}}\right).\end{aligned}$$

Note that $R^*(\lambda) = R(-\bar{\lambda})$, i.e., for the Green's function $G(x, y, \lambda)$ (the Schwartz kernel) we have $\overline{G(y, x, \lambda)} = G(x, y, -\bar{\lambda})$. Since $[\Delta, \chi]$ is formally skew self-adjoint in L^2, for λ real we get

$$\begin{aligned}\int e^{-\mathrm{i}\lambda\theta\cdot y} \left(R(\lambda)f\right)(y)\, \mathrm{d}y &= \int f(y) \left(\overline{R(-\lambda)[\Delta, \chi] e^{\mathrm{i}\lambda\theta\cdot\bullet}}\right)(y)\, \mathrm{d}y \\ &= -\int f(y) \bar{u}_{\mathrm{sc}}(x, -\theta, -\lambda)\, \mathrm{d}y,\end{aligned}$$

where u_{sc} is related to χ instead of χ_1. We can replace $\bar{u}_{\mathrm{sc}}$ by $\bar{u}$, see (5.7), if χ has support separated from that of f. □

Note that if P preserves real functions, i.e., $\overline{Pf} = P\bar{f}$, then $G(x, y, \lambda) = G(y, x, \lambda)$ and we can replace $\bar{u}(x, -\theta, -\lambda)$ by $u(x, -\theta, \lambda)$.

What needs to be added: the general properties of the scattering amplitude, etc., are the same as in the potential scattering case. Wave operators and their kernels, etc.

CHAPTER III

Introduction to the Time-Dependent Scattering Theory for the perturbed wave equation in $\mathbf{R} \times \mathbf{R}^n$

1. Introduction

I will present some basic facts about the time-dependent scattering theory for the perturbed wave equation in $\mathbf{R} \times \mathbf{R}^n$. I will not get into the whole Lax-Phillips theory. I am following here Friedlander, Lax-Phillips and Cooper-Strauss.

The time-dependent theory is most of its part equivalent to the stationary one, roughly speaking. In some sense, it is more intuitive — one has the notion of time and there is finite speed of propagation. Some phenomena like time-depending perturbations are naturally better described by the time-dependent theory.

The analog of the Helmholtz equation, describing propagation of "free waves", is the wave equation

$$(\partial_t^2 - \Delta)u = 0, \quad t \in \mathbf{R},\ x \in \mathbf{R}^n. \tag{1.1}$$

One of the basic examples of a perturbed system is the wave equation with a compactly supported potential

$$(\partial_t^2 - \Delta + V(x))u = 0. \tag{1.2}$$

Another one, in obstacle scattering, is equation (1.1) in the exterior Ω of a compactly supported obstacle with, say, Dirichlet boundary conditions on it. An important example is the acoustic wave equation

$$(\partial_t^2 - c^2(x)\Delta)u = 0 \tag{1.3}$$

with $c > 0$ equal to 1 for large $|x|$. One can involve a metric equal to the Euclidean for large $|x|$, etc.

Another class of examples involve time-depending perturbations, for example,

$$(\partial_t^2 - \Delta + q(t,x))u = 0, \tag{1.4}$$

where $q(t,x) = 0$ for $|x| > R_0$, $R_0 > 0$. In the previous examples, one can take an inverse Fourier transform w.r.t. t to get a stationary problem. Here, we can still do this but we will get a convolution w.r.t. the spectral variable λ. This is related to the fact that time-dependent perturbations do not preserve the frequency, in general. Another example is the obstacle problem for an obstacle which moves and changes shape as well.

2. Scattering for the "free" wave equation

2.1. Basic facts about the wave equation. The natural Cauchy problem for the wave equation is the following

$$(\partial_t^2 - \Delta)u = 0, \quad (u, u_t)|_{t=0} = (f_1, f_2). \tag{2.1}$$

This problem is solvable for any $(f_1, f_2) \in \mathcal{D}'(\mathbf{R}^n) \times \mathcal{D}'(\mathbf{R}^n)$, and there are those explicit textbook formulas.

One basic property is *finite speed of propagation*:

$$\operatorname{supp} u(t, \cdot) \subset (\operatorname{supp} f \cup \operatorname{supp} g) + B(0, |t|). \tag{2.2}$$

Physically, this means that "signals" propagate with speed one or less. Another basic property is the *Huygens' Principle* valid in dimensions $n \geq 3$, odd:

$$\operatorname{supp} u(t, \cdot) \subset (\operatorname{supp} f \cup \operatorname{supp} g) + S(0, |t|), \tag{2.3}$$

where $S(0, r)$ stands for the sphere $|x| = r$. Physically, this means that "signals" propagate with speed one exactly.

Another basic property is energy preservation: the energy

$$E(u(t, \cdot)) := \frac{1}{2} \int \left(|\nabla_x u|^2 + |u_t|^2\right) \mathrm{d}x \tag{2.4}$$

(if finite), is independent of t. We will make this more precise below.

The following formalism is very convenient. We convert the wave equation into a system by setting $\mathbf{u}(t) = (u, u_t)$; then

$$\partial_t \mathbf{u} = A\mathbf{u}, \quad A := \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}. \tag{2.5}$$

The natural energy space of states of finite energy is defined as the completion of $C_0^\infty \times C_0^\infty$ under the energy norm

$$\|\mathbf{f}\|_{\mathcal{H}}^2 = \frac{1}{2} \int \left(|\nabla f_1|^2 + |f_2|^2\right) \mathrm{d}x, \quad \mathbf{f} := (f_1, f_2).$$

In particular, the first term defines the Dirichlet space $H_D(\mathbf{R}^n)$ with norm $\|\nabla f\|_{L^2}$. Surprisingly, when $n = 2$, that space contains elements that are not distributions [**8**]. When $n \geq 3$, they are locally in L^2, as it follows from the following Poincaré inequality

$$\int_{|x|<R} |g|^2 \, \mathrm{d}x \leq \frac{R^2}{2(n-2)} \int |\nabla g|^2 \, \mathrm{d}x.$$

The operator A naturally extends to a skew-selfadjoint one (i.e, $\mathrm{i}A$ is self-adjoint) on $\mathcal{H}$. Then by Stone's theorem, $U_0(t) = e^{tA}$ is a well-defined strongly continuous unitary group, and the solution of (2.1) is given by $\mathbf{u}(t) = U_0(t)\mathbf{f}$. The unitarity means energy conservation, in particular.

We define $\mathcal{H}_{\mathrm{loc}}$ in the usual way. By the finite speed of propagation, the Cauchy problem (2.1) has a well defined solution in $\mathcal{H}_{\mathrm{loc}}$ if the Cauchy data $\mathbf{f}$ is in $\mathcal{H}_{\mathrm{loc}}$ only. We view those solutions as ones with (possibly) infinite energy but locally finite one. Then $\mathbf{u} \in$

$C(\mathbf{R};\ \mathcal{H}_{\rm loc})$ and the wave equation is solved in distribution sense. One can easily extend this to distributions.

2.2. Plane waves, translation representation and asymptotic wave profiles of free solutions. The harmonic plane waves $e^{\mathrm{i}\lambda\omega\cdot x}$ played a fundamental role in the stationary theory. The time dependent analog are the plane waves

$$\delta(t-\omega\cdot x),$$

where δ is the Dirac delta function. They solve the wave equation, obviously. They can be thought of as, well, plane waves propagating in the direction ω with speed one. If we replace t by $t+s$ there, we can think of s as the delay time. The plane wave above is the Schwartz kernel of the Radon transform

$$Rf(s,\omega)=\int\delta(t-\omega\cdot x)f(x)\,\mathrm{d}x=\int_{x\cdot\omega=s}f(x)\,\mathrm{d}S_x.$$

For any density $g(\omega,s)$ (which can be a distribution as well), the superposition

$$u(t,x):=\int_{\mathbf{R}\times S^{n-1}}\delta(t+s-\omega\cdot x)g(s,\omega)\,\mathrm{d}\omega\,\mathrm{d}s=\int_{S^{n-1}}g(\omega\cdot x-t,\omega)\,\mathrm{d}\omega \tag{2.6}$$

is still a solution of the wave equation. The expression above can be recognized as the the transpose R' of the Radon transform applied to $g_t(s,\omega):=g(s-t,\omega)$. The following natural question arises: are all solutions, in some space, at least, given by superpositions of the kind (2.6)? If so, then $(u,u_t)|_{t=0}$ should be able to generate all possible Cauchy data in that space. By (2.6),

$$(u,u_t)|_{t=0}=(R'g,-R'\partial_s g)=(f_1,f_2).$$

We assume $n\geq 3$ odd from now on. The first impression is that those equations form an over-determined system. This not the case however. It is known that with the right choice of the constant, $c_n\partial_s^{(n-1)/2}R:L^2(\mathbf{R}^n)\to L^2_e(\mathbf{R}\times S^{n-1})$ is unitary and surjective, where the subscript e stands for the even functions in that space. Then so is $c_n\partial_s^{(n-1)/2}R':L^2_e(\mathbf{R}\times S^{n-1})\to L^2(\mathbf{R}^n)$. This suggests that we should look for g of the kind $g=c_n\partial_s^{(n-3)/2}(k,-\partial_s k)$. The removal of one s-derivative can be explained by the fact that we need the second component to be in L^2, not the first one. Since $R'g=R'g_e$, the equation $c_nR'\partial_s^{(n-3)/2}k=f_1$ determines uniquely the even part of k. The equation $-c_nR'\partial_s^{(n-1)/2}k=f_2$ determines uniquely the odd part. All this motivates the choice of $\mathcal{R}$ below.

One of the first things we need to understand is the behavior of the solutions of the wave equation at infinity. In [**8**], Lax and Phillips defined the *free translation representation* $\mathcal{R}:\mathcal{H}\to L^2(\mathbf{R}\times S^{n-1})$ as follows

$$k(s,\omega)=\mathcal{R}\mathbf{f}(s,\omega)=c_n(-\partial_s^{(n+1)/2}Rf_1+\partial_s^{(n-1)/2}Rf_2), \tag{2.7}$$

where R is the Radon transform and $c_n=2^{-1}(2\pi)^{(1-n)/2}$, $c_n^-=2^{-1}(-2\pi)^{(1-n)/2}$. The inverse is given by

$$\mathcal{R}^{-1}k(x)=2c_n^-\int_{S^{n-1}}\big(-\partial_s^{(n-3)/2}k(x\cdot\omega,\omega),\ \partial_s^{(n-1)/2}k(x\cdot\omega,\omega)\big)\,\mathrm{d}\omega. \tag{2.8}$$

The map $\mathcal{R}$ is unitary, and $(\mathcal{R}U_0(t)\mathcal{R}^{-1}k)(s,\omega) = k(s-t,\omega)$, which explains the name. We also set

$$u^\sharp(s,\omega) = (-1)^{(n-1)/2}k(s,\omega) \tag{2.9}$$

and call $u^\sharp$ the *asymptotic wave profile* of the solution $\mathbf{u}(t) = U_0(t)\mathbf{f}$. This name is justified by the theorem below, and it is the analog of the far free pattern for solutions of the free wave equation.

THEOREM 2.1 (Lax-Phillips, [**8**]). *Let* $\mathbf{u}(t) = U_0(t)\mathbf{f}$, $\mathbf{f} \in \mathcal{H}$. *Then*

$$\int \left| u_t - |x|^{-(n-1)/2} u^\sharp\Big(|x| - t, \frac{x}{|x|}\Big)\right|^2 \mathrm{d}x \to 0, \quad \textit{as } |t| \to \infty. \tag{2.10}$$

SKETCH OF THE PROOF. Let k be the translation representation of $\mathbf{u}$, see (2.7). Then $U_0(t)\mathbf{f} = \mathcal{R}^{-1}k(\cdot - t, \cdot)$. In particular,

$$u_t = \int_{S^{n-1}} h(\omega\cdot x - t, \omega)\,\mathrm{d}\omega, \quad h := c_n^- \partial_s^{(n-1)/2}k. \tag{2.11}$$

We will assume in the rest of the proof that h is regular enough to justify the limits below, and in particular that h has compact support. We need to compute the limit

$$\lim_{t\to\infty} t^{(n-1)/2}u_t(t,(t+s)\theta),$$

i.e.,

$$\lim_{t\to\infty} t^{(n-1)/2}\int_{S^{n-1}} h((t+s)\omega\cdot\theta - t, \omega)\,\mathrm{d}\omega.$$

As $|t| \to \infty$, the leading term in the argument of h is $t(\omega\cdot\theta - 1)$, which vanishes when $\omega \neq \theta$. Then the behavior when θ is close to ω would determine the limit (this is our "stationary phase argument"). It is straightforward to see that if we just replace ω by theta in the second argument of h, we get an $o(1)$ term as $|t| \to \infty$. Make this substitution and set $\rho = \omega\cdot\theta$ to get that the limit above equals the limit of

$$|S^{n-2}|\int_{-1}^{1} h((t+s)(\rho-1)+s,\theta)(1-\rho^2)^{(n-3)/2}\,\mathrm{d}\rho.$$

Set $\tau = |t+s|(\rho - 1)$ to get

$$2^{(n-3)/2}|S^{n-2}||t+s|^{(1-n)/2}\int_0^\infty h(s\mp\tau,\theta)\tau^{(n-3)/2}(1+\eta)\,\mathrm{d}\tau,$$

where $\eta \to 0$ as $|t+s| \to \infty$. Recall the definition (2.11) of h, and integrate by parts $(n-1)/2$ times to complete the proof. □

REMARK 2.1. In [**8**], the factor $(-1)^{(n-1)/2}$ is missing from (2.9), i.e., $u^\sharp = k$. Cooper and Strauss in [**2**] claim that this factor must be present in (2.9). I follow their suggestion but I cannot (yet) guarantee that they are right. One has to check whether c_n^- is the right constant in (2.8) first.

2.3. Outgoing solutions and their asymptotic wave profiles. We follow here [**1, 2**]. Given $u(t,x)$, we will use the notation $\mathbf{u}(t) := (u(t,\cdot), u_t(t,\cdot))$.

DEFINITION 2.2. The function $\mathbf{u}(t) \in C(\mathbf{R};\ \mathcal{H}_{\rm loc})$ is called outgoing if $\lim_{t\to-\infty}(\mathbf{u}(t), U_0(t)\mathbf{g}) = 0$ for each $\mathbf{g} \in C_0^\infty(\mathbf{R}^n) \times C_0^\infty(\mathbf{R}^n)$.

In this definition, $\mathbf{u}(t)$ does not need to be a solution of the wave equation (anywhere). On the other hand, if $u(t,x)$ solves the wave equation in $|x| > \rho$ for some $\rho > 0$, then, see [**1, 2**], u is outgoing if and only if for any $T \in \mathbf{R}$, $U_0(t-T)\mathbf{u}(T) = 0$ in the forward cone $|x| < t - T - \rho$.

On simple example of non-trivial outgoing solutions (for $|x| > \rho$) is the following. The time-dependent analog of the non-homogeneous Helmholtz equation $(\Delta - \lambda^2)u = f$ is the source equation

$$(\partial_t^2 - \Delta)u = p(t,x) \quad \text{in } \mathbf{R}\times\mathbf{R}^n. \tag{2.12}$$

Solve that equation with Cauchy data

$$(u, u_t)|_{t=t_0} = (0,0),$$

where the source p satisfies $p \in L^1(\mathbf{R};\ L^2(\mathbf{R}^n))$. By Duhamel's formula,

$$\mathbf{u}(t) = \int_{t_0}^t U_0(t-s)\mathbf{p}(s)\,\mathrm{d}s, \quad \mathbf{p}(s) := (0, p(s,\cdot)).$$

The latter is well-defined in $\mathcal{H}_{\rm loc}$ by finite speed of propagation. The solution for $t < 0$ is just zero. Then $\mathbf{u}$ is outgoing in a trivial way. Moreover, this is the unique outgoing solution of (2.12). Indeed, take the difference v of any two. Then $\mathbf{v}(t) = U_0(t)\mathbf{f}$, where $\mathbf{f}$ is the initial condition. Then $0 = \lim_{t\to-\infty}(\mathbf{v}(t), U_0(t)\mathbf{g}) = (\mathbf{f},\mathbf{g})$, for any test function $\mathbf{g}$; therefore, $\mathbf{f} = 0$ and then $\mathbf{v} = 0$.

This can be generalized as follows.

THEOREM 2.3 ([**1, 2**]). *Let $p \in L^1_{\rm loc}(\mathbf{R};\ L^2(\mathbf{R}^n))$ and assume that for each t,*

$$\lim_{T\to-\infty}\int_T^t U_0(-s)\mathbf{p}(s)\,\mathrm{d}s \quad \textit{exists in } \mathcal{H}_{\rm loc} \tag{2.13}$$

Then there exists a unique outgoing solution $\mathbf{u} \in C(\mathbf{R};\ \mathcal{H}_{loc})$ of (2.12) given by

$$\mathbf{u}(t) = \int_{-\infty}^t U_0(t-s)\mathbf{p}(s)\,\mathrm{d}s, \quad \mathbf{p}(s) := (0, p(s,\cdot).$$

REMARK 2.2. Clearly, $p \in L^1((-\infty,a);\ L^2(\mathbf{R}^n))$ for any a would guarantee the regularity assumption on p and (2.13).

PROOF. The absolute convergence of the integral in $H^0_{\rm loc}$ follows from the assumptions. To show that u is outgoing, for $\mathbf{g} \in C_0^\infty \times C_0^\infty$, consider

$$(\mathbf{u}(t), U_0(t)\mathbf{g}) = \int_{-\infty}^t (U_0(t-s)\mathbf{p}(s), U_0(t)\mathbf{g})\,\mathrm{d}s = \int_{-\infty}^t (U_0(-s)\mathbf{p}(s),\mathbf{g})\,\mathrm{d}s.$$

The latter converges to 0 by assumption. □

The following theorem is the analog of the asymptotics of $R_0(\lambda)f$ at infinity.

THEOREM 2.4. *Let $n \geq 3$ be odd. Let $p \in L^1_{\mathrm{loc}}(\mathbf{R};\ L^2(\mathbf{R}^n))$ with $p(t,x) = 0$ for $|x| > \rho$. Let u be the unique outgoing solution of* (2.12).

(a) Then there is a unique function $u^\sharp \in L^2_{\mathrm{loc}}(\mathbf{R} \times S^{n-1})$ such that for all $R_1 < R_2$ we have

$$\int_{R_1+t<|x|<R_2+t} \left| u_t(t,x) - |x|^{-(n-1)/2} u^\sharp \left(|x| - t, \frac{x}{|x|} \right) \right|^2 \mathrm{d}x \to 0, \quad \text{as } t \to \infty.$$

(b) If $p \in C_0^\infty$,

$$u^\sharp(s,\omega) = c_n^- \partial_s^{(n-1)/2} \int p(\omega \cdot x - s, x)\, \mathrm{d}x \tag{2.14}$$

(c) The map $p \to u^\sharp$ is continuous.

REMARK 2.3. For general p as in the theorem, $u^\sharp$ is still given by (2.14) but the derivative is in distribution sense; by (b), the result is in $L^2_{\mathrm{loc}}(\mathbf{R} \times S^{n-1})$. Another way to write (2.14) is

$$\int u^\sharp(s,\omega)\phi(s)\, \mathrm{d}s = c_n \iint p(t,x)(\partial_s^{(n-1)/2}\phi)(\omega \cdot x - s, x)\, \mathrm{d}t\, \mathrm{d}s, \quad \forall \psi(s) \in C_0^\infty(\mathbf{R}). \tag{2.15}$$

PROOF. Motivated by Theorem 2.3, for fixed $R_1 < R_2$, set

$$\mathbf{f} = \int_{-R_2-\rho}^{-R_1+\rho} U_0(-\tau)\mathbf{p}(\tau)\, \mathrm{d}\tau, \quad \mathbf{v}(t) = U_0(t)\mathbf{f}.$$

By Huygens' principle, $\mathbf{v}(t) = \mathbf{u}(t)$ for $R_1 + t < |x| < R_2 + t$. Therefore, $\mathbf{v}$ does have an asymptotic wave profile, and $v^\sharp(s,\omega) = u^\sharp(s,\omega)$ for $R_1 < s < R_2$. On the other hand, we have a formula for $v^\sharp$, (2.8) and (2.9) which say

$$\begin{aligned} v^\sharp(s,\omega) &= (-1)^{(n-1)/2}(\mathcal{R}f)(s,\omega) \\ &= c_n^- \partial_s^{(n-1)/2} \int_{-R_2-\rho}^{-R_1+\rho} \int_{x\cdot\omega = s+\tau} p(x \cdot \omega - s, x)\, \mathrm{d}S_x\, \mathrm{d}\tau. \end{aligned}$$

Then

$$u^\sharp(s,\omega)|_{R_1<s<R_2} = v^\sharp(s,\omega) = c_n^- \partial_s^{(n-1)/2} \int p(x \cdot \omega - s, x)\, \mathrm{d}x.$$

Since $R_1 < R_2$ are arbitrary, this, combined with Theorem 2.3, proves (a); and (b) for $p \in C_0^\infty$.

The proof of (c) is straightforward: use (2.14) and take Fourier transform w.r.t. s. In particular, we get that the map $p \to u^\sharp$ can be extended continuously in those spaces. □

3. Scattering for time-dependent potentials

3.1. Introduction. We consider now the wave equation with time dependent potential $q(t,x)$

$$(\partial_t^2 - \Delta + q(t,x))u = 0. \tag{3.1}$$

Scattering theory for it is interesting even when q is time independent despite having the stationary approach developed in the previous chapter. We assume $q \in C^\infty$ (this assumption is too strong) and that

$$q(t,x) = 0 \quad \text{for } |x| > \rho$$

for some $\rho > 0$.

Similarly to the stationary case, we send a plane wave $\delta(t+s-x\cdot\omega)$ to the perturbation, let it interact with the potential, and measure that asymptotic wave profile $u^\sharp_{\text{sc}}(s',\omega';s,\omega)$ of the scattered wave u_{sc} in the direction ω' with "delay" s'.

We should then solve the problem

$$(\partial_t^2 - \Delta + q(t,x))u = 0, \quad u|_{t<-s-\rho} = \delta(t+s-x\cdot\omega) \tag{3.2}$$

first with $(s,\omega) \in \mathbf{R}\times S^{n-1}$ parameters. Then we set

$$u_{\text{sc}} = u - \delta(t+s-x\cdot\omega). \tag{3.3}$$

The distribution u_{sc} would be automatically outgoing since it vanishes for $t \ll 0$. Then we could compute the asymptotic wave profile of u_{sc}, which would give us the analog of the scattering amplitude. As in the stationary case, we expect this to be "essentially" the kernel of the scattering operator minus identity.

The difficulty with this program is that u is necessarily a distribution, and although u_{sc} is a function, it does not belong to the energy space even locally. This is not such a major problem — we can think as $(u(t,x;s,\omega), u_t(t,x;s,\omega)$ as distribution in the (s,ω) variables with values in $\mathcal{H}_{\text{loc}}$. It is more convenient however to do the following. Let $h_j(t) = h(t)t^j/j!$, $j = 1,2\dots$, where h is the Heaviside function; and we also set $h_{-1} = \delta$. Then $h_j' = h_{j-1}$, $j = 0,1,2,\dots$. We solve

$$(\partial_t^2 - \Delta + q(t,x))\Gamma = 0, \quad u|_{t<-s-\rho} = h_1(t+s-x\cdot\omega) \tag{3.4}$$

first (notice that $h_1(t+s-x\cdot\omega)$ is locally in the energy space now), set

$$\Gamma_{\text{sc}} = \Gamma - h_1(t+s-x\cdot\omega), \tag{3.5}$$

compute the asymptotic wave profile $\Gamma^\sharp(s',\omega';s,\omega)$ of Γ_{sc}, and differentiate the result twice w.r.t. s to get the analog of the scattering amplitude. In particular, then

$$u(t,x;s,\omega) = \partial_s^2\Gamma(t,x;s,\omega), \quad u_{\text{sc}}(t,x;s,\omega) = \partial_s^2\Gamma_{\text{sc}}(t,x;s,\omega). \tag{3.6}$$

will be well defined as distributions.

3.2. Existence of dynamics. By [**7**], the solution to

$$(\partial_t^2 - \Delta + q(t,x))u = 0, \quad (u,u_t)|_{t=s} = (f_1,f_2) \tag{3.7}$$

is given by $\mathbf{u}(t) = U(t,s)\mathbf{f}$, where $\mathbf{f} = (f_1,f_2)$ and $U(t,s)$ is a two-parameter strongly continuous group of bounded operators with the properties

(i) $U(t,s)U(s,r) = U(s,r)$ for all t,s,r; and $U(t,t) = \mathrm{I}$,

(ii) $\|U(t,s)\| \le \exp\left\{C|t-s|\sup_{s\le\tau\le t,\, x\in\mathbf{R}^n}|q(\tau,x)|\right\}$,

(iii) for any $\mathbf{f}\in D(A)$, we have $U(t,s)\mathbf{f}\in D(A)$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}U(t,s)\mathbf{f} = (A-Q(t))U(t,s)\mathbf{f}, \quad \frac{\mathrm{d}}{\mathrm{d}s}U(t,s)\mathbf{f} = -U(t,s)(A-Q(t))\mathbf{f},$$

where $Q(t)\mathbf{f} = (0, q(t,\cdot)f_1)$ (and $Q(t)$ is clearly bounded).

The two-parameter semi-group admits the expansion

$$U(t,s) = U_0(t-s) + \sum_{k=1}^{\infty} V_k(t,s), \tag{3.8}$$

where

$$\begin{aligned} V_k(t,s)\mathbf{f} = (-1)^k \int_s^t \mathrm{d}s_1 \int_s^{s_1} \mathrm{d}s_k \dots \int_s^{s_{k-1}} \mathrm{d}s_k \\ \times U_0(t-s_1)Q(s_1)\dots U_0(s_{k-1}-s_k)Q(s_k)U_0(s_k-s)\mathbf{f}, \quad k \gg 1. \end{aligned}$$

This expansion is an iterated version of the Duhamel's formula

$$\begin{aligned} U(t,s) &= U_0(t-s) + \int_s^t U(t,\sigma)Q(\sigma)U_0(\sigma-s)\,\mathrm{d}\sigma \\ &= U_0(t-s) + \int_s^t U_0(t-\sigma)Q(\sigma)U(\sigma,s)\,\mathrm{d}\sigma. \end{aligned} \tag{3.9}$$

The convergence of (3.8) follows from the estimate

$$\|V_k(t,s)\| \le \frac{|t-s|^k}{k!}\left(\sup_{s\le\tau\le t}\|Q(\tau)\|\right)^k.$$

In particular, we get that we still have the finite speed of propagation property:

$$\operatorname{supp} U(t,s)\mathbf{f} \subset \operatorname{supp}\mathbf{f} + B(0,|t-s|).$$

As before, the finite speed of propagation allows us to can extend $U(t,s)$ to the space $\mathcal{H}_{\mathrm{loc}}$ by a partition of unity.

Finally, notice that when q is time independent, then $U(t,s)$ depends on the difference $t-s$ only, i.e., $U(t,s) = U(t-s)$ where U is a group. It is not unitary however (unless $q=0$) in the space $\mathcal{H}$. If we redefine the energy norm by

$$\|\mathbf{f}\|^2_{\mathcal{H}^q} = \int \left(|\nabla f_1|^2 + q|f_1|^2 + |f_2|^2\right)\mathrm{d}x, \tag{3.10}$$

(we need to know that it is a norm however, and $q \ge 0$ suffices for that), then $U(t)$ is unitary in $\mathcal{H}^q$.

3.3. The scattering amplitude and the scattering kernel. We are ready to fulfill our program now. Let Γ solve (3.4). Since the Cauchy data $(h_1(t+s-c\cdot\omega), h_0(t+s-c\cdot\omega))$, for say, $t=-s-\rho-1$, is in $\mathcal{H}_{\mathrm{loc}}$, a solution (Γ,Γ_t) with locally finite energy exists. Then Γ_{sc} is clearly outgoing. It solves the Cauchy problem

$$(\partial_t^2 - \Delta + q(t,x))\Gamma_{\mathrm{sc}} = -q\Gamma, \quad \Gamma_{\mathrm{sc}}|_{t<-s-\rho} = 0. \tag{3.11}$$

By Theorem 2.4, Γ_{sc} has an asymptotic wave profile $\Gamma^\sharp_{\mathrm{sc}}$ given by

$$\begin{aligned} \Gamma^\sharp_{\mathrm{sc}}(s',\omega';s,\omega) &= -c_n^-\partial_{s'}^{(n-1)/2}\int q(x\cdot\omega'-s',x)\Gamma(x\cdot\omega'-s',x;s,\omega)\,\mathrm{d}x \\ &= -c_n^-\partial_{s'}^{(n-1)/2}\int q(t,x)\Gamma(t,x;s,\omega)\delta(t+s'-x\cdot\omega')\,\mathrm{d}t\,\mathrm{d}x. \end{aligned}$$

Differentiate twice w.r.t. s, see (3.6), to get $u^\sharp$

$$u^\sharp_{\rm sc}(s',\omega';s,\omega) = -c_n^- \partial_{s'}^{(n-1)/2} \int q(t,x)u(t,x;s,\omega)\delta(t+s'-x\cdot\omega')\,\mathrm{d}t\,\mathrm{d}x.$$

DEFINITION 3.1. The scattering amplitude A is given by

$$A^\sharp(s',\omega';s,\omega) = \int q(t,x)u(t,x;s,\omega)\delta(t+s'-x\cdot\omega')\,\mathrm{d}t\,\mathrm{d}x, \tag{3.12}$$

where u solves (3.2).

By the finite speed of propagation, $u(t,x;s,\omega)=0$ for $x\cdot\omega > t+s$. Therefore, the integrand vanishes outside of the region $x\cdot(\omega-\omega')\le s-s'$. The l.h.s. has a lower bound -2ρ on $\operatorname{supp} q$; therefore,

$$\operatorname{supp} A^\sharp \subset \{s'\le s+\rho|\omega-\omega'|\}\subset\{s'\le s+2\rho\}. \tag{3.13}$$

Note that A and $u^\sharp$ can be reconstructed from each other thanks to that support property.

We turn our attention now to the analog if *scattering operator* S, see (2.2) in Chapter I. Since the perturbed dynamics is a two-parameter group, we need to generalize the notion of the wave operators and the scattering scattering operator.

DEFINITION 3.2. We wave operators Ω_- and W_+ in $\mathcal{H}$ are defined as the strong limits

$$\Omega_- = \operatorname{s-}\lim_{t\to-\infty} U(0,t)U_0(t);\quad W_+\mathbf{f} = \lim_{t\to\infty} U_0(-t)U(t,0)\mathbf{f};\quad \mathbf{f}\in \operatorname{Ran}\Omega_-$$

if they exist and define continuous operators. In the latter case, the scattering operator S is defined by

$$S = W_+\Omega_-.$$

This definition also makes sense on $\mathcal{H}_{\rm loc}$.

THEOREM 3.3.
(a) The wave operator $\Omega_-:\mathcal{H}_{\rm comp}\to\mathcal{H}$ exists and

$$U(t,0)\Omega_-\mathbf{f} = 2c_n^- \int_{\mathbf{R}\times S^{n-1}} \mathbf{u}(t,x;s,\omega)\partial_s^{(n-3)/2}(\mathcal{R}\mathbf{f})(s,\omega)\,\mathrm{d}s\,\mathrm{d}\omega. \tag{3.14}$$

(b) The wave operator $W_+:\mathcal{H}\to\mathcal{H}_{\rm loc}$ exists
(c) The scattering operator $S:\mathcal{H}_{\rm comp}\to\mathcal{H}_{\rm loc}$ exists and

$$\mathcal{R}(S-\mathrm{I})\mathcal{R}^{-1} = -2^{-1}(2\pi)^{1-n}\partial_{s'}^{(n-1)/2}\partial_s^{(n-3)/2}A, \tag{3.15}$$

where A is the operator with kernel $A^\sharp$.

PROOF. Let $k\in L^2_{\rm comp}(\mathbf{R}\times S^{n-1})$, with $k=0$ for $|s|>R$ with some R, and let $\mathbf{f}=\mathcal{R}^{-1}k$. Then for $t<-R-\rho:=t_0$, $U(0,t)U_0(t)\mathbf{f}=U(0,t_0)U_0(t_0)\mathbf{f}$. In particular, the limit defining $\Omega_-\mathbf{f}$ exists trivially and $U(t,0)\Omega_-\mathbf{f} = U(t,t_0)U_0(t_0)\mathbf{f}$. The r.h.s. of the latter solves the perturbed wave equation and equals $U_0(t_0)\mathbf{f}=\mathcal{R}^{-1}k(\cdot-t_0,\cdot)$ for $t=t_0$. To prove (3.14), we need to show that the r.h.s. of (3.14), call it $\mathbf{v}(t)$, has the same initial condition for $t\le t_0$.

For $t\le t_0$, $u(t,x;s,\omega)=\delta(t+s-x\cdot\omega)$. Then by (2.8),

$$v(t) = 2c_n^-\int_{\mathbf{R}\times S^{n-1}}\delta(t+s-x\cdot\omega)\partial_s^{(n-3)/2}k(s,\omega)\,\mathrm{d}s\,\mathrm{d}\omega = (\mathcal{R}^{-1}k)_1(\cdot-t,\cdot),$$

which proves (a).

Express u as in (3.3) and plug in the formula above:

$$\mathbf{v}(t) = \mathbf{v}_0(t) - 2c_n \int_{\mathbf{R}\times S^{n-1}} \mathbf{u}_{\mathrm{sc}}(t,x;s,\omega)\partial_s^{(n-3)/2} v_0^\sharp(s,\omega)\,\mathrm{d}s\,\mathrm{d}\omega,$$

where $\mathbf{v}(t) = U(t,0)\Omega_-\mathbf{f}$, $\mathbf{v}_0(t) = U_0(t)\mathbf{f}$. Take asymptotic wave profiles of both sides to get

$$v^\sharp(s',\omega') = v_0^\sharp(s',\omega') + 2c_n^- \int_{\mathbf{R}\times S^{n-1}} \partial_s^{(n-3)/2} u_{\mathrm{sc}}^\sharp(s',\omega';s,\omega) v_0^\sharp(s,\omega)\,\mathrm{d}s\,\mathrm{d}\omega \tag{3.16}$$

in $L^2_{\mathrm{loc}}(\mathbf{R}\times S^{n-1})$.

To prove (b), fix $R>0$ and let $\mathbf{1}_{B(0,R)}$ be the characteristic function of that ball. By (3.9)

$$\mathbf{1}_{B(0,R)}U_0(-t)U(t,s) = \mathbf{1}_{B(0,R)}U_0(-s) + \mathbf{1}_{B(0,R)}\int_s^t U_0(-\sigma)Q(\sigma)U(\sigma,s)\,\mathrm{d}\sigma. \tag{3.17}$$

By Huygens' principle, $\mathbf{1}_{B(0,R)}U_0(-\sigma)Q(\sigma) = 0$ for $\sigma > R+\rho$. For $t > R+\rho$ then the integral above is independent of t and therefore the strong limit $\mathbf{1}_{B(0,R)}W_+$ exists in a trivial way.

Next, for any $R_1 < R_2$, $\|\mathbf{1}_{B(0,R_2)\setminus B(0,R_2)}(U_0(t)S\mathbf{f} - \mathbf{v}(t))\| \to 0$, as $t\to\infty$. Then $U_0(t)S\mathbf{f}$ and $\mathbf{v}(t)$ have the same asymptotic wave profiles. Then, by (2.9) and (3.16),

$$\mathcal{R}S\mathbf{f}(s',\omega') = \mathcal{R}\mathbf{f}(s',\omega') + 2c_n^- \int_{\mathbf{R}\times S^{n-1}} \partial_s^{(n-3)/2} u_{\mathrm{sc}}^\sharp(s',\omega';s,\omega)(\mathcal{R}\mathbf{f})(s,\omega)\,\mathrm{d}s\,\mathrm{d}\omega.$$

The proof now follows from the definition of the scattering amplitude. □

The theorem shows that even though S may not always exist in the energy space, it always does as $S:\mathcal{H}_{\mathrm{comp}}\to\mathcal{H}_{\mathrm{loc}}$. The scattering amplitude $A^\sharp$ is "essentially" the kernel of $S-\mathrm{I}$ in the translation representation. The kernel of S itself is called the scattering kernel and is given by

$$S(s',\omega';s,\omega) = \delta(s'-s)\delta_\omega(\omega') - 2^{-1}(2\pi)^{1-n}\partial_{s'}^{(n-1)/2}\partial_s^{(n-3)/2}A(s',\omega';s,\omega).$$

3.4. Time-independent potentials; connection to the stationary theory. When $q=q(x)$ is independent of t, the solution is represented by a unitary group $\tilde U(t)$. Assume for simplicity that $-\Delta+q$ is a positive operator ($q\ge 0$ would suffice for that). Then $A-Q$ is skew-self-adjoint in $\mathcal{H}^q$, see (3.10), and therefore generates a unitary group $\tilde U(t)$. Clearly, $U(t,s) = U(t-s)$.

The scattering kernel and the scattering amplitude are then of convolution type w.r.t. the time variable, i.e.,

$$S(s',\omega';s,\omega) = \tilde S(s'-s,\omega',\omega), \tag{3.18}$$

where $\tilde S(s',\omega',\omega) = S(s',\omega';0,\omega)$; and similarly for the (time-dependent) scattering amplitude $A^\sharp$. Another way to say this is that the scattering operator commutes with time translations.

THEOREM 3.4. *Let $q(x)$ be time-independent. Then the wave operators*

$$\Omega_\pm : \mathcal{H}\to\mathcal{H}^q$$

are isometries and the scattering operator S defined in Theorem 3.3 extends to a unitary operator on $\mathcal{H}$.

Proof. The groups $U_0(t)$ and $U(t)$ are unitary but in their own spaces only, $\mathcal{H}$ and $\mathcal{H}^q$. The norms in those spaces however coincide outside $B(0,\rho)$. Therefore, by the proof of Theorem 3.3, $\|\Omega_-\mathbf{f}\|_{\mathcal{H}^q} = \|\mathbf{f}\|_{\mathcal{H}}$. Therefore, $\Omega_- : \mathcal{H} \to \mathcal{H}^q$ extends to an isometry (not necessarily surjective).

The scattering operator in Theorem 3.3(c) clearly extends to a bounded operator on $\mathcal{H}$. We have

$$\|U_0(-t)U(t)\Omega_-\mathbf{f} - S\mathbf{f}\|_{\mathcal{H}} \to \infty, \quad \text{as } t \to \infty,$$

therefore,

$$\|U(t)\Omega_-\mathbf{f} - U_0(t)S\mathbf{f}\|_{\mathcal{H}} \to \infty, \quad \text{as } t \to \infty$$

Since $S\mathbf{f} \in \mathcal{H}$, it is easy to get by the Huygens' principle that the energy of $U_0(t)S\mathbf{f}$ in $B(0,\rho)$ decays as $t \to \infty$. Then the same must be true for $U(t)\Omega_-\mathbf{f}$. This yields

$$\|U(t)\Omega_-\mathbf{f}\|_{\mathcal{H}} = \|U_0(t)S\mathbf{f}\|_{\mathcal{H}^q} + o(1), \quad \text{as } t \to \infty,$$

therefore, $\|\Omega_-\mathbf{f}\|_{\mathcal{H}^q} = \|S\mathbf{f}\|_{\mathcal{H}}$. □

We now connect the scattering solutions $u_{\mathrm{sc}}(x,\theta,\lambda)$, see (2.12) in Chapter II to the their time-dependent analogs $u_{\mathrm{sc}}(t,x;s,\omega)$, see (3.3). Since we assume now that q is time-independent, then u_{sc} will depend on $t-s$ only, i.e., $u_{\mathrm{sc}}(t,x;s,\omega) = v_{\mathrm{sc}}(t-s,x,\omega)$ with $v_{\mathrm{sc}}(t,x,\omega) = u_{\mathrm{sc}}(t-s,x;0,\omega)$. We have

Proposition 3.5.

$$u_{\mathrm{sc}}(x,\theta,\lambda) = \int e^{\mathrm{i}\lambda t} v_{\mathrm{sc}}(t,x,\theta)\,\mathrm{d}t. \tag{3.19}$$

Proof. Recall that $v = \delta(t - x\cdot\theta) + v_{\mathrm{sc}}(t,x,\theta)$ with $v_{\mathrm{sc}} = 0$ for $t \ll 0$. Therefore, the r.h.s. U_{sc} of (3.19) extends analytically to complex λ with $\Im\lambda > 0$. Clearly,

$$e^{\mathrm{i}\lambda\theta\cdot x} = \int e^{\mathrm{i}\lambda t}\delta(t - x\cdot\theta)\,\mathrm{d}t.$$

Therefore, for the distribution U defined as in (3.19) with v_{sc} replaced by v, we get that U solves the $(-\Delta + q - \lambda^2)U = 0$, and $U_{\mathrm{sc}} = U - e^{\mathrm{i}\lambda\theta\cdot x}$ is outgoing in stationary sense. Such a solution is unique however, so we get $U_{\mathrm{sc}} = u_{\mathrm{sc}}$. □

In the stationary case, the time-dependent scattering amplitude $A^\sharp$ depends on $s'-s$ only (as a function of (s',s). Set $A^\sharp(s',\omega';s,\omega) = a^\sharp(s',\omega',\omega)$. Since the stationary scattering amplitude a and the time-dependent one $a^\sharp$ can be expressed in terms of the outgoing solutions, see (2.19) in Chapter II and (3.12), we also get

Proposition 3.6.

$$a_0(\omega,\theta,\lambda) = \int e^{\mathrm{i}\lambda t} a^\sharp(s,\omega,\theta)\,\mathrm{d}s, \tag{3.20}$$

where a_0 is the normalized stationary scattering amplitude.

Warning: poor notation for the various scattering amplitudes!

Let us apply the proposition to the representation (3.15) which we rewrite in our case ($q = q(x)$) as

$$\mathcal{R}(S-\mathrm{I})\mathcal{R}^{-1} = -2^{-1}(2\pi)^{1-n}(-1)^{(n-3)/2}\partial_{s'}^{n-2}a^\sharp,$$

where (another bad notation!), $a^\sharp$ is the operator with kernel $a^\sharp(s'-s,\omega',\omega)$. We get a convolution w.r.t. the s variable which can be expected, see (3.18). Let $\mathcal{F}_1^*$ be adjoint the Fourier transform in the s-variable, as in (3.20). Since convolution becomes multiplication in the Fourier domain, we get

$$\left[\mathcal{F}_1^*\mathcal{R}(S-\mathrm{I})\mathcal{R}^{-1}\mathcal{F}_1 h\right](\theta,\lambda) = -2^{-1}(2\pi)^{2-n}(-1)^{(n-3)/2}(-\mathrm{i}\lambda)^{n-2}\int_{S^{n-1}} a_0(\omega,\theta,\lambda)h(\theta,\lambda)\,\mathrm{d}\theta.$$

The kernel of $\mathcal{R}$ is "essentially" that of the Radon transform, $\delta(s-x\cdot\theta)$. Composed with $\mathcal{F}_1^*$, it becomes "essentially" $e^{\mathrm{i}\lambda x\cdot\theta}$. This is the kernel of the spectral projections E_+, see (5.9) in Chapter II. Therefore, ignoring the monomial (in λ) factor above and the matrix structure of S, we get

$$E_+(\lambda)(S-\mathrm{I})E'_+(\lambda)h \quad \text{"="} \quad \int_{S^{n-1}} a_0(\omega,\theta,\lambda)h(\theta)\,\mathrm{d}\theta.$$

In particular, the scattering operator S and the scattering matrix $S(\lambda)$ are related by

$$E_+(\lambda)SE'_+(\lambda) \quad \text{"="} \quad S(\lambda).$$

This connects the fundamental objects in the time dependent theory, the scattering operator S to the fundamental object in the stationary one, the scattering matrix $S(\lambda)$.

Note to myself: rewrite this.

4. Scattering for the time-dependent Schrödinger equation (a few remarks only)

Consider the time-dependent Schrödinger equation

$$\frac{1}{\mathrm{i}}\partial_t u = (-\Delta + V)u, \tag{4.1}$$

where V is a potential as above. The natural energy space now is $L^2(\mathbf{R}^n)$ (and in particular, the norm is independent of V). The "Hamiltonian" $H = -\Delta + V$ extends naturally to a self-adjoint one (V is real), Set $H_0 = -\Delta$. By Stone's theorem, the following two unitary groups are well defined

$$U_0(t) = e^{\mathrm{i}tH_0}, \quad U(t) = e^{\mathrm{i}tH}$$

and represent the solution groups of (4.1) (with $V = 0$ for U_0). The scattering operators are defined as above.

Scattering theory for (4.1) can be related to that of the acoustic equation. Note that for the generator A, see (2.5), we have

$$A^2 := \begin{pmatrix} 0 & \Delta \\ \Delta & 0 \end{pmatrix}. \tag{4.2}$$

We get a similar expression if we replace Δ by $\Delta + V$. This allow to prove that (see [**8**], Chapter VI)

$$S_{\text{Schr}}(z) = S(\sqrt{z}),$$

where S_{Schr} is the scattering matrix for (4.1) (which needs to be defined first), and $S(\lambda)$ is the scattering matrix for the acoustic equation. We will not go into details. Another way to get that relation is to relate (4.1) to the Helmholtz type of equation $(-\Delta + V - z)v = 0$ by taking the Fourier transform $\mathcal{F}_{t\to\ z}$ (which requires some work, of course) and then setting $z = \lambda^2$ to use the link between the time-dependent theory for the acoustic equation and the stationary one for $(-\Delta + V - \lambda^2)v = 0$. In fact, the stationary theory can be considered as a stationary version of both the acoustic and the Schrödinger equations. This is a bit surprising because the latter two PDEs have very different properties in terms of finite speed of propagation, etc.

CHAPTER IV

Inverse Scattering

1. Introduction

Inverse scattering tries to recover the perturbation: the potential, the obstacle, etc., from scattering data. The latter means either the (whole) scattering operator/matrix/amplitude, or some partial information about it; like $S(\lambda)$ for a fixed λ.

The main questions are uniqueness, stability and reconstruction. Another important question is range characterization — what functions can be the scattering amplitude of a C_0^∞ potential, for example.

2. Inverse potential scattering

2.1. High frequency asymptotics and Born approximation. Uniqueness and recovery. A basic question is to understand the high-frequency behavior $\lambda \to \infty$ of the scattering amplitude. This is the semi-classical regime as well, when $h = 1/\lambda \to \infty$.

THEOREM 2.1. *Let $V \in L^\infty_{\mathrm{comp}}(\mathbf{R}^n)$. Then*

$$a_0(\omega, \theta, \lambda) = \hat{V}(\lambda(\omega - \theta)) + O(1/\lambda), \quad \text{as } \lambda \to \infty$$

with the remainder uniform in θ, ω.

PROOF. Follows directly from (2.20) and Theorem 2.1 (or from (2.18), (2.16) and Theorem 1.3). □

Note first that the asymptotic above would not be so interesting if ω and θ are kept fixed because for $V \in C_0^\infty$, first term on the right would decay faster that the remainder, if $\omega \neq \theta$. On the other hand, we can choose sequences $(\omega_k, \theta_k, \lambda_k)$ with $\lambda \to \infty$ so that $\xi_k = \lambda_k(\omega_k - \theta_k)$ stays bounded. Moreover, for each ξ we can choose sequences so that $\xi = \lambda_k(\omega_k - \theta_k)$, $\lambda_k \to \infty$, and then necessarily, $\theta_k - \omega_k \to 0$. Therefore, we can recover $\hat{q}$ from knowing a_0.

COROLLARY 2.2.

$$\hat{V}(\xi) = \lim_{\substack{\xi = \lambda(\omega - \theta) \\ \lambda \to \infty}} a_0(\omega, \theta, \lambda).$$

In particular, this solves the inverse scattering problem: recover V from a. The compact support assumption is too strong here; short range is enough.

Let us look closer at the estimates leading to the theorem. By (2.10) in Chapter II, if $V = 0$ outside the ball $B(0, \rho)$, then

$$\|\mathbf{1}_{B(0,\rho)} R(\lambda) \mathbf{1}_{B(0,\rho)}\| \leq \frac{C}{\lambda}\left(1 - C\lambda^{-1}\|V\|_{L^\infty}\right)^{-1}, \quad \text{when } \|V\|_{L^\infty} \leq \lambda/C,$$

with some absolute constant $C = C(n)$. In particular,

$$\|\mathbf{1}_{B(0,\rho)}R(\lambda)\mathbf{1}_{B(0,\rho)}\| \le \frac{C}{\lambda}, \quad \text{when } \|V\|_{L^\infty} \le \lambda/C$$

with C replaced by $2C$. Then, see (2.14) in Chapter II,

$$\|u_{\mathrm{sc}}\|_{L^2(B(0,\rho))} \le \frac{C}{\lambda}\|V\|_{L^\infty}, \quad \text{when } \|V\|_{L^\infty} \le \lambda/C. \tag{2.1}$$

Then by (2.19) in Chapter II,

$$\max_{\omega,\theta}\left|a_0(\omega,\theta,\lambda) - \hat{V}(\lambda(\omega-\theta))\right| \le \frac{C}{\lambda}\|V\|_{L^\infty}^2, \quad \text{when } \|V\|_{L^\infty} \le \lambda/C. \tag{2.2}$$

This is actually a linearization of a_0 as a function of V, known as the *Born approximation.* Note that λ can be fixed there. Also, note that we actually proved that estimate with $\|V\|_{L^\infty}\|V\|_{L^2}$ on the right, which is a slightly stronger estimate.

2.2. Stability. Since we have an explicit recovery formula "of stable type", stability follows easily. The following lemma is very useful.

LEMMA 2.3. *Let V, $\tilde{V}$ be two potentials in $L^\infty_{\mathrm{comp}}(\mathbf{R}^n)$. Then for the corresponding scattering amplitudes a_0, $\tilde{a}_0$, we have*

$$a_0(\omega,\theta,\lambda) - \tilde{a}_0(\omega,\theta,\lambda) = \int \left(V(x) - \tilde{V}(x)\right)\tilde{u}(x,-\omega,\lambda)u(x,\theta,\lambda)\,\mathrm{d}x,$$

where u, $\tilde{u}$ are the corresponding "perturbed plane waves".

PROOF. This is essentially a consequence of the resolvent identity

$$R(\lambda) - \tilde{R}(\lambda) = R(\lambda)(V - \tilde{V})\tilde{R}(\lambda)$$

written in terms of the Schwartz kernels G and $\tilde{G}$ of the resolvents:

$$G(x,y,\lambda) - \tilde{G}(x,x,\lambda) = \int G(x,z,\lambda)(V(z) - \tilde{V}(z))\tilde{G}(z,x,\lambda)\,\mathrm{d}z.$$

Next, we take the asymptotics $x = r\omega$, $r \to \infty$ and $x = r'\theta$, $r' \to \infty$. [**this needs to be done**]

We will give a more direct but less intuitive proof, following [**19**]. With $u_0(x,\theta,\lambda) := e^{\mathrm{i}\lambda\theta\cdot x}$, write

$$\begin{aligned} a_0(\omega,\theta,\lambda) &= \int u_0(x,-\omega,\lambda)V(x)u(x,\theta,\lambda)\,\mathrm{d}x \\ &= \int (\tilde{u} - \tilde{u}_{\mathrm{sc}})(x,-\omega,\lambda)V(x)u(x,\theta,\lambda)\,\mathrm{d}x \\ &= \int \tilde{u}(x,-\omega,\lambda)V(x)u(x,\theta,\lambda)\,\mathrm{d}x + M, \end{aligned} \tag{2.3}$$

where, by (2.15),

$$M = -\int \tilde{u}_{\mathrm{sc}}(x, -\omega, \lambda) V(x) u(x, \theta, \lambda)\, \mathrm{d}x$$
$$= \int [R_0(\lambda)\tilde{V}\tilde{u}(\cdot, -\omega, \lambda)] V(x) u(x, \theta, \lambda)\, \mathrm{d}x.$$

Since the kernel of $R_0(\lambda)$ is symmetric, when $V = \tilde{V}$, we get $a(\omega, \theta, \lambda) = a(-\theta, -\omega, \lambda)$. In the same way, we get

$$\tilde{a}_0(\omega, \theta, \lambda) = \tilde{a}(-\theta, -\omega, \lambda) = \int u(x, \theta, \lambda)\tilde{V}(x)\tilde{u}(x, -\omega, \lambda)\, \mathrm{d}x + \tilde{M}, \tag{2.4}$$

where

$$\tilde{M} = \int [R_0(\lambda) V u(\cdot, -\theta, \lambda)]\tilde{V}(x)\tilde{u}(x, -\omega, \lambda)\, \mathrm{d}x = M.$$

Subtract (2.4) from (2.3) to complete the proof. □

We now apply (2.1) to u and $\tilde{u}$. Set $q = V - \tilde{V}$. Then

$$\hat{q}(\lambda(\omega - \theta)) = a_0 - \tilde{a}_0 + R,$$

where

$$|R| \le \frac{C}{\lambda}\|q\|_{L^\infty}$$

and $C > 0$ depends on a-priori upper bounds on $\|V\|_{L^\infty}$ and $\|\tilde{V}\|_{L^\infty}$. Given $\xi \neq 0$, choose $\lambda_n \to \infty$, ω_n and θ_n so that $\xi = \lambda_n(\omega_n - \theta_n)$. One way to do this is to fix a unit $\omega_0 \perp \xi$ and to set

$$\omega_n = \sqrt{1 - 1/(4n^2)}\,\omega_0 + \frac{1}{2n|\xi|}\xi,$$
$$\theta_n = \sqrt{1 - 1/(4n^2)}\,\omega_0 - \frac{1}{2n|\xi|}\xi,$$
$$\lambda_n = n|\xi|.$$

Then

$$|\hat{q}(\xi)| \le \|a_0 - \tilde{a}_0\|_{L^\infty} + \frac{C}{n|\xi|}\|q\|_{L^\infty}.$$

We can take the limit $n \to \infty$ here but the norm $\|a - a_0\|_{L^\infty}$ is not such a good candidate — it can be even infinite if $\lambda = 0$ is a pole! Instead, let us assume that we want to obtain an estimate with a_0 restricted to $\lambda \ge \lambda_0$ for some $\lambda_0 > 0$. Then $n|\xi| \ge \lambda_0$, and

$$|\hat{q}(\xi)| \le \|a_0 - \tilde{a}_0\|_{L^\infty(\lambda > \lambda_0)} + \frac{C}{n|\xi|}\|q\|_{L^\infty}, \quad n|\xi| \ge \lambda_0, \quad \xi \neq 0.$$

Now, we for any fixed $\xi \neq 0$ we can take the limit $n \to \infty$ to obtain the following.

THEOREM 2.4. *For V, $\tilde{V}$ as above, for any $\lambda_0 > 0$,*

$$\|\mathcal{F}(V - \tilde{V})\|_{L^\infty} \le \|a_0 - \tilde{a}_0\|_{L^\infty(\lambda > \lambda_0)}.$$

The l.h.s. above is clearly a norm. If we need a more conventional one, we need to use interpolation and a priori compactness assumption. Then we will get a conditional Hölder stability estimate.

Note that the Lemma 2.3 was not really needed here but it will be needed later.

2.3. Inverse potential scattering at a fixed frequency. Calderón's problem. Assume that we know the scattering amplitude $a_0(\lambda,\omega,\theta)$ in potential scattering for a fixed frequency $\lambda>0$ only, and all θ, ω. It turns out that we can still recover V but the proof is much more delicate and the recovery is unstable.

If we just count the number of the variables, $a_0(\lambda,\omega,\theta)$ depends on $2n-2$ (one-dimensional) variables, while V depends on n. Then $2n-2>n$ if $n\geq 3$. This make the problem *formally overdetermined* in dimension $n\geq 3$. When $n=2$, $2n-2=n$. The problem is then *formally determined*. We can expect $n=2$ to be a harder case, and that is actually true. We will restric the exposition to $n\geq 3$.

The main uniqueness result of this section is the following.

THEOREM 2.5. *Let V and $\tilde V$ be in $L^\infty_{\mathrm{comp}}(\mathbf{R}^n)$, $n\geq 3$. For a fixed $\lambda_0>0$, assume*

$$a_0(\lambda_0,\omega,\theta)=\tilde a_0(\lambda_0,\omega,\theta).$$

Then $V=\tilde V$.

2.3.1. *Calderón's problem.* This problem is closely related to the Calderón's inverse boundary value problem. We will formulate the version of this problem related to the Schrödinger operator only. Let Ω be a bounded domain with a smooth boundary $\partial\Omega$. Assume that λ_0^2 is not a Dirichlet eigenvalue for the operator $-\Delta+V-\lambda_0^2$ in Ω. Given f, let v be the unique solution of the IVP

$$\begin{aligned}(-\Delta+V-\lambda_0^2)v&=0 \quad \text{in } \Omega,\\ v&=f \quad \text{on } \partial\Omega.\end{aligned}$$

Then we define the Dirichlet-to-Neumann map Λ by

$$\Lambda f=\partial_\nu v|_{\partial\Omega},$$

where ν is the exterior unit normal. Then $\Lambda: H^{3/2}(\partial\Omega)\to H^{1/2}(\partial\Omega)$.

The Calderón's problem in this setting is: does Λ uniquely determine V? Note that one can replace V there by $V-\lambda_0^2$ and require that 0 is not an Dirichlet eigenvalue. The latter requirement can be avoided if we replace Λ with the set of Cauchy data.

The next theorem is due to Sylvester and Uhlmann [**23**].

THEOREM 2.6. *Let V and $\tilde V$ be in $L^\infty(\Omega)$, $n\geq 3$, and assume that $\lambda_0>0$ is not a Dirichlet eigenvalue associated with neither potential. If*

$$\Lambda=\tilde\Lambda,$$

then $V=\tilde V$.

We will prove Theorem 2.6 first and then reduce the proof of Theorem 2.5 to it.

The first step of the proof is the following "Alessndrini identity" (compare with Lemma 2.3).

LEMMA 2.7. *Let V and $\tilde V$ be two potentials in $L^\infty(\Omega)$ so that 0 is not a Dirichlet eigenvalue for $-\Delta+V$ nor for $-\Delta+\tilde V$. Let Λ, $\tilde\Lambda$ be the corresponding DN maps. Then for any f, $\tilde f$ in $H^{3/2}(\partial\Omega)$,*

$$\int_{\partial\Omega} f(\Lambda-\tilde\Lambda)\tilde f\,\mathrm{d}S = \int_\Omega (V-\tilde V)u\tilde u\,\mathrm{d}x,$$

where u is the solution of $(\Delta+V)u=0$, $u=f$ on $\partial\Omega$; and $\tilde u$ is the solution of $(\Delta+\tilde V)\tilde u=0$, $\tilde u=\tilde f$ on $\partial\Omega$.

PROOF. By the Green's formula,

$$\int_\Omega (V-\tilde V)u\tilde u\,\mathrm{d}x = -\int_\Omega ((\Delta u)\tilde u - u\Delta\tilde u)\,\mathrm{d}x = -\int_{\partial\Omega}((\Lambda f)\tilde f - f\tilde\Lambda\tilde f)\mathrm{d}S.$$

If we take $\tilde V=V$, we see that Λ has a symmetric kernel. This observation, and the formula above complete the proof. □

2.3.2. *Complex geometric optics.* To prove Theorem 2.6, it remains to show that the products $u\tilde u$ of various solutions u and $\tilde u$ form a dense set. This fact is based on the so-called complex geometric optics, developed in [**23**]. As explained above, we can assume $\lambda_0=0$ but in fact the construction below can be done for any fixed $\lambda_0\in\mathbf{R}$. The idea goes back to Calderón. He linearized the problem near $V=0$ to reduce the problem to the following: let $q=V-\tilde V$ be orthogonal (in $L^2(\Omega)$) to the product of any pair of harmonic functions; is it true that $q=0$? He suggested the following: choose $u=e^{\zeta_1\cdot x}$, $\tilde u=e^{\zeta_2\cdot x}$, where ζ_1, ζ_2 are in $\mathcal{C}^n$ and $\zeta_1^2=\zeta_2^2=0$. Here, $\zeta^2=\zeta_1^2+\cdots+\zeta_n^2$. Clearly, u and $\tilde u$ are harmonic. Then $u\tilde u=e^{(\zeta_1+\zeta_2)\cdot x}$, and we get

$$\int q(x)e^{(\zeta_1+\zeta_2)\cdot x}\mathrm{d}x=0$$

for any such ζ_1, ζ_2. Fix $\xi\in\mathbf{R}^n$ and choose

$$\zeta_{1,2}=\frac{1}{2}(\pm\eta-\mathrm{i}\xi),\quad \text{where } \eta\in\mathbf{R}^n,\ \eta\perp\xi,\ |\eta|=|\xi|. \tag{2.5}$$

Then $\zeta_1^2=\zeta_2^2=0$ and $\zeta_1+\zeta_2=-\mathrm{i}\xi$, and we get $\hat q(\xi)=0$, $\forall\xi$, therefore, $q=0$.

When $V\neq0$, $u=e^{\zeta\cdot x}$ do not solve $(-\Delta+V)u=0$. Uhlmann and Sylvester's idea was to construct solutions which look like the Calderón's harmonic ones only asymptotically in the sense

$$u=e^{\zeta\cdot x}(1+O(|\zeta|^{-1}),\quad \zeta\to\infty. \tag{2.6}$$

THEOREM 2.8. *Let $\rho>0$, $n\geq2$. There is a constant $C_0=C_0(n)>0$ so that if $\|V\|_{L^\infty}<C_0|\zeta|$, $\operatorname{supp}V\subset B(0,\rho)$ and $\zeta^2=0$, then there exists a solution $u(x,\zeta)$ of the equation*

$$(-\Delta+V)u=0\quad \text{in } \mathbf{R}^n \tag{2.7}$$

satisfying

$$u(x,\zeta)=e^{\zeta\cdot x}\left(1+u_0(x,\zeta)\right),\quad \text{where}\quad \|u_0\|_{L^2(B(0,2\rho)}\leq\frac{C}{|\rho|}.$$

We will only sketch the proof. First, conjugate $-\Delta+V$ with the operator of multiplication by $e^{\zeta\cdot x}$ to get

$$e^{-\zeta\cdot x}(-\Delta+V)e^{\zeta\cdot x} = -\Delta - 2\zeta\cdot\partial + V.$$

Then u_0 solves

$$(-\Delta - 2\zeta\cdot\partial + V)u_0 = -V. \tag{2.8}$$

To "invert" the operator $-\Delta - 2\zeta\cdot\partial + V$, we will first "invert" $-\Delta_\zeta := -\Delta - 2\zeta\cdot\partial$. More precisely, we want to define an operator $G(\zeta)$ so that $-\Delta_\zeta G(\zeta) = \mathrm{I}$. Clearly, $G(\zeta)$ is not uniquely defined, so we have to make some choices. The symbol of $-\Delta_\zeta$ is given by

$$\sigma(-\Delta_\zeta) = |\xi|^2 - 2\mathrm{i}\zeta\cdot\xi$$

Assuming that we work with tempered distributions, we can use Fourier transform, and attempt to define $G(\zeta)$ as the Fourier multiplier with $(|\xi|^2 - 2\mathrm{i}\zeta\cdot\xi)^{-1}$. The problem here is that $|\xi|^2 - 2\mathrm{i}\zeta\cdot\xi$ has zeros. Let us look at them closely.

One example of ζ with $\zeta^2 = 0$ is $\zeta = \lambda(1, \mathrm{i}, 0, \dots, 0)$. In fact, all other ζ's are equivalent to this, after a rotation. Indeed, write $\zeta = \Re\zeta + \mathrm{i}\Im\zeta$. Then $\zeta^2 = |\Re\zeta|^2 - |\Im\zeta|^2 + 2\mathrm{i}\Re\zeta\cdot\Im\zeta$. Therefore, $\zeta^2 = 0$ is equivalent to

$$|\Re\zeta| = |\Im\zeta|, \quad \Re\zeta\cdot\Im\zeta = 0.$$

Now, if $\zeta \neq 0$, we can choose a unitary linear transformation so that $\Re\zeta/|\Re\zeta|$ and $\Im\zeta/|\Im\zeta|$ are first two vectors of the new basis, thus proving our claim. Next, under such unitary transformations U, $\zeta\cdot x = (U\zeta)\cdot(Ux)$, therefore we can always assume that $\zeta = \lambda(1, \mathrm{i}, 0, \dots, 0)$.

Then

$$-\Delta_\zeta = -\Delta - 2\lambda\partial_1 - 2\lambda\mathrm{i}\partial_2, \quad \sigma(-\Delta_\zeta) = |\xi|^2 - 2\mathrm{i}\lambda\xi_1 + 2\lambda\xi_2.$$

The characteristic variety $\Sigma = \{\sigma(-\Delta_\zeta) = 0\}$ is then given by the codimension 2 manifold

$$\Sigma = \{\xi_1 = 0,\ (\xi_2+\lambda)^2 + \xi_2^3 + \dots \xi_n^2 = \lambda^2\}.$$

This is the intersection of the plane $\xi_1 = 0$ and the ball $B((0, \lambda, 0, \dots), \lambda)$. In $\mathbf{R}^3$, for example, it the 1D circle $\xi_1 = 0$, $(\xi_2+\lambda)^2 + \xi_3^2 = \lambda^2$.

Compare this with the Helmholtz operator $-\Delta + \lambda^2$. Its symbol is $|\xi|^2 - \lambda^2$ and its characteristic variety is the (codimension 1) sphere $|\xi| = \lambda$.

Near Σ, in local coordinates, $\sigma(-\Delta_\zeta)$ is given by $\xi_1 + \mathrm{i}\xi_2$. In fact, this is the so-called normal form of $\sigma(-\Delta_\zeta)$. On the other hand, the symbol of the Helmholtz operator is ξ_1 in local coordinates. This reveals the differences between the two cases: $(\xi_1 + \mathrm{i}\xi_2)^{-1}$ is locally integrable near $\xi_1 = \xi_2 = 0$: indeed, for $n = 2$, the modulus of this function is $|\xi|^{-1}$; and for $n \geq 3$, we first integrate w.r.t. (ξ_1, ξ_2). On the other hand, ξ_1^{-1} is not locally integrable and we needed to consider the regularizations in (1.14), Chapter II.

Note also that in both cases, λ is a large parameter, and the semi-classical calculus is a more appropriate tool, with $h = 1/\lambda$. Then the principal (and the full) symbol of $-\Delta_\zeta$ is $|\xi|^2 - 2\mathrm{i}\xi_1 + 2\lambda\xi_2$, while its classical principal symbol would be $|\xi|^2$ only. The principal symbol of the Helmholtz operator is then $|\xi|^2 - 1$. The characteristic varieties are as above with λ substituted by $\lambda = 1$.

This discussion brings us to the following candidate for a right inverse of $-\Delta_\zeta$, called in the literature also the complex geometric optics Green's function (also identified with its kernel), is

$$G(\zeta) = \mathcal{F}^{-1}\left(|\xi|^2 - 2\mathrm{i}\zeta\cdot\xi\right)^{-1}\mathcal{F}.$$

This Green's function was also studied by Faddeev. The multiplier in the middle of the r.h.s. is locally L^1 and globally tempered; and therefore $G(\zeta) : \mathcal{S} \to \mathcal{S}'$, at least. Next theorem proves something more; and provides a bound of the kind $O(|\zeta|^{-1})$ of the norm. Recall that the weighted L^2 spaces $L^2_\delta(\mathbf{R}^n)$ are defined through the norm

$$\|f\|^2_{L^2_\delta} = \int \left(1+|x|^2\right)^\delta |f(x)|^2\,\mathrm{d}x.$$

Similarly, one defines the weighted Sobolev spaces $H^m_\delta(\mathbf{R}^n)$, $m = 0, 1, \dots$ by the norm

$$\|f\|^2_{H^m_\delta} = \sum_{|\alpha|\le m} \|D^\alpha f\|^2_{L^2_\delta}.$$

Theorem 2.9 ([**23**]). *Let $\zeta^2 = 0$, $0 < \beta \le |\zeta|$, $-1 < \delta < 0$. Then*
(a) There exists $C(\beta,\delta) > 0$ so that

$$\|G(\zeta)f\|_{L^2_\delta} \le \frac{C(\beta,\delta)}{|\zeta|}\|f\|_{L^2_{\delta+1}}. \tag{2.9}$$

(b) For any $f \in L^2_\delta$, there is a unique $w \in L^2_{\delta+1}$ solving

$$-\Delta_\zeta w = f.$$

Moreover, $w = G(\zeta)f$, and

$$\|w\|_{L^2_\delta} \le \frac{C(\beta,\delta)}{|\zeta|}\|f\|_{L^2_{\delta+1}}.$$

Sketch of the proof. We will leave part (a) without a proof (see [**23**]]). The main idea is to prove (2.9) first for the normal form of $-\Delta_\zeta$ first; i.e., for the Fourier multiplier $(\xi_1 + \mathrm{i}\xi_2)^{-1}$. The latter (in 2D) is a convolution with the inverse Fourier transform of $(\xi_1+\mathrm{i}\xi_2)^{-1}$; and a direct computation shows this to be $(x^1+\mathrm{i}x^2)^{-1}$. Next, convolution with the latter can be estimated directly. The next step of the proof if to localize $G(\zeta)f$, for $\zeta = \lambda(1, \mathrm{i}, 0\dots)$ in the Fourier domain near Σ; to apply the normal form estimate there; and to estimate the part away from Σ. The resulting estimate would be $O(1/\lambda)$, which also implies $O(1/|\zeta|)$.

The uniqueness part (b) follows from [**3**, Thm 7.1.27]. Indeed, under the assumption $w \in L^2_{\delta+1}$ (the uniqueness is claimed there only and not true in general, for example, we can add any constant to w), $\hat{w}$ is well defined. Then $\hat{w}(\xi) = 0$ on Σ. Hörmander's lemma then implies $w = 0$. □

Let us return now to the proof of Theorem 2.8. We want to solve (2.8). It is not uniquely solvable for sure but we only need to find a solution $u_0 = O(|\zeta|^{-1})$. We look for a solution u_0 of the form $u_0 = G(\zeta)w$. Then

$$\left(\mathrm{I} + VG_0(\zeta)\right)w = -V.$$

Then

$$VG_0(\zeta): L^2_\delta \to L^2_\delta = O(|\zeta|^{-1}), \quad |\zeta| \gg 1. \tag{2.10}$$

We can therefore use Neumann series to invert $(\mathrm{I}+VG_0(\zeta))$ for $|\zeta| \gg 1$, the resolvent would be $O(1)$, and

$$w = -(\mathrm{I}+VG_0(\zeta))^{-1}V.$$

A priori, $w \in L^2_\delta$ with no other control on the large $|x|$ behavior but we can see directly from (2.10) that w is supported in $B(0,\rho)$. Note that resemblance with (2.13), (2.14) in Chapter II! In potential stationary scattering, to define u_{sc}, we need to invert the operator $(\mathrm{I}+VR_0(\lambda)$, see (2.7) in Chapter II. When $\lambda \gg 1$, this follows from the a priori estimate of $R_0(\lambda)$ of the kind $O(1/\lambda)$.

Now, we can set

$$u_0 = -G(\zeta)(\mathrm{I}+VG_0(\zeta))^{-1}V,$$

and this proves Theorem 2.8.

2.3.3. *Uniqueness for Calderón's problem.*

Proof of Theorem 2.6 . We will use the geometric optics solutions constructed in Theorem 2.8 in the identity of Lemma 2.7. Since $\Lambda = \tilde\Lambda$, with $q := V - \tilde V$, we get

$$\int q(x)u\tilde u \,\mathrm{d}x = 0$$

for any $H^2(\Omega)$ solution u of $(-\Delta+V)u = 0$ in Ω, and any $H^2(\Omega)$ solution $\tilde u$ of $(-\Delta+\tilde V)\tilde u = 0$. Note that the geometric optics solutions constructed in Theorem 2.8 are in $H^2(\Omega)$ since they solve (2.7). Choose ζ_1, ζ_2 with

$$\zeta_1^2 = \zeta_2^2 = 0 \tag{2.11}$$

and let $u(x,\zeta_1)$, $\tilde u(x,\zeta_2)$ be the corresponding solutions. Then

$$\int q(x)e^{(\zeta_1+\zeta_2)\cdot x}(1+u_0(x,\zeta_1))(1+\tilde u_0(x,\zeta_2))\,\mathrm{d}x = 0.$$

Therefore,

$$\int q(x)e^{(\zeta_1+\zeta_2)\cdot x}\,\mathrm{d}x = -\int q(x)e^{(\zeta_1+\zeta_2)\cdot x}\left(u_0(x,\zeta_1)+\tilde u_0(x,\zeta_2)+u_0(x,\zeta_1)\tilde u_0(x,\zeta_2)\right)\mathrm{d}x. \tag{2.12}$$

We would like to choose appropriate ζ_1, ζ_2 and take the limits

$$|\zeta_1| \to \infty, \quad |\zeta_2| \to \infty. \tag{2.13}$$

On the left, we want to obtain the Fourier transform of q, i.e., for any fixed in advance ξ, we need

$$\zeta_1 + \zeta_2 = -\mathrm{i}\xi. \tag{2.14}$$

If we choose them as in (2.5), then we cannot control $|\zeta_j|$, $j=1,2$ because there, $|\zeta_j| = |\xi|^2/2$, which is fixed. It is easy to see that in $\mathbf{R}^2$, this is the only choice up to the change of the sign of η there. This is the reason we required $n \geq 3$. This does not mean, of course, that the theorem is not true in 2D (it is). It only means that we cannot prove it that way.

To choose ζ_1, ζ_2 satisfying (2.11), (2.13) and (2.14), we proceed as follows. Since $n \geq 3$, given $\xi \neq 0$, we can choose real vectors η and p so that (ξ, η, p) form an orthogonal triple and $|\eta|^2 = |p|^2 + |\xi|^2/4$. Then we set

$$\zeta_{1,2} = \pm\eta \pm \mathrm{i}p - \mathrm{i}\xi/2.$$

Then (2.11) holds; to satisfy (2.13), we will take a sequence of p's (and hence of η's) so that $|p| \to \infty$. Then we would also have (2.14), as desired, and by (2.12),

$$|\hat{q}(\xi)| \leq C\left(|\zeta_1|^{-1} + |\zeta_2|^{-1} + |\zeta_1|^{-1}|\zeta_2|^{-1}\right).$$

Take the limit (2.13) to get $\hat{q}(\xi) = 0$ for any $\xi \neq 0$. This implies $q = 0$. □

2.3.4. *Proof of Theorem 2.5.* Follows from Theorem 2.6 (uniqueness for the Calderón problem) and Theorem 5.1 (the inverse scattering problem can be reduced, and is actually equivalent, to the Calderón problem).

2.3.5. *Stability analysis.* The Calderón problem is very unstable, and the inverse scattering one at a fixed energy is even worse. Alessandrini proved the following stability estimate

$$\|V - \tilde{V}\|_{L^\infty} \leq C\left|\log\|\Lambda - \tilde{\Lambda}\|_{H^{3/2}\to H^{1/2}}\right|^{-\mu}, \quad 0 < \mu < 1 \tag{2.15}$$

for all V and $\tilde{V}$ in $L^\infty(B(0,R_0)$ near (in $L^\infty(B(0,R_0))$ some fixed V_0 under the assumptions that 0 is not a Dirichlet eigenvalue for V_0; and also assuming the following a priori bound

$$\|V\|_{C^k} + \|\tilde{V}\|_{C^k} \leq A \tag{2.16}$$

with some fixed $k \gg 1$ and some $A > 0$. Moreover, it has been shown [ref] that this is the best estimate one can get. The condition (2.16) is of compactness type since it guarantees that the set of all V satisfying it, supported in $B(0,R_0)$, is compact in $L^\infty(B(0,R_0)$. By a functional analysis argument, an injective map A between two Banach spaces, restricted to a compact set K, has a bounded inverse form $A(K)$ to K. The stability estimate above is a qualitative result; estimating the modulus of continuity to be at least

$$\phi(t) = \left(\log\frac{1}{\epsilon}\right)^{-\mu}, \quad t > 0.$$

This function tends to 0 as $t \to 0$ very slowly, slower than t^α for any $\alpha > 0$. Since this rate cannot be improved, this analysis actually proves that Calderón's problem is a very (exponentially) unstable one. Stability estimates requiring an additional a priori estimate like (2.16) are called *conditional stability estimates*. They are the most common ones for many non-linear inverse problems. Unconditional estimates are rare, and the most common stability for "stable" non-linear problems is a conditional stability of Hölder type, when the modulus of continuity is

$$\phi(t) = t^\alpha, \quad 0 < \alpha \leq 1.$$

When $\alpha = 1$, it is called Lipschitz stability.

2.4. Concluding remarks. While the high-frequency asymptotic, and thus the uniqueness result (for all frequencies) can, and has been, be easily generalized to non-compactly supported short-range potentials, the fixed energy problem is quite different. There are counter-examples of polynomially decaying short-range potentials for which the uniqueness fails. The uniqueness has been extended to exponentially decaying potentials by Eskin-Ralston, see also Vasy-Uhlmann. [ref]

2.5. Inverse back-scattering. The inverse back-scattering problem in potential scattering consists of the following: can we recover the potential V from the scattering amplitude a at $\omega = -\theta$, for all θ and $\lambda > 0$? In other words, is the map

$$V(x) \quad \longmapsto \quad a(-\theta, \theta, \lambda)$$

injective? This corresponds to measuring the scattered field at the same point (at infinity) at which the signal is sent. As usual, we start with counting the variables. The potential depends on n variables, and so does the back-scattering amplitude. This makes the problem formally determined and it is an indication that it would be harder than many formally overdetermined problems. In fact, uniqueness for the problem is still an open question, even for $V \in C_0^\infty$. Partial results exist however.

Perhaps the second thing to do is to try the Born approximation (2.2). Ignoring the error term, we get

$$a_0(-\omega, \omega, \lambda) \approx \hat{V}(-2\lambda\theta).$$

Since we "know" that for every $\lambda > 0$ and every $\theta \in S^{n-1}$, setting $\xi = -2\lambda\omega$, we get the Fourier transform of V for every $\xi \neq 0$, and that determines V uniquely. In some circles, this actually passes for a proof.

For small V, one can do the following. Let $n = 3$. Then the explicit form of the kernel G_0 of $R_0(\lambda)$, see (1.12) in Chapter II, yields

$$\|\chi R_0(\lambda)\chi\| \leq C$$

for any compactly supported cut-off χ, uniformly in λ. Then the series (2.9) in the same chapter converge for all $\lambda > 0$, as an operator from $L^2(K)$ to itself for any compact set K. Then by the representation formulas (2.18), (2.20) there,

$$a_0(\omega, \theta, \lambda) = \int e^{-\mathrm{i}\lambda\omega\cdot x} V(x) \left(\mathrm{I} - R_0(\lambda)V - R_0(\lambda)VR_0(\lambda)V - \dots\right) e^{\mathrm{i}\lambda\theta\cdot\bullet}\, \mathrm{d}x.$$

Set $\omega = -\theta$ to get the following.

THEOREM 2.10. *For any $R_0 > 0$ there exists a constant $C > 0$ so that if $V \in L^\infty(\mathbf{R}^3)$ and* $\operatorname{supp} V \subset B(0, R_0)$*, then*

$$\hat{V}(-2\lambda\theta) = a_0(-\theta, \theta, \lambda) + O(\|V\|^2_{L^\infty}),$$

with the estimate on the remainder uniform in $\lambda > 0$ and θ.

This does not prove uniqueness, and does not even imply that if the back-scattering amplitude vanishes, then $V = 0$ because the natural norm for $\hat{V}$ is the L^2 one but on the right, we have the L^∞ one, which is stronger.

We will sketch here the approach in [**19**] to prove that for generic V's including small ones, there is local uniqueness. A more general result, which a much longer proof, was presented by Eskin and Ralston. They actually characterized the range of the inverse back-scattering transform, assuming that V belongs to a suitable space of short range potentials.

Let $\mathcal{W} := W^{4,\infty}_{(0)}(B(0, R_0)$ be the Banach space of all functions in the real $W^{4,\infty}_{(0)}(\mathbf{R}^3)$ space (the theorem is true for complex potentials as well) supported in $B(0, R_0)$.

The main result is the following.

THEOREM 2.11. *There exists and open and dense set $\mathcal{O}$ in $\mathcal{W}$ including 0, so that for any $V_0 \in \mathcal{O}$, there exists $\varepsilon > 0$ with the following property. If*

$$\|V - V_0\|_{\mathcal{W}} < \varepsilon, \quad \|\tilde{V} - V_0\|_{\mathcal{W}} < \varepsilon,$$

and if

$$a_0(-\theta, \theta, \lambda) = \tilde{a}_0(-\theta, \theta, \lambda),$$

then $V = \tilde{V}$.

Our starting formula point is the formula in Lemma 2.3. Assuming the same data, we get

$$0 = \int e^{2i\lambda\theta\cdot x}(V - \tilde{V})(x)\,dx + \int \left[(u\tilde{u})(x, \theta, \lambda) - e^{2i\lambda\theta\cdot x}\right](V - \tilde{V})(x)\,dx.$$

Set $\xi = -2\lambda\theta$ and let $\chi \in C_0^\infty$ be such that

$$\chi(x) = 1 \text{ for } |x| \le 1, \;\; \chi(x) = 0 \text{ for } |\chi| \ge 2, \;\text{ and } 0 \le \chi \le 1. \tag{2.17}$$

Set $\chi_a(\xi) = \chi(\xi/a)$, $\chi_{R_0}(x) = \chi(x/R_0)$. Denote $q := V - \tilde{V}$. We get

$$0 = \mathcal{F}q(\xi) + \int \left[(u\tilde{u})\Big(x, -\frac{\xi}{|\xi|}, \frac{|\xi|}{2}\Big) - e^{-i\xi\cdot x}\right]\chi_{R_0}(x)q(x)\,dx.$$

Apply the operator $\chi_{R_0}\mathcal{F}^{-1}(1 - \chi_a)$, with some $a > 0$, to both sides. The low-frequency cut-off $1 - \chi_a$ can be explained by the fact that we have high-frequency control of u but no low-frequency one. Note that

$$\chi_{R_0}\mathcal{F}^{-1}(1 - \chi_a)q = q - \chi_{R_0}\mathcal{F}^{-1}\chi_a\mathcal{F}q.$$

We therefore get $\mathcal{A}_{V,\tilde{V}}q = 0$, where

$$\begin{aligned}\mathcal{A}_{V,\tilde{V}}q = q &- \chi_{R_0}\mathcal{F}^{-1}\chi_a\mathcal{F}q \\ &+ \chi_{R_0}\mathcal{F}^{-1}(1 - \chi_a)\int \left[(u\tilde{u})\Big(x, -\frac{\xi}{|\xi|}, \frac{|\xi|}{2}\Big) - e^{-i\xi\cdot x}\right]\chi_{R_0}(x)q(x)\,dx.\end{aligned}$$

We therefore reduced the non-linear problem to a pseudo-linear one. If we can show that $\mathcal{A}_{V,\tilde{V}}$ is injective, say on L^2, we proved uniqueness. Note that the latter problem is not equivalent to the original one. We cannot exclude the possibility that $\mathcal{A}_{V,\tilde{V}}$ is not injective but we still have uniqueness — i.e., that $V - \tilde{V}$ is not in the kernel.

LEMMA 2.12. *The operator*

$$\mathcal{A}_{0,0} := \mathrm{I} - \chi_{R_0}\mathcal{F}^{-1}\chi_a\mathcal{F}$$

is invertible on $L^2(\mathbf{R}^n)$.

PROOF. Clearly, the operator $\chi_{R_0}\mathcal{F}^{-1}\chi_a\mathcal{F}$ is compact. Then it is enough to show that $\mathcal{A}_{0,0}$ has a trivial null-space. If $\mathcal{A}_{0,0}f = 0$, then

$$\|f\| = \|\chi_{R_0}\mathcal{F}^{-1}\chi_a\mathcal{F}f\| \le \|\mathcal{F}^{-1}\chi_a\mathcal{F}f\| \le \|\chi_a\mathcal{F}f\| \le \|\hat{f}\| = \|f\|,$$

where $\mathcal{F}$ is normalized so that it is unitary. Then all the inequalities above are actually equalities. Then $\chi_a\hat{f} = \hat{f}$, and $\chi_{R_0}f = f$. Then both f and $\hat{f}$ have compact support; therefore, $f = 0$. □

Next, we show that the operator

$$Kf := \chi_{R_0}\mathcal{F}^{-1}(1-\chi_a)\int \Big[(u\tilde{u})\Big(x,-\frac{\xi}{|\xi|},\frac{|\xi|}{2}\Big) - e^{-\mathrm{i}\xi\cdot x}\Big]\chi_{R_0}(x)q(x)\,\mathrm{d}x$$

is compact. This would show that the operator $\mathcal{A}_{V,\tilde{V}}$ is Fredholm. While this would not prove injectivity, it would show at least that the kernel is finitely dimensional.

The operator K is actually a ΨDO of order -1, when V and $\tilde{V}$ are in C_0^∞, and the cutoff χ_{R_0} makes it compact. The proof of that would require to study an infinite expansion of u and would require infinite smoothness. For our purposes, a finite expansion would be enough.

We have

$$u\Big(x,-\frac{\xi}{|\xi|},\frac{|\xi|}{2}\Big) = e^{-\mathrm{i}\xi\cdot x} + u_{\mathrm{sc}}\Big(x,-\frac{\xi}{|\xi|},\frac{|\xi|}{2}\Big).$$

The estimate we have on $u_{\mathrm{sc}} = O(|\xi|^{-1})$ however does not seem to be not enough to show compactness. Indeed, $K = \chi_{R_0}\mathcal{F}^{-1}\tilde{K}$, where

$$f \quad \longmapsto \tilde{K}f(\xi) := (1-\chi_a)\int \Big[(u\tilde{u})\Big(x,-\frac{\xi}{|\xi|},\frac{|\xi|}{2}\Big) - e^{-\mathrm{i}\xi\cdot x}\Big]\chi_{R_0}(x)q(x)\,\mathrm{d}x.$$

We would get that the kernel $\tilde{K}(\xi,x)$ of $\tilde{K}$ satisfies the estimate

$$|\tilde{K}(x,\xi)| \le \frac{(1-\chi_a(\xi))\chi_{R_0}(x)}{|\xi|^2}.$$

That estimate alone is not enough to claim compactness, or even boundedness (which does not mean it is not true). One of the ways to prove that we have a compact operator is to show that the kernel is Hilbert-Schmidt. The $O(|\xi|^{-2})$ decay is not sufficient for that; but $O(|\xi|^{-k})$, with $k > n/2$ would be. Actually, the reason for our troubles is that we neglect the oscillatory behavior of u_{sc}, and we are trying to estimate it with brute force.

The strategy now is the following. We will compute explicitly the second term in the expansion of u, i.e., the leading one in the expansion of u_{sc}, as $\lambda \to \infty$. We will estimate its contribution using its explicit form, which would have an oscillatory behavior. For the remainder, we gained a power of $|\xi|^{-1}$, which is enough to claim that it contributes a Hilbert-Schmidt operator when $n = 2$ and $n = 3$. In higher dimensions, we need to study higher order terms.

2.5.1. *High-frequency expansion of* $u(x,\theta,\lambda)$. We stick to the 3D case here, but generalizations to higher dimensions are immediate.

Define the so-called *beam transform* Bf of f:

$$Bf(x,\theta) = \int_{-\infty}^{0} f(x+s\theta)\,\mathrm{d}s.$$

The next proposition is a typical geometric optics construction.

Proposition 2.13. *Let* $V \in W^{4,\infty}(\mathbf{R}^3)$ *and* $V = 0$ *for* $|x| > R_0$*. Then*

$$u(x,\theta,\lambda) = e^{\mathrm{i}\lambda\theta\cdot x} - \frac{\mathrm{i}}{2\lambda}e^{\mathrm{i}\lambda\theta\cdot x}BV(x,\theta) + R(x,\theta,\lambda),$$

where

$$\|R(x,\theta,\lambda)\|_{L^2(B(0,2R_0))} \le \frac{C}{\lambda^2}, \quad \|R(x,\theta,\lambda)\|_{L^\infty(B(0,2R_0))} \le \frac{C}{\lambda}.$$

Moreover, if V belongs to the ball $\|V\|_{W^{4,\infty}} \le M$ with some fixed $M > 0$, and $\lambda > C_0 M$ with some absolute constant $C_0 > 0$, then the constant C above can be chosen uniformly, depending on R_0 and M only.

Proof. By the Lippmann-Schwinger equation (2.16) in Chapter II, $u = u_0 - R_0(\lambda)Vu$, where $u_0 := e^{\mathrm{i}\lambda\theta\cdot x}$. Iterate this to get

$$u = u_0 - R_0(\lambda)Vu_0 + R_0(\lambda)VR_0(\lambda)Vu.$$

Therefore,

$$R = -R_0(\lambda)Vu_0 + \frac{\mathrm{i}}{2\lambda}BVu_0 + R_0(\lambda)VR_0(\lambda)Vu.$$

The last term is easy to estimate because $\chi R_0(\lambda)\chi = O(\lambda^{-1})$ for any compactly supported cutoff χ. Moreover,

$$\|R_0(\lambda)\|_{L^2(K)\to L^\infty(K)} = O(1)$$

for any compact set K, which proves the second estimate for that term.

It remains to estimate

$$-R_0(\lambda)Vu_0 + \frac{\mathrm{i}}{2\lambda}BVu_0.$$

Let $\chi \in C_0^\infty$ be a cut-off function so that $\chi = 1$ on $B(0,3R_0)$. Note that BV solves the transport equation $\theta\cdot\partial_x BV = V$. Since $\chi V = V$, we get

$$(-\Delta-\lambda^2)\left(-R_0(\lambda)Vu_0 + \frac{\mathrm{i}}{2\lambda}\chi BVu_0\right) = -\frac{\mathrm{i}}{2\lambda}\Delta(\chi BV)u_0 + (\theta\cdot\chi)(BV)u_0.$$

Since $-R_0(\lambda)Vu_0 + \frac{\mathrm{i}}{2\lambda}\chi BVu_0$ is outgoing,

$$-R_0(\lambda)Vu_0 + \frac{\mathrm{i}}{2\lambda}\chi BVu_0 = R_0(\lambda)\left(-\frac{\mathrm{i}}{2\lambda}\Delta(\chi BV)u_0 + (\theta\cdot\partial_x\chi)(BV)u_0\right).$$

The first term on the right can now be estimated right away as before. The second term is

$$R_0(\lambda)[(\theta\cdot\partial_x\chi)(BV)u_0] = \frac{1}{4\pi}\int \frac{e^{\mathrm{i}\lambda(|x-y|+y\cdot\theta)}}{|x-y|}(\theta\cdot\partial_x\chi)(y)(BV)(y,\theta)\,\mathrm{d}y.$$

The phase is $\phi = |x-y| + y\cdot\theta$. To apply the stationary phase, compute

$$\partial_y\phi = \frac{y-x}{|y-x|} + \theta.$$

On the support of the integrand, that differential does not vanish, and in fact, $\theta\cdot(y-x) > 0$. Then we can integrate by parts once to estimate the integral by $O(\lambda^{-2})$ in the $L^\infty(B(0,2R_0)$ norm. □

Warning: poor notation. R_0 stands both for a radius of a ball where $\operatorname{supp} V$ is included, and for the free resolvent. Another poor notation: R stands for the remainder term above, and for the resolvent.

Proposition 2.14. *$\mathcal{A}_{V,\tilde V} : L^2(\mathbf{R}^3) \to L^2(\mathbf{R}^3)$ is bounded; moreover, $\mathcal{A}_{V,\tilde V} - \mathrm{I}$ is compact.*

SKETCH OF THE PROOF. It is enough to show that K is compact. The representation above allows us to estimate each term in $(u\tilde{u})\Big(x, -\frac{\xi}{|\xi|}, \frac{|\xi|}{2}\Big) - e^{-\mathrm{i}\xi\cdot x}$ and show that they contribute compact operators. The $R\tilde{R}$ term decays like $|x\|^{-4}$, which is enough to claim that it gives us a Hilbert-Schmidt operator. Terms involving BV are not Hilbert-Schmidt. A typical operator that we need to study is the following

$$f \quad \longmapsto \quad \mathcal{F}^{-1}\frac{1-\chi_a(\xi)}{|\xi|}\int e^{-\mathrm{i}x\cdot\xi}(BV)\Big(x, -\frac{\xi}{|\xi|}\Big)\chi_{R_0}(x)f(x)\,\mathrm{d}x.$$

To prove compactness of the latter, it is enough to prove that the operator

$$f \quad \longmapsto \quad \mathcal{F}^{-1}(1-\chi_a)\int e^{-\mathrm{i}x\cdot\xi}(BV)\Big(x, -\frac{\xi}{|\xi|}\Big)\chi_{R_0}(x)f(x)\,\mathrm{d}x$$

is bounded. The operator adjoint to it is a formal ΨDO with symbol $C\chi_{R_0}(x)(B\bar{V})(x, -\xi/|\xi|)(1-\chi_a(\xi))$. If V is smooth, it is an actual ΨDO of order -1, and the cut-off χ_{R_0} makes it compact. Theorem 18.1.11' in [**5**] however guarantees boundedness even under the finite regularity conditions we have. □

The progress we made so far shows that the uniqueness question can be resolved if we show that

$$\mathcal{A}_{V,\tilde{V}}q = 0 \quad \Longrightarrow \quad q = 0,$$

and this is a Fredholm equation. In particular, we get immediately that the kernel must be finite dimensional. The next step is to apply the analytic Fredholm theorem. For that, we need an analytic dependence of a parameter.

PROPOSITION 2.15. *The map*

$$\mathcal{W}\times\mathcal{W} \ni (V, \tilde{V}) \mapsto \mathcal{A}_{V,\tilde{V}} \in \mathcal{L}(L^2, L^2)$$

is analytic.

SKETCH OF THE PROOF. The notion of analyticity in this setting can be found in Reed and Simon. We need to prove that (a) $\|\mathcal{A}_{V+zh,\tilde{V}+z\tilde{h}}\|$ is uniformly bounded for $z \in \mathbf{C}$ small enough and all h, $\tilde{h}$ with $\|h\|_{\mathcal{W}} \leq 1$, $\|\tilde{h}\|_{\mathcal{W}} \leq 1$ (easy to see); and (b) that the map $z \mapsto \mathcal{A}_{V+zh,\tilde{V}+z\tilde{h}}$ is analytic for any such h, $\tilde{h}$ and z near $z = 0$.

Inspecting the proof above, we see that we need to prove analyticity of u_{V+zh}, where the subscript indicates the dependence on the potential. Let χ be the characteristic function of $B(0, R_0)$, and denote for a moment χu by u again, and $\chi\mathbf{R}_0(\lambda)\chi$ by $R_0(\lambda)$ again. By the Lippmann-Schwinger equation,

$$u = u_0 - R_0(\lambda)Vu.$$

For u_{V+zh} we have

$$u_{V+zh} = \left[\mathrm{I} + z(\mathrm{I} + R_0(\lambda)V)^{-1}R_0(\lambda)h\right]^{-1}(\mathrm{I} + R_0(\lambda)V)^{-1}u_0.$$

We will use our freedom to control a now. Choose a so that for $\lambda > a/2$, we have $\|R_0(\lambda)\| \leq (2M)^{-1}$, $\|(\mathrm{I} + R_0(\lambda)V)^{-1}\| \leq 2$, where M is an a priori bound of $\|V\|_{\mathcal{W}}$ and $\tilde{|}V\|_{\mathcal{W}}$. Then we can use a Neumann series above, and prove analyticity. □

Sketch of the proof of Theorem 2.11. Let

$$\mathcal{O} = \{V \in \mathcal{W};\ \mathcal{A}_{V,V} \text{ is an isomorphism}\}.$$

First, $\mathcal{O}$ is open because $\mathcal{A}_{V,V}$ depends continuously on V. Next, $\mathcal{O}$ is dense by the analytic Fredholm theorem. Indeed, the operator

$$\mathcal{A}_{zV,zV}$$

depends analytically on z for $|z| < 2$ a fixed V with $\|V\|_{\mathcal{W}} \le M$ and $a = a(M)$. When $z = 0$, it is invertible by Lemma 2.12. By the analytic Fredholm theorem, it is invertible with a possible exception of a discrete set of z's. If $z = 1$ is not one of those exceptional point, we are done. If it is, for any $\varepsilon > 0$, there is a real z with $|z - 1| < \varepsilon$ not in that set. Then $\mathcal{A}_{zV,zV}$ with that z will be invertible, i.e., $zV \in \mathcal{O}$. Note that we showed in particular that $0 \in \mathcal{O}$. □

2.5.2. *Stability analysis.* This is a stable problem in the sense, that there is Hölder type of conditional stability. This, for example, follows from the fact that the linearization is stable (Fredholm), and by an abstract approach to stability, see [**22**].

2.6. Inverse scattering for time-dependent potentials.

3. Inverse Obstacle Scattering

3.1. Uniqueness with full all data. Recall that an obstacle $\mathcal{O} \subset \mathbf{R}^n$ is a compact set with smooth (C^2 is enough) boundary and connected complement.

The following theorem by Schiffer can be found in [**8**].

THEOREM 3.1. *Let $\mathcal{O}$, $\tilde{\mathcal{O}}$ be two obstacles with $a(\omega, \theta, \lambda) = \tilde{a}(\omega, \theta, \lambda)$ for all $\lambda > 0$, ω and θ in S^{n-1}. Then $\mathcal{O} = \tilde{\mathcal{O}}$.*

Of course, it would be enough to require that the scattering amplitudes are equal for an open set of $(\omega, \theta, \lambda)$; then we can use analytic continuation.

PROOF. Let $u(x, \theta, \lambda)$, $\tilde{u}(x, \theta, \lambda)$ be the scattering solutions defined before. The Rellich uniqueness theorem, applied for any fixed λ and θ implies that

$$u(x, \theta, \lambda) = \tilde{u}(x, \theta, \lambda), \quad |x| \gg 1.$$

This can even be done in a constructive (but unstable) way, as shown in section 5.2. Let Ω_+ be the connected unbounded component of $\mathbf{R}^n \setminus (\mathcal{O} \cup \tilde{\mathcal{O}})$. Set $\Omega_- = \mathbf{R}^n \setminus \bar{\Omega}_+$. Then $\bar{\Omega}_- \supset \mathcal{O} \cup \tilde{\mathcal{O}}$. Note that Ω_- is an open set containing the interior of $\mathcal{O} \cup \tilde{\mathcal{O}}$ but also all components of $\mathbf{R}^n \setminus (\mathcal{O} \cup \tilde{\mathcal{O}})$ disconnected from infinity. Since u is an analytic function of x outside Ω_+, we get

$$u(x, \theta, \lambda) = \tilde{u}(x, \theta, \lambda), \quad x \in \partial\Omega_+$$

Assume $\mathcal{O} \neq \tilde{\mathcal{O}}$. Then either $\Omega_- \setminus \mathcal{O}$ is a nonempty open set, or $\Omega_- \setminus \tilde{\mathcal{O}}$ is. Assume the former. Let G be any connected component of $\Omega_- \setminus \mathcal{O}$. Then $u = 0$ on $\partial\Omega$. Therefore, u solves the problem

$$(-\Delta - \lambda^2)u = 0 \quad \text{in } G, \quad u|_{\partial G} = 0.$$

Note that ∂G may not be smooth but this is still a well posed problem. We know that u cannot be identically zero in G because otherwise, it should be zero for large $|x|$. Therefore, λ^2 is an eigenvalue of the Dirichlet Laplacian in D, and that is true for any λ. That could only happen for a discrete number of λ's, however. □

3.2. Uniqueness with partial data. There are several refinements of the uniqueness result. It is enough to know the scattering amplitude for all ω, a fixed $\lambda_0 > 0$, and N incident directions θ; or for all ω, a fixed θ_0, and N frequencies $\lambda \leq \lambda_0$, where N is greater than the number of the Dirichlet eigenvalues $\lambda^2 \leq \lambda_0^2$ of the Laplacian in a ball containing the obstacle. [refs] This requires an a priori knowledge of the size of the obstacle. In particular, if we fix either θ or λ, we get uniqueness for small enough obstacles. In 3D, the upper bound is given by $\lambda_R < \pi$, where R is the radius of such a ball. Those results are based on the following observation: We want to exclude the possibility that λ^2 is a Dirichlet eigenvalue in some domain $D \subset (\mathcal{O} \setminus \tilde{\mathcal{O}}) \cup (\tilde{\mathcal{O}} \setminus \mathcal{O})$, as above. If we have an a priori bound on the diameter of the smallest ball including both obstacles, we can use the monotonicity of the Dirichlet eigenvalues w.r.t. the domain (the smaller the domain, the larger the eigenvalues, as follows from the mini-max principle).

One open inverse problem in obstacle scattering is the following: Can we determine $\mathcal{O}$ from $a(\omega, \theta_0, \lambda_0)$ known for all outgoing directions ω but for a fixed incident one θ and a fixed frequency $\lambda_0 > 0$? As we mentioned above, if the obstacle is a priori small enough, we can. In [**21**], we proved that there is local uniqueness result.

THEOREM 3.2. *Fix $\lambda_0 > 0$, $\theta_0 \in S^{n-1}$. Let $\mathcal{O}_- \subset \mathcal{O}_+$ be two obstacles and assume that $\mathrm{Vol}(\mathcal{O}_+ \setminus \mathcal{O}_-) < \omega_n \lambda_0^{-n}$. Let $\mathcal{O}_- \subset \mathcal{O}_j \subset \mathcal{O}_+$, $j = 1, 2$, be two other obstacles and assume that $A_{\mathcal{O}_1}(\omega, \theta_0, \lambda_0) = A_{\mathcal{O}_2}(\omega, \theta_0, \lambda_0)$. Then $\mathcal{O}_1 = \mathcal{O}_2$.*

Here, ω_n is the volume of the unit ball in $\mathbf{R}^n$. The proof is based on the observation that we only need the volume of the domain to be small to claim that the eigenvalues are large, and therefore exceed λ_0; the smallness of the diameter is not needed.

3.3. Inverse scattering for strictly convex obstacles and high-frequency asymptotics. The theorems above are not constructive, and the problem is unstable. Assuming strict convexity (this condition can be relaxed a bit), we have an explicit and stable reconstruction. This was first done by Majda.

The starting point is Theorem 3.3 in Chapter II for the scattering amplitude for the Dirichlet problem:

$$a(\omega, \theta, \lambda) = \frac{-1}{4\pi \mathrm{i}} \left(\frac{\lambda}{2\pi}\right)^{\frac{1}{2}(n-3)} e^{\frac{1}{4}\pi i(n-1)} \int_{\partial\Omega} \left(\mathrm{i}\lambda\omega \cdot \nu(y) e^{-\mathrm{i}\lambda y\cdot(\omega-\theta)} + e^{-\mathrm{i}\lambda y\cdot\omega} \partial_\nu u_{\mathrm{sc}}(y, \theta, \lambda)\right) \mathrm{d}S_y.$$

If we knew $\partial_\nu u_{\mathrm{sc}}$, we would have an expression for a which we could analyze and try to recover $\partial\Omega$. In the applied literature, one uses the Kirchhoff approximation:

$$\partial_\nu u_{\mathrm{sc}}|_{\partial\mathcal{O}} \approx -\mathrm{i}\lambda |\nu \cdot \theta| e^{\mathrm{i}\lambda y\cdot\theta}, \quad \text{as } \lambda \to \infty.$$

Assume $\nu \cdot \theta < 0$. This approximation is based on the fact that u_{sc} satisfies the boundary condition $u_{\mathrm{sc}} = -e^{\mathrm{i}\lambda y\cdot\theta}$ on $\partial\Omega$. If we formally extend this away from the boundary and differentiate, we would almost get the approximation , but with the wrong sign. The reason for the sign to be exactly the opposite of what this naive (and wrong) arguments suggests is that the wave gets reflected, and that reflection inverts the sign of the normal component of θ. It does more than this, of course, because the boundary is curved and the reflected wave is not that simple anymore. We will justify the Kirchhoff approximation below.

The function u_{sc} is the outgoing solution of

$$(-\Delta - \lambda^2) u_{\mathrm{sc}} = 0 \quad \text{in } \Omega = \mathbf{R}^n \setminus \mathcal{O}, \quad u_{\mathrm{sc}}|_{\partial\Omega} = -e^{-\mathrm{i}\lambda x\cdot\theta}.$$

Assume $\theta_0 \cdot \nu(x_0) < 0$ (a ray with incident direction θ_0 hits $\partial\Omega$ transversely at x_0), and we are going to work with (x, θ) near (x_0, θ_0). The geometric optics construction now works in the following way. We are looking for a parametrix of the form

$$u_{\mathrm{sc}} = e^{\mathrm{i}\lambda\phi(x,\theta)}(a_0 + \lambda^{-1} a_1 + \dots),$$

with a phase function $\phi(x, \theta)$ and amplitudes $a_j = a_j(x, \theta)$ which we will compute. Plug into the equation and equate the leading powers of λ to get the eikonal equation

$$|\partial\phi|^2 = 1, \quad \phi|_{\partial\Omega} = x \cdot \theta,$$

and the first transport one:

$$(2(\partial\phi)\cdot\partial+\Delta\phi)a_0=0,\quad a_0|_{\partial\Omega}=-1.$$

The boundary condition for ϕ allows us to compute the tangential gradient $\partial'\phi=\theta'$ on $\partial\Omega$, where $\theta'=\theta'(x)$ is the tangential projection of θ on $\partial\Omega$ at x. Since the total gradient is a unit vector, we have two choices for $\partial\phi|_{\partial\Omega}$: one of them is just θ, which points inside; and the other one is θ reflected off $\partial\Omega$ by the laws of geometric optics: same tangential projection and the opposite normal one. We chose the second one to be able to satisfy the outgoing condition. In particular, we get

$$\partial_\nu\phi|_{\partial\Omega}=-\partial_\nu x\cdot\theta|_{\partial\Omega}=-\nu\cdot\theta.$$

The parametrix can be justified as usual: solve the PDE asymptotically and estimate the error by standard estimates. Then the discussion above implies

$$\partial_\nu u_{\rm sc}|_{\partial\Omega}=\mathrm{i}\lambda(\partial_\nu\phi)e^{\mathrm{i}\lambda\phi(x,\theta)}a_0|_{\partial\Omega}+O(\lambda^{-1})=-\mathrm{i}\lambda\nu\cdot\theta e^{\mathrm{i}\lambda x\cdot\theta}+O(1),$$

as claimed.

The Kirchhoff approximation allows us to express the reduced scattering amplitude a_0 (the integral terms above) as

$$a_0=\int_{\partial\Omega}\left(\mathrm{i}\lambda\omega\cdot\nu(y)e^{-\mathrm{i}\lambda y\cdot(\omega-\theta)}-e^{-\mathrm{i}\lambda y\cdot\omega}\mathrm{i}\lambda\nu(y)\cdot\theta e^{\mathrm{i}\lambda y\cdot\theta}\right)\,\mathrm{d}S_y+O(1),$$

therefore,

$$a_0=\mathrm{i}\lambda\int_{\partial\Omega}e^{-\mathrm{i}\lambda y\cdot(\omega-\theta)}(\omega-\theta)\cdot\nu(y)\,\mathrm{d}S_y+O(\lambda^{-1}),$$

[needs to be completed]

4. Inverse Scattering by metrics

5. Inverse Boundary Value Problems and Inverse Black Box Scattering

5.1. From the far filed to the near field and back; uniqueness. Let P be an operator satisfying the black-box scattering assumptions in section 5, Chapter II. We want to link the inverse problem of "finding the black box" to a certain related inverse boundary value problem.

Let $\Omega \subset \overline{B(0,R)}$ be a bounded open set with smooth boundary, diffeomorphic to a ball. It would be enough to choose $\Omega = B(0,R_1)$, $R_1 > R_0$. By (5.3) in Chapter II, $\mathbf{1}_\Omega(P+\mathrm{i})^{-1}$ is compact on $\mathcal{H}$. Using standard elliptic theory, we can show that the boundary value problem

$$(P-\lambda^2)u = 0 \quad \text{in } \Omega, \quad u|_{\partial\Omega} = f \tag{5.1}$$

is of Fredholm type and it is uniquely solvable, if λ^2 is not a Dirichlet eigenvalue of P in Ω. Recall that here "in Ω" is actually means that we work in the Hilbert space $\mathcal{H}_{R_0} \oplus L^2(\Omega \setminus B(0,R_0))$ which we intuitively think of as being $L^2(\Omega)$. The non-eigenvalue condition can be achieved by perturbing the boundary of Ω a bit by the strict monotonicity of the eigenvalues w.r.t. the domain [needs to be expanded]. Then we have a well defined DN map $\Lambda = \Lambda(\lambda)$ on $\partial\Omega$.

The inverse boundary value problem in this setting can be formulated as follows. Given $(\mathcal{H}_{R_0}, P)$ and $(\tilde{\mathcal{H}}_{R_0}, \tilde{P})$, does $\Lambda = \tilde{\Lambda}$ (for a fixed or for a range of frequencies λ) imply that $(\mathcal{H}_{R_0}, P)$ and $(\tilde{\mathcal{H}}_{R_0}, \tilde{P})$ are unitarily equivalent? We will leave that definition a bit vague on purpose. In potential scattering, we want to show that $V = \tilde{V}$. In obstacle scattering, we want to show that $g = \psi^*\tilde{g}$, where ψ is a diffeomorphism in Ω fixing $\partial\Omega$. If P is a general second order elliptic operator, we want to show invariance under gauge transformations, etc.

The goal of this section is to show that the inverse scattering problem of recovery of $(\mathcal{H}_{R_0}, P)$ from the scattering amplitude $a_0(\omega,\theta,\lambda)$ and from the DN map are equivalent, without attempting to solve either one. Moreover, we show that those two problems are related in an explicit way.

THEOREM 5.1. *Fix λ^2 not a Dirichlet eigenvalue of P in Ω. Then the DN map $\Lambda(\lambda)$ determines the scattering amplitude $a_0(\lambda,\cdot,\cdot)$ uniquely and vice versa.*

PROOF. We will give a short uniqueness but not constructive proof. Let P and $\tilde{P}$ be two black-box operators, and let Λ and $\tilde{\Lambda}$; and a_0 and $\tilde{a}_0$ be the corresponding DN maps and scattering amplitudes.

Assume $\Lambda = \tilde{\Lambda}$ first. Let u, $\tilde{u}$ be the corresponding scattering solutions. Let v be the solution of the BVP

$$(\tilde{P}-\lambda^2)v = 0 \quad \text{in } \Omega, \quad v|_{\partial\Omega} = u.$$

If we replace $\tilde{P}$ by P, we would get u, of course. Then u and v have the same Dirichlet data on $\partial\Omega$; but they also have the same (interior) Neumann one since $\Lambda = \tilde{\Lambda}$. Set

$$w := v \quad \text{in } \Omega; \qquad w := u \quad \text{in } \mathbf{R}^n \setminus \Omega.$$

Then $(\tilde{P}-\lambda^2)w = 0$ away from $\partial\Omega$ but since the Cauchy data of w of both sides of $\partial\Omega$ is the same, we also get $(\tilde{P}-\lambda^2)w = 0$ globally. Then we get two solutions, $\tilde{u}$ and w, of the scattering problem for $\tilde{P}$ of the kind $e^{i\lambda\theta\cdot x}$ + outgoing; and we know that there is unique one. Therefore, $w = \tilde{u}$, hence $u = \tilde{u}$ for large $|x|$. The asymptotic of that gives us $a_0 = \tilde{a}_0$,

Assume now $a_0 = \tilde{a}_0$. Then $u - \tilde{u} = O(|x|^{-(n+1)/2})$ as $|x| \to \infty$, see (2.17) in Chapter II. Then by Lemma 1.9 there, $\tilde{u} - u = 0$ for $|x| > R_0$ (the situation is even simpler here because $\tilde{u} - u = 0$ is outgoing; in some papers, this is the actual formulation of the Rellich theorem). Then $\Lambda = \tilde{\Lambda}$ on the span of $u(\cdot, \theta, \lambda)$ in, say, $H^{3/2}(\partial\Omega)$. The proof would be complete if we show that that span is dense. □

LEMMA 5.2. *Fix $\lambda \neq 0$ so that λ^2 is not a Dirichlet eigenvalue of P in Ω. Then the set $\{u(\cdot, \theta, \lambda),\ \theta \in S^{n-1}\}$ is dense in $H^s(\partial\Omega)$ for any s.*

PROOF. Take $\phi \in C^\infty(\partial\Omega)$. It is enough to prove the following

$$\int_{\partial\Omega} u(x, \theta, \lambda)\phi(x)\, \mathrm{d}S_x = 0, \quad \forall\theta \in S^{n-1} \quad \Longrightarrow \quad \phi = 0.$$

This would prove density in $L^2(\Omega)$ because $\phi \in C^\infty(\partial\Omega)$ is dense there. It would also prove density in $H^s(\Omega)$ if we replace ϕ by $(1-\Delta_{\partial\Omega})^s\phi$, where $\Delta_{\partial\Omega}$ is the Laplace-Beltrami operator on $\partial\Omega$. Then we would get $(1 - \Delta_{\partial\Omega})^s\phi = 0$, hence $\phi = 0$.

Without loss of generality we may assume that we have $\bar{u}(x, -\theta, -\lambda)$ above. Then we can think about the integral above as the far field pattern of $v := R(\lambda)\phi\delta_{\partial\Omega}$, see Theorem (5.4), i.e., the far field pattern of

$$v := \int_{\partial\Omega} G(x, y, \lambda)\phi(y)\, \mathrm{d}S_y.$$

This is a simple layer potential with kernel G. Since $G - G_0$ is smooth (actually, analytic) near $\partial\Omega$, then the jump relations (3.4) still hold. In particular, v is continuous across $\partial\Omega$. By the Rellich uniqueness theorem, Theorem 1.11, v has compact support. Since it is harmonic outside Ω, it actually vanishes there. By continuity, $v|_{\partial\Omega} = 0$. Inside Ω, v solves $(P - \lambda^2)v = 0$. Since λ^2 is not a Dirichlet eigenvalue of P in Ω, $v = 0$ in Ω, as well. Then, in $\mathbf{R}^n$,

$$0 = (P - \lambda^2)v = \phi\delta_{\partial\Omega},$$

therefore, $\phi = 0$. □

5.2. Constructive Rellich theorem. The Rellich uniqueness theorem, Theorem 1.11 says that for every outgoing u, the far field pattern of u determines uniquely u away from the black box. This can be done in a constructive way, using the decomposition in spherical harmonics and Hankel functions, see Chapter V.

Recall that by Corollary 1.10, for any outgoing u, we have

$$u(r\omega) = \frac{e^{i\lambda r}}{r^{(n-1)/2}} g(\omega) + O(r^{-(n+1)/2}), \tag{5.2}$$

where $g(\theta)$ is the far field pattern of u. On the other hand, by Theorem 1.1 in Chapter V,

$$u(r\omega) = \sum_{l,m} a_{lm}(\lambda) h_l^{(1)}(\lambda r) Y_l^m(\omega), \tag{5.3}$$

with some coefficients $a_{lm}(\lambda)$. Let us expand g in spherical harmonics:

$$g(\omega) = \sum_{l,m} g_{lm} Y_l^m(\omega), \quad g_{lm} = (g, Y_l^m)_{L^2(S^{n-1})}.$$

Then by (5.2),

$$u(r\omega) = \frac{e^{\mathrm{i}\lambda r}}{r^{(n-1)/2}} \sum_{l,m} g_{lm} Y_l^m(\omega) + O(r^{-(n+1)/2}), \tag{5.4}$$

On the other hand, for each (l,m), and $\lambda > 0$ fixed,

$$\begin{aligned}(u(r\cdot), Y_l^m)_{L^2(S^{n-1})} &= a_{lm}(\lambda) h_l^{(1)}(\lambda r) \\ &= a_{lm}(\lambda) e^{-\mathrm{i}(2l+n-1)\pi/4} \frac{e^{\mathrm{i}\lambda r}}{(\lambda r)^{(n-1)/2}} + O(r^{-(n+1)/2}), \quad r \to \infty,\end{aligned}$$

by (1.9) in Chapter V. Compare this with (5.4) to get the following

$$g_{lm} = e^{-\mathrm{i}(2l+n-1)\pi/4} \lambda^{-(n-1)/2} a_{lm}. \tag{5.5}$$

PROPOSITION 5.3. *Fix $\lambda > 0$ and let u be outgoing. Then u can be reconstructed from its far field pattern h by means of* (5.3) *, with coefficients a_{lm} obtained from the Fourier coefficients g_{lm} of g as in* (5.5).

Note that this reconstruction is highly unstable. For (5.3) to converge at any fixed r, the coefficients a_{lm} have to decay super-exponentially to be able to compensate for the faster than exponential growth of $h_l(\lambda r)$ as $l \to \infty$, see (1.1) in Chapter V for the 3D case. If h is a far field pattern of some outgoing solution in $|x| > R_0$, then this must be true for $r > R_0$. This shows first that the Fourier coefficients of possible far field patterns decay super-exponentially. This should not be so unexpected because possible far field patterns must be analytic functions. Next, small changes in those coefficients (say, in l^2), when $l \gg 1$, would not only lead to big changes in u, but they may destroy the convergence of (5.3) in the first place. On the other hand, the stability of the map $h \mapsto u|_{|x|>R_0}$ holds if we consider g_{lm} as members of some Sobolev space of sequences with an exponentially decaying weight, dictated by (1.1) in Chapter V. For more details, we refer to [**18**].

The step $h \mapsto u|_{|x|>R_0}$ can be stabilized under some a priori assumptions, for example that u extends to $|x| \le R_{-1}$ with $R_{-1} < R_0$ as a solution of the Helmholtz equation and is a priori bounded there. We refer also to [**6**].

5.3. From the scattering amplitude to the Green's function away from the black box. The analysis above allows us to reconstruct explicitly Green's function $G(x,y,\lambda)$ (the Schwartz kernel of $R(\lambda)$ for $|x| > R_0$, $|y| > R_0$ from the scattering amplitude $a(\omega,\theta,\lambda)$. Recall that we already have an explicit way to get a from $G(x,y,\lambda)$ known near the black box, see (5.6) in Chapter II. The Green's function can be also explicitly connected to the DN map Λ on any domain $\Omega \supset B(0,R_0)$ as above [**10**]. This would make to step $a_0 \mapsto \Lambda$ constructive.

Consider $G(x,y,\lambda) - G_0(x,y,\lambda)$. It is an analytic function in $B(0,R_0) \times B(0,R_0)$. We can view it as $(R(\lambda) - R_0(\lambda))\delta_y$. For $R(\lambda)\delta_y$, we have the expansion of Theorem 5.4 in

Chapter II. For $R_0(\lambda)\delta_y$, we have the expansion in Theorem 1.5 in that chapter ($\hat f = 1$ in our case now). Subtract those two asymptotics to get

$$
\begin{aligned}
G(x,y,\lambda) - G_0(x,y,\lambda) &= c_n(\lambda)\frac{e^{\mathrm{i}\lambda|y|}}{|y|^{(n-1)/2}}\bar u_{\mathrm{sc}}\Big(x, -\frac{y}{|y|}, -\lambda\Big)(1+O(1/|y|)),\\
c_n(\lambda) &:= \frac{1}{4\pi}\left(\frac{\mathrm{i}\lambda}{2\pi}\right)^{\frac12(n-3)}.
\end{aligned}
\tag{5.6}
$$

We can now use the asymptotic of u_{sc} as $|x|\to\infty$, see (5.6) and (2.17) in Chapter II, to get the following.

THEOREM 5.4. *For $|x| > R_0$, $|y| > R_0$,*

$$
\begin{aligned}
G(x,y,\lambda) - G_0(x,y,\lambda) =& c_n(\lambda)\frac{e^{\mathrm{i}\lambda|x|}}{|x|^{(n-1)/2}}\frac{e^{\mathrm{i}\lambda|y|}}{|y|^{(n-1)/2}}\\
&\times \bar a\Big(\frac{x}{|x|}, -\frac{y}{|y|}, -\lambda\Big)(1+O(1/|y|))(1+O(1/|x|)).
\end{aligned}
\tag{5.7}
$$

This theorem shows that G, known outside the black-box, determines a uniquely by taking the asymptotics $|x|\to\infty$, $|y|\to\infty$. The converse can be done in an explicit way as well.

Let $a_{lml'm'}$ be the Fourier coefficients of $\bar a(\theta,-\theta',-\lambda)$. For operators P with the property $\overline{Pf} = P\bar f$, those coefficients are the same as those of $a(\theta,-\theta',\lambda)$. In other words.

$$
a_{lml'm'} = \iint_{S^{n-1}\times S^{n-1}} \bar a(\theta,-\theta',-\lambda)\bar Y_l^m(\theta)\bar Y_{l'}^{m'}(\theta')\,\mathrm{d}\theta\mathrm{d}\theta'.
$$

By (5.6) and Proposition 5.3,

$$
G(x,y,\lambda) - G_0(x,y,\lambda) = \sum_{lm}\gamma_{lm}h_l^{(1)}(\lambda|y|)Y_l^m(y/|y|)
$$

with

$$
e^{-\mathrm{i}(2l+n-1)\pi/4}\lambda^{-(n-1)/2}\gamma_{lm}(x,\lambda) = c_n(\lambda)\int_{S^{n-1}} u_{\mathrm{sc}}(x,-\theta',-\lambda)\bar Y_{l'}^{m'}(\theta')\,\mathrm{d}\theta'.
$$

Apply Proposition 5.3 again in the x variable to get

$$
\gamma_{lm}(x,\lambda) = \sum_{l'm'}\gamma_{lml'm'}h_{l'}^{(1)}(\lambda|x|)Y_{l'}^{m'}(x/|x|),
$$

where

$$
\gamma_{lml'm'} = c_n'(\lambda)\iint_{S^{n-1}\times S^{n-1}} \bar a(\theta,-\theta',-\lambda)\bar Y_l^m(\theta)\bar Y_{l'}^{m'}(\theta')\,\mathrm{d}\theta\mathrm{d}\theta' = c_n'(\lambda)a_{lml'm'}.
$$

with $e^{-\mathrm{i}(2l+n-1)\pi/4}\lambda^{-(n-1)/2}c_n'(\lambda) = c_n^2(\lambda)$. All this implies the following.

THEOREM 5.5. *For $|x| > R_0$, $|y| > R_0$,*

$$
\begin{aligned}
&G(x,y,\lambda) - G_0(x,y,\lambda)\\
&\qquad = c_n'(\lambda)\sum_{lml'm'} a_{lml'm'}h_{l'}^{(1)}(\lambda|x|)Y_{l'}^{m'}(x/|x|)h_l^{(1)}(\lambda|y|)Y_l^m(y/|y|),
\end{aligned}
\tag{5.8}
$$

where $a_{lml'm'}$ *are the Fourier coefficients of the scattering amplitude* $\bar{a}(\theta, -\theta', -\lambda)$.

This gives an explicit formula for recovering $G(x, y, \lambda)$ for x, y outside the black box, knowing the scattering amplitude, for any fixed $\lambda > 0$.

[Next: Nachman's formula connecting G and Λ]

CHAPTER V

Appendix

1. Spherical harmonics, Hankel functions and radial separation of variables for the Helmholtz equation

1.1. Spherical harmonics and Hankel functions. We will recall some facts about separation of variables in polar coordinates for the Laplace operator, see e.g., Folland's book on PDEs. If $n = 2$, then the Laplacian (with negative sign in front) on S^1 is just $-\mathrm{d}^2/\mathrm{d}\theta^2$, where θ is the polar angle, and the eigenvalues are l^2, $l = 0, 1, \dots$ with eigenfunctions 1 for $l = 0$ and $e^{\lambda l\theta}$, $e^{-\lambda l\theta}$ for $\lambda > 0$. Clearly, all positive eigenvalues are of multiplicity two, and the eigenfunction expansion is just the classical Fourier series.

Let $n \geq 3$ now. Denote by Y_l^m, $l = 0, 1, \dots$, $m = 1, \dots m(l)$, an orthonormal set of spherical harmonics on S^{n-1}. They are the eigenfunctions of the Laplacian $\Delta_{S^{n-1}}$ on S^{n-1}. We have

$$-\Delta_{S^{n-1}} Y_l^m = l(l+n-2)Y_l^m, \quad l = 0, 1, \dots; \ m = 1, \dots, m(l).$$

For each l, the multiplicity of the eigenvalue

$$\mu(l) = l(l+n-2) \tag{1.1}$$

is given by

$$m(l) = \frac{2l+n-2}{n-2}\binom{l+n-3}{n-3} = \frac{2l^{n-2}}{(n-2)!}\left(1 + O(l^{-1})\right). \tag{1.2}$$

If $n = 3$, then the eigenvalues are $l(l+1)$ of multiplicities $m(l) = 2l + 1$.

Let u solve the Helmholtz equation

$$(-\Delta - \lambda^2)u = 0 \tag{1.3}$$

in some radially symmetric domain. We want to find representation of u in polar coordinates $r > 0$, $\omega \in S^{n-1}$. We need to keep in mind that $r = 0$ is a singular point for the polar change of variables, and all traps that this can possibly cause.

We are looking first for solutions of the form $u = R(r)\Omega(\omega)$. Plug in the Helmholtz equation and use the standard separation of variables techniques to get that possible solutions are of the form $u = R(r)Y_l^m(\omega)$, where Y_l^m is any of the spherical harmonics, and $R(r)$ (depending on l but not on m) solves the ODE

$$R'' + \frac{n-1}{r}R' + \left(\lambda^2 - \frac{\mu(l)}{r^2}\right)R = 0 \tag{1.4}$$

in view of the formula for Δ in polar coordinates

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{S^{n-1}}. \tag{1.5}$$

Equation (1.4) (after we multiply by r^2) resembles the Bessel equation

$$z^2\frac{\mathrm{d}^2y}{\mathrm{d}z^2}+z\frac{\mathrm{d}y}{\mathrm{d}z}+(z^2-\alpha^2)y=0. \tag{1.6}$$

To put it in that form, notice first that there is certain homogeneity there w.r.t. r: if we replace r by λr, λ becomes 1 and the rest does not change. So is natural to look for a solution which is a function of λr. Next, we really want the second term to be $(1/r)\mathrm{d}/\mathrm{d}r$ after the change, not $(n-1)$ times that. This suggests the change of variables $R(r)=(\lambda r)^{\alpha}v(\lambda r)$, and a simple computation yields $\alpha=1-n/2$. So we look for a solution $R(r)=(\lambda r)^{1-n/2}v(\lambda r)$ to get

$$z^2v''+zv'+\left(z^2-\big(\mu(l)+n^2/4-n+1\big)\right)v=0$$

with $v=v(z)$, $z=\lambda r$. By (1.1), we can write the latter as

$$z^2v''+zv'+\left(z^2-(l+n/2-1)^2\right)v=0.$$

Therefore, the general solution is any linear combination of two linearly independent Bessel functions like $(\lambda r)^{1-n/2}H^{(1,2)}_{l+n/2-1}(\lambda r)$, $(\lambda r)^{1-n/2}J_{l+n/2-1}(\lambda r)$, etc. It is convenient to set (note that the definition depends on n)

$$h_l^{(k)}(z)=\sqrt{\frac{\pi}{2}}z^{1-n/2}H^{(k)}_{l+n/2-1}(z),\quad k=1,2,\quad j_l(z)=\sqrt{\frac{\pi}{2}}z^{1-n/2}J_{l+n/2-1}(z). \tag{1.7}$$

It is well known that

$$j_l=\frac12 h_l^{(1)}+\frac12 h_l^{(2)}. \tag{1.8}$$

Of those three Bessel functions, only j_l is not singular at 0; and this is in fact a defining property of j_l. On the other hand, when ν is a half-integer, $H^{(1)}_\nu(z)$ and $H^{(2)}_\nu(z)$ are rational functions multiplied by $e^{\pm \mathrm{i}z}$ with pole at $z=0$ (only). The Hankel functions have the asymptotic

$$H^{(1)}_\nu(z)=\left(\frac{2}{\pi z}\right)^{\frac12}e^{\mathrm{i}(z-\nu\pi/2-\pi/4)}+O(z^{-3/2}),\quad \text{as } z\to\infty.$$

Then

$$h_l^{(1)}(z)=e^{-\mathrm{i}(2l+n-1)\pi/4}\frac{e^{\mathrm{i}z}}{z^{(n-1)/2}}\left(1+O\left(\frac1z\right)\right),\quad \text{as } z\to\infty, \tag{1.9}$$

and $h_l^{(2)}(z)$ is just the conjugate of $h_l^{(1)}(z)$ (for z real). In particular, when $n=3$ (remember that $h_l^{(1)}$ depends on n as well),

$$h_l^{(1)}(z)=(-\mathrm{i})^{l+1}\frac{e^{\mathrm{i}z}}{z}+O\left(\frac{1}{z^2}\right),\quad \text{as } z\to\infty. \tag{1.10}$$

1.2. Radial separation of variables for the Helmholtz equation.

THEOREM 1.1. *Let $n \geq 3$.*

(a) Let u solve the Helmholtz equation in the ball $|x| < R_0$ with some R_0. Then

$$u(r\omega) = \sum_{l,m} c_{lm}(\lambda) j_l(\lambda r) Y_l^m(\omega), \tag{1.11}$$

$0 < r < R_0$, $\omega \in S^{n-1}$, *with some coefficients* $c_{lm}(\lambda)$.

(b) Let u solve the Helmholtz equation in the annulus $R_0 < |x| < R_1$ with some $0 \leq R_0 < R_1$. Then

$$u(r\omega) = \sum_{l,m} \left(a_{lm}(\lambda) h_l^{(1)}(\lambda r) + b_{lm}(\lambda) C_2 h_l^{(2)}(\lambda r) \right) Y_l^m(\omega), \tag{1.12}$$

$R_0 < r < R_1$, $\omega \in S^{n-1}$, *with some coefficients* $a_{lm}(\lambda)$, $b_{lm}(\lambda)$.

(c) If $R_1 = \infty$, and u is outgoing, then $b_{lm}(\lambda) = 0$; if u is incoming, then $a_{lm}(\lambda) = 0$.

PROOF. Let u solve the Helmholtz equation for $|x| < R_0$ with some $R_0 > 0$. Fix l and m and set

$$f(r) = \int_{S^{n-1}} u(r\omega) \bar{Y}_l^m(\omega)\, \mathrm{d}\omega.$$

Then f solves the ODE (1.4) for $0 < r < R_0$ and can therefore be represented as a linear combination of $j_l(\lambda r)$ and another linearly independent modified Bessel function of λr, say $h^{(1)}(\lambda r)$. Only the first term remains bounded when $r \to 0$. Therefore, $f(r) = c_{lm}(\lambda) j_l(\lambda r)$. Thus any solution u of the Helmholtz equation $(-\Delta - \lambda^2)u = 0$ near 0 has the form (1.11). The convergence follows from the fact that $\omega \mapsto u(r\omega)$ is clearly in $L^2(S^{n-1})$, for $0 < r < R_0$; and the above series is just an eigenfunction expansion for any such r.

On the other hand, let u solve the Helmholtz equation for $|x| > R_0$ with some $R_0 > 0$. Then f defined as above solves the ODE (1.4) again for $r > R_0$ and can therefore be represented as the following linear combination

$$f(r) = a_{lm}(\lambda) h_l^{(1)}(\lambda r) + b_{lm}(\lambda) C_2 h_l^{(2)}(\lambda r).$$

Neither term can be discarded without additional a priori knowledge. Then we get (1.12) with the same remark about the convergence.

Assume now that u above is outgoing, see Definition 1.6. Then for any m, l, the function f defined above satisfies $f(r) = e^{\mathrm{i}r}/r^{(n-1)/2} + O(r^{-(n+1)/2})$, as $r \to \infty$, by Theorem 1.5. By (1.9), only $h_l^{(1)}(\lambda r)$ has this behavior and no other non-trivial combination of the two Hankel functions has this asymptotic. Therefore, $b_{lm} = 0$. Similarly, if u is incoming, $a_{lm} = 0$. □

REMARK 1.1. Just because the series (1.12) converges for $R_0 < r < R_1$ does not mean that it will converge for all or even some $r > R_1$. Very strong, super-exponential decay of the coefficients a_{lm} and b_{lm} with respect to l is needed to guarantee convergence for all $r > R_1$, see, e.g., [**18**].

REMARK 1.2. The previous remark is related to the following. The behavior of the Hankel functions when $l \to \infty$, which determines the convergence of the series, and the behavior as

$z \to \infty$ "cannot be commuted". Another way to say the that is that the asymptotic (1.9) is not uniform w.r.t. l as $l \to \infty$. In fact, when $n = 3$, we have the following

$$h_l^{(1)}(z) \sim -(2/e)^{1/2} z^{-1} \left(\frac{2l+1}{ez} \right)^l, \quad \text{as } l \to \infty. \tag{1.13}$$

This is super-exponentially increasing with l and it can not be obtained from the leading term in (1.10) by taking the limit $l \to \infty$. What we have is a family of functions $h_l^{(1)}(z)$ (and $H_\alpha^{(1)}(z)$) with two parameters, z and l, respectively α. The latter solves the Bessel equation (1.6). If you consider α there as a large parameter, then that equation is elliptic in semi-classical sense when $|\alpha| > z > 0$; and it is of hyperbolic type when $|\alpha| < z$. The asymptotic (1.10) is then in the hyperbolic regime, and the leading term is oscillating; while the asymptotic (1.13) is in the elliptic one; and the leading term is of exponential type. Detailed two-parameter asymptotic expansions of the Bessel functions (for z complex!) can be found in works by Olver, see [**11**] and the references there, written in the 50s(!!).

Bibliography

[1] J. Cooper and W. Strauss. Representations of the scattering operator for moving obstacles. *Indiana Univ. Math. J.*, 28(4):643–671, 1979.

[2] J. Cooper and W. Strauss. Scattering of waves by periodically moving bodies. *J. Funct. Anal.*, 47(2):180–229, 1982.

[3] L. Hörmander. *The analysis of linear partial differential operators. I*, volume 256. Springer-Verlag, Berlin, 1983. Distribution theory and Fourier analysis.

[4] L. Hörmander. *The analysis of linear partial differential operators. II*, volume 257. Springer-Verlag, Berlin, 1983. Differential operators with constant coefficients.

[5] L. Hörmander. *The analysis of linear partial differential operators. III*, volume 274. Springer-Verlag, Berlin, 1985. Pseudodifferential operators.

[6] V. Isakov. *Inverse problems for partial differential equations*, volume 127 of *Applied Mathematical Sciences*. Springer, New York, second edition, 2006.

[7] T. Kato. Linear evolution equations of "hyperbolic" type. *J. Fac. Sci. Univ. Tokyo Sect. I*, 17:241–258, 1970.

[8] P. D. Lax and R. S. Phillips. *Scattering theory*, volume 26 of *Pure and Applied Mathematics*. Academic Press Inc., Boston, MA, second edition, 1989. With appendices by Cathleen S. Morawetz and Georg Schmidt.

[9] R. B. Melrose. *Geometric scattering theory*. Stanford Lectures. Cambridge University Press, Cambridge, 1995.

[10] A. I. Nachman. Reconstructions from boundary measurements. *Ann. of Math. (2)*, 128(3):531–576, 1988.

[11] F. W. J. Olver. *Asymptotics and special functions*. Academic Press, 1974. Computer Science and Applied Mathematics.

[12] V. Petkov. *Scattering theory for hyperbolic operators*, volume 21 of *Studies in Mathematics and its Applications*. North-Holland Publishing Co., Amsterdam, 1989.

[13] V. Petkov and M. Zworski. Semi-classical estimates on the scattering determinant. *Ann. Henri Poincaré*, 2(4):675–711, 2001. MR1852923.

[14] M. Reed and B. Simon. *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness*. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1975.

[15] A. Ruiz. Harmonic Analysis and Inverse Problems. http://www.uam.es/gruposinv/inversos/publicaciones/Inverseproblem

[16] J. Sjöstrand. Lectures on resonances. http://sjostrand.perso.math.cnrs.fr/Coursgbg.pdf, 2002.

[17] J. Sjöstrand and M. Zworski. Complex scaling and the distribution of scattering poles. *J. Amer. Math. Soc.*, 4(4):729–769, 1991.

[18] P. Stefanov. Stability of the inverse problem in potential scattering at fixed energy. *Ann. Inst. Fourier (Grenoble)*, 40(4):867–884 (1991), 1990.

[19] P. Stefanov. Generic uniqueness for two inverse problems in potential scattering. *Comm. Partial Differential Equations*, 17(1-2):55–68, 1992.

[20] P. Stefanov. Estimates on the residue of the scattering amplitude. *Asymptot. Anal.*, 32(3-4):317–333, 2002.

[21] P. Stefanov and G. Uhlmann. Local uniqueness for the fixed energy fixed angle inverse problem in obstacle scattering. *Proc. Amer. Math. Soc.*, 132(5):1351–1354 (electronic), 2004.

[22] P. Stefanov and G. Uhlmann. Linearizing non-linear inverse problems and an application to inverse backscattering. *J. Funct. Anal.*, 256(9):2842–2866, 2009.

[23] J. Sylvester and G. Uhlmann. A global uniqueness theorem for an inverse boundary value problem. *Ann. of Math. (2)*, 125(1):153–169, 1987.

[24] M. E. Taylor. *Partial differential equations. I*, volume 115 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1996. Basic theory.

[25] M. E. Taylor. *Partial differential equations. II*, volume 116 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1996. Qualitative studies of linear equations.

[26] M. Zworski. Lectures on scattering resonances. http://math.berkeley.edu/ zworski/res.pdf.