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Stability of resonances under smooth perturbations of the boundary *

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Abstract

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We prove that the resonances related to a second order elliptic differential operator with Dirichlet boundary conditions are stable in each compact of the complex plane under small C^2-perturbations of the boundary and small changes of the coefficients.

1. Introduction

Let

$$H = -\sum_{i,j=1}^{3} a_{ij}(x)\partial_{x_i}\partial_{x_j} + \sum_{i=1}^{3} b_i(x)\partial_{x_i} + c(x)$$

be a symmetric second order elliptic differential operator in a domain $\Omega \subset \mathbf{R}^3$ with compact complement $\mathbf{R}^3 \setminus \Omega$. Assume that the coefficients a_{ij}, b_i, c are Lipshitz functions, a_{ij} is real and positively definite matrix and suppose that H is a compactly supported perturbation of the Laplacian, i.e., $a_{ij}(x) = \delta_{ij}$, $b_i(x) = 0$, $c(x) = 0$ for $|x| > R$, with some $R > 0$. The operator H can be defined as a self-adjoint one in $L^2(\Omega)$ by imposing the Dirichlet boundary condition on the boundary. It is known [8, 9] that if χ is a C_0^∞ cut-off function equal to 1 in a neighborhood of $B_R \cap \Omega$, then the cut-off resolvent

$$R_\chi(z) := \chi(H - z^2)^{-1}\chi$$

admits a meromorphic continuation from $\operatorname{Im} z < 0$, $z^2 \notin \sigma_{\text{point}}(H)$ to the complex plane $\mathbf{C}$. The poles of this continuation are called resonances. The situation we study can be considered as a

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generalization of the classical cases $H = -\Delta_D$ (the Dirichlet Laplacian in Ω) or $H = -\Delta + q(x)$ in $\mathbf{R}^3$ with compactly supported q when the resonances are exactly the poles of the scattering matrix (see [5]).

The aim of this work is to prove that in any fixed compact in $\mathbf{C}$ the resonances are stable under small C^2-perturbations of the boundary and small perturbations of the coefficients of H. Let $(p,t) \in \partial\Omega \times [-\delta,\delta]$ be the normal coordinates in a neighborhood of $\partial\Omega$, i.e., for x sufficiently close to the boundary $|t| = \mathrm{dist}(x,\partial\Omega)$, $p \in \partial\Omega$ is the nearest point on the boundary. The interior of Ω is given locally by $t > 0$. Let $\widetilde{\Omega}$ be another domain given by the equation $t = g(p)$, $p \in \partial\Omega$. We assume that

$$\|g\|_{C^2(\partial\Omega)} \leqslant \varepsilon, \tag{1}$$

where $0 < \varepsilon < \delta/2$. Further, let

$$\widetilde{H} = \sum_{i,j=1}^{3} \widetilde{a}_{ij}(x)\partial_{x_i}\partial_{x_j} + \sum_{i=1}^{3} \widetilde{b}_i(x)\partial_{x_i} + \widetilde{c}(x)$$

be an operator of the same kind as H in $\widetilde{\Omega}$, such that

$$\|\widetilde{a}_{ij} - a_{ij}\|_{L^\infty(\Omega\cap\widetilde{\Omega})} \leqslant \varepsilon, \qquad \|\widetilde{b}_i - b_i\|_{L^\infty(\Omega\cap\widetilde{\Omega})} \leqslant \varepsilon, \qquad \|\widetilde{c} - c\|_{L^\infty(\Omega\cap\widetilde{\Omega})} \leqslant \varepsilon. \tag{2}$$

Let $K \subset \mathbf{C}$ be a fixed compact set and denote by $\{\lambda_j\}_{j=1}^n$ the resonances of $R_X(z)$ in K (we suppose that there are no resonances on ∂K). Denote by d_j and m_j respectively the order and multiplicity of λ_j. We have the following.

Theorem 1. *There exist* $M = M(K) > 0$, $\varepsilon_0 = \varepsilon_0(K) > 0$, *such that if* $0 < \varepsilon < \varepsilon_0$, *then in any disk*

$$\{z \in \mathbf{C};\ |z - \lambda_j| < M\varepsilon^{1/(2d_j-1)}\}$$

there are exactly m_j *resonances of* $\widetilde{H}$ *counted according to their multiplicities and all the resonances of* $\widetilde{H}$ *in* K *lie in the union of these disks. The constants* M *and* ε_0 *depend only on* H *and on the Lipshitz constant related to* $\widetilde{H}$.

We note that a result of the similar kind has been obtained by Petras [6] for one-parameter perturbations of the Laplace operator in $\mathbf{R}^3$ generated by a compactly supported metric. Results about one-parameter perturbations of the boundary can be found in [7]. On the other hand we would like to mention the more difficult and very interesting case of singularly perturbed domains – the so called resonators where the resonances approach the interior eigenvalues of the cavity and the resonances of the exterior (see [1–3] and the references herein).

2. Proof of Theorem 1

First we will construct a C^2-diffeomorphism h in $\mathbf{R}^3$ such that $h(\partial\widetilde{\Omega}) = \partial\Omega$ and

$$\|h - I\|_{C^2} \leqslant C\varepsilon, \tag{3}$$

with a constant C depending only on Ω. For this reason for $\operatorname{dist}(x, \partial\Omega) < \delta$ we set

$$h(x) = x - \varphi(t)g(p)n,$$

where (p, t) are the normal coordinates defined above, n is the inner normal to the boundary and φ is a smooth cut-off function such that $\varphi(t) = 1$ near $t = 0$, $\varphi(t) = 0$ for $|t| > 3\delta/4$. For $\operatorname{dist}(x, \partial\Omega) > \delta$ we set $h(x) = x$. It is not hard to see that if ε is sufficiently small, h is invertible. Assumption (1) implies that (3) is fulfilled. Moreover, h preserves the Dirichlet boundary condition. Under the change of coordinates $x = h(y)$ the operator $\widetilde{H}$ becomes

$$-\sum_{i,j,k,l} \widetilde{a}_{kl} \frac{\partial h_i}{\partial y_k} \frac{\partial h_j}{\partial y_l} \partial_{x_i}\partial_{x_j} + \sum_{i,k,l} \widetilde{a}_{kl} \frac{\partial^2 h_i}{\partial y_k \partial y_l} \partial_{x_i} + \sum_{i,k} \widetilde{b}_k \frac{\partial h_i}{\partial y_k} \partial_{x_i} + \widetilde{c},$$

where

$$\widetilde{a}_{kl} = \widetilde{a}_{kl}(y), \quad \widetilde{b}_k = \widetilde{b}_k(y), \quad \widetilde{c} = \widetilde{c}(y), \quad y = h^{-1}(x).$$

In order to simplify the notations we will keep the same notation $\widetilde{H}$ for the operator above. Note that $D(\widetilde{H}) = D(H) \subset H^2(\Omega)$. Moreover, $\widetilde{H}$ is self-adjoint in $L^2(\Omega; J(x)\,\mathrm{d}x)$, where $J(x)$ is the Jacobian $J = |\det(\mathrm{d}y/\mathrm{d}x)|$. For the difference $V := \widetilde{H} - H$ we have

$$V = -\sum_{i,j} \left(\sum_{k,l} \widetilde{a}_{kl} \frac{\partial h_i}{\partial y_k} \frac{\partial h_j}{\partial y_l} - a_{ij} \right) \partial_{x_i}\partial_{x_j} + \sum_{i,k,l} \widetilde{a}_{kl} \frac{\partial^2 h_i}{\partial y_k \partial y_l} \partial_{x_i}$$
$$+ \sum_{i} \left(\sum_{k} \widetilde{b}_k \frac{\partial h_i}{\partial y_k} - b_i \right) \partial_{x_i} + \widetilde{c} - c,$$

where $y = h^{-1}(x)$. It is a straightforward consequence of (2) and (3) that the coefficients of V can be estimated by $C\varepsilon$, where C depends only on H and on the Lipshitz constant related to $\widetilde{H}$.

Now, let $\lambda_j \in K$ be a pole of $R_\chi(z)$ and consider the circle $\Gamma_j = \{z \in \mathbf{C};\ |z - \lambda_j| = r_j\}$, where the radius $r_j > 0$ will be specified latter. We assume that

$$r_j < c_1, \tag{4}$$

where c_1 is chosen so that (4) guarantees that the disks $D_j = \{z \in \mathbf{C};\ |z - \lambda_j| \leqslant r_j\}$ lie in K and they do not intersect each other. Let us estimate $R_\chi(z) - \widetilde{R}_\chi(z)$ for $z \in \Gamma_j$, where $R_\chi(z)$ and $\widetilde{R}_\chi(z)$ are related to H and $\widetilde{H}$, respectively. For z belonging to some neighborhood of z_0 we have

$$\widetilde{H} - z^2 = \left[I + V\left(H - z^2\right)^{-1}\right]\left(H - z^2\right).$$

Since $\|V(H - z_0^2)^{-1}\| \leqslant C\varepsilon$, we find that for ε sufficiently small the resolvent $(\widetilde{H} - z^2)^{-1}$ exists and for such z

$$\left(\widetilde{H} - z^2\right)^{-1} = \left(H - z^2\right)^{-1}\left[I + V\left(H - z^2\right)^{-1}\right]^{-1}.$$

From the above equality we get

$$R_\chi(z) - \widetilde{R}_\chi(z) = \widetilde{R}_\chi(z) V\left(H - z^2\right)^{-1}\chi. \tag{5}$$

Here we used the fact that $\chi V = V$. Hence,

$$\widetilde{R}_\chi(z)\left[I + V\left(H - z^2\right)^{-1}\chi\right] = R_\chi(z). \tag{6}$$

It can be seen from [8; 10, Section 3] that for any differential operator P of order not greater than 2 with compactly supported coefficients $P(H - z^2)^{-1}\chi$ is a meromorphic function of z with the same poles with orders not greater than those of $R_\chi(z)$. Therefore, if χ_1 is a cut-off function equal to 1 on the support of χ, then $\chi_1 \partial^\alpha (H - z^2)^{-1}\chi$ has the same poles as does $R_\chi(z)$ for $|\alpha| \leqslant 2$. Thus the same is true for $V(H - z^2)^{-1}\chi$. The equation (6) is therefore valid for all z. Assume in what follows that $z \in \Gamma_j$. Since the operator V can be written in the form

$$V = \sum_{ij} V_{ij}(x)\chi_1(x)\partial_{x_i}\partial_{x_j} + \sum_i V_i(x)\chi_1(x)\partial_{x_i} + V(x)\chi_1(x)$$

with $|V_{ij}(x)| < C\varepsilon$, $|V_i(x)| < C\varepsilon$, $|V(x)| < C\varepsilon$, we have

$$\left\|V\left(H - z^2\right)^{-1}\chi\right\| \leqslant C_1 \varepsilon r_j^{-d_j}.$$

The constant C_1 depends only on H and c_1. Therefore, if

$$\varepsilon r_j^{-d_j} < \frac{1}{2C_1}, \tag{7}$$

$I + V(H - z^2)^{-1}\chi$ is invertible and from (6) we get

$$\left\|\widetilde{R}_\chi(z)\right\| \leqslant 2\left\|R_\chi(z)\right\|.$$

Let us estimate $R_\chi(z) - \widetilde{R}_\chi(z)$ (recall that $z \in \Gamma_j$). According to (5) we have

$$\left\|R_\chi(z) - \widetilde{R}_\chi(z)\right\| \leqslant \left\|\widetilde{R}_\chi(z)\right\| \left\|V\left(H - z^2\right)^{-1}\chi\right\| \leqslant C_2 \varepsilon r_j^{-2d_j}.$$

Since λ_j is a pole of $R_\chi(z)$, the operator

$$P_j = \frac{1}{2\pi \mathrm{i}} \oint_{z \in \Gamma_j} R_\chi(z) z \,\mathrm{d}z$$

is not trivial. In fact, P_j is a finite rank operator and its rank by definition is the multiplicity of λ_j (see [8]). If $\widetilde{P}_j$ is related to $\widetilde{H}$ by the same way, we get

$$\left\|P_j - \widetilde{P}_j\right\| \leqslant C_3 \varepsilon r_j^{1-2d_j}. \tag{8}$$

Since P_j is a finite rank operator (its rank is m_j), there exists a m_j-dimensional subspace $\mathcal{H}_j$ such that $\mathcal{H}_j$ generates $\operatorname{Im} P_j$, i.e., $P_j(\mathcal{H}_j) = \operatorname{Im} P_j$, P_j is an isomorphism between these spaces. Let us choose r_j, so that (4), (7) hold and

$$C_3 \varepsilon r_j^{1-2d_j} < \frac{1}{2} \inf\left\{\left\|P_j f\right\|;\ f \in \mathcal{H}_j,\ \|f\| = 1\right\}. \tag{9}$$

It is easy to see that if we put $r_j = M\varepsilon^{1/(2d_j-1)}$, then the second and the third condition will be fulfilled for M sufficiently large and $0 < \varepsilon < \varepsilon_0$ with $\varepsilon_0 < 1$. Fix such a number M and choose $\varepsilon_0 < 1$ such that for $0 < \varepsilon < \varepsilon_0$ we have (4). Assume that $\operatorname{Rank}\widetilde{P}_j < \operatorname{Rank}P_j$. Then there exists $f \in \mathcal{H}_j$ with $\|f\| = 1$, such that $P_j f$ is orthogonal to $\operatorname{Im}\widetilde{P}_j$, therefore,

$$\|(\widetilde{P}_j - P_j)f\|^2 = \|P_j f\|^2 + \|\widetilde{P}_j f\|^2 \geqslant \|P_j f\|^2,$$

which contradicts (8), (9). Thus $\operatorname{Rank}\widetilde{P}_j \geqslant \operatorname{Rank}P_j$ and in D_j there are at least m_j resonances of $\widetilde{H}$ (counted according to their multiplicities). Let us note that we cannot just exchange the roles of H and $\widetilde{H}$ in the proof above in order to prove that in fact we have equality because these arguments hold in a neighborhood of $\widetilde{H}$ with a size depending on $\widetilde{H}$ and H may not belong to that neighborhood. So in order to complete the proof it remains to show that there are exactly m_j resonances in D_j and there are no other resonances in $K \setminus (D_1 \cup D_2 \ldots \cup D_n)$. In view of what we have already shown, this will be proven if we prove that the total multiplicity of all the resonances of $\widetilde{H}$ in K (the sum of the multiplicities of all the resonances in K) is the same as that of H. To this end we will make use of the characterization of the resonances as eigenvalues of a certain non-selfadjoint operator obtained by the so-called complex scaling method. Without loss of generality we can assume that K does not contain any resonances which are negative eigenvalues of H. Indeed, for the eigenvalues of H we have classical perturbation theorems (see [4] and the arguments below), that show that for ε sufficiently small the total multiplicity of the negative eigenvalues in K for both operators is the same. Moreover, since the resonances are symmetric with respect to the imaginary axis, we can assume that K is included in $\{0 \leqslant \arg z \leqslant \pi/2+\delta\}$, for some $\delta > 0$ sufficiently small. By applying the complex scaling method (we refer to [8] and the references herein for more details), we see that there exists a family of operators H_θ, $\theta \in [0,\pi)$, acting on a Hilbert space $\mathcal{H}_\theta$, such that $H_0 = H$ and if $\arg\lambda < \theta$ and λ is a resonance, then λ^2 is an eigenvalue of H_θ with the same multiplicity. So if we set $\theta = \pi/2 + 2\delta$, all the resonances in K correspond to eigenvalues of the non-selfadjoint operator H_θ in the set

$$K_1 = \{z;\ z = \lambda^2,\ \lambda \in K\}$$

and the same is true for $\widetilde{H}_\theta$. Moreover, the spectrum of H_θ, $\widetilde{H}_\theta$ in K_1 consists only of squares of resonances. Now we are in position to apply a classical theorem about the perturbation of the spectrum of a closed operator in a compact set (see [4, Theorem IV.3.18]). Let us consider the pair $\widetilde{H}_\theta$, H_θ in the space $\mathcal{H}_\theta$, related to H_θ. The operator $\widetilde{H}_\theta$ will be no more self-adjoint on this space, but its spectrum remains the same because $\mathcal{H}_\theta$, $\widetilde{\mathcal{H}}_\theta$ coincide as topological spaces. For the difference $V_\theta = \widetilde{H}_\theta - H_\theta$ we see that it can be naturally identified with V. The fact that the coefficients of V can be estimated by $C\varepsilon$ makes possible to arrange the estimate

$$\|V_\theta f\| \leqslant C\varepsilon\left(\|H_\theta f\| + \|f\|\right).$$

Therefore, by [4, Theorem IV.3.18] for ε sufficiently small the total multiplicities of the eigenvalues of H_θ and $\widetilde{H}_\theta$ in K are the same. This proves that the total multiplicities of the resonances of H and $\widetilde{H}$ in K are the same if ε is properly chosen. Since we have already shown that in each D_j the multiplicity of the perturbed resonances is at least the same as that of the unperturbed ones, this completes the proof of the theorem.

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