THE GEODESIC X-RAY TRANSFORM WITH FOLD CAUSTICS

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ABSTRACT. We give a detailed microlocal study of X-ray transforms over geodesics-like families of curves with conjugate points of fold type. We show that the normal operator is the sum of a pseudodifferential operator and a Fourier integral operator. We compute the principal symbol of both operators and the canonical relation associated to the Fourier integral operator. In two dimensions, for the geodesic transform, we show that there is always a cancellation of singularities to some order, and we give an example where that order is infinite; therefore the normal operator is not microlocally invertible in that case. In the case of three dimensions or higher if the canonical relation is a local canonical graph we show microlocal invertibility of the normal operator. Several examples are also studied.

1. INTRODUCTION

The purpose of this paper is to study X-ray type of transforms over geodesics-like families of curves with caustics (conjugate points). We concentrate on the most common type of caustics — those of fold type. Let γ_0 be a fixed geodesic segment on a Riemannian manifold, and let f be a function which support does not contain the endpoints of γ_0. The question that we are trying to answer is the following: what information about the wave front set $\mathrm{WF}(f)$ of f can be obtained from the assumption that (possibly weighted) integrals

$$Xf(\gamma) = \int_\gamma f \, \mathrm{d}s \tag{1.1}$$

of f along all geodesics γ close enough to γ_0 vanish (or depend smoothly on γ)? We actually study more general geodesic-like curves. Since X has a Schwartz kernel with singularities of conormal type, Xf could only provide information for $\mathrm{WF}(f)$ near the conormal bundle $\mathcal{N}^*\gamma_0$ of γ_0. If there are no conjugate points along γ_0, then we know that $\mathrm{WF}(f) \cap \mathcal{N}^*\gamma_0 = \emptyset$. This has been shown, among the other results, in [9, 25] in this context. It also follows from the microlocal approach to Radon transforms initiated by Guillemin [10] when the Bolker condition (in our case that means no conjugate points) is satisfied. Then the localized normal operator $N_\chi := X^*\chi X$, where χ is a standard cut-off near γ_0 is a pseudo-differential operator (ΨDO), elliptic at conormal directions to γ_0. If there are conjugate points along γ_0, then N_χ is no longer a ΨDO. One of the goals of this work is first to study the microlocal structure of N_χ in presence of fold conjugate points, and then use it to see what singularities can be recovered. That would also allow us to tell whether the problem of inverting X is Fredholm or not, and would help us to determine the size of the kernel, and to analyze the stability and the possible instability of this problem.

In some applications like geophysics, recovery of singularities is actually the primary goal. The effect of possible conjugate points is treated there as "artifacts" in the reconstruction, creating multiple images of the same object. Our analysis provides in particular a microlocal way to understand those "artifacts", and in same cases, to shed light on the possibility to resolve the singularities. We are also motivated by the non-linear boundary and lens rigidity problems, and their applications to seismology, where the X-ray transform appears as a linearization, see e.g., [17, 3, 4, 24, 22, 26].

The simplest possible X-ray transform is that over lines in $\mathbf{R}^n$:

$$Xf(x,\theta) = \int f(x+t\theta)\, \mathrm{d}t,$$

where $\theta \in S^{n-1}$. Parameterization by $x \in R^n$ is overdetermined, of course, and we need to think of (x,θ) as a way to parameterize a line. It is well known to be injective, on $L^1_{\mathrm{comp}}(\mathbf{R}^n)$, for example. It is easy to see, for example by

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the Fourier Slice Theorem, that Xf, known for a fixed θ_0 and all x, determines the Fourier transform $\hat{f}(\xi)$ for $\xi \perp \theta_0$. We refer to [11, 20] for more details about Euclidean X-ray and Radon transforms. Using relatively simple microlocal techniques, one can show that Xf, known in a neighborhood of some line ℓ, determines $\mathrm{WF}(f)$ near $\mathcal{N}^*\ell$. A positive smooth weight in the definition of X would not change that. Those facts are well known and serve as a basis for local tomography methods, see e.g., [7, 8] where the microlocal point of view is implicit.

Geodesic X-ray transforms have a long history, generalizing the Radon type X-ray transform in the Euclidean space, see, e.g., [11]. When the weight is constant, and (M, g) is a simple manifold with boundary, uniqueness and non-sharp stability estimates have been proven in [18, 19, 2], using the energy method. Simple manifolds are compact manifolds diffeomorphic to a ball with convex boundary and no conjugate points. The uniqueness result has been extended to not necessarily convex manifolds under the no-conjugate points assumption in [6]. The authors used microlocal methods to prove a sharp stability estimate in [23] for simple manifolds and uniqueness and stability estimates for more general weighted geodesic-like transforms without conjugate points in [9]. The X-ray transform over magnetic geodesics with the simplicity assumption was studied in [5]. Many of those and other works study integrals of tensors as well and the results for tensors of order two or higher are less complete. For an overview of the microlocal approach to the geodesic X-ray transform, we refer to [22].

The authors considered in [25] the X-ray transform of functions and tensors on manifolds with possible conjugate points. Using the overdeterminacy of the problem in dimensions $n \geq 3$, we showed that if there exists a family of geodesics without conjugate points with a conormal bundle covering T^*M, then we still have generic uniqueness and stability. In dimension two however that family has to be the set of all geodesics, and even in higher dimensions, [25] does not answer the question what is the contribution of the conjugate points to Xf.

We first show in Theorem 2.1 that the normal operator N_χ can be represented as a sum of a ΨDO and a Fourier Integral Operator (FIO). The FIO part comes from the conjugate point and represents the "artifact". An essential part of the proof of Theorem 2.1 is to understand well the geometry of the conjugate locus Σ of pairs $(p, q) \in M \times M$ conjugate to each other. We show that the Lagrangian of the FIO is $N^*\Sigma$. To prove Theorem 2.1, we analyze the singularities of the Schwartz kernel of N_χ in Theorem 6.1, that is interesting by itself.

In section 9, we study whether we can invert N_χ microlocally, when the curves are geodesics. It turns out that in some cases we can, and in some, we cannot. In two dimensions, some cancellation of singularities always occurs, at least to a finite order, see Theorem 9.2. In dimensions three and higher, there are examples (not all geodesic though) where we cannot resolve singularities, and where we can. If the canonical relation of the FIO part is a local graph, then we can but that is not always the case.

In section 10, we present a few examples, some of them mentioned above. Most of them are based on the transform of integrating a function over circles of a fixed radius in $\mathbf{R}^2$. Those circles are actually geodesics of a magnetic system with an Euclidean metric and a constant magnetic field. This example has the advantage that we can compute explicitly the kernel of X^*X, we can get an explicit full expansion of the latter as an FIO, etc. In this case, the singularities cancel to infinite order. We can construct more or less explicit singular distributions f with the property their singularities are invisible for X localized near a single circle, i.e., $Xf \in C^\infty$ locally.

2. Formulation of the problem

Let (M, g) be an n-dimensional Riemannian manifold. Let $\exp_p(v)$, where $(p, v) \in TM$, be a regular exponential map, see section 3, where we recall the definition given by Warner in [29]. The main example is the exponential map of g or that of another metric on M or other geodesic-like curves, for example magnetic geodesics, see also [5]. Let κ be a smooth function on $TM \setminus 0$. We define the weighted X-ray transform Xf by

$$Xf(p,\theta) = \int \kappa\big(\exp_p(t\theta), \dot{\exp}_p(t\theta)\big) f(\exp_p(t\theta))\, \mathrm{d}t, \quad (p,\theta) \in SM, \tag{2.1}$$

where we used the notation

$$\dot{\exp}(tv) = \frac{\mathrm{d}}{\mathrm{d}t}\exp(tv).$$

The t integral above is carried over the maximal interval, including $t = 0$, where $\exp(t\theta)$ is defined. The assumptions that we make below guarantee that this interval remains bounded.

Let $(p_0, v_0) \in TM$ be such that $v = v_0$ is a critical point for $\exp_{p_0}(v)$ (that we call a conjugate vector) of fold type, see the definition below. Let $q_0 = \exp_{p_0}(v_0)$. Then our goal is to study Xf for p close to p_0 and θ close to $\theta_0 := v_0/|v_0|$ under the assumption that the support of f is such that v_0 is the only conjugate vector v at p_0 so that $\exp_{p_0}(v) \in \operatorname{supp} f$. Note that v_0 can be written in two different ways as $t\theta_0$, $|\theta_0| = 1$, with $\pm t > 0$, and we chose the first one. The contribution of the second one can be easily derived from our results by replacing θ_0 by $-\theta_0$.

Instead of studying X directly, we study the operator

$$\begin{aligned} Nf(p) &= \int_{S_pM} \kappa^\sharp(p,\theta) Xf(p,\theta)\, \mathrm{d}\sigma_p(\theta) \\ &= \int_{S_pM} \int \kappa^\sharp(p,\theta)\kappa\left(\exp_p(t\theta), \dot{\exp}_p(t\theta)\right) f(\exp_p(t\theta))\, \mathrm{d}t\, \mathrm{d}\sigma_p(\theta), \end{aligned} \tag{2.2}$$

with some smooth $\kappa^\sharp$ localized in a neighborhood of (p_0, θ_0). Here $\mathrm{d}\sigma_p(\theta)$ is the induced Riemannian surface measure on $S_p(M)$. When exp is the geodesic exponential map, there is a natural way to give a structure of a manifold to all non-trapping geodesics with a natural choice of a measure, see section 5. The operator X can be viewed as a map from functions or distributions on M to functions or distributions on the geodesics manifold. Then one can define the adjoint X^* with respect to that measure. Then the operator X^*X is of the form (2.2) with $\kappa^\sharp = \bar{\kappa}$, see (5.1). The condition that $\operatorname{supp}\kappa^\sharp$ should be contained in a small enough neighborhood of (p_0, θ_0) can be easily satisfied by localizing p near p_0, and choosing $\operatorname{supp}\kappa$ to be near $(\gamma_{p_0,\theta_0}, \dot\gamma_{p_0,\theta_0})$. In the case of general regular exponential maps N is not necessarily X^*X.

A direct calculation, see [23] and Theorem 5.1, shows that the Schwartz kernel of X^*X in the geodesic case (see also [9] for general families of curves), is singular at the diagonal, as can be expected, and that singularity defines a ΨDO of order -1 similarly to the integral geometry problem for geodesics without conjugate points. We refer to section 5 for more details. Next, singularities away from the diagonal exist at pairs (p, q) so that $q = \exp_p(v)$ for some v, and $\mathrm{d}_v \exp_p$ is not an isomorphism (p and q are conjugate points). The main goal of this paper is to study the contribution of those conjugate points to the structure of X^*X and the consequences of that. We actually study a localized version of this; for a global version on a larger open set, under the assumption that all conjugate points are of fold type, one can use a partition of unity.

Let $\mathcal{U}$ be a small enough neighborhood of (p_0, θ_0) in SM. Let U be a small neighborhood of p_0 so that $U \subset \pi(\mathcal{U})$, where π is the natural projection on the base. Fix $\kappa^\sharp \in C_0^\infty(\mathcal{U})$. Let Nf be as in (2.2), related to $\kappa^\sharp$, where κ is a smooth weight. We will apply X to functions f supported in an open set $V \ni p_0$ satisfying the conjugacy assumption of the theorem below, see Figure 1. Our goal is to study the contribution of a single fold type of singularity. Let $\Sigma \subset M \times M$ be the conjugate locus in a neighborhood of (p_0, q_0), see section 3. Finally, let $\gamma_0 = \gamma_{p_0,\theta_0}(t)$, $t \in I$, be the geodesic through (p_0, θ_0) defined in the interval $I \ni 0$, with endpoints outside V.

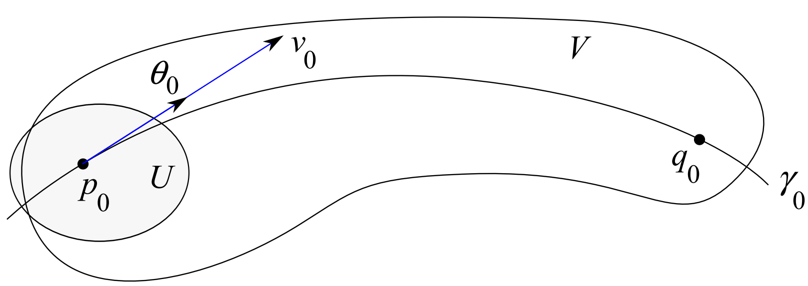

FIGURE 1

The first main result of this paper is the following.

Theorem 2.1. *Let $v_0 = |v_0|\theta_0$ be a fold conjugate vector at p_0, and let N be as in (2.2). Let v_0 be the only singularity of $\exp_{p_0}(v)$ on the ray $\{\exp_p(t\theta_0),\ t \in I\} \cap V$. Then if $\mathcal{U}$ (and therefore, U) is small enough, the operator*

$$N : C_0^\infty(V) \longrightarrow C_0^\infty(U)$$

admits the decomposition

$$N = A + F, \tag{2.3}$$

where A is a ΨDO of order -1 with principal symbol

$$\sigma_p(A)(x,\xi) = 2\pi \int_{S_x M} \delta(\xi(\theta))\, (\kappa^\sharp \kappa)(x,\theta)\, \mathrm{d}\sigma_x(\theta), \tag{2.4}$$

and F is an FIO of order $-n/2$ associated to the Lagrangian $\mathcal{N}^*\Sigma$. In particular, the canonical relation $\mathcal{C}$ of F in local coordinates is given by

$$\mathcal{C} = \left\{(p,\xi,q,\eta),(p,q)\in\Sigma,\ \xi = -\eta_i \partial \exp_p^i(v)/\partial p, \eta \in \operatorname{Coker} \mathrm{d}_v \exp_p(v),\ \det \mathrm{d}_v \exp_p(v) = 0\right\}. \tag{2.5}$$

If exp *is the exponential map of g, then $\mathcal{C}$ can also be characterized as $\mathcal{N}^*\Sigma'$, where $\mathcal{N}\Sigma$ is as in (4.17), and the prime means that we replace η by $-\eta$.*

It is easy to check that $\mathcal{C}$ above is invariantly defined.

In section 9 we show that in dimension 3 or higher in the case that $\mathcal{C}$ is a local canonical graph the operator N is microlocally invertible. In two dimensions, in the geodesic case, we show that there is always a loss of some derivatives at least when the curves are geodesics. We study in detail the case of the circular Radon transform in two dimensions in section 10, and show that then N is not microlocally invertible.

3. Regular exponential maps and their generic singularities

3.1. **Regular exponential maps.** Let M be a fixed n-dimensional manifold. We will recall the definition of Warner [29] of a regular exponential map at $p\in M$. We think of it as a generalization of the exponential map on a Riemannian manifold, by requiring only those properties that are really necessary for what follows. For that reason, we use the notation $\exp_p(v)$. In addition to [29] , we will require $\exp_p(v)$ to be smooth in p as well. Let $N_p(v)\subset T_vT_pM$ denote the kernel of $\mathrm{d}\exp_p$. Unless specifically indicated, d is the differential w.r.t. v. The radial tangent space at v will be denoted by r_v. It can be identified with $\{sv,\ s\in\mathbf{R}\}$, where v is considered as an element of T_vT_pM.

Definition 3.1. *A map $\exp_p(v)$ that for each $p\in M$ maps $v\ni T_pM$ into M is called a* ***regular exponential map****, if*

- **(R1)** exp *is smooth in both variables, except possibly at $v=0$. Next, $\mathrm{d}\exp_p(tv)/\mathrm{d}t\neq 0$, when $v\neq 0$.*
- **(R2)** *The Hessian $\mathrm{d}^2\exp_p(v)$ maps isomorphically $r_v\times N_p(v)$ onto $T_{\exp_p(v)}M/\mathrm{d}\exp_p(T_vT_pM)$ for any $v\neq 0$ in T_pM for which $\exp_p(v)$ is defined.*
- **(R3)** *For each $v\in T_pM\setminus 0$, there is a convex neighborhood U of v such that the number of singularities of $\exp_p$, counted with multiplicities, on the ray tv, $t\in\mathbf{R}$ in U, for each such ray that intersects U, is constant and equal to the order of v as a singularity of $\exp_p$.*

An example is the exponential map on a Riemannian (or more generally on a Finsler manifold), see [29]. Then (R1) is clearly true. Next, (R2) follows from the following well known property. Fix p and a geodesic through it. Consider all Jacobi fields vanishing at p. Then at any q on that geodesic, the values of those Jacobi fields that do not vanish at q and the covariant derivatives of those that vanish at q span T_qM. Also, those two spaces are orthogonal. Finally, (R3) represents the well known continuity property of the conjugate points, counted with their multiplicities that follows from the Morse Index Theorem (see, e.g., [15, Thm 4.3.2]).

We would need also an assumption about the behavior of the exponential map at $v=0$.

- **(R4)** $\exp_p(tv)$ is smooth in p,t,v for all $p\in M$, $|t|\ll 1$, and $v\neq 0$. Moreover,

$$\exp_p(0) = p, \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}\exp_p(tv) = v \quad \text{for } t=0.$$

Given a regular exponential map, we define the "geodesic" $\gamma_{p,v}(t)$, $v\neq 0$, by $\gamma_{p,v}(t)=\exp_p(tv)$. We will often use the notation

$$q = \exp_p(v) = \gamma_{p,v}(1), \quad w = -\dot{\exp}_p(v) := -\dot\gamma_{p,v}(1), \quad \theta = v/|v|. \tag{3.1}$$

Note that the "geodesic flow" does not necessarily obey the group property. We will assume that

- **(R5)** For q, w as in (3.1), we have $\exp_q(w)=p$, $\dot{\exp}_q(w)=-v$.

This shows that in particular, $(p,v) \mapsto (q,w)$ is a diffeomorphism. If exp is the exponential map of a Riemannian metric, then (R5) is automatically true and that map is actually a symplectomorphism (on T^*M).

Remark 3.1. In case of magnetic geodesics, or more general Hamiltonian flows, (R5) is equivalent to time reversibility of the "geodesics." This is not true in general. On the other hand, one can define the reverse exponential map $\exp_q^-(w) = \gamma_{q,-w}(-1)$ in that case, see e.g. [5], near (q_0, w_0), and replace exp by $\exp^-$ in that neighborhood. Then (R5) would hold. In other words, (R5) really says that $(p,v) \mapsto (q,w)$ is assumed to be a local diffeomorphism with an inverse satisfying (R1) – (R4).

3.2. **Generic properties of the conjugate locus.** We recall here the main result by Warner [29] about the regular points of the conjugate locus of a fixed point p. The ***tangent conjugate locus*** $S(p)$ of p is the set of all vectors $v \in T_pM$ so that $\mathrm{d}\exp_p(v)$ (the differential of $\exp_p(v)$ w.r.t. v) is not an isomorphism. We call such vectors ***conjugate vectors*** at p (called conjugate points in [29]). The kernel of $\mathrm{d}\exp_p(v)$ is denoted by $N_p(v)$. It is part of T_vT_pM that we identify with T_pM. In the Riemannian case, by the Gauss lemma, $N_p(v)$ is orthogonal to v. In the general case, by (R1), it is always transversal to v. The images of the conjugate vectors under the exponential map $\exp_p$ will be called **conjugate points** to p. The image of $S(p)$ under the exponential map $\exp_p$ will be denoted by $\Sigma(p)$ and called the **conjugate locus of** p. Note that $S(p) \subset T_pM$, while $\Sigma(p) \subset M$. We always work with p near a fixed p_0 and with v near a fixed v_0. Set $q_0 = \exp_{p_0}(v_0)$. Then we are interested in $S(p)$ restricted to a small neighborhood of v_0, and in $\Sigma(p)$ near q_0. Note that $\Sigma(p)$ may not contain all points near q_0 conjugate to p along some "geodesic"; and may not contain even all of those along $\exp_{p_0}(tv_0)$ if the later self-intersects — it contains only those that are of the form $\exp_p(v)$ with v close enough to v_0.

Normally, $\mathrm{d}\exp_p(v)$ stands for the differential of $\exp_p(v)$ w.r.t. v. When we need to take the differential w.r.t. p, we will use the notation d_p for it, We write d_v for the differential w.r.t. v, when we want to distinguish between the two.

We denote by Σ the set of all conjugate pairs (p,q) localized as above. In other words, $\Sigma = \{(p,q);\ q \in \Sigma(p)\}$, where p runs over a small neighborhood of p_0. Also, we denote by S the set (p,v), where $v \in S(p)$.

A ***regular conjugate vector*** v is defined by the requirement that there exists a neighborhood of v, so that any radial ray of T_pM contains at most one conjugate point there. The regular conjugate locus then is an everywhere dense open subset of the conjugate locus that has a natural structure of an $(n-1)$-dimensional manifold. The order of a conjugate vector as a singularity of $\exp_p$ (the dimension of the kernel of the differential) is called an order of the conjugate vector.

In [29, Thm 3.3], Warner characterized the conjugate vectors at a fixed p_0 of order at least 2, and some of those of order 1, as described below. Note that in B_1, one needs to postulate that $N_{p_0}(v)$ remains tangent to $S(p_0)$ at points v close to v_0 as the latter is not guaranteed by just assuming that it holds at v_0 only.

(F) Fold conjugate vectors: Let v_0 be a regular conjugate vector at p_0, and let $N_{p_0}(v_0)$ be one-dimensional and transversal to $S(p_0)$. Such singularities are known as fold singularities. Then one can find local coordinates ξ near v_0 and y near q_0 so that in those coordinates, $\exp_{p_0}$ is given by

$$y' = \xi', \quad y^n = (\xi^n)^2. \tag{3.2}$$

Then

$$S(p_0) = \{\xi^n = 0\}, \quad N_{p_0}(v_0) = \operatorname{span}\{\partial/\partial\xi^n\}, \quad \Sigma(p_0) = \{y^n = 0\}. \tag{3.3}$$

Since the fold condition is stable under small C^∞ perturbations, as follows directly from the definition, those properties are preserved under a small perturbation of p_0.

(B_1) Blowdown of order 1: Let v_0 be a regular conjugate vector at p_0 and let $N_{p_0}(v)$ be one-dimensional. Assume also that $N_{p_0}(v)$ is tangent to $S(p_0)$ for all regular conjugate v near v_0. We call such singularities blowdown of order 1. Then locally, $\exp_{p_0}$ is represented in suitable coordinates by

$$y' = \xi', \quad y^n = \xi^1\xi^n. \tag{3.4}$$

Then

$$S(p_0) = \{\xi^1 = 0\}, \quad N_{p_0}(v_0) = \operatorname{span}\{\partial/\partial\xi^n\}, \quad \Sigma(p_0) = \{y^1 = y^n = 0\}. \tag{3.5}$$

Even though we postulated that the tangency condition is stable under perturbations of v_0, it is not stable under a small perturbation of p_0, and the type of the singularity may change then. In some symmetric cases, one can check directly that the type is locally preserved.

(B_k) Blowdown of higher order: Those are regular conjugate vectors in the case where $N_{p_0}(v_0)$ is k-dimensional, with $2 \le k \le n-1$. Then in some coordinates, $\exp_{p_0}$ is represented as

$$\begin{aligned} y^i &= \xi^i, \qquad i = 1,\dots,n-k \\ y^i &= \xi^1\xi^i, \quad i = n-k+1,\dots,n. \end{aligned} \tag{3.6}$$

Then

$$\begin{aligned} S(p_0) &= \{\xi^1 = 0\}, \quad N_{p_0}(v_0) = \operatorname{span}\left\{\partial/\partial\xi^{n-k+1},\dots,\partial/\partial\xi^n\right\}, \\ \Sigma(p_0) &= \{y^1 = y^{n-k+1} = \dots = y^n = 0\}. \end{aligned} \tag{3.7}$$

In particular, $N_{p_0}(v_0)$ must be tangent to $S(p_0)$, see also [29, Thm 3.2]. This singularity is unstable under perturbations of p_0, as well. A typical example are the antipodal points on S^n, $n \ge 3$; then $k = n-1$.

The purpose of this paper is to study the effect of fold conjugate points to X.

4. Geometry of the fold conjugate locus

In this section, we study the geometry of the tangent conjugate locus $S(p)$, and S respectively; and the conjugate locus $\Sigma(p)$ and Σ, respectively. Recall that we work locally, and everywhere below, even if not stated explicitly, (p,v) belongs to a small enough neighborhood of (p_0,v_0); (q,v) is near (q_0,w_0). We assume throughout the section that v_0 is conjugate vector at p_0 of fold type. We also fix a non-zero covector η_0 at q_0 as in (2.5), and let ξ_0 be the corresponding ξ as in (2.5). We will see later that $\xi_0 \ne 0$. We refer to Figure 2, where w is not shown, and the zero subscripts are omitted.

We start with properties of $S(p)$ and S.

Lemma 4.1.
(a) Let $v \in S(p)$ be a fold conjugate vector. Then near $q = \exp_p(v)$, $\Sigma(p)$ is a smooth surface of codimension one, tangent to $w := -\dot\gamma_{p,v}(1)$.

(b) S is a smooth $(2n-1)$-dimensional surface in TM that can be considered as the bundle $\{S(p),\, p \in M\}$ with fibers $S(p)$.

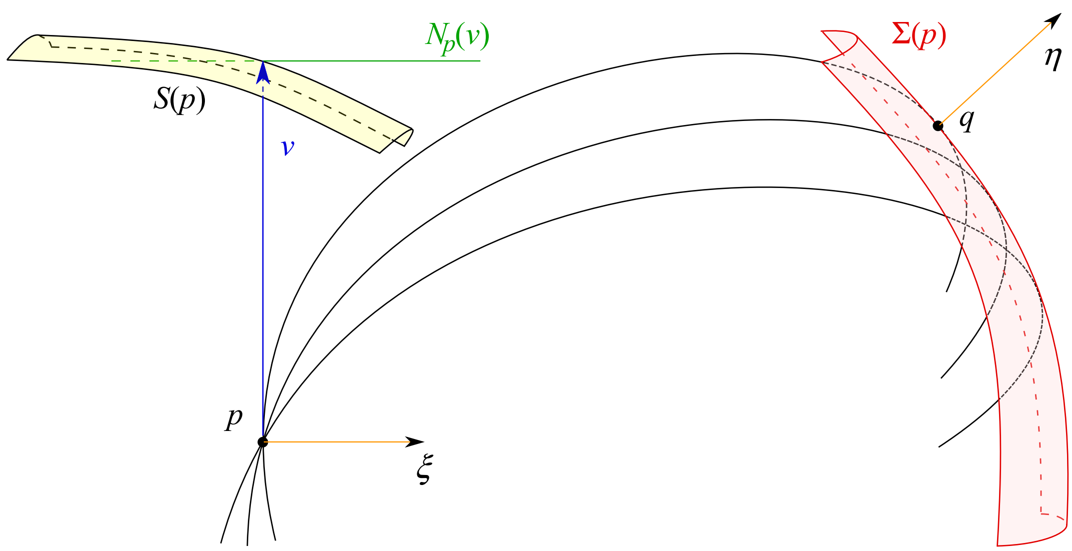

FIGURE 2. A typical fold conjugate locus

Proof. Consider (a) first. The representation (3.2) implies that locally, $\Sigma(p) = \exp_p(S(p))$ is a smooth surface of codimension one (given by $y^n = 0$). Next, for $v \in S(p)$, the differential $\mathrm{d}\exp_p$ sends any vector to a vector tangent to $S(p)$, as it follows from (3.2) again. In particular, this is true for the radial vector v (considered as a vector in $T_v T_p M$). This proves that w is tangent to $\Sigma(p)$.

The statement (b) follows from the fact that S is defined by $\det \mathrm{d}\exp_p(v) = 0$, and that $\det \mathrm{d}\exp_p(v)$ has a non-vanishing differential w.r.t. v. □

Remark 4.1. It is easy to show that in (a), $\gamma_{p,v}$ is tangent to $\Sigma(p)$ of order 1 only.

We define "Jacobi fields" along $\gamma_{p,v}$ vanishing at p as follows. For any $\alpha \in T_v T_p M$, set

$$J(t) = \mathrm{d}[\exp_p(tv)](\alpha) = \alpha^k \frac{\partial}{\partial v^k} \exp_p(tv).$$

Then $J(0) = 0$, $\dot J(0) = \alpha$, where $\dot J(T) = \mathrm{d}J(t)/\mathrm{d}t$. If $J(1) = 0$, then a direct computation shows that

$$\dot J(1) = \mathrm{d}^2 \exp_p(v)(\alpha \times v). \tag{4.1}$$

When exp is the exponential map of a Riemannian metric, it is natural to work with the covariant derivative $D_t J(t) =: J'(1)$ instead of $\dot J(t)$. While they are different in general, they coincide at points where $J(t) = 0$.

The next lemma shows that the fold/blowdown conditions are symmetric w.r.t. p and q.

Lemma 4.2. *The vector v_0 is a conjugate vector at p_0 of fold type, if and only if w_0 is a conjugate vector at q_0 of fold type.*

Proof. Set $w_0 = -\dot\gamma_{p_0,v_0}(1)$, as in (3.1). Then $p_0 = \exp_{q_0}(w_0)$. Assume now that $\alpha \in N_{p_0}(v_0)$. In some local coordinates, differentiate $p = \exp_q(w)$ w.r.t. v in the direction of α; here q, w are viewed as functions of p, v. Then, using the Jacobi field notation introduced above in (4.1), we get

$$0 = \mathrm{d}\exp_{q_0}(w_0)\left(\alpha^k \frac{\partial w}{\partial v^k}(p_0, v_0)\right) = \mathrm{d}\exp_{q_0}(w_0)\dot J(1)$$

because

$$\alpha^k \frac{\partial w}{\partial v^k}(p_0, v_0) = \alpha^k \frac{\partial}{\partial v^k} \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=1} \exp_p(tv)(p_0, v_0) = \dot J(1).$$

By (R2), $\dot J(1) \neq 0$, so in particular, this shows that w_0 is conjugate at q_0, and $\dot J(1) \in N_{q_0}(w_0)$. Moreover, by (R2), the linear map

$$N_p(v) \ni \alpha = \dot J(0) \mapsto \dot J(1) := \beta \in N_q(w), \quad J(0) = J(1) = 0 \tag{4.2}$$

defines an isomorphism between $N_p(v)$ and $N_q(w)$. Then (4.2) shows that w_0 is conjugate at q_0 of multiplicity one. By (R3), applied to w_0, it is also regular.

We will prove now that w_0 is of fold type. Since it is regular and of multiplicity one, $S(q_0)$ near w_0 is a smooth $(n-1)$ dimensional surface either of type F, as in (3.3) or of type B_1, as in (3.5). Assume the latter case first, then $\Sigma(q_0)$ is of codimension two, as follows from (3.5). In particular, using the normal form (3.4), we see that in this case, one can find a non-trivial one-parameter family of vectors $w(s)$ so that $w(0) = w_0$ and $\exp_{q_0}(w(s)) = p_0$. Then the corresponding tangent vectors at p_0 would form a non-trivial one-parameter family of vectors $v(s)$ so that $\exp_{p_0}(v(s)) = q_0$. That cannot happen, if v_0 is of type F, see (3.2), since the equation $\exp_{p_0}(v) = q_0$ has (near v_0) at most two solutions. □

For $(p, v) \in S$, let $\alpha = \alpha(p, v) \in N_p(v)$ be a unit vector. To fix the direction, assume that the derivative of $\det \mathrm{d}\exp_p(v)$ in the direction of α, for v a conjugate vector, is positive. Here we identify $T_v T_p M$ and $T_p M$. In the fold case, $N_p(v)$ is clearly a smooth vector bundle on TM near (p_0, v_0), and α is a smooth vector field.

Lemma 4.3. *For any fixed p near p_0, the map*

$$S(p) \ni v \mapsto \alpha(p, v) \in N_p(v) \tag{4.3}$$

is a local diffeomorphism, smoothly depending on p if and only if

$$\mathrm{d}^2 \exp_{p_0}(v_0)\left(N_{p_0}(v_0) \setminus 0 \times \cdot\,\right)\big|_{T_{v_0} S(p_0)} \quad \textit{is of full rank.} \tag{4.4}$$

Proof. In local coordinates, we want to find a condition so that the equation

$$\alpha^i \partial_{v^i} \exp_p(v) = 0$$

can be solved for v so that $v = v_0$ for $(p,\alpha) = (p_0,\alpha_0)$, where $\alpha_0 = \alpha(p_0, v_0)$. Then v would automatically be in $S(p)$. By the implicit function theorem, this is equivalent to

$$\det\left(\partial_v \alpha_0^i \partial_{v^i} \exp_{p_0}(v)\right) \neq 0 \quad \text{at } v = v_0.$$

Choose a coordinate system near v_0 so that $\partial/\partial v^n$ spans $N_{p_0}(v_0)$, and $\{\partial/\partial v^1, \dots \partial/\partial v^{n-1}\}$ span $T_{v_0}S(p_0)$. Denote $F(v) = \exp_{p_0}(v)$ and denote by F_i, F_{ij} the corresponding partial derivatives. Greek indices below run from 1 to $n-1$. We have

$$\partial_n F(v_0) = 0, \qquad \text{because } \partial/\partial v^n \in N_{p_0}(v_0), \tag{4.5}$$

$$\partial_\alpha \det(\partial F)(v_0) = 0, \qquad \text{because } \partial/\partial v^\alpha \text{ is tangent to } S(p_0) \text{ at } v_0, \tag{4.6}$$

$$\partial_n \det(\partial F)(v_0) \neq 0, \qquad \text{by the fold condition,} \tag{4.7}$$

$$c^\alpha \partial_\alpha F(v_0) \neq 0, \quad \forall c \neq 0, \qquad \text{because } c^\alpha \partial/\partial v^\alpha \notin N_{p_0}(v_0). \tag{4.8}$$

We want to prove that $\det(\partial_n \partial F)(v_0) \neq 0$ if and only if (4.4) holds. That determinant equals

$$\det(F_{1n}, F_{2n}, \dots, F_{nn})(v_0). \tag{4.9}$$

Perform the differentiation in (4.6). By (4.5), (4.8),

$$\det(F_1, \dots, F_{n-1}, F_{n\alpha})(v_0) = 0, \quad \forall\alpha \quad \Longrightarrow \quad F_{n\alpha}(v_0) \in \operatorname{span}(F_1(v_0), \dots, F_{n-1}(v_0)).$$

Similarly, (4.7) implies

$$\det(F_1, \dots, F_{n-1}, F_{nn})(v_0) \neq 0 \quad \Longrightarrow \quad 0 \neq F_{nn}(v_0) \notin \operatorname{span}(F_1(v_0), \dots, F_{n-1}(v_0)). \tag{4.10}$$

Those two relations show that (4.9) vanishes if and only if $(F_{n1}(v_0), \dots F_{n,n-1}(v_0))$ form a linearly dependent system, that is equivalent to (4.4). □

We study the structure of the conjugate loci $\Sigma(p)$, $\Sigma(q)$ and Σ next. Recall again that we work locally near p_0, v_0 and q_0.

Theorem 4.1. *Let v_0 be a fold conjugate vector at p_0.*

(a) Then for any p near p_0, $\Sigma(p)$ is a smooth hypersurface of dimension $n-1$ smoothly depending on p. Moreover for any $q = \exp_p(v) \in \Sigma(p)$, $T_q M$ is a direct sum of the linearly independent spaces

$$T_q M = T_q\Sigma(p) \oplus N_q(w), \tag{4.11}$$

and

$$T_q\Sigma(p) = \operatorname{Im} \mathrm{d}\exp_p(v), \quad N_q^*\Sigma(p) = \operatorname{Coker} \mathrm{d}_v \exp_p(v).$$

Next, those statements remain true with p and q exchanged.

(b) Σ is a smooth $(2n-1)$-dimensional hypersurface in $M \times M$ near (p_0, q_0), that is also a fiber bundle $\Sigma = \{\Sigma(p),\ p \in M\}$ with fibers $\Sigma(p)$ (and also $\Sigma = \{\Sigma(q),\ q \in M\}$). Moreover, the conormal bundle $\mathcal{N}^*\Sigma$ is given by

$$\begin{aligned}\mathcal{N}^*\Sigma = \big\{&(p,q,\xi,\eta);\ (p,q) \in \Sigma,\ \xi = \eta_i \partial \exp_p^i(v)/\partial p, \eta \in \operatorname{Coker} \mathrm{d}_v \exp_p(v)\\ &\textit{where } v = \exp_p^{-1}(q) \textit{ with } \exp_p \textit{ restricted to } S(p)\big\}.\end{aligned} \tag{4.12}$$

Proof. We start with (a). By the normal form (3.2), also clear from the fold condition, the image of $S(p)$ under $\mathrm{d}\exp_p(v)$ coincides with $T_q\Sigma(p)$. In particular, $\mathrm{d}\exp_p(v)$, restricted to $S(p)$ is a diffeomorphism to its image. Relation (4.11) follows from (4.2) and (R2).

Consider (b). We have $(p,q) \in \Sigma$ if and only if there exists v (near v_0) so that

$$q = \exp_p(v), \quad \det \mathrm{d}_v \exp_p(v) = 0. \tag{4.13}$$

In some local coordinates, we view this as $n+1$ equations for the $3n$-dimensional variable (p,q,v) near (p_0,q_0,v_0). We show first that the solution that we denote by L, is a $(2n-1)$-dimensional submanifold. To this end, we need to show that the following differential has rank $n+1$ at (p_0,q_0,v_0):

$$\begin{pmatrix} \mathrm{d}_p \exp_p(v) & -\mathrm{Id} & \mathrm{d}_v \exp_p(v) \\ \mathrm{d}_p \det \mathrm{d}_v \exp_p(v) & 0 & \mathrm{d}_v \det \mathrm{d}_v \exp_p(v) \end{pmatrix}. \tag{4.14}$$

The elements of the first "row" are $n\times n$ matrices, while the second row consists of three n-vectors. That the rank of the differential above is full follows from the fact that $\mathrm{d}_v \det \mathrm{d}_v \exp_p(v) \neq 0$ at (p_0,v_0), guaranteed by the fold condition.

Set $\pi(p,q,v)=(p,q)$. We show next that $\pi(L)$ is a $(2n-1)$-dimensional submanifold, too. To this end, we need to show that $\mathrm{d}\pi$ is injective on TL. The tangent space to L is given by the orthogonal complement to the rows of (4.14). Let us denote any vector in TL by $\rho=(\rho_p,\rho_q,\rho_v)$. Then $\mathrm{d}\pi(\rho)=(\rho_p,\rho_q)$. Our goal is therefore to show that $\rho_p=\rho_q=0$ implies $\rho_v=0$. Then $(0,0,\rho_v)$ is orthogonal to the rows of (4.14), therefore,

$$\rho_v^i \partial_{v^i} \exp_p^k(v)=0, \quad k=1,\dots,n, \quad \rho_v^i \partial_{v^i} \det \mathrm{d}_v \exp_p(v)=0.$$

The latter identity shows that $\rho_v\in N_p(v)$, while the first one shows that $\rho_v\in \operatorname{Ker} \mathrm{d}_v \exp_p(v)$. By the fold condition, $\rho_v=0$.

This analysis also shows that the covectors ν orthogonal to Σ are of the form $\nu=(\nu_p,\nu_q)$ with the property that $(\nu_p,\nu_q,0)$ is conormal to L. Since the conormals to L are spanned by the rows of (4.14), in order to get the third component to vanish, we have to take a linear combination with coefficients a_i, $i=1,\dots,n$ and b so that

$$a_i \frac{\partial q^i}{\partial v^j} + b\frac{\partial \det \mathrm{d}_v \exp_p(v)}{\partial v^j} = 0, \quad \forall j, \tag{4.15}$$

where $q=\exp_p(v)$. Let $0\neq\alpha\in N_p(v)$. Multiply by α^j and sum over j above to get that the v-derivative of $b\det \mathrm{d}_v \exp_p(v)$ in the direction of $N_p(v)$ vanishes. According to the fold assumption, this is only possible if $b=0$. Then we get that $a\in \operatorname{Coker} \mathrm{d}_v \exp_p(v)$. Therefore the normal covectors to Σ are of the form

$$\nu=\left(\left\{a_i \frac{\partial q^i}{\partial p^j}\right\}, -a\right), \quad a\in \operatorname{Coker} \mathrm{d}_v \exp_p(v), \tag{4.16}$$

that proves (4.12).

□

Theorem 4.2. *Let v_0 be a fold conjugate vector at p_0. Let $\exp_p$ be the exponential map of a Riemannian metric.*

(a) Then the sum in (4.11) is an orthogonal one, i.e.,

$$N_q\Sigma(p)=N_q(w).$$

(b) Next, (4.17) also admits the representation

$$\begin{aligned} \mathcal{N}\Sigma = \big\{&(p,q,\alpha,\beta);\ (p,q)\in\Sigma,\ \alpha=J'(0), \beta=-J'(1), \text{ where } J \text{ is any Jacobi field} \\ &\text{along the locally unique geodesic connecting } p \text{ and } q \text{ with } J(0)=J(1)=0\big\}. \end{aligned} \tag{4.17}$$

(c) $\mathcal{N}\Sigma$ is a graph of a smooth map $(p,\alpha)\mapsto(q,\beta)$ if and only if condition (4.4) is fulfilled. Then that map is a local diffeomorphism.

Remark 4.2. Note that for $(p,q)\in\Sigma$, the geodesic connecting p and q is unique, as follows from the normal form (3.2), only among the geodesics with $\dot\gamma(0)$ close to v_0. Also, J is determined uniquely up to a multiplicative constant. Next, once we prove that Σ is smooth, then $\alpha\in N_p(v)$ and $\beta\in N_q(w)$ by (a) (see also (3.2)), but (4.17) gives something more than that — it restricts (α,β) to an one-dimensional space.

Remark 4.3. It is a natural question whether $|J'(0)|=|J'(1)|$. One can show that generically, this is not the case.

Proof. By [16, Lemma IX.3.5], the conjugate of $\mathrm{d}\exp_p(v)$, w.r.t. the metric form is given by

$$\big(\mathrm{d}\exp_p(v)\big)^* = \mathrm{d}\exp_q(w), \tag{4.18}$$

where we use the notation (3.1). The normal to $\Sigma(p)$ at q is in the orthogonal complement to the image of $\mathrm{d}\exp_p(v)$, that by (4.18) is $\operatorname{Ker} \mathrm{d}\exp_q(w) = N_q(w)$. This proves (a).

Then we get by (4.18), (4.15) (where $b = 0$) that $a \in N_q(w)$, where we identify the covector a with a vector by the metric.

We will use now [16, Lemma IX.3.4]: for any two Jacobi fields J_1, J_2 along a fixed geodesic, the Wronskian $\langle J_1', J_2\rangle - \langle J_1, J_2'\rangle$ is constant. Along the geodesic connecting p and q, in fixed coordinates near p, let $\tilde J$ be determined by $\tilde J(0) = e_j$, $\tilde J'(0) = 0$. Here e_j has components δ^i_j. If p and q are conjugate to each other, then $\tilde J(1)$ is the equal to the variation $\partial q/\partial p^j$, and this is independent on the choice of the local coordinates, as long as e_j is considered as a fixed vector at p. Define another Jacobi field by $J(1) = 0$, $J'(1) = a$, where a is as in (4.16) but considered as a vector. Denote the field in the brackets in (4.16) by X_j. Then

$$
\begin{aligned}
X_j &= \langle a, \tilde J(1)\rangle \\
&= \langle J'(1), \tilde J(1)\rangle \\
&= \langle J'(1), \tilde J(1)\rangle - \langle J(1), \tilde J'(1)\rangle \\
&= \langle J'(0), \tilde J(0)\rangle - \langle J(0), \tilde J'(0)\rangle \\
&= J_j'(0).
\end{aligned}
\tag{4.19}
$$

This proves (4.17).

The proof of (c) follows directly from Lemma 4.3. □

5. The Schwartz kernel of N near the diagonal and mapping properties of X and N

5.1. **The geodesic case.** Let exp be the exponential map of the metric g. Then X is the weighted geodesic ray transform. One way to parametrize the geodesics is the following. Let H be any orientable hypersurface with the property that it intersects transversally, at one point only, any geodesic in M issued from a point in $\mathcal{U}$. For our local analysis, H can be an arbitrarily small surface intersecting transversally γ_{p_0,v_0}, so let us fix that choice. Let $\mathrm{d}\,\mathrm{Vol}_H$ be the induced measure in H, and let ν be a smooth unit normal vector field on H consistent with the orientation of H. Let $\mathcal{H}$ consist of all $(p,\theta) \in SM$ with the property that $p \in H$ and θ is not tangent to H, and positively oriented, i.e., $\langle \nu, \theta\rangle > 0$. Introduce the measure $\mathrm{d}\mu = \langle n, \theta\rangle\, \mathrm{d}\,\mathrm{Vol}_H(p)\, \mathrm{d}\sigma_p(\theta)$ on $\mathcal{H}$. Then one can parametrize all geodesics intersecting H transversally by their intersection p with H and the corresponding direction, i.e., by elements in $\mathcal{H}$. An important property of $\mathrm{d}\mu$ is that it introduces a measure on that geodesics set that is invariant under a different choice of H by the Liouville Theorem, see e.g., [23].

The weighted geodesic transform X can be defined as in (2.1) for $(p,\theta) \in \mathcal{H}$ instead of $(p,\theta) \in \mathcal{U}$ because transporting (p, v) along the geodesic flow does not change the integral. Since we assumed originally that κ is localized near a small enough neighborhood of γ_{p_0,v_0}, we get that κ is supported in a small neighborhood of (p_0, θ_0) in $\mathcal{H}$. We view X as the following map

$$X : L^2(M) \to L^2(\mathcal{H}, \mathrm{d}\mu),$$

restricted to a neighborhood of (p_0, θ_0). This map is bounded, see [21], and this also follows from our analysis of N. By the proof of Proposition 1 in [23], X^*X is given by

$$X^*Xf(p) = \frac{1}{\sqrt{\det g(p)}} \int_{S_pM} \int \bar\kappa(p,\theta)\kappa\big(\exp_p(t\theta), \dot{\exp}_p(t\theta)\big) f(\exp_p(t\theta))\, \mathrm{d}t\, \mathrm{d}\sigma_p(\theta). \tag{5.1}$$

We therefore proved the following.

Proposition 5.1. *Let* exp *be the geodesic exponential map. Let X be the weighted geodesic ray transform (2.1), and let N be as in (2.2), depending on $\kappa^\sharp$. Then*

$$X^*X = N \quad \textit{with } \kappa^\sharp = \bar\kappa.$$

Split the t integral in (5.1) in two: for $t > 0$ and for $t < 0$, and make a change of variables $(t, \theta) \mapsto (-t, -\theta)$ in the second one to get

$$X^*Xf(p) = \frac{1}{\sqrt{\det g(p)}} \int_{T_pM} W(p, v) f(\exp_p(v)) \,\mathrm{d}\,\mathrm{Vol}(v), \tag{5.2}$$

where

$$\begin{aligned} W = |v|^{-n+1} \Big(&\bar\kappa(p, v/|v|)\kappa\big(\exp_p(v), \dot{\exp}_p(v)/|v|\big) \\ &+ \bar\kappa(p, -v/|v|)\kappa\big(\exp_p(v), -\dot{\exp}_p(v)/|v|\big)\Big). \end{aligned} \tag{5.3}$$

Note that $|\dot{\exp}_p(v)| = |v|$ in this case.

Next we recall a result in [23]. Part (a) is based on formula (5.2) after a change of variables. We denote by ρ the distance in the metric g.

Theorem 5.1 ([23])**.** *Let* exp *be the exponential map of M. Assume that $\exp_p : \exp_p^{-1}(M) \to M$ is a diffeomorphism for p near p_0.*

(a) Then for p in the same neighborhood of p_0,

$$X^*Xf(p) = \frac{1}{\sqrt{\det g(p)}} \int A(p, q) \frac{f(q)}{\rho(p, q)^{n-1}} \left| \det \frac{\partial^2(\rho^2/2)}{\partial p \partial q} \right| \mathrm{d}q, \tag{5.4}$$

where

$$A(p, q) = \bar\kappa(p, -\operatorname{grad}_p \rho)\kappa(q, \operatorname{grad}_q \rho) + \bar\kappa(p, \operatorname{grad}_p \rho)\kappa(q, -\operatorname{grad}_q \rho).$$

(b) X^*X is a classical ΨDO of order -1 with principal symbol

$$\sigma_p(X^*X)(x, \xi) = 2\pi \int_{S_xM} \delta(\xi(\theta)) |\kappa(x, \theta)|^2 \,\mathrm{d}\sigma_x(\theta), \tag{5.5}$$

where $\xi(\theta) = \xi_i \theta^i$, and δ is the Dirac delta function.

Note that the integral (5.4) is not written in an invariant form but one can easily check that writing it w.r.t. the volume form, the kernel is invariant. We also note that in the proof of Theorem 2.1, we apply the theorem above by restricting supp f and the region where we study Nf to a small enough neighborhood of p_0, where there will be no conjugate points. This gives the ΨDO part A of N in Theorem 2.1. Finally, note that Theorem 5.2 provides a proof of part (b) even in the context of general exponential maps.

Mapping properties of X. Let (x', x^n) be semigeodesic coordinates on H near x_0. Then (x', ξ') parameterize the vectors near (x_0, θ_0). We define the Sobolev space $H^1(\mathcal{H})$ of functions constant along the flow, supported near the flow-out of (x_0, θ_0) as the H^s norm in those coordinates w.r.t. the measure $\mathrm{d}\mu$. We can choose another such surface H near q_0 with some fixed coordinates on it; the resulting norm will the be equivalent to that on $\mathcal{H}$.

Proposition 5.2. *With the notation and the assumptions above, for any $s \geq 0$, the operators*

$$X : H_0^s(V) \longrightarrow H^{s+1/2}(\mathcal{H}), \tag{5.6}$$

$$X^*X : H_0^s(V) \longrightarrow H^{s+1}(V) \tag{5.7}$$

are bounded.

Proof. Recall first that the weight κ localizes in a small neighborhood of $(\gamma_0, \dot\gamma_0)$. Let first f have small enough support in a set that we will call M_0. Then M_0 will be a simple manifold if small enough. Then we can replace H by another surface H_0 that lies in M_0, and denote by $\mathcal{H}_0$ the corresponding $\mathcal{H}$. This changes the original parameterization to a new one, that will give us an equivalent norm.

Then, if s is a half-integer,

$$\|Xf\|^2_{H^{s+1/2}(\mathcal{H}_0)} \leq C \sum_{|\alpha| \leq 2s+1} \left| \left(\partial^\alpha_{x', \xi'} Xf, Xf \right)_{L^2(\mathcal{H}_0)} \right| = C \sum_{|\alpha| \leq 2s+1} \left| \left(X^* \partial^\alpha_{x', \xi'} Xf, f \right)_{L^2(\mathcal{H}_0)} \right|.$$

The term $\partial^\alpha_{x',\xi'} Xf$ is a sum of weighted ray transforms of derivatives of f up to order $|\alpha|$. Then $X^*\partial^\alpha_{x',\xi'}X$ is a ΨDO of order $|\alpha|-1$ because M_0 is a simple manifold. That easily implies

$$\|Xf\|_{H^{s+1/2}(\mathcal{H}_0)} \le C\|f\|_{H^s}.$$

The case of general $s\ge 0$ follows by interpolation, see, e.g., [27, Sec 4.2].

To finish a proof, we cover γ_0 with open sets so that the closure of each one is a simple manifold. Choose a finite subset and a partition of unity $1=\sum \chi_j$ related to that. Then we apply the estimate above to each $X\chi_j f$ on the corresponding $\mathcal{H}_j$. We then have finitely many Sobolev norms that are equivalent, and in particular equivalent to the one on $\mathcal{H}$. This proves (5.6).

To prove the continuity of X^*X, we need to estimate the derivatives of X^*X. We have that $\partial^\alpha X^*Xf$ is sum of operators X_{κ_α} of the same kind but with possibly different weights applied to derivatives of Xf up to order $|\alpha|$, see (5.1). Let first $s=0$. For f, h in $C_0^\infty(V)$, $|\beta|=1$, we have

$$\left|\left(f, X^*_{\kappa_\beta}\partial^\beta_{x',\xi'}Xh\right)_{L^2(V)}\right| \le C\|X_{\kappa_\beta} f\|_{H^{1/2}}\|Xh\|_{H^{1/2}} \le C\|f\|_{L^2(V)}\|h\|_{L^2(V)}.$$

In the last inequality, we used (5.6) that we proved already. This proves (5.7) for $s=0$.

For $s\ge 1$, integer, we can "commute" the derivative in $\partial^\alpha X^*X$ with X^*X by writing it as a finite sum of operators of the type $X^*_{\tilde\beta}X_\beta P_\beta f$, $|\beta|\le|\alpha|$, where P_β are differential operators of order β. To this end, we first "commute" it with X^*, as above, and then with X. Then we apply (5.7) with $s=0$. The case of general $s\ge 0$ follows by interpolation. □

Remark 5.1. We did not use the fold condition here. In fact, Proposition 5.2 holds without any assumptions on the type of the conjugate points, as long as V is contained in a small enough neighborhood of a fixed geodesic segment that extends to a larger one with both endpoints outside V. Note that proving the mapping properties of X^*X based on its FIO characterization is not straightforward, and we would get the same conclusion under some assumptions only, for example that the canonical relation is a canonical graph; that is not always true.

Remark 5.2. A global version of Proposition 5.2 can easily be derived by a partition of unity in the phase space. Let (M,g) be a compact non-trapping Riemannian manifold with boundary, i.e., all maximal geodesics in M have a uniform finite bound on their length. Let M_1 be another such manifold which interior includes M, and assume that ∂M_1 is strictly convex. Such M_1 always exists if ∂M is strictly convex. Let ∂_-SM_1 denote the vectors with base point on ∂M pointing into M_1. Then we can parameterize all (directed) geodesics with points in ∂_-SM_1, that plays the role of $\mathcal{H}$ above. Then for $s\ge 0$,

$$X: H^s_0(M)\longrightarrow H^{s+1/2}(\partial_-SM_1),\quad X^*X: H^s_0(M)\longrightarrow H^{s+1}(M_1)$$

are bounded.

5.2. **General regular exponential maps.** Let now exp be a regular exponential map. As above, we split the t-integral in the second line below into two parts to get

$$\begin{aligned} Nf(p) &= \int \kappa^\sharp(p,\theta)Xf(p,\theta)\,\mathrm{d}\sigma_p(\theta)\\ &= \int_{S_pM}\int \kappa^\sharp(p,\theta)\kappa\left(\exp_p(t\theta), \dot{\exp}_p(t\theta)\right) f(\exp_p(t\theta))\,\mathrm{d}t\,\mathrm{d}\sigma_p(\theta)\\ &= \int_{T_pM} W(p,v)f(\exp_p(v))\,\mathrm{d}\,\mathrm{Vol}(v), \end{aligned}\tag{5.8}$$

where

$$\begin{aligned} W = |v|^{-n+1}\Big(&\kappa^\sharp(p,v/|v|)\kappa\big(\exp_p(v), \dot{\exp}_p(v)/|v|\big)\\ &+\kappa^\sharp(p,-v/|v|)\kappa\big(\exp_p(v), -\dot{\exp}_p(v)/|v|\big)\Big). \end{aligned}\tag{5.9}$$

Theorem 5.2. *Let $\exp_p(v)$ satisfy (R1) and (R4) and assume that for any $(p,\theta) \in \operatorname{supp}\kappa^\sharp$, $t\theta$ is not a conjugate vector at p for t such that $\exp_p(t\theta) \in \operatorname{supp} f$. Then N is a classical ΨDO of order -1 with principal symbol*

$$\sigma_p(N)(x,\xi) = 2\pi \int_{S_xM} \delta(\xi(\theta))(\kappa^\sharp\kappa)(x,\theta)\,\mathrm{d}\sigma_x(\theta), \tag{5.10}$$

where $\xi(\theta) = \xi_i\theta^j$, and δ is the Dirac delta function.

Proof. The theorem is essentially proved in Section 4 of [9], where the exponential map is related to a geodesic like family of curves. We will repeat the arguments there in this more general situation.

Notice first that it is enough to study small enough $|t|$. Fix local coordinates x near p_0. By (R4),

$$\exp_x(t\theta) = x + tm(t,\theta;x), \quad m(0,\theta;x) = \theta,$$

with a smooth function m near $(0,\theta_0,p_0)$. Introduce new variables $(r,\omega) \in \mathbf{R} \times S_xM$ by

$$r = t|m(t,\theta;x)|, \quad \omega = m(t,\theta;x)/|m(t,\theta;x)|,$$

where $|\cdot|$ is the norm in the metric $g(x)$. Then (r,ω) are polar coordinates for $\exp_x(t\theta) - x = r\omega$ with r that can be negative, as well, i.e.,

$$\exp_x(t\theta) = x + r\omega.$$

The functions (r,ω) are clearly smooth for $|t| \ll 1$, and x close to p_0. Let

$$J(t,\theta;x) = \det \mathrm{d}_{t,v}(r,\omega)$$

be the Jacobi determinant of the map $(t,v) \mapsto (r,\omega)$. By (R4), $J|_{t=0} = 1$, therefore that map is a local diffeomorphism from $(-\varepsilon,\varepsilon) \times S_xM$ to its image for $0 < \varepsilon \ll 1$. It is not hard to see that for $0 < \varepsilon \ll 1$ it is also a global diffeomorphism, because it is clearly injective. Let $t = t(x,r,\omega)$, $\theta = \theta(x,r,\omega)$ be the inverse functions defined by that map. Then

$$t = r + O(|r|), \quad \theta = \omega + O(|r|), \quad \dot{\exp}(t\theta) = \omega + O(|r|).$$

Assume that the weight κ in (2.2) vanishes for p outside some small neighborhood of p_0. Then after a change of variables, we get

$$Nf(x) = \int_{S_xM}\int A(x,r,\omega) f(x+r\omega)\,\mathrm{d}r\,\mathrm{d}\sigma_x(\omega),$$

where

$$A(x,r,\omega) = \kappa^\sharp(x,\theta(x,r,\omega))\kappa(x+r\omega,\omega+rO(1))J^{-1}(x,r,\omega)$$

with J as before, but written in the variables (x,r,ω). By [9, Lemma 4.2], N is a classical ΨDO of order -1 with a principal symbol

$$2\pi\int_{S_xM}\delta(\xi(\omega))A(x,0,\omega)\,\mathrm{d}\sigma_x(\omega) = 2\pi\int_{S_xM}\delta(\xi(\omega))\kappa^\sharp(x,\omega)\kappa(x,\omega)\,\mathrm{d}\sigma_x(\omega). \tag{5.11}$$

The proof in [9] starts with the change of variables $y = x + r\omega$. Then we write the Schwartz kernel of N as a singular one with a leading part $2A_{\text{even}}(x,0,\omega)|x-y|^{-1}$, $\omega = (y-x)/|y-x|$, where A_{even} is the even part of A w.r.t. ω. It then follows that N is a ΨDO of order -1 with a principal symbols as claimed. □

Remark 5.3. Formulas (5.2) and (5.8) are valid regardless of possible conjugate points. In our setup, the supports of κ, $\kappa^\sharp$ guarantee that $\exp_p(t\theta)$, for (p,θ) close to (p_0,θ_0) reaches a conjugate point for $t > 0$ but not for $t < 0$. Therefore, near the conjugate point q of p, the second term on the r.h.s. of (5.3), and (5.9), respectively, vanishes.

6. The Schwartz kernel of N near the conjugate locus Σ

We will introduce first three invariants. Let $F : M \to N$ be a smooth orientation preserving map between two orientable Riemannian manifolds (M, g) and (N, h). Then one defines $\det \mathrm{d}F$ invariantly by

$$F^*(\mathrm{d}\,\mathrm{Vol}_N) = (\det \mathrm{d}F)\,\mathrm{d}\,\mathrm{Vol}_M, \tag{6.1}$$

see also [16, X.3]. In local coordinates,

$$\det \mathrm{d}F(x) = \sqrt{\frac{\det h(F(x))}{\det g(x)}}\det\frac{\partial F(x)}{\partial x}. \tag{6.2}$$

We choose an orientation of $S(p_0)$ near v_0, as a surface in $T_{p_0}M$ by choosing a unit normal field so that the derivative of $\det \mathrm{d}\exp_{p_0}(v)$ along it is positive on $S(p)$. Then we extend this orientation to $S(p)$ for p close to p_0 by continuity. On Figure 2, the positive side is the one below $S(p)$, if v is the first conjugate vector along the geodesic through (p, v). Then we choose an orientation of $\Sigma(p)$ so that the positive side is that in the range of $\exp_p$. On Figure 2, the positive side is to the left of $\Sigma(p)$. The so chosen orientations conform with the signs of ξ^n and y^n in the normal form (3.2).

Next we synchronize the orientations of T_pM and M near q by postulating that $\exp_p$ is an orientation preserving map from the positive side of $S(p)$, as described above, to the positive side of $\Sigma(p)$.

For each $p \in M$, the transformation laws in TT_pM under coordinate changes on the base show that T_pM has the natural structure of a Riemannian manifold with the constant metric $g(p)$. Then one can define $\det \mathrm{d}\exp_p$ invariantly as above. Let $\mathrm{d}\,\mathrm{Vol}_p$ be the volume form in T_pM, and let $\mathrm{d}\,\mathrm{Vol}$ be the volume form in M. Then $\det \mathrm{d}\exp_p$ is defined invariantly by

$$\exp_p^*\mathrm{d}\,\mathrm{Vol} = \left(\det \mathrm{d}\exp_p\right)\mathrm{d}\,\mathrm{Vol}_p\,. \tag{6.3}$$

In local coordinates,

$$\det \mathrm{d}\exp_p = \sqrt{\frac{\det g(\exp_p v)}{\det g(p)}}\det\frac{\partial}{\partial v}\exp_p(v),$$

where, with some abuse of notation, $g(p)$ is the metric g in fixed coordinates near a fixed p_0, and $g(\exp_p v)$ is the metric g in a possibly different system of fixed coordinates near $q_0 = \exp_{p_0} v_0$. Set

$$A(p, v) := |\mathrm{d}\det \mathrm{d}\exp_p(v)|. \tag{6.4}$$

Since $\det \mathrm{d}\exp_p(v)$ is a defining function for $S(p)$, its differential is conormal to it. By the fold condition, $A \neq 0$. One can check directly that A is invariantly defined on Σ.

By (3.3), for $(p, v) \in S$, the differential of $\exp_p$ maps isomorphically $T_vS(p)$ (equipped with the metric on that plane induced by $g(p)$) into $T_q\Sigma$, with the induced metric. Let D be the determinant of $\exp_p|_{S(p)}$, i.e.,

$$D := \det\left(\mathrm{d}\exp_p|_{T_vS(p)}\right), \tag{6.5}$$

defined invariantly by (6.1). We synchronize the orientations of $S(p)$ and $\Sigma(p)$ so that $D > 0$.

We express next the weight $W(p, v)$ restricted to S in terms of the variables (p, q). For $(p, q) \in \Sigma$, $v = \exp_p^{-1}(q)$, where we inverted $\exp_p$ restricted to S. Let $w = w(p, q)$ be defined as in (3.1) with v as above. Then we set, see also (5.9), and Remark 5.3,

$$W_\Sigma(p, q) := W\left((p, \exp_p^{-1}(q))\right|_\Sigma = |v|^{1-n}\kappa^\sharp(p, v/|v|)\kappa(q, -w/|v|) \tag{6.6}$$

For p close to p_0, $\Sigma(p)$ divides M in a neighborhood of q_0 into two parts: one of them is in the range of $\exp_p(v)$ for v near v_0, that is the positive one w.r.t. the chosen orientation; the other is not. Let $z'(p, q)$ be the distance from q to $\Sigma(p)$ with a positive sign in the first region, and with a negative sign in the second one. Then for a fixed p, $z' = z'(p, q)$ is a normal coordinate to $\Sigma(p)$ depending smoothly on p, and Σ is given locally by $z' = 0$. Then z' is a defining function for Σ, i.e., $\Sigma = \{z' = 0\}$ and $\mathrm{d}_{p,q}z' \neq 0$ because $\mathrm{d}_q z' \neq 0$. Let $z'' = z''(p, q) \in \mathbf{R}^{2n-1}$ be such that its differential restricted to $T\Sigma$ is an isomorphism at (p_0, q_0). Since $\mathrm{d}z''$ and $\mathrm{d}z'$ are linearly independent, $z = z(z', z'')$ are coordinates near (p_0, q_0). One way to construct z'' is the following. Choose $(z_{n+1}, \dots, z_{2n})$, depending on p only, to be local coordinates for p, and to choose $(z', z_2, \dots, z_n)$, depending on p and q, to be semi-geodesic coordinates of q near $\Sigma(p)$.

The next theorem shows that near Σ, the operator N has a singular but integrable kernel with a conormal singularity of the type $1/\sqrt{z'}$.

Theorem 6.1. *Near $\Sigma(p)$, the Schwartz kernel $N(p,q)$ of N (with respect to the volume measure) near (p_0, q_0) is of the form*

$$N = W_\Sigma \frac{\sqrt{2}}{\sqrt{ADz'}}(1 + \sqrt{z'}R(\sqrt{z'}, z'')), \tag{6.7}$$

where $W_\Sigma = W_\Sigma(z'')$, $A = A(z'')$, $D = D(z'')$, and R is a smooth function.

Proof. We start with the representation (5.8). We will make the change of variables $y = \exp_p(v)$ for (p, v) close to (p_0, v_0) as always. Then y will be on the positive side of $\Sigma(p)$, and the exponential map is 2-to-1 there. We split the integration in (5.8) in two parts: one, where v is on the positive side of $S(p)$, that we call $N_+ f$, and the other one we denote by $N_- f$. Then

$$N_\pm f(p) = \int_{S_pM} \int W f(y) \big(\det \mathrm{d} \exp_p^\pm(v)\big)^{-1} \mathrm{d}\,\mathrm{Vol}(y), \tag{6.8}$$

where W is as in (6.6) but not restricted to Σ, and $(\exp_p^\pm)^{-1}$ there is the corresponding inverse in each of the two cases.

To prove the theorem, we need to analyze the singularity of the Jacobian determinant $\det \mathrm{d}\exp_p(v)$ near $\Sigma(p)$. It is enough to do this at (p_0, v_0).

Let $y = (y', y^n)$ be semi-geodesic coordinates near $\Sigma(q_0)$, $q_0 = \exp_{p_0}(v_0)$, and let y_0 correspond to q_0. We assume that $y^n > 0$ on the positive side of $\Sigma(p)$. In other words, $y^n = z'(p_0, q)$.

We have

$$\mathrm{d}\,\mathrm{Vol}(y) = \det\big(\mathrm{d}_v \exp_p(v)\big)\, \mathrm{d}\,\mathrm{Vol}(v)$$

The form on the left can be written as $\mathrm{d}\,\mathrm{Vol}_{\Sigma(p)}(y')\,\mathrm{d}y^n$; while the one on the right, restricted to $S(p)$, equals $\mathrm{d}\,\mathrm{Vol}_{S(p)}(v')\,\mathrm{d}v^n$ in boundary normal coordinates to $S(p)$, where $v^n > 0$ gives the positive side of $S(p)$. On the other hand, by (6.5),

$$\mathrm{d}\,\mathrm{Vol}_{\Sigma(p)}(y') = D\,\mathrm{d}\,\mathrm{Vol}_{S(p)}(v').$$

We therefore get

$$D\,\mathrm{d}y^n = \det\big(\mathrm{d}\exp_p(v)\big)\,\mathrm{d}v^n.$$

By the definition of A, we have

$$\det \mathrm{d}_v \exp_p(v) = Av^n(1 + O(v^n)). \tag{6.9}$$

Therefore,

$$D\,\mathrm{d}y^n = A(1 + O(v^n))\, v^n \mathrm{d}v^n.$$

Since $y^n = 0$ for $v^n = 0$, we get

$$y^n = (v^n)^2 \frac{A}{2D}(1 + O(v^n)).$$

Solve this for v^n and plug into (6.9) to get

$$\det \mathrm{d}\exp_p(v) = \pm\sqrt{2ADy^n}\left(1 + O_\pm\big(\sqrt{y^n}\big)\right). \tag{6.10}$$

Here $O_\pm\big(\sqrt{y^n}\big)$ denotes a smooth function of $\sqrt{y^n}$ near the origin with coefficients smooth in y', that vanishes at $y^n = 0$. The positive/negative sign corresponds to v belonging to the positive/negative side of $S(p)$. By (6.8),

$$N_\pm f(p) = \int W f(y) \frac{1}{\sqrt{2ADy^n}}\left(1 + O_\pm\big(\sqrt{y^n}\big)\right) \mathrm{d}\,\mathrm{Vol}(y). \tag{6.11}$$

We replace A_0, D_0 in (6.11) by their values at $y^n = 0$; the error will then just replace the remainder term above by another one of the same type. Similarly, $W = W(p, v)$, where $\exp_p(v) = q$. Solving the latter for $v = v(p,q)$ provides a function having a finite Taylor expansion in powers of $\sqrt{y^n}$ of any order, with smooth coefficients. The leading term is what we denoted by W_Σ that is a smooth function on Σ.

With the aid of (6.2), it is easy to see that (6.11) is a coordinate representation of the formula (6.7) at the so fixed p. When p varies near p_0, it is enough to notice that since we already wrote the integral in invariant form, y^n then becomes the function $z'(p,q)$ introduced above. For z'' we then have $z''(p,q) = (x(p), y'(p,q))$. Finally, we note that another choice of z'' so that (z', z'') are coordinates would preserve (6.7) with a possibly different R. □

7. N AS A FOURIER INTEGRAL OPERATOR. PROOF OF THEOREM 2.1

We are ready to finish the proof of Theorem 2.1. By Theorem 6.1, near Σ, the Schwartz kernel of N has a conormal singularity at Σ, supported on one side of it, that admits a singular expansion in powers of $\sqrt{z'_+}$, with a leading singularity $1/\sqrt{z'_+}$. The Fourier transform of the latter is

$$\sqrt{\pi}e^{-\mathrm{i}\pi/4}(\zeta_+^{-1/2} + \mathrm{i}\zeta_-^{-1/2}) \tag{7.1}$$

where $\zeta_+ = \max(\zeta, 0)$, $\zeta_- = (-\zeta)_+$. The singularity near $\zeta = 0$ can be cut off, and we then get a symbol of order $-1/2$, depending smoothly on the other $2n-1$ variables. Therefore, near Σ, the kernel of N belongs to the conformal class $I^{-n/2}(M \times M, \Sigma; \mathbf{C})$, see e.g., [13, 18.2]. It is elliptic when $\kappa^\sharp(p_0,\theta_0)\kappa^\sharp(q_0,-w_0) \neq 0$ by (5.9), (6.6). Therefore, the kernel of N near Σ is a kernel of an FIO associated to the Lagrangian $T^*\Sigma$. Moreover, the amplitude of the conormal singularity at Σ is in the class $S^{-1/2,1/2}_{\mathrm{phg}}$ (polyhomogeneous of order $-1/2$, having an asymptotic expansion in integer powers of $|\zeta|^{1/2}$), see also (9.13) and (9.14).

8. THE TWO DIMENSIONAL CASE

Theorem 8.1. *Let* $\dim M = 2$. *Assume that (R1) – (R5) are fulfilled. Then* $\mathcal{N}^*\Sigma \setminus 0$, *near* $(p_0, \xi_0, q_0, \eta_0)$, *is the graph of a local diffeomorphism* $T^*M \setminus 0 \ni (p,\xi) \mapsto (q,\eta) \in T^*M \setminus 0$, *homogeneous of order one in its second variable (a canonical graph).*

Proof. For (p,ξ) near (p_0,ξ_0), there are exactly two smooth maps that map ξ to a unit normal vector. We choose the one that maps ξ_0 to $v_0/|v_0|$. Then we map the latter to $v \in S(p)$. Since the radial ray through v is transversal to $S(p)$, that map is smooth. Knowing v, then we can express $q = \exp_p(v) \in \Sigma(p)$ and $w = -\dot{\exp}_p(v)$ as smooth functions of (p,ξ) as well. Then in local coordinates, $\eta = \xi_i \partial \exp^i_q(w)/\partial q$, see (4.12), that in particular proves the homogeneity.

By (R5), this map is invertible. □

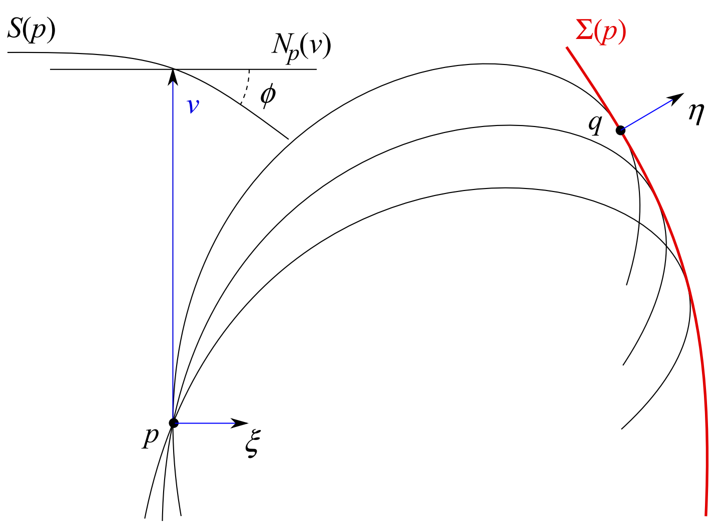

FIGURE 3. The 2D case

The principal symbol of X^*X in the geodesics case, see Theorem 5.1, and (5.5), is given by

$$\sigma_{\mathrm{p}}(X^*X)(x,\xi) = 2\pi|\kappa(x, \xi^\perp/|\xi^\perp|)|^2, \tag{8.1}$$

where $\xi^\perp$ is a continuous choice of a vector field normal to ξ and of the same length so that at $p = p_0$, $\xi_0^\perp/|\xi_0^\perp| = \theta_0$, $-\xi_0^\perp/|-\xi_0^\perp| = \theta_0$; therefore, the sign of the angle of rotation is different near ξ_0 and near $-\xi_0$. Notice that (5.5) in the two dimensional case is a sum of two terms but we assumed that κ is supported near (p_0, θ_0), therefore only one of the terms is non-trivial. A similar remark applies to (5.10).

Theorem 6.1 takes the following form in two dimensions, in the Riemannian case.

Corollary 8.1. *Let $n = 2$ and let $\exp$ be the exponential map of a Riemannian metric. With the notation of Theorem 6.1, we then have*

$$N = W_\Sigma \frac{\sqrt{2}}{\sqrt{Bz'}}(1 + \sqrt{z'}R(\sqrt{z'}, z'')), \tag{8.2}$$

where

$$B = \left|\frac{\mathrm{d}}{\mathrm{d}N} \det \mathrm{d} \exp_p(v)\right|$$

is evaluated at $v \in S(p)$ such that $q = \exp_p(v)$, and $\mathrm{d}/\mathrm{d}N$ stands for the derivative in the direction of $N_p(v)$.

Proof. Note first that $B \neq 0$ by the fold condition. Let ϕ be the (acute) angle between $S(p)$ and $N_p(v)$ at v. Since $N_p(v)$ is orthogonal to the radial ray at v, we can introduce an orthonormal coordinate system at v with the first coordinate vector being $v/|v|$, and the second one: the positively oriented unit vector along $N_p(v)$, that we call ξ. Let us parallel transport this frame along the geodesic $\gamma_{p,v}$; and invert the direction of the tangent vector to conform with our choice of w at q. In particular, this introduces a similar coordinate system near the corresponding vector w at q in the conjugate locus. In these coordinates then

$$\mathrm{d}\exp_p(v) = \begin{pmatrix} -1 & 0 \\ 0 & j/|v| \end{pmatrix}, \tag{8.3}$$

where j is uniquely determined by $J(t) = j(t)\Xi(t)$, where $J(t)$ is the Jacobi field with $J(0) = 0$, $J'(0) = \xi$, and $\Xi(t)$ is the parallel transport of ξ, compare that with (4.1). The extra factor $1/|v|$ comes from the fact that we normalize v now in our basis, so that the result would be the Jacobian determinant. Then the Jacobi determinant $\det \mathrm{d}\exp_p(v)$ is given by $-j/|v|$. In particular, for $(p, v) \in S$ we have $\mathrm{d}\exp_p(v) = \operatorname{diag}(-1, 0)$. Note that j depends on v as well, therefore its differential that essentially gives $\mathrm{d}\det\mathrm{d}\exp_p(v)$ depends on the properties of the Jacobi field under a variation of the geodesic.

Now, it easily follows from the definition (6.5) of D that

$$D = \sin\phi.$$

On the other hand, $\mathrm{d}\det\mathrm{d}\exp_p(v)$ is conormal to $S(p)$, therefore, the derivative of $\det\mathrm{d}\exp_p(v)$ in the direction of $N_p(v)$ satisfies

$$\left|\frac{\mathrm{d}}{\mathrm{d}N} \det \mathrm{d} \exp_p(v)\right| = |\mathrm{d}\det\mathrm{d}\exp_p(v)|\sin\phi = A\sin\phi = AD.$$

□

9. Resolving the singularities in the geodesic case

Let, as before, (p_0, q_0) be a pair of fold conjugate points along γ_0, and X be the ray transform with a weight that localizes near γ_0. We want to see whether we can resolve the singularities of f near p_0 and near q_0 knowing that $Xf \in C^\infty$, and more generally, whether we can invert X microlocally. Assume for simplicity that $p_0 \neq q_0$.

We will restrict ourselves to the geodesic case only but the same analysis holds without changes to the case of magnetic geodesics as well. We avoid the formal introduction of magnetic geodesics for simplicity of the exposition. Assume also that

$$\kappa(p, \theta)\kappa(q, -w/|w|) \neq 0, \quad \text{for } (p, \theta) \in \mathcal{U}_0, \tag{9.1}$$

where (q, w) are given by (3.1), and $\mathcal{U} \Supset \mathcal{U}_0 \ni (p_0, \theta_0)$. This guarantees the microlocal ellipticity of the ΨDO A near $\mathcal{N}^*(p_0, v_0)$ and $\mathcal{N}^*(q_0, w_0)$ in Theorem 2.1, see Theorem 5.1.

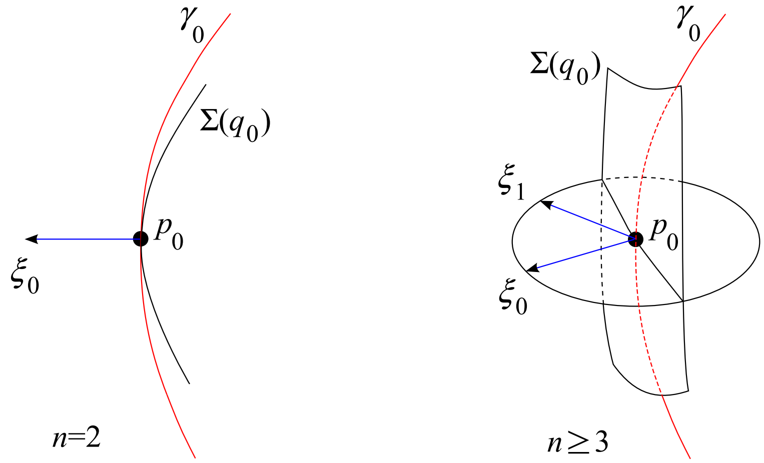

FIGURE 4. Two geometric objects can detect singularities at p_0 in the geodesic case: a geodesic γ_0 through p_0, and the conjugate locus $\Sigma(q_0)$ of q_0 conjugate to p_0. By Theorem 4.2, γ_0 is parallel to $\Sigma(q_0)$.

9.1. **Sketch of the results.** We explain the results before first in an informal way. As we pointed out in the Introduction, $Xf(\gamma)$ for geodesics near γ_0 can only provide information for WF(f) near $\mathcal{N}^*\gamma_0$, and does not "see" the other singularities. The analysis below based on Theorem 2.1, shows that on a principal symbol level, the operator $|D|^{1/2}F$ behaves as a Radon type of transform on the curves (when $n=2$) or the surfaces (when $n\geq 3$) $\Sigma(p)$. Similarly, its adjoint behaves as a Radon transform on the curves/surfaces $\Sigma(q)$. Therefore, there are two geometric objects that can detect singularities at p_0 conormal to v_0: the geodesic $\gamma_0=\gamma_{p_0,v_0}$ (and those close to it) and the conjugate locus $\Sigma(q_0)$ through p_0 (and those corresponding to perturbations of v_0). We refer to Figure 4.

When $n=2$, the information coming from integrals along the two curves (and their neighborhoods) may in principle cancel; and we show in Theorem 9.2 that this actually happens, at least to order one. When $n\geq 3$, the Radon transform over $\Sigma(q)\ni p$ competes with the geodesic transform over geodesics through p. Depending on the properties of that Radon transform, the information that we get for $\pm\xi_0$ may or may not cancel because ξ_0 is conormal both to γ_0 and $\Sigma(q_0)$. On the other hand, for any other ξ_1 conormal to v_0 but not parallel to ξ_0, the geodesic γ_0 (and those close to it) can detect whether it is in WF(f) but the Radon transform restricted to small perturbations of v_0 (and therefore of q_0) will not. Thus, we can invert N microlocally at such (p_0,ξ_1).

Now, when $n\geq 3$, we may try to invert N even at ξ_0 by choosing v's close to v_0 but normal to ξ_0. If ξ_0 happens not to be conormal to the corresponding conjugate locus $\Sigma(q(p_0,v))$ at p_0, we can just use the argument above with the new v. In particular, if the map (4.3) is a local diffeomorphism, this can be done.

This suggests the following sufficient condition for inverting N at (p_0,ξ_1):

(9.2) $\exists\theta_1\in S_{p_0}M$, so that $\kappa(p_0,\theta_1)\neq 0$, $\xi_1(\theta_1)=0$, and ξ_1 is not conormal to $\Sigma(q(p_0,\theta_1))$ at p_0.

Above, $\Sigma(q(p_0,\theta_1))$ is the conjugate locus to the point q that is conjugate to p_0 along γ_{p_0,θ_1}. We normally denote that point by $q(p_0,v_1)$, where $v_1\in S(p_0)$ has the same direction as θ_1.

In case of the geodesic transform, one could formulate (9.2) in terms of the map (4.3) as follows:

(9.3) $\exists v_1\in S(p_0)$, so that $\kappa(p_0,v_1/|v_1|)\neq 0$, $\xi_1(v_1)=0$, and ξ_1 is not the image of v_1 under the map (4.3) at p_0.

In Section 10.3, we present an example where (4.3) is a local diffeomorphism, therefore (9.2) holds. In Section 10.4 we present another example, where (9.2) fails.

9.2. **Recovery of singularities in all dimensions.** We proceed next with analysis of the recovery of singularities.

Let $\chi_{1,2}$ be smooth functions on M that localize near p_0, and q_0, respectively, i.e., supp $\chi_1\subset U_1$, supp $\chi_2\subset U_2$, where $U_{1,2}$ are small enough neighborhoods of p and q, respectively. Assume that χ_1, χ_2 equal 1 in smaller neighborhoods of p_0, q_0, where f_1, f_2 are supported. Then $f := f_1 + f_2$ is supported in $U_1 \cup U_2$ and we can write

$$\chi_1 N f = A_1 f_1 + F_{12} f_2, \tag{9.4}$$

where $A_1 = \chi_1 N \chi_1$ is a ΨDO by Theorem 5.2, while $F_{12} = \chi_1 N \chi_2$ is the FIO that we denoted by F in Theorem 2.1. By (R5), we can do the same thing near q_0 to get

$$\chi_2 N f = A_2 f_2 + F_{21} f_1, \tag{9.5}$$

where $A_2 = \chi_2 N \chi_2$, $F_{21} = \chi_2 N \chi_1$. It follows immediately that $F_{21} = F_{12}^*$. Recall that $F_{12} = F$ in the notation of Theorem 2.1. Assuming $X^* X f \in C^\infty$, we get

$$A_1 f_1 + F f_2 \in C^\infty, \quad A_2 f_2 + F^* f_1 \in C^\infty. \tag{9.6}$$

Solve the first equation for f_2, plug into the second one to get

$$\left(\mathrm{Id} - A_2^{-1} F^* A_1^{-1} F\right) f_2 \in C^\infty \quad \text{near } (q_0, \pm\eta_0)\ , \tag{9.7}$$

where A_1^{-1}, A_2^{-1}, denote parametrices of A_1, A_2 near $(p_0, \pm\xi_0)$, and $(q_0, \pm\eta_0)$, respectively. The operator in the parentheses is a ΨDO of order 0 if the canonical relation is a graph, that is true in particular when $n = 2$, by Theorem 8.1. In that case, if $\mathrm{Id} - A_2^{-1} F^* A_1^{-1} F$ is an elliptic (as a ΨDO of order 0), near $(q_0, \pm\eta_0)$, then we can recover the singularities. Without the canonical graph assumption, if it is hypoelliptic, then we still can.

Another way to express the arguments above is the following. Since $\chi_{1,2}$ together with κ restrict to conic neighborhoods of $(p_0 \pm \xi_0)$, and $(q_0 \pm \eta_0)$, respectively, and $A_{1,2}$, F, F^* have canonical relations of graph type that preserve the union of those neighborhoods, we may think of $f = f_1 + f_2$ as a vector $f = (f_1, f_2)$, and then

$$\mathcal{F} = \begin{pmatrix} A_1 & F \\ F^* & A_2 \end{pmatrix}. \tag{9.8}$$

The operator $\mathrm{Id} - A_2^{-1} F^* A_1^{-1} F$ can be considered then as the "determinant" of $\mathcal{F}$, up to elliptic factors.

Theorem 9.1. *Let the canonical relation of F be a canonical graph. With the assumptions and the notation above, if the zeroth order ΨDO*

$$Id - A_2^{-1} F^* A_1^{-1} F \tag{9.9}$$

is elliptic in a conic neighborhood of $(q_0, \pm\eta_0)$, then $Xf \in C^\infty$ near (p_0, θ_0) (or more generally, $Nf \in C^\infty$ near p_0 and q_0) implies $f \in C^\infty$.

In the geodesic case in two dimensions, the principal symbol of $A_2^{-1} F^* A_1^{-1} F$ is always 1, see the Proposition 9.1 below.

When $n \geq 3$ and F is of graph type, then $A_2^{-1} F^* A_1^{-1} F$ is of negative order, therefore we can resolve the singularities.

Corollary 9.1. *Let $n \geq 3$ and assume that the canonical relation of F is a canonical graph. Then the conclusions of Theorem 9.1 hold, i.e., $Xf \in C^\infty$ near (p_0, θ_0) (or more generally, $Nf \in C^\infty$ near p_0, q_0) implies $f \in C^\infty$.*

Proof. In this case, $A_1^{-1} F$ is an FIO of order $1 - n/2$ with the same canonical relation is F. Similarly $A_2^{-1} F^*$ is an FIO of order $1 - n/2$ with a canonical relation that is a graph of the inverse canonical map. Their composition is therefore a ΨDO of order $2 - n < 0$. Its principal symbol as a ΨDO of order 0 is zero. The corollary now follows from Theorem 9.1. □

In Section 10.3, we give an example where the assumptions of the corollary hold. Note that those assumptions are stable under small perturbations of the dynamical system.

When the graph condition does not hold, the analysis is harder. Then (4.3) is not a local diffeomorphism. If its range is a lower dimension submanifold, for example, we can at least recover the conormal singularities to θ_0 away from it, as the corollary below implies. Note that below, (b) implies (a). Also, (9.1) is not needed; only ellipticity of κ at (p_0, θ_0) suffices.

Corollary 9.2. *Let $Xf \in C^\infty$ for γ near γ_0. Then*

(a) If $\xi_1 \in T_{p_0}M \setminus 0$ is conormal to v_0 but not conormal to $\Sigma(q_0)$ (not parallel to ξ_0), then

$$(p_0, \xi_1) \not\in \mathrm{WF}(f).$$

(b) The same conclusion holds if condition (9.2) or the equivalent (9.3) is fulfilled.

Proof. Note first that A_1 is elliptic at (p_0, ζ) by (9.1) and Theorem 5.1(b). By the first relation in (9.6), $(p_0, \xi_1) \in \mathrm{WF}(f_1)$ if and only if $(p_0, \xi_1) \in \mathrm{WF}(Ff_2)$. To analyze the latter, we will use the relation $\mathrm{WF}(Ff_2) \subset \mathrm{WF}'(F) \circ \mathrm{WF}(f_2)$, see [12, Thm 8.5.5]. Note also that in the notation in [12, Thm 8.5.5], $\mathrm{WF}(F)_X$ is empty. By Theorem 6.1, $\mathrm{WF}'(F)$ consists of those points in the canonical relation $\mathcal{C}$, see (2.5), for which the conormal singularity in (6.7) is not canceled by a zero weight.

Now, let ξ_1 be as in (a). Since ξ_1 is separated by $\pm\xi_0$ by a conic neighborhood, one can choose a weight χ on SM that is constant along the geodesic flow, non-zero at (p_0, θ_0) and supported in a flow-out of a neighborhood $\mathcal{V}$ of it small enough such that the conormals to the corresponding conjugate loci at p_0 stay away from a neighborhood of ξ_1. In the geodesics case, the condition is that the map (4.3) restricted to $\mathcal{V}$, does not intersect a chosen small enough conic neighborhood of $\pm\xi_0$. This can always be done by continuity arguments. Then left projection of $\mathrm{WF}'(F)$ will not be singular at (p_0, ξ_1), and therefore, Ff_2 will have the same property regardless of the singularities of f_2.

Statement (b) follows from (a) by varying v near v_0 in directions normal to ξ_1. □

9.3. **Calculating the principal symbol of (9.9) in case of Riemannian surfaces.** Let exp be the exponential map of g, and let $n \geq 2$. We will take $n = 2$ later. Recall that the leading singularity of the kernel of N near Σ is of the type $(z'_+)^{-1/2}$, by Theorem 6.1. We will compose F with a certain ΨDO R so that this singularity becomes of the type $\delta(z')$. Then modulo lower order terms, $FRf(p)$ will be a weighted Radon transform over the surface $\Sigma(p)$. In 2D, that will be an X-ray type of transform. We are only interested in this composition acting on distributions with wave front sets in a small conic neighborhood $\mathcal{W}$ of $(q_0, \pm\eta_0)$.

The Fourier transform of $(z'_+)^{-1/2}$ is given by (7.1). Its reciprocal is

$$\pi^{-1/2} e^{\mathrm{i}\pi/4} \left(h(\zeta)\zeta^{1/2} - \mathrm{i}h(-\zeta)(-\zeta)^{1/2} \right) = \pi^{-1/2} e^{\mathrm{i}\pi/4} \big(h(\zeta) - \mathrm{i}h(-\zeta) \big) |\zeta|^{1/2},$$

where h is the Heaviside function, and $|\zeta|$ is the norm in $T^*_y M$. We fix p near p_0 and local coordinates $x = x(p)$ there, and we work in semi-geodesic coordinates $y = y(p, q)$ near q_0 normal to $\Sigma(p)$ oriented as in section 6. Let x denote local coordinates near q_0. Let R be a properly supported ΨDO of order $1/2$ with principal symbol, equal to

$$r(y, \eta) = \pi^{-1/2} e^{\mathrm{i}\pi/4} \big(h(\eta_n) - \mathrm{i}h(-\eta_n) \big) |\eta|^{1/2} r_0(y, \eta), \tag{9.10}$$

in $\mathcal{W}$, outside some neighborhood of the zero section, where r_0 is a homogeneous symbol of order 0, an even function of η. Note that

$$|r|^2 = \pi^{-1} |\eta| r_0^2. \tag{9.11}$$

The appearance of the Heaviside function here can be explained by the fact that $N^*\Sigma$ has two connected components: near $(p_0, q_0, -\xi_0, \eta_0)$ and near $(p_0, q_0, \xi_0, -\eta_0)$; and the constants needs to be chosen differently in each component.

We start with computing the composition

$$FR. \tag{9.12}$$

Since the kernel of (9.12) is the transpose of that of RF', we will compute the latter; and we only need those singularities that belong to $\mathcal{W}$. Denote by $F(p, q)$ the Schwartz kernel of F. Then the kernel $F'(q, p) = F(p, q)$ of F' (with the notation convention $F'f(q) = \int F'(q, p) f(p)\, \mathrm{d}\,\mathrm{Vol}(p)$) can be written as $F'(q(x, y), p(x))$ that with some abuse of notation we denote again by $F'(y, x)$. Then

$$F'(y, x) := (2\pi)^{-1} \int e^{\mathrm{i}y^n \eta_n} \tilde{F}'(y', \eta_n, x)\, \mathrm{d}\eta_n, \tag{9.13}$$

where $\tilde{F}'$ is the partial Fourier transform of F w.r.t. y^n, and there is no summation in $y^n \eta_n$. By Theorem 6.1 and (7.1),

$$\tilde{F}'(y', \eta_n, x) = \pi^{1/2} e^{-\mathrm{i}\pi/4} \big(h(\eta_n) + \mathrm{i}h(-\eta_n) \big) |\eta_n|^{-1/2} G(x, y', \eta_n) \tag{9.14}$$

where G is a symbol w.r.t. η_n, smoothly depending on (x, y') with principal part

$$G_0 := W_\Sigma \frac{\sqrt{2}}{\sqrt{AD}}.$$

Moreover, by Theorem 6.1, G has an expansion it terms of positive powers of $|\eta_n|^{-1/2}$. In particular, $G - G_0$ is an amplitude of order $-1/2$ that contributes a conormal distribution in the class $I^{-n/2-1/2}(M \times M, \Sigma; \mathbf{C})$, see, e.g., [13, Thm 18.2.8]. By the calculus of conormal singularities, see e.g., [13, Theorem 18.2.12], the kernel of FR is of conormal type at $y^n = 0$ as well, with a principal symbol given by that of F multiplied by $r|_{y^n=0,\eta'=0}$. That principal symbol coincides with the full one modulo conormal kernels of order 1 less that the former, see the expansions in [13] preceding Theorem 18.2.12. Since we assumed that r_0 is an even homogeneous function of η of order 0, $r_0(y', 0, 0, \eta_n)$ is a function of y' only for η in a conic neighborhood of $(0, \pm 1)$, equal to $r(y, 0, 0, 1)$. Therefore, the principal part of $r(y, D_y)F'(\cdot, x)$ is

$$(2\pi)^{-1} \int e^{\mathrm{i}y^n\eta_n} G_0(x, \eta') r_0(y', 0, 0, 1)\, \mathrm{d}\eta_n = W_\Sigma \frac{\sqrt{2}}{\sqrt{AD}} r_0(y', 0, 0, 1)\delta(y^n), \tag{9.15}$$

and the latter is in $I^{-n/2+1/2}(M \times M, \Sigma; \mathbf{C})$. The "error" is determined by the next term of the principal symbol of the composition FR with G replaced by G_0, that is of order 1 lower and by the contribution of $G = G_0$ that is of order $-1/2$ lower. Since the coordinates (y', y^n) depend on p, as well, $r_0(y', 0, 0, 1)$ is actually the restriction of r_0 to $\mathcal{N}^*\Sigma(p)$. So we proved the following.

Lemma 9.1. *Let r_0 be as in (9.10). Then modulo $I^{-n/2}(M \times M, \Sigma; \mathbf{C})$, $FR \in I^{1/2-n/2}(M \times M, \Sigma; \mathbf{C})$ reduces to the Radon transform*

$$FRf(p) \simeq \int_{\Sigma(p)} af\, dS, \quad a := r_0|_{\mathcal{N}^*\Sigma(p)}\, W_\Sigma \frac{\sqrt{2}}{\sqrt{AD}},$$

where $\mathrm{d}S$ *is the Riemannian surface measure on $\Sigma(p)$ that we previously denoted by* $\mathrm{d}\,\mathrm{Vol}_{\Sigma(p)}$.

In two dimensions, this is an X-ray type of transform. In higher dimensions, this is a Radon type of transform on the family of codimension one surfaces $\Sigma(p)$.

In what follows, $n = 2$.

We will compute RF^*FR next. We have

$$\int FRf\,\overline{FRh}\, \mathrm{d}\,\mathrm{Vol} \simeq \int_M \int_{\Sigma(p)} (af)(z')\, dS(z') \int_{\Sigma(p)} (\bar{a}\bar{h})(q)\, dS(q)\, \mathrm{d}\,\mathrm{Vol}(p) \tag{9.16}$$

modulo terms of the kind (Pf, h), where P is a ΨDO of order $-3/2$ or less.

In the latter integral, p parameterizes the curve $\Sigma(p)$, while $q \in \Sigma(p)$ parameterizes a point on it. Another parameterization is by p and $\xi \in S_p^*M$ with ξ oriented positively; then $q = \exp_p(v)$, where $v \in \Sigma(p)$ and $\xi(v) = 0$. For the Jacobian of that change we have

$$\mathrm{d}S(q)\, \mathrm{d}\,\mathrm{Vol}(p) = D\, \mathrm{d}\,\mathrm{Vol}_{S(p)}(v)\, \mathrm{d}\,\mathrm{Vol}(p) = \frac{|v|D}{\cos\phi}\, \mathrm{d}\sigma_p(\xi)\, \mathrm{d}\,\mathrm{Vol}(p), \tag{9.17}$$

and we recall that $\mathrm{d}\sigma_p$ denotes the surface measure on S_pM, that in this case is a circle. The canonical map $(p, \xi) \to (q, \eta)$ is symplectic, and therefore preserves the volume form $\mathrm{d}p\, \mathrm{d}\xi$. Set

$$K := |\eta(p, \xi)|/|\xi|. \tag{9.18}$$

Then this map takes S^*M into $\{(q, \eta) \in T^*M;\ |\eta| = K\}$. Project that bundle to the unit circle one, and set $\hat{\eta} = \eta/|\eta|$. Then we have the map $(p, \xi) \to (q, \hat{\eta})$, and $\mathrm{d}\,\mathrm{Vol}(p)\, \mathrm{d}\sigma_p(\xi) = K^2 \mathrm{d}\,\mathrm{Vol}(q)\, \mathrm{d}\sigma_q(\hat{\eta})$.

When we perform those changes of variables in (9.16), we will have

$$\mathrm{d}S(q)\, \mathrm{d}\,\mathrm{Vol}(p) = \frac{|w|DK^2}{\cos\phi} \mathrm{d}\,\mathrm{Vol}(q)\, \mathrm{d}\sigma_q(\eta), \tag{9.19}$$

where $p \in M$, $q \in \Sigma(p)$, $(q, \eta) \in S^*M$, and we removed the hat over η. Let w be the corresponding vector in $S(q)$ normal to η. That parameterizes the curves $\Sigma(p)$ over which we integrate by initial points q and unit conormal vectors

η. The latter can be replaced by unit tangent vectors $\hat w = w/|w|$; then $\mathrm{d}\,\mathrm{Vol}(q)\,\mathrm{d}\sigma_q(\eta) = \mathrm{d}\,\mathrm{Vol}(q)\,\mathrm{d}\sigma_q(\hat w)$. Let us denote the so parameterized curves by $c_{q,\hat w}(s)$, where s is an arc-length parameter.

It remains to notice that the integral w.r.t. $z' \in \Sigma(p)$ is an integral w.r.t. the arc-length measure on $\Sigma(p)$, that we denote by s. Then performing the change of the variables $(p,q,z') \mapsto (q,\hat w, z')$ in (9.16), we get

$$\int FRf\,\overline{FRh}\,\mathrm{d}\,\mathrm{Vol} \simeq \int_{\mathbf{R}\times S_qM\times M} (af)(c_{q,\hat w}(s))\bar a(q,-\hat w)\bar h(q)\,\mathrm{d}s\,\frac{|w|DK^2}{\cos\phi}\,\mathrm{d}\sigma_q(\hat w)\,\mathrm{d}\,\mathrm{Vol}(q). \tag{9.20}$$

Therefore, we get as in (5.2), (5.4),

$$\begin{aligned} R^*F^*FRf(q) &\simeq \frac{1}{\sqrt{\det(g(q))}}\int a\bar a\frac{|w|DK^2}{\cos\phi}\frac{f(q')}{\rho(q,q')}\,\mathrm{d}\,\mathrm{Vol}(q')\\ &\simeq \frac{1}{\sqrt{\det(g(q))}}\int \left|r_0|_{\mathcal{N}^*\Sigma(p)}\right|^2|W_\Sigma|^2\frac{2|w|K^2}{A\cos\phi}\frac{f(q')}{\rho(q,q')}\,\mathrm{d}\,\mathrm{Vol}(q'). \end{aligned} \tag{9.21}$$

For the directional derivatives of $\det \mathrm{d}\exp_p(v) = -J'/|v|$, see (8.3), we have that the derivative along the radial ray is $|J'(1)|/|v|$ by absolute value, while the derivative in the direction of $S(p)$ vanishes. That implies

$$A\cos\phi = |J'(1)|/|w| = K/|w|.$$

Therefore,

$$R^*F^*FRf(q) \simeq \frac{1}{\sqrt{\det(g(q))}}\int 2K\left|r_0|_{\mathcal{N}^*\Sigma(p)}\right|^2|W_\Sigma|^2|w|^2\frac{f(q')}{\rho(q,q')}\,\mathrm{d}\,\mathrm{Vol}(q'). \tag{9.22}$$

Here (p,v) is defined as follows. It is the point in SM that lies on the continuation of the geodesic through q, q' to its conjugate point near p_0, The weight κ restricts q' to a small neighborhood of γ_0. Next, A_2 restricts q' near q_0.

We compare (9.22) with (5.4) and (5.5). Notice that the Jacobian term in (5.4) at the diagonal equals $\sqrt{\det g}$ and therefore cancels the factor in front of the integral in the calculation of the principal symbol. We therefore proved the following.

Lemma 9.2. *Let $n=2$. Then R^*F^*FR is a ΨDO of order -1 with principal symbol modulo $S^{-3/2}$ at (q,η) near (q_0,η_0) given by*

$$4\pi K|\eta|^{-1}\left|r_0|_{\mathcal{N}^*\Sigma(p)}\right|^2|\kappa(p,v/|v|)|^2|\kappa(q,-w/|w|)|^2$$

Here $w/|w|$ is a continuous choice of a unit vector normal to η at q, so that $(q,w/|w|) = (q_0,w_0/|w_0|)$ when $(q,\eta)=(q_0,\eta_0)$, and $v/|v|$ is a parallel transport of $-w/|w|$ from q to its conjugate point p along the geodesic $\gamma_{q,w}$.

Later we use the notation $w = \eta^\perp/|\eta^\perp|$, and $v = \xi^\perp/|\xi^\perp|$.

Proposition 9.1. *Let $n=2$. Then*

$$Id - A_2^{-1}F^*A_1^{-1}F$$

is a ΨDO of order $-1/2$.

Proof. We apply Lemma 9.2 with $\pi^{-1/2}e^{\mathrm{i}\pi/4}|\eta|^{1/2}r_0$ being the principal symbol of $A_2^{-1/2}$, see (9.10), where $A_2^{-1/2}$ is a parametrix of $A_2^{1/2}$ near $(q_0,\pm\eta_0)$. To this end, choose

$$\pi^{-1/2}e^{\mathrm{i}\pi/4}(2\pi)^{-1/2}r_0(q,\eta) = (2\pi)^{-1/2}|\kappa(q,\eta^\perp/|\eta^\perp|)|^{-1},$$

see (8.1). Note that $\kappa(q,w/|w|) = \kappa(p,-v/|v|) = 0$ because of the assumption on $\operatorname{supp}\kappa$. Then $\left|r_0|_{\mathcal{N}^*\Sigma(p)}\right| = 2^{-1/2}|\kappa(q,-w/|w|)|^{-1}$, where w is as in (3.1). The choice of r_0 yields $RR^* = A_2^{-1/2} \mod \Psi^{-1}$. So Lemma 9.2 implies that R^*F^*FR, and therefore RR^*F^*F and $A_2^{-1}F^*F$, have principal symbol

$$\sigma_{\mathrm{p}}(A_2^{-1}F^*F)(q,\eta) = 2\pi K|\kappa(p,\xi^\perp/|\xi^\perp|)|^2/|\eta|$$

We only need to insert A_1^{-1} between F^* and F. By [14, Thm 25.3.5], modulo ΨDOs of order 1 lower, the principal symbol of $A_2^{-1}F^*A_1^{-1}F$ is given by that of $A_2^{-1}F^*F$ multiplied by the principal symbol $\left(2\pi|\kappa(p,v)|^2/|\xi|\right)^{-1}$ of A_1^{-1} pushed forward by the canonical map of F. In other words,

$$\sigma_p(A_2^{-1}F^*A_1^{-1}F)(q,\eta) = \frac{2\pi|\kappa(p,\xi^\perp/|\xi^\perp|)|^2}{|\eta|}K\left[2\pi|\kappa((p,\xi^\perp/|\xi^\perp|)|^2/|\xi(q,\eta)|)\right]^{-1} = 1.$$

□

The following lemma is needed below for the proof of Theorem 9.2.

Lemma 9.3. *Let κ_1 and κ both satisfy the assumptions for κ in the Introduction, and let $\kappa(p_0,\theta_0)\neq 0$. Let $\chi\in\Psi^0$ have essential support near $(p_0,\pm\xi_0)\cup(q_0,\pm\eta_0)$ and Schwartz kernel in $(U_1\times U_1)\cup(U_2\times U_2)$. Then there exists a zero order classical ΨDO Q with the same support properties so that*

$$QX_\kappa^*X_\kappa\chi = X_{\kappa_1}^*X_\kappa\chi, \quad \textit{mod } I^{-3/2}(M\times M,\ \Delta\cup\mathcal{N}^*\Sigma,\ \mathbf{C}),$$

where Δ is the diagonal. In particular, $QX_\kappa^*X_\kappa\chi - X_{\kappa_1}^*X_\kappa\chi : H^s\to H^{s+3/2}$ is bounded for any s.

Proof. We define $Q = Q_1+Q_2$ where $Q_{1,2}$ have Schwartz kernels in $U_1\times U_1$ and $U_2\times U_2$, respectively. Following the notation convention in (9.8), $Q = \operatorname{diag}(Q_1,Q_2)$.

Then we choose Q_1 to have principal symbol

$$\bar\kappa_1(p,\xi^\perp/|\xi^\perp|)/\bar\kappa(p,\xi^\perp/|\xi^\perp|) \tag{9.23}$$

in a conic neighborhood of $(p_0,\pm\xi_0)$ with the same choice of $\xi^\perp$ as in (8.1). Next, we choose Q_2 with a principal symbol

$$\bar\kappa_1(q,\eta^\perp/|\eta^\perp|)/\bar\kappa(q,\eta^\perp/|\eta^\perp|) \tag{9.24}$$

in a conic neighborhood of $(q_0,\pm\eta_0)$. Then

$$QX_\kappa^*X_\kappa = \begin{pmatrix} Q_1A_1 & Q_1F\\ Q_2F^* & Q_2A_2\end{pmatrix}.$$

Then, see (8.1),

$$\sigma_{\mathrm{p}}(Q_1A_1) = 2\pi(\bar\kappa_1\kappa)(p,\xi^\perp/|\xi^\perp|),\quad \sigma_{\mathrm{p}}(Q_2A_2) = 2\pi(\bar\kappa_1\kappa)(q,\eta^\perp/|\eta^\perp|).$$

For Q_1F, Q_2F^*, we use the arguments used in the proof of Lemma 9.1. A representation of the Schwartz kernel of F' as a conormal distribution is given by (9.13). The composition Q_2F^* then is of the same conormal type with a principal symbol equal to the complex conjugate of that of F' multiplied by the symbol (9.24) restricted to $\mathcal{N}^*\Sigma$. This replaces $\kappa^\sharp=\bar\kappa$ in (6.6) by $\bar\kappa_1$. Since in (6.6), $\kappa^\sharp=\bar\kappa$ we get that Q_2F^* is of the same conormal type with leading singularity as in Theorem 6.1, with

$$W_\Sigma = |v|^{-1}\bar\kappa(p,v/|v|)\kappa_1(q,-w/|w|).$$

This is however the leading singularity of $\chi_2X_{\kappa_1}^*X_\kappa\chi_1$.

The proof for Q_1F is the same with the roles of p and q replaced. □

9.4. **Cancellation of singularities on Riemannian surfaces.** Assume in all dimensions that there are no conjugate points on the geodesics in M, and that ∂M is strictly convex. Let $M_1\supset M$ be an extension of M so that the interior of M_1 contains M be as in Remark 5.2. Then if $\kappa\neq 0$,

$$\|f\|_{L^2(M)} \le C\|X^*Xf\|_{H^1(M_1)} + C_k\|f\|_{H^{-k}(M)},\quad \forall f\in L^2(M), \tag{9.25}$$

for all $k\ge 0$, see [23, 9], and [25] for a class of manifolds with conjugate points. When we know that X is injective, for example when the weight is constant; then we can remove the H^{-k} term. The same arguments there show that for any $s\ge 0$,

$$\|f\|_{H^s(M)} \le C\|X^*Xf\|_{H^{s+1}(M_1)} + C_k\|f\|_{H^{-k}(M)},\quad \forall f\in H_0^s(M). \tag{9.26}$$

Consider Xf parameterized by points in ∂_+SM_1, that defines Sobolev spaces for Xf as in section 5.1. Then

$$\|f\|_{H^s(M)} \le C\|Xf\|_{H^{s+1/2}(\partial_+SM_1)} + C_k\|f\|_{H^{-k}(M)},\quad \forall f\in H_0^s(M),\ s\ge 0. \tag{9.27}$$

Indeed, in Proposition 5.2, one can complete M_1 and $\mathcal{H}$ to closed manifolds, and then we would get that $X^*: H^s \to H^{s+1/2}$ is bounded. Then (9.27) follows by (9.26). Estimate (9.27) is sharp in view of Proposition 5.2. In the following theorem, we show that (9.25), (9.27) fail in the 2D case, with a loss at least of one derivative in the first one, and $1/2$ derivative in the second one.

Theorem 9.2. *Let $n = 2$, and let γ_0 be a geodesic of g with conjugate points satisfying the assumptions in section 2. Then for each $f_2 \in H^s(M)$, $s \ge 0$, with* WF(f_2) *in a small neighborhood of $(q_0, \pm\eta^0)$, there exists $f_1 \in H^s(M)$ with* WF(f_1) *in a some neighborhood of $(p_0, \pm\xi^0)$ so that*

$$Xf \in H^{s+3/4} \quad \text{and} \quad X^*Xf \in H^{s+3/2}, \quad \text{where } f := f_1 + f_2.$$

In particular, if (M, g) is a non-trapping Riemannian surface with boundary with fold type of conjugate points on some geodesics, neither of the inequalities (9.25), (9.27) can hold.

Remark 9.1. It is an open problem whether we can replace $H^{s+3/4}$ and $H^{s+3/2}$ above with C^∞. See Section 10.1 for an example where this can be done.

Remark 9.2. If there are no conjugate points, one has $Xf \in H^{s+1/2}$, $X^*Xf \in H^{s+1}$. Therefore, the conjugate points are responsible for an $1/4$ derivative smoothing for Xf, and an $1/2$ derivative smoothing for X^*Xf

Proof. Let f_2 be as in the theorem. Set

$$f_1 = -A_1^{-1}Ff_2,$$

where, as before, A_1^{-1}, A_2^{-1} are parametrices of $A_{1,2}$ in conic neighborhoods of $(p_0, \pm\xi_0)$ and $(q_0, \pm\eta_0)$, respectively. Then f_1 belongs to H^s and has a wave front set in small neighborhood of $(p_0\pm, \xi_0)$, by Theorem 2.1. By construction and by (9.4),

$$\chi_1 X^*Xf \in C^\infty. \tag{9.28}$$

Next, by (9.28),

$$A_2f_2 + F^*f_1 = A_2f_2 - F^*A_1^{-1}Ff_2 = (A_2 - F^*A_1^{-1}F)f_2.$$

The operator in the parentheses is a ΨDO of order $-3/2$ by Proposition 9.1. Therefore, see (9.5),

$$\chi_2X^*Xf = A_2f_2 + F^*f_1 \in H^{s+3/2}.$$

We therefore get $X^*Xf \in H^{s+3/2}(U_1 \cup U_2)$.

To prove $Xf \in H^{s+3/4}$, note first that above we actually proved that

$$X^*X(\mathrm{Id} - A_1^{-1}F)\chi : H^s(U_2) \longrightarrow H^{s+3/2}(U_1 \cup U_2) \tag{9.29}$$

is bounded, being a ΨDO of order $-3/2$, where χ denotes a zero order ΨDO with essential support in a small neighborhood of $(p_0, \pm\eta_0)$ and Schwartz kernel supported in $U_2 \times U_2$.

Our goal is to show that

$$X(\mathrm{Id} - A_1^{-1}F)\chi : H^s(U_2) \longrightarrow H_0^{s+3/4}(\mathcal{H})$$

is bounded. It is enough to prove that

$$\chi^*(\mathrm{Id} - A_1^{-1}F)^*X^*P_{2s+3/2}X(\mathrm{Id} - A_1^{-1}F)\chi : H^s(U_2) \longrightarrow H^{-s}(U_2) \tag{9.30}$$

for any ΨDO $P_{2s+3/2}$ of order $2s + 3/2$ on $\mathcal{H}$. All adjoints here are in the corresponding L^2 spaces. By (9.29),

$$Q_{2s+3/2}X^*X(\mathrm{Id} - A_1^{-1}F)\chi : H^s(U_2) \longrightarrow H^{-s}(U_2) \tag{9.31}$$

is bounded for any ΨDO $Q_{2s+3/2}$ of order $2s + 3/2$.

To deduce (9.30) from (9.31), it is enough to "commute" X^* with $P_{2s+3/2}$ in (9.30). Let $2s+3/2$ be a non-negative integer first. As in the proof of Proposition 5.2, we use the fact that $X^*P_{2s+3/2} = (P^*_{2s+3/2}X)^*$, and $P^*_{2s+2}Xf$ is a finite sum of X-ray transforms with various weights of derivatives of f of order not exceeding $2s + 2$. Thus we can write

$$X^*P_{2s+2} = \sum \tilde{Q}_jX_j^*, \tag{9.32}$$

where Q_j are differential operators on $\mathcal{H}$ of degree $2s + 3/2$ or less, and X_j are like X in (2.1) but with different weights still supported where κ is supported. By Lemma 9.3, $\tilde{Q}_jX_j^*X = R_jX^*X$, where R_j is a ΨDO of the same order as $\tilde{Q}_j$. The proof of (9.30) is then completed by the observation that $\chi^*(\mathrm{Id} - A_1^{-1}F)^*$ maps continuously H^s into itself, since the canonical relation of F is canonical graph. □

10. EXAMPLES

In this section, we present a few examples. We start in Section 10.1 with the fixed radius circular transform in the plane, where we can have cancellation of singularities similarly to Theorem 9.2 but we show that this happens to any order. Then we consider in Section 10.2 the geodesic X-ray transform on the sphere, where the conjugacy is not of fold type, but a similar result holds. Next, in Section 10.3, we study an example of magnetic geodesics in the Euclidean space $\mathbf{R}^3$ with a constant magnetic field. We show that then the canonical relation of F is a canonical graph, and therefore, one can resolve the singularities. Finally, in Section 10.4, we present an example of a Riemannian manifold of product type where the graph condition is violated.

10.1. **The fixed radius circular transform in the plane.** Let R be the integral transform in $\mathbf{R}^2$ of integrating functions over circles of radius 1. We fix the negative orientation on those circles; then for each $(x,\xi) \in S\mathbf{R}^2$, there is a unique unit circle passing through x in the direction of θ. It is very easy to see, see below, that the first conjugate point appears at "time" π. The next one is at 2π, that equals the period of the curve. If one originally chooses f supported near, say $(0,0)$ and $(2,0)$; and chooses γ_0 to be the arc of the circle that is a small extension of $\{|x_1-1|^2 + x_2^2 = 1,\ x_2 \geq 0\}$, then we are in the situation studied above. On the other hand, if we do not impose any assumptions on supp f, we will get contributions that are smoothing operators only. Therefore, we do not need to restrict supp f.

Those circles are also magnetic geodesics w.r.t. the Euclidean metric and a constant non-zero magnetic field, see e.g., [5]. Let us use the following parameterization first. We temporarily denote vectors θ by $\overrightarrow{\theta} := (\sin\theta, \cos\theta)$ to reserve θ for their (non-standard) polar angles. The circle through x in the direction of $\overrightarrow{\theta}$ is given by

$$\gamma_{x,\theta}(t) = x + (\cos\theta - \cos(\theta+t), -\sin\theta + \sin(\theta+t)). \tag{10.1}$$

Then $\gamma_{x,\theta}(0) = x$, $\dot\gamma_{x,\theta}(0) = \overrightarrow{\theta}$. Let J_1 be the Jacobi matrix $\partial\gamma_{x,\theta}(t)/\partial(t,\theta)$. We have

$$J_1 = \begin{pmatrix} \sin(\theta+t) & -\sin\theta + \sin(\theta+t) \\ \cos(\theta+t) & -\cos\theta + \cos(\theta+t) \end{pmatrix}. \tag{10.2}$$

Then $\det J_1 = -\sin(\theta+t)\cos\theta + \sin\theta\cos(\theta+t) = -\sin t$. It vanishes when $t=\pi$ (see the remarks above why the other zeros do not matter). Therefore, in the (t,θ) coordinates, the tangent conjugate locus $S(x)$ is given by $\{t=\pi\}$, for any x. The conjugate locus of x then is the circle $\Sigma(x) = \{\gamma_{x,\theta}(\pi)\} = \{x + 2(\cos\theta, -\sin\theta);\ \theta\in\mathbf{R}\}$, i.e.,

$$\Sigma(x) = \{y;\ |y-x| = 2\}$$

that is the envelope of all circles of radius 1 passing through x, see Figure 5. Next,

$$J_1|_{t=\pi} = \begin{pmatrix} -\sin\theta & -2\sin\theta \\ -\cos\theta & -2\cos\theta \end{pmatrix}. \tag{10.3}$$

The null-space consist of multiples of $2\partial/\partial t - \partial/\partial\theta$. That null-space is transversal to $\{t=\pi\}$, therefore, we have a fold conjugate locus.

To write this in the Cartesian coordinates $x = (x^1, x^2)$, set

$$v = t(\sin\theta, \cos\theta),$$

i.e., $v = t\overrightarrow{\theta}$. Set also $\exp_x(v) = \gamma_{x,\theta}(t)$, i.e., the endpoint of the magnetic geodesic originating at x in the direction $v/|v|$, of length $|v|$. Then

$$S(x) = \{v;\ |v| = \pi\}.$$

We compute next $N_x(v)$ for $v = (0,\pi)$. By the rotational symmetry, this would determine $N_x(v)$ for any $v\in S_x(v)$ in a trivial way. For the Jacobi matrix $J_2 := \partial v/\partial(t,\theta)$ we get

$$J_2 = \begin{pmatrix} \sin\theta & t\cos\theta \\ \cos\theta & -t\sin\theta \end{pmatrix}. \tag{10.4}$$

To find the Jacobi matrix $J := \partial \exp_x(v)/\partial v = \partial\gamma_{x,\theta}(t)/\partial v$ at $v = (0, \pi)$, we write $J = J_1 J_2^{-1}$ at $\theta = 0$, $t = \pi$, to get

$$J|_{v=(0,\pi)} = \begin{pmatrix} 0 & 0 \\ -2/\pi & -1 \end{pmatrix}. \tag{10.5}$$

The null space is spanned by $(-1, 2/\pi)$. For general θ it follows immediately that

$$N_x(v) = \mathbf{R}e^{-\mathrm{i}\theta}(-1, 2/\pi),$$

where we used complex identification to denote rotation by the angle $-\theta$. We could have obtained this as $J = J_1 J_2^{-1}$ for $t = \pi$, and general θ's, of course. In particular, for $\theta = 0$, i.e., for $v = (0, \pi)$, we get $N_x(v) = \mathbf{R}(-1/2, \pi)$. We see again that S is a fold conjugate locus. The other assumptions of the dynamical system are easy to check.

It is much more natural to parametrize those circles by their centers, we use the notation $C(x)$. Then the circular integral transform is defined by

$$Xf(y) = \int_{C(y)} f\,\mathrm{d}\ell = \int_{|\omega|=1} f(y+\omega)\,\mathrm{d}\ell_\omega = \int_0^{2\pi} f(z+e^{\mathrm{i}\alpha})\,\mathrm{d}\alpha. \tag{10.6}$$

The connection to the natural parametrization by x and θ that we used above is as follows. As in [5], for all circles in neighborhood of a given one, for example the one with $x = 0$ and $\theta = 0$, we choose a curve S through $x = 0$, transversal to that circle. Let z be the point of intersection of those circles with S, close to 0. Then we use z and θ as parameters, and the natural measure is $\mathrm{d}\mu = |\theta \cdot \nu(z)|\mathrm{d}\ell_z\,\mathrm{d}\theta$, where $\mathrm{d}\ell_z$ is the Euclidean length measure on S, $\nu(z)$ is the unit normal at z. This measure has the property to be independent of the choice of S. Choose $S = \{x^2 = 0\}$. Then the natural measure on those circles is $\mathrm{d}\mu = \cos\theta\,\mathrm{d}z^1\,\mathrm{d}\theta$, near $z^1 = 0$, $\theta = 0$. The center of each such circle is given by $y := (z^1 + \cos\theta, -\sin\theta)$, see (10.1). Using y as a new parameter, and computing the Jacobian of the map $(z^1, \theta) \mapsto y$, we see that $\mathrm{d}\mu = \mathrm{d}y$ in the new variables. Therefore, with the parameterization by its center as in (10.6), X is unitarily equivalent to its previous definition, and if we define X^* w.r.t. the inner product $L^2(\mathbf{R}^2, \mathrm{d}y)$, X^*X will not change.

10.1.1. *X as a convolution.* It is well known and easy to see that X is a convolution with the delta function δ_{S^1} of the unit circle

$$Xf = \delta_{S^1} * f.$$

Fourier transforming, we get

$$X = 2\pi\mathcal{F}^{-1}J_0(|\xi|)\mathcal{F}, \tag{10.7}$$

where J_0 is the Bessel function of order 0. This shows that

$$X^*X = (2\pi)^2\mathcal{F}^{-1}J_0^2(|\xi|)\mathcal{F}. \tag{10.8}$$

Note that $J_0^2(|\xi|)$ is not a symbol because it oscillates. In principle, one can use this representation to analyze X^*X but this is not so convenient when we want to analyze X locally.

10.1.2. *Integral representation.* We write

$$\begin{aligned}(Xf, Xh) &= \int\int_{|\omega|=1} f(x+\omega)\,\mathrm{d}\ell_\omega \int_{|\theta|=1} \bar{h}(x+\theta)\,\mathrm{d}\ell_\theta\,dx \\ &= \int\int_{|\omega|=1}\int_{|\theta|=1} f(y+\omega-\theta)\bar{h}(y)\,\mathrm{d}\ell_\omega\,\mathrm{d}\ell_\theta\,\mathrm{d}y.\end{aligned} \tag{10.9}$$

Therefore,

$$X^*Xf(x) = \int_{|\omega|=1}\int_{|\theta|=1} f(x+\omega+\theta)\,\mathrm{d}\ell_\omega\,\mathrm{d}\ell_\theta, \tag{10.10}$$

compare with (5.1).

We will make the change of variables $z = \omega + \theta$. For $0 < |z| < 2$, there are exactly two ways z can be represented this way. Write $\omega = e^{i\alpha}$, $\theta = e^{i\beta}$. Since $d\ell_\omega = d\alpha$, $d\ell_\theta = d\beta$, and $dz_1 \wedge dz_2 = (-2i)^{-1} dz \wedge d\bar{z}$, we get

$$dz_1 \wedge dz_2 = \frac{1}{-2i}\left(ie^{i\alpha}d\alpha + ie^{i\beta}d\beta\right) \wedge \left(-ie^{-i\alpha}d\alpha - ie^{-i\beta}d\beta\right) = \sin(\beta-\alpha)\, d\alpha \wedge d\beta$$
$$= \sin(\beta-\alpha)\, d\ell_\omega \wedge d\ell_\theta.$$

It is easy to see that $|\beta - \alpha|$ equals twice the angle between $z = \omega + \theta$ and θ. Let $r = |z|$. Then $r/2 = \cos\frac{|\alpha-\beta|}{2}$. Elementary calculations then lead to

$$\sin|\alpha - \beta| = \frac{r}{2}\sqrt{4-r^2}.$$

Therefore, (10.10) yields the following.

Proposition 10.1. *Let X be the circular transform defined above. Then*

$$X^*Xf(x) = \int_{r<2} \frac{4}{r\sqrt{4-r^2}} f(y)\, dy, \quad r := |x-y|. \tag{10.11}$$

10.1.3. *X^*X as an FIO.* The kernel has singularities near the diagonal $x = y$, and also near

$$\Sigma = \{|x-y| = 2\}.$$

That singularity is of the type $(2-|x-y|)^{-1/2}$, and for a fixed x the expression $2-|x-y|$ measures the distance from the circle $\Sigma(x)$ to the point y inside that circle. We therefore get the same singularity as in Theorem 6.1. Note also that

$$\mathcal{N}^*\Sigma = \{(x, x \pm 2\xi/|\xi|, \xi, -\xi);\ \xi \in \mathbf{R}^2 \setminus 0\}. \tag{10.12}$$

Based on Proposition 10.1, and Theorem 2.1, we conclude that X^*X is an FIO of order -1 with a canonical relation $\mathcal{C}$ of the following type. We have that $(x, \xi, y, \eta) \in \mathcal{C}$ if and only if $(y, \eta) = (x, \xi)$ (that gives us the ΨDO part), or $(y, \eta) = (x \pm 2\xi/|\xi|, \xi)$.

This can also be formulated also in the following form.

Theorem 10.1. *Let X be the circular transform defined above. Then, modulo $\Psi^{-\infty}$,*

$$X^*X = A_0 + F_+ + F_-, \tag{10.13}$$

where A_0, $F+$ and F_- are Fourier multipliers with the properties

(a) $A_0 = 4\pi|D|^{-1}$ mod Ψ^{-2};

(b) $F_\pm$ are elliptic FIOs of order -1 with canonical relations of a graph type given by

$$\mathcal{F}_\pm : (x, \xi) \mapsto (x \pm 2\xi/|\xi|, \xi). \tag{10.14}$$

(c) $F_- = F_+^*$.

Proof. We start with the Fourier multiplier representation (10.7). The leading term of $(2\pi)^2 J_0^2(|\xi|)$ is

$$\frac{8\pi}{|\xi|}\cos^2(|\xi| - \pi/4) = \frac{8\pi}{|\xi|}(1 + \sin(2|\xi|)) = 2\pi\left(\frac{2}{|\xi|} + \frac{e^{2i|\xi|}}{i|\xi|} - \frac{e^{-2i|\xi|}}{i|\xi|}\right). \tag{10.15}$$

Those three terms are the principal parts of the operators in (10.13). The first one gives $4\pi|D|^{-1}$, while the second and the third one are FIOs with phase functions $\phi_\pm = (x-y)\cdot\xi \pm 2|\xi|$. A direct calculation show that the canonical relations of $F_\pm$ are given by (10.14), indeed. For the complete proof of the theorem, we need the full asymptotic expansion of J_0.

We recall the well known expansion of $J_0(z)$ for $z \to \infty$:

$$J_0(z) \sim \sqrt{2/(\pi z)}\left(P(z)\cos(z - \pi/4) - Q(z)\sin(z - \pi/4)\right),$$

where

$$P(z) \sim \sum_{k=0}^{\infty} p_k z^{-2k}, \quad Q(z) \sim \sum_{k=0}^{\infty} q_k z^{-2k-1},$$

with some (explicit) coefficients p_k, q_k. In particular, $p_1 = 1, q_1 = -1/8$. Then

$$(2\pi)^2 J_0^2(z) \sim \frac{2\pi}{z}\left((P+\mathrm{i}Q)e^{\mathrm{i}(z-\pi/4)} + (P-\mathrm{i}Q)e^{-\mathrm{i}(z-\pi/4)}\right)^2$$
$$\sim \frac{2\pi}{z}\left(-\mathrm{i}(P+\mathrm{i}Q)^2 e^{2\mathrm{i}z} + \mathrm{i}(P-\mathrm{i}Q)^2 e^{-2\mathrm{i}z} + 2P^2 + 2Q^2\right).$$

We set

$$A_0 = 4\pi|D|^{-1}\left(P^2(|D|) + Q^2(|D|)\right), \quad F_\pm = \mp 2\pi\mathrm{i}|D|^{-1}\big(P(|D|) \pm \mathrm{i}Q(|D|)\big)^2 e^{\pm 2\mathrm{i}|D|}. \tag{10.16}$$

This completes the proof. □

We will now connect this to Theorem 2.1. Let $p_0 = (0,0)$, $q_0 = (2,0)$, $v_0 = (0,\pi)$, $w_0 = (0,\pi)$. Then $v_0 \in S(p_0)$. Choose $\xi_0 = (1,0)$, conormal to the conjugate locus $\Sigma(q_0) = \{|x - q_0| = 2\}$ at p_0; and choose $\eta_0 = (1,0)$, conormal to the conjugate locus $\Sigma(p_0) = \{|x-p_0| = 2\}$ at q_0. The directions of ξ_0, η_0 reflects the choice of the orientation we made earlier. We refer to Figure 5.

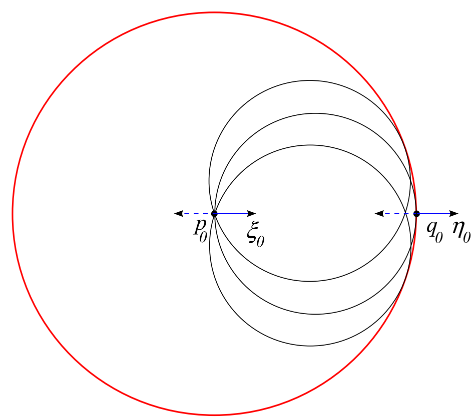

FIGURE 5

If we localize X near $v = v_0$, then the pseudo-differential part of $X^*\chi X$ is $(1/2)A_0$, see (5.10). Therefore, in the notation of Theorem 2.1,

$$A = \frac{1}{2}A_0, \quad F = F_+ + F_-.$$

The canonical relation of F_+ maps (p_0, ξ_0) into (q_0, η_0), see Figure 5, while that of F_- maps $(p_0, -\xi_0)$ into $(q_0, -\eta_0)$. This is consistent with the results in Theorem 2.1, where the Lagrangian has two disconnected components located near $(p_0, q_0, \pm\xi_0, \mp\eta_0)$.

To analyze the operator (9.9), note first that $A_1 = A_2 = A_0/2$. Let us first analyze this operator applied to distributions with wave front set near (q_0, η_0) but not near $(q_0, -\eta_0)$. Then F reduces to F_+ only, and we have, modulo $\Psi^{-\infty}$,

$$A_2^{-1}F^*A_1^{-1}F = \frac{1}{4}A^{-2}F_+^*F_+ = \mathrm{Id},$$

see (10.16). The analysis near $(q_0, -\eta_0)$ is similar. Therefore, we have a stronger version of Theorem 9.2 in this case: singularities can cancel to any order.

Theorem 10.2. *Let f_1 be any distribution with* WF(f_1) *supported in a small conic neighborhood of some $(x_0, \xi^0) \in T^*\mathbf{R}^2 \setminus 0$. Then there exists a distribution f_2 with* WF(f_2) *supported in a small conic neighborhood of $(x_0 \pm 2\xi^0/|\xi^0|, \xi^0)$, that is an image of* WF(f_1) *under the map $\mathcal{F}_\pm$, so that $X(f_1 + f_2) \in C^\infty$ for all unit circles in a neighborhood of the unit circle $C(x_0 \pm \xi^0)$.*

In other words, for a fixed circle C_0 of radius 1, there is a rich set of distributions f, with any order of singularity at $\mathcal{N}^*C_0$, so that those singularities are invisible by X localized near C_0, i.e., $Xf \in C^\infty$. Explicit examples can be constructed by choosing $f_2(x) = \delta(x - q_0)$, then Ff_2 near p_0 is just given by the Schwartz kernel of X^*X, see (10.11). To obtain f_1, we apply $2A_0^{-1}$ to the result.

We would like to emphasize on the fact that the theorem provides an example of cancellation of singularities for the localized transform only. As we will see below, $Xf \in C^\infty$ (globally) for $f \in \mathcal{E}'$ implies $f \in C^\infty$. On the other hand, without the compact support assumption, one can construct singular distributions in the kernel of X, using the Fourier transform.

10.1.4. *The wave front set of a distribution in* Ker X. Now, if $Xf = 0$ or more generally, if $Xf \in C^\infty$, one easily gets that

$$\forall f \in \operatorname{Ker} X, \operatorname{WF}(f) \text{ is invariant under the action of the group } \{\mathcal{F}_+^m,\ m \in \mathbf{Z}\}. \tag{10.17}$$

Then, if f is compactly supported (or more generally, smooth outside some compact set), we get that WF(f) must be empty, i.e., $f \in C^\infty(\mathbf{R}^2)$. In other words, even though recovery of WF(f) is impossible by knowing Xf locally, as we saw above; the condition $Xf \in C^\infty$ globally, together with the compact support assumption yielded a global recovery of singularities. Here an important role is played by the fact that X is translation invariant, and in particular, our assumptions are valid for any $(p_0, \theta_0) \in TS\mathbf{R}^2$ that cannot be guaranteed in the general case. Also, the dynamics is not time reversible; therefore for each $(x_0, \xi^0) \in T^*M \setminus 0$ there are two different curves through x_0 in our family. The latter is true for general magnetic systems with a non-zero magnetic field, see [5].

Remark 10.1. One can see that X is invertible on $L^2(M)$ by using Fourier transform, see (10.7). The formal inverse is $1/J_0(|\xi|)$, and conjugating a compactly supported χ with the Fourier transform, one gets a convolution in the ξ variable that will smoothen out the zeros of $J_0(|\xi|)$, thus producing a Fourier multiplier with asymptotic $\sim |\xi|^{1/2}$. In $L^p(\mathbf{R}^2)$ with $p > 4$ however it is not invertible, and elements of the kernel include functions with Fourier transforms supported on the circles $J_0(|\xi|) = 0$, see also [28, 1].

Finally, we remark that in this case, one can study X directly, instead of $X^*X = X^2$, with the same methods. Our goal however is to connect the analysis of this transform with our general results.

10.2. **The X-ray transform on the sphere.** Consider the geodesic ray transform on the sphere S^n. The conjugate points are not of fold type, instead they are of blow-down type. Let J be the antipodal map.

Without going into details, we will just mention that then (2.3) still holds with

$$CN = |D|^{-1} - |D|^{-1}J,$$

with some constant C, where the canonical relation of F is the graph of the antipodal map, lifted to T^*S^2. Then $CN|D| = \mathrm{Id} - J$. The canonical graph is an involution, however (its square is identity), so arguments similar to that in the previous example do not apply. That means that singularities may cancel. In fact, it is known that X has an infinite dimensional kernel — all odd functions with respect to J.

In this case Σ consists of all antipodal pairs (x, y), and has dimension 2 (and codimension 2), unlike the case above (dimension 3 and codimension 1). On the other hand, $\mathcal{N}^*\Sigma$ still has the same dimension (that is 2n=4, and this is always the case as long as Σ is smooth submanifold). One can see that the Lagrangian in this case is still $\mathcal{N}^*\Sigma$.

10.3. **Magnetic geodesics in $\mathbf{R}^3$.** Consider the magnetic geodesic system in the Euclidean space $\mathbf{R}^3$ with a constant magnetic potential $(0, 0, \alpha)$, $\alpha > 0$. The geodesic equation is then given by

$$\ddot{\gamma} = \dot{\gamma} \times (0, 0, \alpha), \tag{10.18}$$

where $\times$ denotes the vector product in $\mathbf{R}^3$. The r.h.s. above is the Lorentz force that is always normal to the trajectory and in particular does not affect the speed. We restrict the trajectories on the energy level 1 that is preserved under the flow. Then we get

$$\ddot{\gamma}^1 = \alpha\dot{\gamma}^2, \quad \ddot{\gamma}^2 = -\alpha\dot{\gamma}^1, \quad \ddot{\gamma}^3 = 0.$$

The magnetic geodesics are then given by

$$\gamma(t) = \gamma(0) + \Big(\frac{r}{\alpha}(\sin(\alpha t + \theta) - \sin\theta), \frac{r}{\alpha}(-\cos(\alpha t + \theta) + \cos\theta), tz\Big),$$

where (r,θ,z) are the cylindrical coordinates of $\dot\gamma(0)$. The unit speed requirement means that

$$r^2+z^2=1.$$

The geodesics are then spirals; when $z=0$ then they reduce to closed circles, and when $r=0$ they are vertical lines.

The parameterization by cylindrical coordinates is singular when $r=0$. Away from that we can use θ, z to parametrize unit speeds. Then in $\exp_p(v)$, we use the coordinates (t,θ,z) to parametrize v, i.e.,

$$v=t\Big(\sqrt{1-z^2}(\cos\theta,\sin\theta),z\Big).$$

At $t=0$ we may have additional singularity but this is irrelevant for our analysis since we know that the exponential map has an injective differential near $v=0$. An easy computation yields that the conjugate locus is given by the condition $\alpha t=\pi$, i.e.,

$$S_p(v)=\{v;\ |v|=\pi/\alpha\},$$

and this is true for any $p\in\mathbf{R}^3$. This is a sphere in $T\mathbf{R}^3$. For $\Sigma(p)$ we then get

$$\gamma(\pi/\alpha)=p+\alpha^{-1}(-2r\sin\theta,2r\cos\theta,\pi z) \tag{10.19}$$

with $p=\gamma(0)$. This shows that $\Sigma(p)$ is an ellipsoid

$$\Sigma=\left\{(p,q);\ \frac14(q_1-p_1)^2+\frac14(q_2-p_2)^2+\frac{1}{\pi^2}(q_3-p_3)^2=\alpha^{-2}\right\}.$$

Then

$$\mathcal{N}^*\Sigma=\left\{(p,q,\xi,\eta);\ (p,q)\in\Sigma;\ \xi=c\Big(p_1-q_1,p_2-q_2,\frac{4}{\pi^2}(p_3-q_3)\Big),\ \eta=-\xi,\ 0\neq c\in\mathbf{R}\right\}. \tag{10.20}$$

Therefore, given p,ξ, we can immediately get q as a smooth function of (p,ξ), and we can obtain v so that $\exp_p(v)=q$ by (10.19), where the l.h.s. is q. Therefore, $(p,\xi)\mapsto v$ is a smooth map, and therefore $(p,\xi)\mapsto(q,\eta)$ is a smooth map, too. The later also directly follows from (10.20), since $\eta=-\xi$.

We therefore get that F is an FIO of order $-3/2$ with a canonical relation

$$(p,\xi)\mapsto(q,\xi), \tag{10.21}$$

where q can be determined as described above. A geometric description of q is the following: q is one of the two points on the ellipsoid Σ, where the normal is given by ξ. The choice of one out of the two points is determined by the choice of the initial velocity v_0 near which we localize; changing v_0 to $-v_0$ would alter that choice. Since (10.21) is a diffeomorphism, F is of canonical graph type, and therefore maps H^s to $H^{s+3/2}$. In contrast, $A_{1,2}$ are elliptic of order -1, thus they dominate over F. By Corollary 9.1, X can be inverted microlocally in the setup described in Section 2.

10.4. **Fold caustics on product manifolds.** Let $(M,g)=(M',g')\times(M'',g'')$ be a product of two Riemannian manifolds. The geodesics on M then have the form

$$\gamma_{p,v}(t)=(\gamma'_{p',v'}(t),\gamma''_{p'',v''}(t)).$$

Consequently,

$$\exp_p(v)=(\exp'_{p'}(v'),\exp''_{p''}(v'')).$$

Assume that in (M',g'), v'_0 is conjugate at p_0 of fold type, and assume that v''_0 is not conjugate at p''_0 in (M'',g''). Then

$$\mathrm{d}\exp_p(v)=\mathrm{diag}(\mathrm{d}\exp'_{p'}(v'),\mathrm{d}\exp_{p''}(v'')).$$

The kernel of $\mathrm{d}\exp_p(v)$ then consists of $N_p(v)=N_{p'}(v')\times 0$. Next, $S(p)=S(p')\times T_{p''}M''$, and $\Sigma(p)=\Sigma'(p')\times M''$. Then $N_p(v_0)$ is transversal to $S(p)$ at $v=v_0$, therefore (v',v'') is a fold conjugate vector for $v'\in S'(p)$ close to v_0 and for any v''. Then the left projection π_{L} of the Lagrangian $\mathcal{N}^*\Sigma$ consists of (p,ξ) with $(p',\xi')\in\pi_{\mathrm{L}}(\Sigma')$ and $\xi''=0$. Thus the rank drops at least by $n''=\dim(M'')$. We get the same conclusion for $\pi_{\mathrm{R}}(\mathcal{N}^*\Sigma)$. Therefore, $\mathcal{N}^*\Sigma$ is not a canonical graph in this case.

Let $n'=\dim(M')=2$. Then the canonical relation in (M',g') is a canonical graph, and we get that $\pi_{\mathrm{L,R}}(\mathcal{N}^*\Sigma)$ have rank $2n'+n''=4+n''$ instead of the maximal possible $2n=4+2n''$; i.e., the loss is exactly n''.

Assume now that $n' = 2$, $n'' = 1$, and the metric in M is given by

$$\sum_{\alpha,\beta=1}^{2} g_{\alpha\beta}(x^1, x^2)\mathrm{d}x^\alpha \mathrm{d}x^\beta + (\mathrm{d}x^3)^2.$$

Assume also that in M', we have a fold conjugate vector $v_0 = (0,1)$ at $x^1 = x^2 = 0$. Then all possible conormals to the conjugate loci at $(0,0)$ corresponding to small perturbations of v_0 will lie in the plane $v^3 = 0$. This is an example where Corollary 9.2 can be applied. We can recover singularities of the kind $\xi = (\xi_1, \xi_2, \xi_3)$ at $p_0 = (0,0,0)$ with $\xi_3 \neq 0$ and (ξ_1, ξ_2) in a conic neighborhood of $(1,0)$. The ones with $\xi_3 = 0$ are the problematic ones.

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