Distribution of resonances for the Neumann problem in linear elasticity outside a strictly convex body

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1 Introduction

Let $\mathcal{O}$ be a strictly convex compact in $\mathbf{R}^3$ with C^∞-smooth boundary Γ and denote by $\Omega = \mathbf{R}^3 \setminus \mathcal{O}$ the exterior domain. Denote by Δ_e the elasticity operator which is a 3×3 matrix-valued differential operator defined by

$$\Delta_e v = \mu_0 \Delta v + (\lambda_0 + \mu_0) \nabla (\nabla \cdot v),$$

$v = {}^t(v_1, v_2, v_3)$. Here λ_0, μ_0 are the Lamé constants and we assume that

$$\mu_0 > 0, \qquad 3\lambda_0 + 2\mu_0 > 0. \tag{1.1}$$

The Neumann boundary conditions for Δ_e are of the form

$$\sum_{j=1}^{3} \sigma_{ij}(v) \nu_j |_\Gamma = 0, \qquad i = 1, 2, 3, \tag{1.2}$$

where $\sigma_{ij}(v) = \lambda_0 \nabla \cdot v \delta_{ij} + \mu_0 \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)$ is the stress tensor, ν is the outer normal to $\Gamma = \partial\Omega$. It is known that $-\Delta_e$ acting on functions $v \in C^\infty_{\rm comp}(\bar\Omega; \mathbf{C}^3)$ satisfying (1.2) can be extended to a self-adjoint operator on $L^2(\Omega; \mathbf{C}^3)$ which will be denoted by L. The operator L is non-negative and has no point spectrum. Then the cut-off resolvent $R_\chi(\lambda) = \chi \left(L - \lambda^2 \right)^{-1} \chi$, $\chi \in C_0^\infty$ being a cut-off function equal to 1 near Γ, can be extended as a meromorphic function from $\mathrm{Im}\, \lambda < 0$ to the whole complex plane $\mathbf{C}$ with possible poles in $\mathrm{Im}\, \lambda > 0$ (see e.g. [Va], [Vo]). The poles of $R_\chi(\lambda)$ are called *resonances* (known also as scattering poles).

There is a lot of works dealing with resonances for the Dirichlet or Neumann Laplacian in an exterior domain. It follows from [MS1] and [MS2] that if there are no trapped rays the singularities of the solution of the wave equation escape to infinity. Thus the method in [LP2] (see also [Va]) gives that for nontrapping obstacles (and in particular for strictly convex ones) for any $C_1 > 0$ there exists $C_2 > 0$ (depending on C_1) so that all the resonances are above the curve $\mathrm{Im}\, \lambda = C_1 \ln |\lambda| - C_2$. In the case of analytic boundary this was improved in [BLR] to a cubic curve $\mathrm{Im}\, \lambda = C_1 |\lambda|^{1/3} - C_2$ with some constants $C_1, C_2 > 0$ which can be

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calculated explicitly. Recently, it has been shown in [SZ] and [HL] that this is the case for any strictly convex obstacle with C^∞- smooth boundary as well but with different constants C_1 and C_2. On the other hand, Lax and Phillips [LP1] conjectured that if the obstacle is trapping, then there should exist an infinite sequence of poles converging to the real axis. As the case of two strictly convex obstacles (see [I1], [I2], [G]) shows however, in general this fails but still there exists a strip $0 < \mathrm{Im}\,\lambda < C$ containing infinitely many poles. Thus, one could modify the Lax and Phillips conjecture asserting that for any trapping obstacle there are infinitely many poles in some strip $0 < \mathrm{Im}\,\lambda < C$, and this, to authors' best knowledge, has not been neither proved nor disproved so far. Note that in the case of two strictly convex obstacles the poles below some logarithmic curve are localized very precisely and they all are close to some explicitly calculated points (pseudopoles) forming a lattice. Ikawa [I1], [I2] found the first series of these pseudopoles and latter Gérard [G] obtained all of them. In [I3] Ikawa gives an example of a trapping obstacle consisting of two (non-strictly) convex bodies for which the poles converge to the real axis.

In the case we study in the present work $\mathcal{O}$ is strictly convex, so there are no classical trapped rays reflected at the boundary according to the usual laws of geometric optics. However, it has been shown in [T1], [Y] that there are three types of rays that carry singularities for L. The first two types consist of classical rays that reflect at the boundary and the singularities propagate along them with the two sound speeds $c_1 = \sqrt{\mu_0}$, $c_2 = \sqrt{\lambda_0 + 2\mu_0}$ of L. There is a third type of trajectories on the boundary along which singularities propagate with a third, slower speed c_R (the Rayleigh speed). So $\mathcal{O}$ is trapping for L because of the existence of singularities propagating along the boundary. Moreover, it is proved in [IN] in the spherical case and in [K] in the general one that the local energy of the corresponding elastic wave equation does not decay uniformly as $t \to \infty$. These phenomena well correspond to the existence of Rayleigh surface waves (see e.g. [R], [A], [CP], [Gr], [Gu]). So, it is natural to expect that the Rayleigh waves generate poles converging to the real axis, i.e. the Lax and Phillips conjecture holds for that problem. In the present work we show that this is precisely what happens when the obstacle is strictly convex. Our main result is the following theorem.

Theorem 1.1

(a) For any $C_1 > 0$ there exists $C_2 > 0$, such that for any $N > 0$ there are no resonances in the domain

$$C_N|\lambda|^{-N} < \mathrm{Im}\,\lambda < C_1 \ln|\lambda|, \qquad |\mathrm{Re}\,\lambda| > C_2$$

with some $C_N > 0$.

(b) There exist two infinite sequences $\{\lambda_j\}$, $\{-\bar{\lambda}_j\}$ of distinct resonances of L, such that

$$0 < \mathrm{Im}\,\lambda_j \le C_N|\lambda_j|^{-N} \qquad \text{for any } N > 0.$$

In the case where $\mathcal{O}$ is a ball, the authors [SV] proved that in fact the sequence λ_j tends to the real axis exponentially fast and the pole-free domain is of the kind $Ce^{-\gamma|\lambda|} < \mathrm{Im}\,\lambda < C_1|\lambda|^{1/3}$, $|\mathrm{Re}\,\lambda| > C_2$. So, it is natural to expect that such a result still holds for any strictly convex obstacle with analytic boundary.

Theorem 1.1 implies immediately existence of "eigenfunctions" corresponding to the resonances λ_j. We refer to Definition 2.1 for a definition of a λ-outgoing function.

Corollary 1.1 *Let $\{\lambda_j\}$ be the sequence of resonances of Theorem 1.1. Then for any j there exists a nontrivial $v_j \in C^\infty(\bar{\Omega})$, such that v_j solves the problem*

$$\begin{cases} (\Delta_e + \lambda_j^2)v_j &= 0 \qquad \text{in } \Omega, \\ \sum_{k=1}^3 \sigma_{ik}(v_j)\nu_k &= 0 \qquad \text{on } \Gamma, \\ v_j &- \lambda_j\text{-outgoing.} \end{cases} \tag{1.3}$$

Moreover, if $\operatorname{dist}(x, \Gamma)$ *is sufficiently small, then for any multiindex α and any integer N we have*

$$|\partial_x^\alpha v_j(x)| \leq C_{N,\alpha} \left(1 + \operatorname{dist}(x,\Gamma)|\lambda_j|\right)^{-N} \|v_j|_\Gamma\|_{L^2(\Gamma)}. \tag{1.4}$$

Corollary 1.1 gives another interpretation of the Rayleigh waves. Namely, we find that for some λ_j with $\operatorname{Im}\lambda_j = O(|\lambda_j|^{-\infty})$ there exist nontrivial exact solutions v_j concentrated on the boundary in the sense that they decay rapidly near Γ. These solutions can be regarded as the Rayleigh waves themselves and then Corollary 1.1 proves the existence of the Rayleigh waves for any strictly convex obstacle $\mathcal{O}$. It should be noted however that for large $|x|$, v_j increase exponentially.

Our approach is based on the following ideas. Below any logarithmic curve resonances are the poles of $\mathcal{N}^{-1}(\lambda)$, where $\mathcal{N}(\lambda)$ is the Neumann operator on Γ related to L mapping the Dirichlet data to the Neumann data of the corresponding outgoing solution. We use the calculus of ΨDO-s and FIO-s with large parameter (see [G], [D]). The large parameter in our calculus is the complex spectral parameter λ (assumed to lie in a logarithmic domain), or $\lambda_1 = \operatorname{Re}\lambda$. We represent the operator $\mathcal{N}(\lambda)$ as a ΨDO with large parameter λ in the hyperbolic and the mixed region in $T^*\Gamma$ and as a ΨDO with large parameter λ_1 in the elliptic one. In the two glancing regions we get $\mathcal{N}(\lambda) = J(A_1Q + A_2)J^{-1}$ (compare with [T2]), where $A_1Q + A_2$ is a hypoelliptic ΨDO with large parameter λ, while J is an elliptic FIO with large parameter λ. It turns out that the characteristic variety of the parametrix for $\mathcal{N}(\lambda)$ is $\Sigma = \{\zeta \in T^*\Gamma;\, c_R\|\zeta\| = 1\}$, where c_R is the Rayleigh speed (see e.g. [K], [CP]), while outside Σ, the parametrix for $\mathcal{N}(\lambda)$ is elliptic in the hyperbolic, mixed and the elliptic region and respectively hypoelliptic in the glancing regions in the sense described above. Thus $\mathcal{N}(\lambda)$ can be microlocally inverted outside Σ. Now, if $\{\lambda_j\}$ are the poles below a logarithmic curve, then there exists $f(x,\lambda)$, $\lambda = \lambda_j$, $j = 1, 2\ldots$, such that $\mathcal{N}(\lambda)f(x,\lambda) = 0$ and $\widetilde{\mathrm{WF}}(f) \subset \Sigma$. Then the solutions v_j appearing in Corollary 1.1 have Dirichlet data $v_j|_\Gamma = f(x,\lambda_j)$. Therefore, up to an error $O(|\lambda|^{-\infty})$, v_j are given by the elliptic parametrix and the properties of this parametrix to decrease rapidly near Γ enable us to prove Theorem 1.1(a). In order to prove Theorem 1.1(b) we apply the Phragmén-Lindelöf principle in the domain $\Lambda_{a,b} = \{\lambda \in \mathbf{C};\, |\operatorname{Im}\lambda| < a\ln(\operatorname{Re}\lambda),\, \operatorname{Re}\lambda > b\}$ as follows. Using the parametrix, we show that $\|\mathcal{N}^{-1}(\lambda)\|_{\mathcal{L}(H^{3/2}(\Gamma))} \leq c/\ln|\operatorname{Re}\lambda|$ on $\partial\Lambda_{a,b}$ for $a > 0$ sufficiently small and $b > 0$ sufficiently large. On the other hand, we show that we have the following a priori estimate $\|\mathcal{N}^{-1}(\lambda)\|_{\mathcal{L}(H^{3/2}(\Gamma))} \leq Ce^{C|\lambda|^4}$ in $\Lambda_{a,b}$ assuming that $\mathcal{N}^{-1}(\lambda)$ is analytic in a slightly larger domain. This a priori estimate is closely related to the problem of finding sharp polynomial bounds on the number of scattering poles in the disk $\{\lambda \in \mathbf{C};\, |\lambda| \leq r\}$ (see e.g. [Vo]). It follows from a similar estimate (see Prop. 5.2) of the cut-off resolvent that was suggested to the authors by M. Zworski. Having this a priori estimate we apply the Phragmén-Lindelöf principle in order to get the bound $\|\mathcal{N}^{-1}(\lambda)\|_{\mathcal{L}(H^{3/2}(\Gamma))} \leq C/\ln \lambda$ for $\lambda \in \mathbf{R}_+$ sufficiently large and then we show that this contradicts the fact that $\mathcal{N}(\lambda)$ is not elliptic in the elliptic region. Hence $\mathcal{N}^{-1}(\lambda)$ cannot be analytic in $\Lambda_{a,b}$.

The paper is organized as follows. In Section 2 we show how one can construct a parametrix for the Dirichlet problem for $(\Delta_e + \lambda^2)v = 0$ by using the parametrix for the Dirichlet problem for $(\Delta + \lambda^2)u = 0$ built in the Appendix. A parametrix for the Neumann operator $\mathcal{N}(\lambda)$ is built in Section 3. In Section 4 we prove the existence of the pole-free domain. The existence of a sequence of resonances tending to the real axis is proved in Section 5. In the Appendix we construct a parametrix for the Dirichlet problem for the equation $(\Delta + \lambda^2)u = 0$ and for the corresponding Neumann operator following Gérard [G] and Taylor [T2].

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2 Parametrix of the Dirichlet problem

We begin with a brief discussion of the notion *outgoing*. Let us first denote the self-adjoint realization of Δ_e in $L^2(\mathbf{R}^3)$ again by Δ_e and set $R_0(\lambda) = (-\Delta_e - \lambda^2)^{-1} \in \mathcal{L}(L^2)$ for $\operatorname{Im} \lambda < 0$. Here and below we denote by $L^2(\mathbf{R}^3)$, $C^\infty(\Gamma)$, etc. the spaces $L^2(\mathbf{R}^3; \mathbf{C}^3)$, $C^\infty(\Gamma; \mathbf{C}^3)$ etc. Then $R_0(\lambda)$ admits an analytic extension $R_0(\lambda) : L^2_{\text{comp}} \to L^2_{\text{loc}}$ in $\mathbf{C}$ (the free outgoing resolvent). Similarly, denote by L_D the Dirichlet realization of Δ_e in $L^2(\Omega)$ and by $R_D(\lambda) : L^2_{\text{comp}}(\bar{\Omega}) \to L^2_{\text{loc}}(\bar{\Omega})$ the outgoing resolvent of L_D. Here *outgoing* means the meromorphic extension from the lower half-plane (where the resolvent is holomorphic with values in $\mathcal{L}(L^2(\Omega))$ to the whole complex plane. We will call the poles of $R_D(\lambda)$ Dirichlet resonances. Similarly one can treat the resolvent of L (the Neumann realization of Δ_e in Ω). Next we will give a definition of a λ-outgoing function.

Definition 2.1 *Given $\lambda \in \mathbf{C}$ we say that the function u is λ-outgoing, if there exists $a > 0$ and $f \in L^2_{\text{comp}}(\mathbf{R}^3)$ such that $u|_{|x|>a} = R_0(\lambda) f|_{|x|>a}$.*

Proposition 2.1

a) For any $f \in L^2_{\text{comp}}(\bar{\Omega})$ and any λ not a Dirichlet resonance, the function $u = R_D(\lambda) f$ is λ-outgoing.

b) If $(\Delta_e + \lambda^2)u = f$ in Ω, $u \in H^2_{\text{loc}}(\bar{\Omega})$, $f \in L^2_{\text{comp}}(\bar{\Omega})$, $u|_\Gamma = 0$, λ is not a Dirichlet resonance and u is λ-outgoing, then $u = R_D(\lambda) f$.

Proof. To prove (a), let $\chi \in C^\infty$ be such that $\chi = 0$ near Γ, $\chi = 1$ for large $|x|$. Then $(\Delta_e + \lambda^2)\chi u = [\Delta_e, \chi]u + \chi f$ is compactly supported. Since $u \in L^2$ for $\operatorname{Im} \lambda < 0$, we see that in the lower half-plane we have $\chi u = R_0(\lambda) [(\Delta_e + \lambda^2)\chi u]$. Both sides of this equality are meromorphic in λ, therefore it holds in the whole complex plane thus proving that u is λ-outgoing for any λ not a Dirichlet resonance.

To prove (b), let $\chi_1 + \chi_2 = 1$, where $\chi_1 \in C_0^\infty$, $\chi_1 = 1$ for $|x| < a$. Here a is such that $u|_{|x|>a} = R_0(\lambda)f|_{|x|>a}$ with some $f \in L^2_{\text{comp}}(\mathbf{R}^3)$, $\lambda \in \mathbf{C}$ and we can assume that $\mathcal{O} \subset \{x;\ |x| < a\}$. Set

$$v(\mu) = \chi_1 u + \chi_2 R_0(\mu) f.$$

Clearly, $v(\lambda) = u$. Note that $(\Delta_e + \mu^2)v(\mu) = g(\mu)$ in Ω, where $g(\mu) = (\Delta_e + \mu^2)\chi_1 u + [\Delta, \chi_2]R_0(\mu)f + \chi_2 f$ is compactly supported. Thus we get that $v(\mu)$ solves the problem $(\Delta_e + \mu^2)v(\mu) = g(\mu)$ in Ω, $v(\mu)|_\Gamma = 0$ and $v(\mu) \in L^2$ in the lower half-plane, $g(\mu) \in L^2_{\text{comp}}$ for all μ. Therefore, in the lower half-plane we have $v(\mu) = R_D(\mu)g(\mu)$ and since both sides of this equality are meromorphic, it holds for any μ not a Dirichlet resonance. In particular, for $\mu = \lambda$ we get $u = R_D(\lambda)f$. □

Next we show how one can construct a parametrix of the Dirichlet problem

$$\begin{cases} (\Delta_e + \lambda^2)v &= 0 \quad \text{in } \Omega, \\ v &= g \quad \text{on } \Gamma, \\ v &- \lambda\text{-outgoing} \end{cases} \tag{2.1}$$

by using the parametrix built in the Appendix for the Dirichlet problem for the Laplacian. We will prove that any λ-outgoing solution v of the equation $(\Delta_e + \lambda^2)v = 0$ in Ω is of the form $-\nabla \times \nabla \times u(x, c_1^{-1}\lambda) + \nabla\nabla \cdot u(x, c_2^{-1}\lambda)$, where $u(x, \lambda)$ is a (vector-valued) solution to $(\Delta + \lambda^2)u = 0$. Let us recall that $c_1 = \sqrt{\mu_0}$, $c_2 = \sqrt{\lambda_0 + 2\mu_0}$. Thus one can use the parametrix for the Laplacian built in the Appendix and substitute it in the above formula. We prefer this approach instead of constructing a parametrix for the elasticity operator directly in order to avoid solving transport systems (instead of transport equations) that could cause difficulties in the glancing regions for example. We assume that $\lambda \in \Lambda$ (see (A.2)).

Lemma 2.1 *Given $f \in C^\infty(\Gamma)$ denote by $u(x, \lambda)$ the solution to (A.1). Then*

$$\mathcal{A}(\lambda) : f \mapsto -\nabla \times \nabla \times u(x, c_1^{-1}\lambda)|_\Gamma + \nabla\nabla \cdot u(x, c_2^{-1}\lambda)|_\Gamma =: \mathcal{A}(\lambda)f \tag{2.2}$$

extends to a bounded invertible operator on $H^s(\Gamma)$, $s \geq 0$ for $\lambda \in \Lambda$ with $|\lambda|$ sufficiently large.

Proof. We will analyze $\mathcal{A}$ in a manner similar to that used for the Neumann operator in the Appendix. Since here we have two sound speeds c_1 and c_2, we have to consider the following five regions in $\hat{T}^*\Gamma$.

hyperbolic region $\{\zeta \in T^*\Gamma;\ \|\zeta\| < c_2^{-1}\}$,
glancing region I $\{\zeta \in T^*\Gamma;\ \|\zeta\| = c_2^{-1}\}$,
mixed region $\{\zeta \in T^*\Gamma;\ c_2^{-1} < \|\zeta\| < c_1^{-1}\}$,
glancing region II $\{\zeta \in T^*\Gamma;\ \|\zeta\| = c_1^{-1}\}$,
elliptic region $\{\zeta \in \hat{T}^*\Gamma;\ \|\zeta\| > c_1^{-1}\}$.

Here $\|\cdot\|$ is the norm in $T^*\Gamma$, while with $|\eta|_x$ in the sequel we will denote the norm of a covector (x, η) written in local coordinates. Choose a point ζ^0 in the *hyperbolic region* and choose local coordinates (see the end of Section A.1), such that ζ^0 is given by $x' = 0$, $x_1 = 0$, $\eta = \eta^0$. Denote by χ any cut-off function with sufficiently small support in the hyperbolic region such that $\chi = 1$ near ζ^0 and denote by $\mathrm{Op}_\lambda(\chi)$ the corresponding ΨDO written in these local coordinates. By using the parametrix for u constructed in the Appendix, (A.30), (A.31), we get that

$$\mathcal{A}(\lambda)\mathrm{Op}_\lambda(\chi)f = A_h\mathrm{Op}_\lambda(\chi)f + Rf, \tag{2.3}$$

where A_h is a local ΨDO with large parameter λ and R has kernel in $\tilde{C}^\infty(\Gamma\times\Gamma)$. Let us compute the principal symbol of A_h. We have

$$\mathcal{A}(\lambda)f = -c_1^{-2}\lambda^2 u(x, c_1^{-1}\lambda)|_\Gamma + \nabla\nabla\cdot(u(x, c_2^{-1}\lambda) - u(x, c_1^{-1}\lambda))|_\Gamma \tag{2.4}$$

It is easy to see that

$$\sigma_p(A_h)|_{x'=0} = -\lambda^2 \begin{pmatrix} c_2^{-2} & (p_2-p_1)\eta_1 & (p_2-p_1)\eta_2 \\ (p_2-p_1)\eta_1 & c_1^{-2} & 0 \\ (p_2-p_1)\eta_2 & 0 & c_1^{-2} \end{pmatrix}, \tag{2.5}$$

where $p_1 = -\sqrt{c_1^{-2} - |\eta|^2}$, $p_2 = -\sqrt{c_2^{-2} - |\eta|^2}$. Thus, computing the determinant of this matrix and writing it in an invariant form, we get

$$\det(\sigma_p(A_h)) = \lambda^6 c_1^{-2}\left[c_1^{-2}c_2^{-2} - \left(\sqrt{c_1^{-2} - |\eta|_x^2} - \sqrt{c_2^{-2} - |\eta|_x^2}\right)^2 |\eta|_x^2\right].$$

Since in the hyperbolic region we have $|\eta|_x < c_2^{-1}$, we get

$$|\lambda|^{-6}\det(\sigma_p(A_h)) \geq c_1^{-2}(c_1^{-2}c_2^{-2} - (c_1^{-2} - |\eta|_x^2)|\eta|_x^2) > c_1^{-2}(c_1^{-2}c_2^{-2} - c_1^{-2}c_2^{-2}) = 0,$$

hence $|\lambda|^{-6}\det(\sigma_p(A_h)) > 0$. Therefore, $\mathcal{A}(\lambda)$ is elliptic here.

Let ζ^0 belong to the *mixed region* and χ be a cut-off function as above. Then we consider the hyperbolic parametrix of $u(x, c_1^{-1}\lambda)$ and the elliptic one for $u(x, c_2^{-1}\lambda)$. The latter can be constructed as a ΨDO with large parameter λ because $\operatorname{supp}\chi$ is compact. Arguing as above we see that for the principal symbol of the corresponding parametrix A_m we have

$$\det(\sigma_p(A_m)) = \lambda^6 c_1^{-2}\left[c_1^{-2}c_2^{-2} - \left(\sqrt{c_1^{-2} - |\eta|_x^2} - i\sqrt{|\eta|_x^2 - c_2^{-2}}\right)^2 |\eta|_x^2\right].$$

and

$$\mathrm{Im}\,\det(\sigma_p(A_m)) = 2|\lambda|^6 c_1^{-2}|\eta|_x^2\sqrt{c_1^{-2} - |\eta|_x^2}\sqrt{|\eta|_x^2 - c_2^{-2}} \neq 0.$$

So A_m is elliptic as well.

Let ζ^0 belong to the *elliptic region* and $\chi \in C^\infty(T^*\Gamma)$ be a cut-off function (with non-compact support) related to ζ^0 as in the beginning of Section A.4. Then, $\mathcal{A}(\lambda)\mathrm{Op}_\lambda f = A_e\mathrm{Op}_\lambda f + Rf$, where (see Sections A.4, A.5) A_e is a ΨDO with large parameter λ_1, while R has kernel in $\tilde{C}^\infty(\Gamma\times\Gamma)$. In this case we have

$$\det(\sigma_p(A_e)) = \lambda_1^6 c_1^{-2}\left[c_1^{-2}c_2^{-2} + \left(\sqrt{|\eta|_x^2 - c_1^{-2}\alpha^2} - \sqrt{|\eta|_x^2 - c_2^{-2}\alpha^2}\right)^2 |\eta|_x^2\right] \neq 0.$$

Therefore, A_e is elliptic at any finite point in $\hat{T}^*\Gamma$ in the elliptic region. The ellipticity in the elliptic region (see [G, p. 102]) however requires also certain estimate as $|\eta| \to \infty$, namely

$$|\sigma(A)^{-1}(x,\eta,\lambda)| \leq C\lambda_1^{-k}(1+|\eta|)^{-m}, \tag{2.6}$$

provided that $A \in L^{m,k}_{0,0}$. The construction of the elliptic parametrix implies that

$$\sigma(A_e) \sim \lambda_1^2 B_2 + \lambda_1 B_1 + \sum_{j=0}^{\infty} \lambda_1^{-j} B_{-j}, \tag{2.7}$$

where $B_j(x',\eta) \in \tilde{S}^j$, $j = 2, 1, 0, \ldots$, $\tilde{S}^j$ being the classical set of symbols [T2]. Note that B_j here depend also on λ via α. So, a priori $A_e \in L^{2,2}_{0,0}$ and $\sigma_p(A_e) = \lambda_1^2 B_2$. Similarly to (2.5),

$$B_2(0,\eta) = -\begin{pmatrix} c_2^{-2} & (p_2^{(\alpha)} - p_1^{(\alpha)})\eta_1 & (p_2^{(\alpha)} - p_1^{(\alpha)})\eta_2 \\ (p_2^{(\alpha)} - p_1^{(\alpha)})\eta_1 & c_1^{-2} & 0 \\ (p_2^{(\alpha)} - p_1^{(\alpha)})\eta_2 & 0 & c_1^{-2} \end{pmatrix},$$

where $p_1^{(\alpha)} = i\sqrt{|\eta|^2 - c_1^{-2}\alpha^2}$, $p_2^{(\alpha)} = i\sqrt{|\eta|^2 - c_2^{-2}\alpha^2}$. Letting $|\eta| \to \infty$, we get $B_2(0,\eta) = B_2^{(0)} + B_2^{(1)}$, where

$$B_2^{(0)} = -\begin{pmatrix} c_2^{-2} & \frac{i\alpha^2}{2}(c_1^{-2} - c_2^{-2})\frac{\eta_1}{|\eta|} & \frac{i\alpha^2}{2}(c_1^{-2} - c_2^{-2})\frac{\eta_2}{|\eta|} \\ \frac{i\alpha^2}{2}(c_1^{-2} - c_2^{-2})\frac{\eta_1}{|\eta|} & c_1^{-2} & 0 \\ \frac{i\alpha^2}{2}(c_1^{-2} - c_2^{-2})\frac{\eta_2}{|\eta|} & 0 & c_1^{-2} \end{pmatrix} \tag{2.8}$$

and $B_2^{(1)} \in \tilde{S}^{-2}$. Moreover, $\det(B_2^{(0)}) = c_1^{-2}\left(c_1^{-2}c_2^{-2} + \alpha^4(c_1^{-2} - c_2^{-2})^2/4\right) \neq 0$, provided that α is close to 1. Therefore, $B_2(0,\eta)$ is elliptic in $\tilde{S}^0$ (but not in $\tilde{S}^2$).

It is not hard to see that for B_1, which is a priori in $\tilde{S}^1$, we have $B_1(0,\eta) \in \tilde{S}^0$. Indeed, by (2.4) one can see that the terms in the expansion of $B_1(0,\eta)$ homogeneous of order 1 in η coming from $\nabla\nabla\cdot(u(x,c_2^{-2}\lambda) - u(x,c_1^{-2}\lambda))$ cancel because they do not depend on c_1, c_2.

Thus regarding A_e as an operator in $L^{2,2}_{0,0}$, we see by (2.8) that $\sigma_p(A_e)$ (defined modulo $S^{1,1}_{0,0}$) belongs in fact to $S^{1,2}_{0,0}$. Therefore, $A_e \in L^{1,2}_{0,0}$. Further, regarding A_e as an operator in $L^{1,2}_{0,0}$, we conclude from (2.8) and the fact $B_1(0,\eta) \in \tilde{S}^0$ that $\sigma_p(A_e) = \lambda_1^2 B_2 \pmod{L^{0,1}_{0,0}}$ and in fact $\sigma_p(A_e) \in S^{0,2}_{0,0}$. Therefore, $A_e \in L^{0,2}_{0,0}$ and then $\sigma_p(A_e) = \lambda_1^2 B_2 + \lambda_1 B_1 \pmod{S^{-1,1}_{0,0}}$. Because of the ellipticity of $B_2(0,\eta)$, $\sigma_p(A_e)$ is elliptic at $x' = 0$ and therefore it is elliptic in the elliptic region in the sense of (2.6) with $k = 2$, $m = 0$. So, A_e is an elliptic ΨDO in $L^{0,2}_{0,0}$.

It remains to consider the two glancing regions. Consider first *glancing region I*. Let ζ^0 belong to *glancing region I*, i.e. in the local coordinates related to ζ^0 we have $\zeta^0 = (0,\eta^0)$ with $|\eta^0| = c_2^{-1}$. Choose a cut-off function χ supported near ζ^0 as above. Then the corresponding glancing parametrix $A_g \mathrm{Op}_\lambda f$ for $\mathcal{A}(\lambda)\mathrm{Op}_\lambda f$ can be divided into two parts — a hyperbolic one for $(\Delta - \nabla\nabla\cdot)u(x,c_1^{-1}\lambda)|_\Gamma$ and a glancing one for $\nabla\nabla\cdot u(x,c_2^{-1}\lambda)|_\Gamma$. The glancing parametrix for $\nabla\nabla\cdot u(x,c_2^{-1}\lambda)|_\Gamma$ can be written microlocally in the form

$$J(A_1 Q + A_2)J^{-1} \tag{2.9}$$

(see the Appendix). Here $A_j \in L^{0,2}_{0,0}$, $j = 1, 2$, $Q \in L^{0,0}_{2/3,0}$ and B_1, B_2 are matrix valued operators, while Q is the same as in the Appendix. Let us represent the (hyperbolic) parametrix for $f \mapsto (\Delta - \nabla\nabla\cdot)u(x, c_1^{-1}\lambda)|_\Gamma$ in the form JA_3J^{-1}, where $A_3 \in L^{0,2}_{0,0}$. Therefore, the parametrix for $\mathcal{A}(\lambda)$ in glancing region I has the form

$$J(A_1Q + A_2')J^{-1}, \qquad A_1, A_2' \in L^{0,2}_{0,0}. \tag{2.10}$$

The properties of q imply that $\lambda^{-2}\sigma(A_1Q)$ is small near $\alpha = 0$. Here $\alpha = |\eta| - c_2^{-1}$. On the other hand A_2' is elliptic near $\alpha = 0$. The easiest way to see that without calculating $\sigma(A_2')$ is the following. Let us note that $\mathcal{A}$ is elliptic in the hyperbolic region and this ellipticity is uniform as $|\eta|_x \to c_2^{-1}$. Suppose that $A_1Q + A_2'$ acts on $f(x, \lambda)$ with $\widetilde{\mathrm{WF}}(f) \subset \{c_2^{-1} - 2\varepsilon < |\eta| < c_2^{-1} - \varepsilon\}$ with ε sufficiently small and assume that A_2' is not elliptic at $\alpha = 0$. Then we have $(A_1Q + A_2')f = J^{-1}A_gJf \mod O(|\lambda|^{-\infty})$ and since $\widetilde{\mathrm{WF}}(Jf)$ is also contained in a set of the kind $\{c_2^{-1} - \delta_2 < |\eta|_x < c_2^{-1} - \delta_1\}$, $0 < \delta_1 < \delta_2$, one can see that in this case A_g can be replaced with A_h. Thus one can conjugate A_h with J and claim that $A_1Q + A_2'$ is a ΨDO with a symbol that can be obtained from the symbol of A_h. Since A_h is uniformly elliptic as $|\eta|_x \to c_2^{-1}$, letting $\varepsilon \to 0$ we see that $\lambda^{-2}\sigma(A_1Q + A_2')$ could not be small for small $\alpha \neq 0$ and for large λ. Since on the other hand, $\lambda^{-2}\sigma(A_1Q)$ is small near $\alpha = 0$, we get that $\lambda^{-2}\sigma(A_2')$ is elliptic at $\alpha = 0$. Therefore, $A_1Q + A_2' \in L^{0,2}_{2/3,0}$ is elliptic and its inverse modulo neglectible operators is $J(A_1Q + A_2')^{-1}J^{-1}$. We note that the situation here is simpler than that for the Neumann operator for the Laplacian considered in the Appendix, because here A_2' is elliptic.

By a similar way one treats *glancing region II.* Then one has a sum of two terms — a glancing parametrix coming from $(\Delta - \nabla\nabla\cdot)u(x, c_1^{-1}\lambda)|_\Gamma$ and an elliptic parametrix (with large parameter λ) coming from $\nabla\nabla\cdot u(x, c_2^{-1}\lambda)|_\Gamma$. Note that $\mathcal{A}$ is again uniformly elliptic in the elliptic (mixed) region as $|\eta| \to c_1^{-1}$, which enables us to proceed as above.

Now, let ζ^0 belong to the hyperbolic (mixed) region and assume that χ, χ', χ'' are three cut-off functions with sufficiently small supports near ζ^0, such that $\chi = \chi' = \chi'' = 1$ near ζ^0, $\chi' = 1$ in a neighborhood of $\operatorname{supp}\chi$, $\chi'' = 1$ in a neighborhood of $\operatorname{supp}\chi'$. By (2.3),

$$\mathcal{A}(\lambda)\mathrm{Op}_\lambda(\chi'')f = A_h\mathrm{Op}_\lambda(\chi'')f + Rf. \tag{2.11}$$

Since A_h is elliptic on $\operatorname{supp}\chi''$, we deduce by Proposition A.1 that

$$\|\mathrm{Op}_\lambda(\chi)\mathrm{Op}_\lambda(\chi'')f\| \le C|\lambda|^{-2}\|\mathrm{Op}_\lambda(\chi')\mathcal{A}\mathrm{Op}_\lambda(\chi'')f\| + C_N|\lambda|^{-N}\|f\|.$$

Here and below $\|\cdot\|$ could be any $H^s(\Gamma)$-norm, $s \ge 0$ and $\mathrm{Op}_\lambda(\chi)$ is the ΨDO with full symbol χ in the special coordinates related to ζ^0 (see Section A.1). Since $\mathrm{Op}_\lambda(\chi)\mathrm{Op}_\lambda(\chi'') = \mathrm{Op}_\lambda(\chi)$ modulo neglectible operators, we get

$$\begin{aligned} \|\mathrm{Op}_\lambda(\chi)f\| &\le C|\lambda|^{-2}\|\mathrm{Op}_\lambda(\chi')\mathcal{A}\mathrm{Op}_\lambda(\chi'')f\| + C_N|\lambda|^{-N}\|f\| \\ &\le C|\lambda|^{-2}\Big(\|\mathrm{Op}_\lambda(\chi')\mathcal{A}f\| + \|\mathrm{Op}_\lambda(\chi')\mathcal{A}(I - \mathrm{Op}_\lambda(\chi''))f\|\Big) \\ &\qquad + C_N|\lambda|^{-N}\|f\|. \end{aligned} \tag{2.12}$$

Note that $\operatorname{supp}\chi' \cap \operatorname{supp}(1 - \chi'') = \emptyset$. This yields

$$\|\mathrm{Op}_\lambda(\chi')\mathcal{A}(I - \mathrm{Op}_\lambda(\chi''))f\| = O(|\lambda|^{-\infty})\|f\|. \tag{2.13}$$

Indeed, up to a neglectible operator (see (A.31), $\mathcal{A}$ is a finite sum of microlocal parametrices, which are ΨDO-s or operators of the form (2.10). Thus, although $\mathcal{A}$ is not a ΨDO on the boundary because of the complications in the glancing zones, any microlocal parametrix used in the construction of $\mathcal{A}$ satisfies (2.13) including the operators of the form (2.10). This yields (2.13). From (2.12) and (2.13) we obtain

$$\|\mathrm{Op}_\lambda(\chi)f\| \le C|\lambda|^{-2}\|\mathcal{A}(\lambda)f\| + C_N|\lambda|^{-N}\|f\| \tag{2.14}$$

for any $N>0$ and for any f. The constants C, C_N depend on ζ^0. The same estimate holds if χ is supported in the elliptic region as in the Appendix because A_e is an elliptic ΨDO (with large parameter λ_1) in $L^{0,2}_{0,0}$. Next, since in the two glancing regions $\mathcal{A}$ is elliptic as well in the sense described above, one has an estimate similar to (2.14) here as well with $|\lambda|^{-2}$ replaced by $|\lambda|^{-2}e^{C|\mathrm{Im}\,\lambda|}$. Picking up a partition of unity and summing up the corresponding estimates we get

$$\|f\| \le Ce^{C|\mathrm{Im}\,\lambda|}|\lambda|^{-2}\|\mathcal{A}(\lambda)f\| + C_N|\lambda|^{-N}\|f\|, \qquad \lambda\in\Lambda.$$

If C_2 (see (A.2)) is sufficiently large, one gets

$$\|f\| \le C'e^{C|\mathrm{Im}\,\lambda|}|\lambda|^{-2}\|\mathcal{A}(\lambda)f\|, \qquad \lambda\in\Lambda.$$

In order to conclude that $\mathcal{A}$ is invertible for large λ, it is enough to show that a similar estimate holds for $\mathcal{A}^*$ as well. This follows immediately from the analysis of $\mathcal{A}$, because $\mathcal{A}^*$ is an operator with similar properies and can be inverted microlocally in all regions. Thus we obtain that $\mathcal{A}(\lambda)$ is bounded and invertible operator in $H^s(\Gamma)$, $s\ge 0$ and for $\lambda\in\Lambda$

$$\|\mathcal{A}\| \le Ce^{C|\mathrm{Im}\,\lambda|}|\lambda|^2, \qquad \|\mathcal{A}^{-1}\| \le Ce^{C|\mathrm{Im}\,\lambda|}|\lambda|^{-2},$$

where $\|\cdot\|$ is the norm in $\mathcal{L}(H^s(\Gamma))$. This completes the proof of the lemma. □

Proposition 2.2 *Let $g\in H^s(\Gamma)$, $s\ge 3/2$. Then for $\lambda\in\Lambda$ and $|\lambda|$ sufficiently large the solution v to (2.1) is of the form*

$$v(x,\lambda) = -\nabla\times\nabla\times u(x,c_1^{-1}\lambda) + \nabla\nabla\cdot u(x,c_2^{-1}\lambda), \tag{2.15}$$

where $u(x,\lambda)$ is the solution to the Dirichlet problem (A.1) for the Laplace operator with $f=\mathcal{A}^{-1}(\lambda)g$.

Proof. It is easy to see that (2.15) gives a solution $v\in H^2_{\mathrm{loc}}(\bar\Omega)$ to $(\Delta_e+\lambda^2)v=0$ in Ω. Next, Lemma 2.1 implies that $v|_\Gamma=g$. It remains to show that v is λ-outgoing. To this end, notice that $u(x,c_1^{-1}\lambda)$, $u(x,c_2^{-1}\lambda)$ belong to H^2 for $\mathrm{Im}\,\lambda<0$, hence $v(\cdot,\lambda)\in L^2$. Therefore, in the lower half-plane v is the unique L^2-solution to (2.1), i.e. $v=Eg-R_D(\lambda)(\Delta_e+\lambda^2)Eg$, $Eg\in H^{s+1/2}(\Omega)$ being an extension of $g\in H^s(\Gamma)$ supported in a fixed compact set. Since v is analytic in $\lambda\in\Lambda$ for large $|\lambda|$, we get that the last equality holds in that part of Λ that lies in the upper half-plane as well and by Proposition 2.1(a) we conclude that v is λ-outgoing. □

An immediate consequence of Proposition 2.2 is that the Dirichlet problem (2.1) for the elasticity system has no resonances in Λ provided that C_2 (see (A.2)) is properly chosen. This is expected because we know that for the Dirichlet problem singularities propagate by a standard way [Y], [T2]. Another consequence of Proposition 2.2 is that one can construct a parametrix for (2.1) by using the parametrix built in the Appendix for the Laplace operator. Indeed, formulas (A.30), (A.31) show that if we substitute in (2.15) the parametrix for $u(x, c_1^{-1}\lambda)$, $u(x, c_2^{-1}\lambda)$ (see (A.29)), then we get a parametrix for (2.1). Therefore, if $H_{\Delta_D}(\lambda)$ is the parametrix appearing in (A.29) and if we denote by $v = \mathcal{H}_{\Delta_{e,D}}(\lambda)g$ the solution to (2.1), then

$$H_{\Delta_{e,D}}(\lambda) := \left[-\nabla \times \nabla \times H_{\Delta_D}(c_1^{-1}\lambda) + \nabla\nabla \cdot H_{\Delta_D}(c_2^{-1}\lambda)\right]\mathcal{A}^{-1}(\lambda) \tag{2.16}$$

differs from $\mathcal{H}_{\Delta_{e,D}}(\lambda)$ by an operator with kernel in $\tilde{C}^\infty(U \times \Gamma)$.

3 The Neumann operator for the elasticity system

In this section we study the Neumann operator $\mathcal{N}(\lambda)$ for (2.1) in a manner similar to that in the Appendix. We will show that $\mathcal{N}$ has properties similar to those of the Neumann operator for the Laplacian with the only difference that $\mathcal{N}$ is not elliptic in the elliptic region (see also [K], [CP]).

The Neumann operator $\mathcal{N}(\lambda)$ is defined by the formula

$$\mathcal{N}(\lambda) : H^s(\Gamma) \ni g \mapsto \sum_{j=1}^{3} \boldsymbol{\sigma}_j(v)\nu_j|_\Gamma \in H^{s-1}(\Gamma), \qquad s \geq \frac{3}{2}, \tag{3.1}$$

where $\boldsymbol{\sigma}_j = {}^t(\sigma_{1j}, \sigma_{2j}, \sigma_{3j})$, σ_{ij} is the stress tensor (see (1.2)) and v solves (2.1). Obviously, $\mathcal{N}(\lambda)$ is a meromorphic family of operators with poles at the Dirichlet resonances of the elasticity system, i.e. the poles of (2.1). In particular, for any $C_1 > 0$ and for C_2 sufficiently large (see (A.2)), $\mathcal{N}(\lambda)$ is analytic in Λ. Let us fix further $s = 3/2$, i.e. $\mathcal{N}(\lambda) : H^{3/2} \to H^{1/2}$. The next assertion is in fact well-known and we give its proof just for the sake of completeness.

Proposition 3.1 *Assume that λ_0 is not a Dirichlet resonance for the elasticity system. Then λ_0 is a Neumann resonance if and only if λ_0 is a pole of $\mathcal{N}^{-1}(\lambda)$. Moreover, if λ_0 is a Neumann resonance, then there exists a non-trivial $g \in H^{3/2}(\Gamma)$ such that $\mathcal{N}(\lambda_0)g = 0$.*

Proof. First note that if λ is not a Neumann resonance, then $\mathcal{N}^{-1}(\lambda)$ is well defined and maps the Neumann data to its Dirichlet data. It remains to prove that if λ_0 is a Neumann resonance, then $\mathcal{N}^{-1}(\lambda)$ has a pole at $\lambda = \lambda_0$. Let $\mathcal{H}_{\Delta_{e,D}}(\lambda)$ be the operator solving (2.1) and denote by $\mathcal{H}_{\Delta_{e,N}}(\lambda)$ the operator solving the corresponding Neumann problem. Then

$$\mathcal{H}_{\Delta_{e,D}}(\lambda)\mathcal{N}^{-1}(\lambda) = \mathcal{H}_{\Delta_{e,N}}(\lambda).$$

Since the Neumann resonances are exactly the poles of $\mathcal{H}_{\Delta_{e,N}}(\lambda) : H^{1/2}(\Gamma) \to L^2_{\mathrm{loc}}(\Omega)$, we see that any Neumann resonance is a pole of $\mathcal{N}^{-1}(\lambda)$.

In order to prove the last assertion of the proposition, consider the operator

$$P := \oint_{|\lambda-\lambda_0|=\varepsilon} \mathcal{N}^{-1}(\lambda)(\lambda-\lambda_0)^{d-1}\, d\lambda \neq 0,$$

where d is the order of the pole λ_0, ε is chosen so that there are no other poles in the disk $|\lambda-\lambda_0| \leq \varepsilon$. Therefore, for some $f \in H^{3/2}$, $Pf \neq 0$. It is easy to check that $\mathcal{N}(\lambda_0)Pf = 0$, i.e. the proposition is satisfied with $g = Pf$. Let us note that if λ_0 is a resonance, we get that there exists a non-trivial λ-outgoing solution v to $(\Delta_e + \lambda^2)v = 0$ satisfying the Neumann boundary conditions. □

Using the parametrix, one can analyze $\mathcal{N}(\lambda)$ in the same way as it was done for $\mathcal{A}(\lambda)$. Namely, $\mathcal{N}$ is a ΨDO with large parameter in the hyperbolic, mixed and elliptic region and has the form (2.9) in the glancing regions.

In the *hyperbolic region* the parametrix of $\mathcal{H}_{\Delta_{e,D}} g$ according to (2.16) is of the form

$$\sum_{j=1}^{2} \left(\frac{\lambda}{2\pi}\right)^2 \iint e^{i\lambda(\psi_j(x,\eta)-y\cdot\eta)} a_j(x,y,\eta,\lambda) f(y,\lambda)\, dy\, d\eta, \tag{3.2}$$

where $f = \mathcal{A}^{-1}(\lambda)g$, $(\nabla\psi_1)^2 = c_1^{-2}$, $(\nabla\psi_2)^2 = c_2^{-2}$, $\psi_1|_\Gamma = \psi_2|_\Gamma = x\cdot\eta$ (see (2.16)). The principal symbols a_j^0 of a_j are

$$a_1^0 = -\lambda^2 \nabla\psi_1 \times \nabla\psi_1 \times, \qquad a_2^0 = -\lambda^2 \nabla\psi_2 \nabla\psi_2 \cdot .$$

By applying $\sum \boldsymbol{\sigma}_j \nu_j$ (see (3.1)) to (3.2) and by setting $x_1 = 0$, we get that the hyperbolic parametrix $N_h(\lambda)$ of $\mathcal{N}(\lambda)$ is a ΨDO with large parameter λ in $L^{0,1}_{0,0}$ and its principal symbol can be computed explicitly. A direct calculation shows (see also [CP], [K], [T1]) that

$$\begin{aligned} \det(\sigma_p(N_h)) &= i\mu_0^3\lambda^3 \left(\sqrt{c_1^{-2}-|\eta|_x^2}\sqrt{c_2^{-2}-|\eta|_x^2} + |\eta|_x^2\right)^{-1} \\ &\quad \times \sqrt{c_1^{-2}-|\eta|_x^2}\left[\left(2|\eta|_x^2 - c_1^{-2}\right)^2 + 4|\eta|_x^2 \sqrt{c_1^{-2}-|\eta|_x^2}\sqrt{c_2^{-2}-|\eta|_x^2}\right] \neq 0, \end{aligned} \tag{3.3}$$

therefore $\mathcal{N}(\lambda)$ is elliptic here.

In the *mixed region* for the parametrix $N_m(\lambda)$ we have

$$\begin{aligned} \det(\sigma_p(N_m)) &= i\mu_0^3\lambda^3 \left(i\sqrt{c_1^{-2}-|\eta|_x^2}\sqrt{|\eta|_x^2 - c_2^{-2}} + |\eta|_x^2\right)^{-1} \\ &\quad \times \sqrt{c_1^{-2}-|\eta|_x^2}\left[\left(2|\eta|_x^2 - c_1^{-2}\right)^2 + 4i|\eta|_x^2 \sqrt{c_1^{-2}-|\eta|_x^2}\sqrt{|\eta|_x^2 - c_2^{-2}}\right] \neq 0, \end{aligned} \tag{3.4}$$

so in this case $\mathcal{N}$ is elliptic as well.

In the *elliptic region* the corresponding parametrix $N_e(\lambda)$ is a ΨDO with large parameter λ_1. Since the Neumann boundary condition is given via a first order differential operator and in the elliptic region the corresponding amplitudes a_1, a_2 (see (3.2)) are in $S^{2,2}_{0,0}$ and on the other hand $A_e^{-1} \in L^{0,-2}_{0,0}$, we get that a priori $N_e \in L^{3,1}_{0,0}$. We will show that in fact $N_e \in L^{1,1}_{0,0}$.

Indeed, applying $\sum \sigma_{ij}(v)\nu_j$ to $v = -\nabla \times \nabla \times u(x, c_1^{-1}\lambda) + \nabla\nabla \cdot u(x, c_2^{-1}\lambda)$, where u solves (A.1), we get on Γ

$$\sum_{j=1}^{3} \sigma_{ij}(v)\nu_j = -\lambda^2 \frac{\lambda_0}{\lambda_0 + 2\mu_0} \nabla \cdot u(x, c_2^{-1}\lambda)\nu_i - \lambda^2 \partial_\nu u_i(x, c_1^{-1}\lambda) - \lambda^2 \nu \cdot \partial_{x_i} u(x, c_1^{-1}\lambda) + 2\mu_0 \partial_\nu \partial_{x_i} \nabla \cdot \left(u(x, c_2^{-1}\lambda) - u(x, c_1^{-1}\lambda) \right). \tag{3.5}$$

The first three terms lead to an operator in $L^{1,3}_{0,0}$. The fourth one gives an operator that is a priori in $L^{3,3}_{0,0}$. However, it is not hard to check that the terms homogeneous of order $(3,3)$, $(2,3)$, $(2,2)$ in (η, λ_1) cancel so in fact the fourth term in (3.5) also gives a boundary ΨDO in $L^{1,3}_{0,0}$. Therefore, $N_e g = \tilde{N}_e f$, where $\tilde{N}_e \in L^{1,3}_{0,0}$ and $f = \mathcal{A}^{-1}(\lambda) g$. Since $A_e^{-1} \in L^{0,-2}_{0,0}$, we get $N_e \in L^{1,1}_{0,0}$ as can be expected.

Let us recall that in the elliptic region when constructing $H_e(\lambda)$ (related to the elasticity Dirichlet problem) as a FIO with large parameter λ_1, the eikonal equations read $(\nabla\psi_1)^2 = \alpha^2 c_1^{-2}$, $(\nabla\psi_2)^2 = \alpha^2 c_2^{-2}$, where $\alpha = 1 + i \tan \arg \lambda$. In this case we have

$$\det(\sigma_p(N_e)) = \mu_0^3 \lambda_1^3 \left(-\sqrt{|\eta|_x^2 - \alpha^2 c_1^{-2}} \sqrt{|\eta|_x^2 - \alpha^2 c_2^{-2}} + |\eta|_x^2 \right)^{-1} |\eta|_x^4 \sqrt{|\eta|_x^2 - \alpha^2 c_1^{-2}} R(\alpha c_1^{-1} |\eta|_x^{-1}), \tag{3.6}$$

where

$$R(s) = \left(s^2 - 2\right)^2 - 4\left(1 - s^2\right)^{1/2} \left(1 - \frac{\mu_0}{\lambda_0 + 2\mu_0} s^2\right)^{1/2}.$$

It is well known that there is only one simple root $s = s_0$ of $R(s) = 0$ in $0 < s < 1$, therefore the equation $R(\alpha c_1^{-1}|\eta|_x^{-1}) = 0$, where $c_1^{-1}|\eta|_x^{-1} < 1$ has no roots if α is non-real and α is sufficiently close to 1 while for $\alpha = 1$ the characteristic variety determined by $\det(\sigma_p(N_e)) = 0$ is given by

$$\Sigma = \left\{ \zeta \in T^*\Gamma;\ \|\zeta\| = c_R^{-1} \right\},$$

where $c_R = \mu_0^{1/2} s_0$ is the Rayleigh speed (see [T1], [K]). Therefore, $N_e(\lambda)$ is elliptic outside Σ and loses its ellipticity at Σ.

Finally, since $N_e \in L^{1,1}_{0,0}$ and by (3.6)

$$\det(\sigma_p(N_e)) = C(\alpha)\lambda_1^3 \left(|\eta|_x^3 + O(|\eta|_x^2) \right), \qquad C(\alpha) \neq 0,$$

we see that N_e is elliptic at any infinite point of $T^*\Gamma$ (i.e. (2.6) holds with $k = m = 1$).

In *glancing region I* we have $\mathcal{N}(\lambda) = \mathcal{M}(\lambda)\mathcal{A}^{-1}(\lambda)$, where $\mathcal{M} : f \mapsto \sum_{j=1}^3 \boldsymbol{\sigma}_j(v)\nu_j|_\Gamma$. Here v is given by (2.15) with u, f solving (A.1). The parametrix M_g of $\mathcal{M}$ in glancing region I has the form

$$M_g = J(A_1 Q + A_2)J^{-1}, \tag{3.7}$$

where $A_1,\ A_2 \in L^{0,3}_{0,0}$. Recall that M_g is produced by a sum of a hyperbolic parametrix with respect to the first wave speed c_1^{-1} and a glancing one with respect to c_2^{-1}. We can analyze the ellipticity (hypoellipticity) of $A_1 Q + A_2$ by using arguments similar to those in the analysis of the Neumann operator in the Appendix and of A_g in Section 2. Assume first that $\lambda_0 \neq 0$. Let us investigate the principal symbol of $\mathcal{M}$ in the hyperbolic region as $|\eta|_x \to c_2^{-1}$. From (3.3) we see that $\lambda_0 \neq 0$ implies that $2c_2^{-2} - c_1^{-2} \neq 0$, therefore $\det(\sigma_p(M_h)) = \det(\sigma_p(N_h))\det(\sigma_p(A_h))$ is uniformly elliptic as $|\eta|_x \to c_2^{-1}$. This implies that (see Section 2) $A_1Q + A_2$ in (3.7) is elliptic, thus one can invert $\mathcal{M}$ and therefore $\mathcal{N}$ here as done for $\mathcal{A}$. The inverse has microlocally the form $N_g^{-1} = JBJ^{-1}$, where $B \in L^{0,-1}_{2/3,0}$. Next assume that $\lambda_0 = 0$. Then $\det(\sigma_p(M_h))$ has simple zero as $|\eta|_x \to c_2^{-1}$ because (3.3) has simple zero at $|\eta|_x = c_2^{-1}$. Therefore, $A_1Q + A_2$ is no longer elliptic, but we can proceed as in the analysis of the Neumann operator for the Laplacian (see the Appendix). Given a matrix B, denote by ${}^{\rm co}B$ the co-matrix of B, i.e. ${}^{\rm co}B_{11} = B_{22}B_{33} - B_{23}B_{32}$ etc. and ${}^{\rm co}BB = B^{\rm co}B = \det B$. Then

$$ {}^{\rm co}\sigma(A_1Q + A_2) \in S^{0,6}_{2/3,0} \tag{3.8} $$

Since $\det(\sigma(M_h))$ has simple zero at $|\eta|_x = c_2^{-1}$, one can apply arguments similar to those in the Appendix to get that

$$ \det(\sigma(A_1Q + A_2)) = a_1Q + a_2, \qquad \text{with } a_1 \in S^{0,9}_{2/3,0},\ a_2 \in S^{0,9}_{0,0}, \tag{3.9} $$

and a_2 is not elliptic at $\alpha = 0$, while a_1 is elliptic. Therefore,

$$ \det(\sigma(A_1Q + A_2))^{-1} = Q^{-1}\left(a_1 + a_2Q^{-1}\right)^{-1} \in S^{0,-9+1/3}_{2/3,0} $$

and

$$ (\sigma(A_1Q + A_2))^{-1} = {}^{\rm co}\sigma(A_1Q + A_2)/\det(\sigma(A_1Q + A_2)) \in S^{0,-9+1/3}_{2/3,0}. $$

Let us recall that the parametrix A_g of $\mathcal{A}$ in glancing region I has similar representation $A_g = J(A_1'Q + A_2')J^{-1}$ with $A_1'Q + A_2'$ elliptic operator in $S^{0,2}_{2/3,0}$. Therefore, $\mathcal{N}^{-1} = \mathcal{A}\mathcal{M}^{-1}$ is microlocally of the form

$$ N_g^{-1} = JBJ^{-1} \qquad \text{with } B \in L^{0,-2/3}_{2/3,0}, \tag{3.10} $$

i.e. N_g is hypoelliptic, result similar to that for the Neumann operator for the Laplacian.

By similar arguments one treats $\mathcal{N}$ in *glancing region II*. Note that in this case we have a boundary operator coming from a glancing parametrix related to the wave speed c_1^{-1} and an elliptic one (as a ΨDO with large parameter λ) coming from the wave speed c_2^{-1}. Then, if $|\eta| \to c_1^{-1}$ in the mixed region for example (i.e. with $|\eta|_x < c_1^{-1}$), we have a simple zero in (3.4) due to the factor $\sqrt{c_1^{-2} - |\eta|_x^2}$ there, so one can apply similar arguments in order to get a microlocal representation of $\mathcal{N}$ in this region similar to (3.10) with B in the same class.

4 The pole-free domain

We are ready to prove Theorem 1.1(a), i.e. to show that for C_2 large enough the domain $\{\lambda \in \Lambda;\ \operatorname{Im}\lambda \geq C_N|\operatorname{Re}\lambda|^{-N}\}$ is free of poles provided that C_N is suitably chosen. Without loss of generality we can deal only with λ with $\operatorname{Re}\lambda > 0$.

Let $\{\lambda_j\}_{j=1}^{\infty}$ be a sequence of resonances in Λ with $\operatorname{Re}\lambda_j > 0$. As shown in Proposition 3.1, there exists a sequence g_j in $H^{3/2}(\Gamma)$, such that $\mathcal{N}(\lambda_j)g_j = 0$. It is convenient to regard g_j as a family $g(x,\lambda)$, $\lambda \in \Theta := \{\lambda_j\}_{j=1}^{\infty}$, i.e. λ takes values in a discrete set. In this section we will deal with ΨDO-s with large parameter $\lambda \in \Theta$. Clearly the calculus we use in the Appendix is valid when λ belongs to a discrete set in Λ as well. Then, from Section 3 it follows that

$$\widetilde{\mathrm{WF}}(g) \subset \Sigma. \tag{4.1}$$

Since $\widetilde{\mathrm{WF}}(g)$ does not contain infinite points from $\hat{T}^*\Gamma$, it follows from [G, Pr. A.I.12] and the remark after it that $g_j \in C^\infty(\Gamma)$, $j = 1, 2, \dots$.

Let $v(x,\lambda)$, $\lambda \in \Theta$ be the family of the solutions to the Dirichlet problem (2.1) corresponding to Dirichlet data $v = g$ on Γ. Since $\mathcal{N}(\lambda_j)g_j = 0$, we have that $v_j := v(\,\cdot\,,\lambda_j)$ solve the problem (1.3). Let $\phi \in C_0^\infty(\bar{\Omega})$ be a cut-off function, such that $\phi(x) = 1$ for x belonging to some neighborhood of Γ. Then

$$\Delta_e(\phi v) = \phi\Delta_e v + [\Delta_e, \phi]v.$$

Here the commutator $[\Delta_e, \phi]$ is a first order differential operator with coefficients in $C_0^\infty(\Omega)$ (vanishing near Γ and for large $|x|$). By (1.3),

$$(\Delta_e + \lambda^2)(\phi v) = [\Delta_e, \phi]v \qquad \lambda \in \Theta. \tag{4.2}$$

Let us multiply (4.2) by ϕv and integrate over Ω. Since Δ_e with Neumann boundary conditions is symmetric, we get

$$\operatorname{Im}\lambda^2 \|\phi v\|_{L^2}^2 = \operatorname{Im}\left([\Delta_e, \phi]v, \phi v\right),$$

hence,

$$\operatorname{Im}\lambda^2 \leq \frac{\|[\Delta_e, \phi]v\|_{L^2}}{\|\phi v\|_{L^2}}, \qquad \lambda \in \Theta. \tag{4.3}$$

Now we will make use of (4.1) combined with the exponential decay in λ near Γ of the parametrix of v in the elliptic region in order to show that the right hand side of (4.3) decays rapidly. First note that (4.3) remains true up to an error $O(|\lambda|^{-\infty})$ if we replace v in the numerator by the parametrix of v. Recall that a parametrix of v is given (see (A.29), (A.30), (A.31) and Section 2) by a finite sum of microlocal parametrices $v^{(n)}$ using a partition of unity. According to (4.1) all terms solving

$$\begin{cases} (\Delta_e + \lambda^2)v^{(n)} = O(|\lambda|^{-\infty})g & \text{near } \Gamma, \\ v^{(n)} = \mathrm{Op}_\lambda(\chi_n)g + O(|\lambda|^{-\infty})g & \text{on } \Gamma \end{cases} \tag{4.4}$$

with $\operatorname{supp}\chi_n \cap \Sigma = \emptyset$ contribute to v a term of the kind $O(|\lambda|^{-\infty})$. Therefore, we can replace v in the numerator in (4.3) by some $\tilde{v}$ which is a finite sum of $v^{(n)}$ solving (4.4) with $\chi_n \in C_0^\infty(T^*\Gamma)$, $\operatorname{supp}\chi_n \cap \Sigma \neq \emptyset$ of the kind

$$v^{(n)} = \sum_{m=1}^{2} \left(\frac{\lambda_1}{2\pi}\right)^2 \iint e^{i\lambda_1(\psi_m(x,\eta) - y\cdot\eta)} A_m(x,y,\eta,\lambda) g(x,\lambda)\, dy\, d\eta, \tag{4.5}$$

where ψ_m solve the eikonal equations $(\nabla\psi_m)^2 = \alpha^2 c_m^{-2}$, $m = 1,2$ and A_1, A_2 are matrix-valued amplitudes in $S^{1,1}_{0,0}$. According to the construction of the elliptic parametrix (see the Appendix and [G]), we have $c_0 x_1 \leq \mathrm{Im}\,\psi_m$ with some $c_0 > 0$, thus

$$\mathrm{Re}\ (i\lambda_1(\psi_m(x,\eta) - y\cdot\eta)) = -\lambda_1 \mathrm{Im}\,\psi_m \leq -c_0\lambda_1 x_1.$$

Set $\Omega_\rho = \{x \in \Omega;\ \mathrm{dist}(x,\Gamma) \leq \rho\}$. Then we have

$$\|\tilde{v}\|_{H^s(\Omega_{2\rho}\setminus\Omega_\rho)} \leq Ce^{-\gamma\lambda_1\rho}\|g\|_{L^2(\Gamma)}, \qquad s \geq 0, \tag{4.6}$$

with some $C = C(s) > 0$, $\gamma > 0$ provided that $\rho > 0$ is sufficiently small. On the other hand, for any $\rho > 0$ we have

$$\|v\|_{L^2(\Omega_\rho)} \geq \frac{C(\rho)}{|\lambda|^2}\|g\|_{L^2(\Gamma)}. \tag{4.7}$$

Estimate (4.7) follows easily by observing that by the trace theorem (assume ρ fixed)

$$\|g\|_{L^2(\Gamma)} \leq C\|v\|_{H^2(\Omega_\rho)},$$

while the right hand side above can be estimated by $C'(|\lambda|^2+1)\|v\|_{L^2(\Omega_\rho)}$ because the Neumann boundary condition is coercive for Δ_e. Combining (4.3) (fulfilled for $\tilde{v}$ up to an error $O(\lambda_1^{-\infty})$), (4.6) and (4.7), we get

$$\mathrm{Im}\,\lambda^2 \leq Ce^{-c\lambda_1} + O(\lambda_1^{-\infty}), \qquad \lambda \in \Theta.$$

Therefore,

$$\mathrm{Im}\,\lambda_j \leq C_N|\lambda_j|^{-N}, \qquad j = 1,2,\ldots, \quad \forall N > 0,$$

which completes the proof of Theorem 1.1(a). □

Proof of Corollary 1.1. Let λ_j be a sequence of resonances as in Theorem 1.1. Then, according to Proposition 3.1, related to any λ_j there exists $g_j \in H^{3/2}(\Gamma)$, such that $\mathcal{N}(\lambda_j)g_j = 0$. As mentioned above, from (4.1) it follows that $g_j \in C^\infty$. Then for the solutions v_j of (1.3) with Dirichlet data $v_j = g_j$ on Γ we have $v_j \in C^\infty(\bar{\Omega})$. The estimate (4.6) for the parametrix of v proves (1.4). □

5 Existence of resonances converging to the real axis

In this section we prove that there exists a sequence λ_j of resonances such that $\mathrm{Im}\,\lambda_j \leq C_N|\mathrm{Re}\,\lambda_j|^{-N}$.

Proposition 5.1 *There exist constants $a_0 > 0$, $b_0 > 0$, such that $\mathcal{N}(\lambda)$ is invertible in $H^{3/2}(\Gamma)$ on the curves $l_\pm = \{\lambda \in \mathbf{C};\ \mathrm{Im}\,\lambda = \pm a\ln(\mathrm{Re}\,\lambda),\ \mathrm{Re}\,\lambda > b\}$, provided that $0 < a \leq a_0$, $b \geq b_0$, and*

$$\|\mathcal{N}^{-1}(\lambda)\|_{\mathcal{L}(H^{3/2}(\Gamma))} \leq \frac{C(a)}{\ln(\mathrm{Re}\,\lambda)}, \qquad \lambda \in l_\pm.$$

Proof. Here we will use the structure of $\mathcal{N}(\lambda)$ established in Section 3. Note that all the arguments above remain true if we work with ΨDO-s and FIO-s with large parameter $\lambda \in l_\pm$. Since in the hyperbolic and in the mixed regions the corresponding parametrices are elliptic ΨDO-s, we get (compare with (2.14)) that

$$\|\mathrm{Op}_\lambda(\chi)f\| \leq C|\lambda|^{-1}\|\mathcal{N}(\lambda)f\| + C_N|\lambda|^{-N}\|f\|, \tag{5.1}$$

provided that χ is supported in a small neighborhood of a point in the hyperbolic (mixed) region. Here and in what follows until the end of the section, $\|\cdot\| := \|\cdot\|_{H^{3/2}(\Gamma)}$. The same estimate holds in the elliptic region provided that χ is supported outside some fixed neighborhood of Σ. In the glancing regions $J^{-1}N_gJ$ is hypoelliptic (see (3.10)). Therefore,

$$\|\mathrm{Op}_\lambda(\chi)f\| \leq C|\lambda|^{-2/3+\varepsilon}\|\mathcal{N}(\lambda)f\| + C_N|\lambda|^{-N}\|f\|, \tag{5.2}$$

where $\varepsilon > 0$ is sufficiently small if a is small and χ is suitably supported near a point in one of the glancing regions. Next, consider the parametrix N_e in a small neighborhood U in the elliptic region of some $\zeta^0 \in \Sigma$. Since we are going to apply it on functions f with a compact wave front set, we can construct $N_e(\lambda)$ as a ΨDO with large parameter λ_1. According to (3.6), $\det(\sigma_p(N_e))$ coincides with $R(\alpha c_1^{-1}|\eta|_x^{-1})$ up to an elliptic factor. For $\lambda \in l_\pm$ we have $\alpha = 1 \pm ia\ln\lambda_1/\lambda_1$ and α is close to 1 when $\lambda_1 \gg 1$. Since the function $R(s)$ has simple zero at $s = s_0$, we get

$$\det(\sigma_p(N_e)) = \lambda_1^3\left(c_R^2|\eta|_x^2 - \alpha^2\right)R_1(x,\eta,\alpha), \tag{5.3}$$

where $R_1 \neq 0$ for $(x,\eta) \in U$ provided that U is sufficiently close to Σ. Moreover, it can be seen that if (x,η) is close to Σ, $\sigma_p(N_e)$ has three distinct eigenvalues of the kind $\lambda_1 a_1 = \lambda_1 a_1'\,(c_R^2|\eta|_x^2 - \alpha^2)$, $\lambda_1 a_2$, $\lambda_1 a_3$, where a_1', a_2, a_3 do not vanish. Here a_1', a_2, a_3 depend also on λ via α. Thus, one can find a unitary matrix $T(x,\eta,\lambda) \in S^{0,0}_{0,0}$, $(x,\eta) \in U$, such that

$$T^*\sigma_p(N_e)T = \lambda_1\mathrm{diag}(c_R^2|\eta|_x^2 - \alpha^2, 1, 1)S \tag{5.4}$$

in U, where $S = \mathrm{diag}(a_1', a_2, a_3)$ is elliptic. Let us observe that $\lambda_1^2\,(c_R^2|\eta|_x^2 - \alpha^2)$ is the principal symbol of $-c_R^2\Delta_\Gamma - \lambda^2$, where Δ_Γ is the Laplacian on Γ. Therefore, with χ a cut-off function supported in a small neighborhood of ζ^0 we get

$$N_e(\lambda)\mathrm{Op}_{\lambda_1}(\chi)f = \mathrm{Op}_{\lambda_1}(T)\mathrm{diag}\left(\lambda_1^{-1}(-c_R^2\Delta_\Gamma - \lambda^2), \lambda_1, \lambda_1\right)\mathrm{Op}_{\lambda_1}(ST^*)\mathrm{Op}_{\lambda_1}(\chi)f + Mf, \tag{5.5}$$

where M, $\mathrm{Op}_{\lambda_1}(T)$, $\mathrm{Op}_{\lambda_1}(ST^*) \in L^{0,0}_{0,0}$ are ΨDO-s with large parameter λ_1. According to our construction, T, ST^* are elliptic in U. Let χ, χ', χ'' be three cut-off functions with sufficiently small supports in U, such that $\chi = \chi' = \chi'' = 1$ near ζ^0, $\chi' = 1$ in a neighborhood of $\operatorname{supp}\chi$, $\chi'' = 1$ in a neighborhood of $\operatorname{supp}\chi'$. Taking into account that

$$\|\lambda_1^{-1}\left(-c_R^2\Delta_\Gamma - \lambda^2\right)g\| \geq 2a\ln\lambda_1\|g\|, \tag{5.6}$$

and replacing $\mathrm{Op}_{\lambda_1}(\chi)f$ by $\mathrm{Op}_{\lambda_1}(\chi'')f$, we get by (5.5) and Proposition A.1

$$\|\mathrm{Op}_{\lambda_1}(\chi')\left(\mathcal{N}(\lambda)\mathrm{Op}_{\lambda_1}(\chi'')f - Mf\right)\| \geq C\ln\lambda_1\|\mathrm{Op}_{\lambda_1}(\chi)f\| - C_N|\lambda|^{-N}\|f\|.$$

Since $\|M\| \leq$ const., as in the proof of (2.14) using the fact that (2.13) holds for $\mathcal{N}$ as well, we get for any f and $|\lambda|$ sufficiently large

$$\|\mathrm{Op}_\lambda(\chi) f\| \leq C'(\ln \lambda_1)^{-1} \left(\|\mathcal{N}(\lambda) f\| + C''\|f\|\right) \tag{5.7}$$

for any cut-off function χ such that $\chi = 1$ near $\zeta^0 \in \Sigma$ and $\operatorname{supp} \chi$ is sufficiently small.

Now, picking up a partition of unity and summing up (5.1), (5.2) and (5.7) we get the desired estimate. Proposition 5.1 is proved. □

Proposition 5.2 *If $R_\chi(\lambda)$ is analytic in the domain Λ_{C_1,C_2} given by (A.2), then for any $C_1' < C_1$, $C_2' > C_2$, we have*

$$\|R_\chi(\lambda)\|_{\mathcal{L}(L^2;H^2)} \leq C e^{C|\lambda|^4}, \qquad \lambda \in \Lambda_{C_1',C_2'}. \tag{5.8}$$

Proof. Here we will use some arguments from [Vo]. The resolvent $R_\chi(\lambda)$ satisfies the relation

$$R_\chi(\lambda)(I - K(\lambda)) = K_1(\lambda), \qquad \lambda \in \mathbf{C},$$

where

$$\begin{aligned} K(\lambda) &= [\chi_1, \Delta_e]\left(R_0(\lambda)\eta - R_0(\lambda_0)\eta\right) K_2 + (\lambda^2 - \lambda_0^2)\chi_2 R_\chi(\lambda_0), \\ K_1(\lambda) &= (1-\chi_1)\left(\chi R_0(\lambda)\eta - \chi R_0(\lambda_0)\eta\right) K_2 + R_\chi(\lambda_0) \\ K_2 &= (1-\chi_2)\chi + [\chi_2, \Delta_e] R(\lambda_0)\chi. \end{aligned}$$

Here λ_0 is an arbitrary point with $\operatorname{Im}\lambda_0 < 0$, say $\lambda_0 = -i$ and χ_1, χ_2, η are cut-off functions in $C_0^\infty(\mathbf{R}^3)$, such that $\chi_1 = 1$ in a neighborhood of the obstacle $\mathcal{O}$, $\chi_2 = 1$ on $\operatorname{supp}\chi_1$, $\chi = 1$ on $\operatorname{supp}\chi_2$, $\eta = 1$ on $\operatorname{supp}(1-\chi_2)\chi$ and $\eta = 0$ on $\operatorname{supp}\chi_1$. Note that $K_2 \in \mathcal{L}(L^2)$ is independent of λ. As in [Vo] we see that $K^2(\lambda)$ is a trace class operator and

$$R_\chi(\lambda)\left(I - K^2(\lambda)\right) = \tilde{K}_1(\lambda), \tag{5.9}$$

where $\tilde{K}_1(\lambda) = K_1(\lambda)(I + K(\lambda))$. It is easy to see that

$$\|\tilde{K}_1(\lambda)\|_{\mathcal{L}(L^2;H^2)} \leq C e^{C|\lambda|}, \qquad \lambda \in \mathbf{C}. \tag{5.10}$$

Let us introduce the function

$$h(\lambda) = \det\left(I - K^2(\lambda)\right).$$

Then $h(\lambda)$ is an entire function, $h(\lambda_0) = 1$ and one can prove as in [Vo] that

$$|h(\lambda)| \leq \prod_{j=1}^{\infty}\left(1 + \mu_j(K^2(\lambda))\right) \leq C e^{C|\lambda|^3}, \qquad \lambda \in \mathbf{C}, \tag{5.11}$$

where $\mu_j(K^2)$ are the characteristic values of K^2. Let λ_j be the zeros of $h(\lambda)$ in $\mathbf{C}$ and denote by V the domain

$$V = \mathbf{C} \setminus \bigcup_{j=1}^{\infty}\{\lambda \in \mathbf{C};\ |\lambda - \lambda_j| \leq |\lambda_j|^{-4}\}.$$

Then by [Ti, Ch. VIII] we conclude from (5.11) that

$$|h(\lambda)|^{-1} \leq C' e^{C'|\lambda|^4}, \qquad \lambda \in V. \tag{5.12}$$

On the other hand, we have (see e.g. [GK, Thm. 5.1])

$$\left|\det\left(I - K^2(\lambda)\right)\right| . \left\|\left(I - K^2(\lambda)\right)^{-1}\right\|_{\mathcal{L}(L^2)} \leq \prod_{j=1}^{\infty}\left(1 + \mu_j(K^2(\lambda))\right) \leq C e^{C|\lambda|^3}. \tag{5.13}$$

By (5.12) and (5.13) we deduce

$$\left\|\left(I - K^2(\lambda)\right)^{-1}\right\|_{\mathcal{L}(L^2)} \leq C' e^{C'|\lambda|^4}, \qquad \lambda \in V. \tag{5.14}$$

Relations (5.9), (5.10) and (5.14) imply

$$\|R_\chi(\lambda)\|_{\mathcal{L}(L^2;H^2)} \leq C'' e^{C''|\lambda|^4}, \qquad \lambda \in V. \tag{5.15}$$

Now, let us observe that $\mathbf{C} \setminus V = \bigcup_{j=1}^{\infty} U_j$, where U_j are disjoint connected sets and each U_j is a union of a finite number of disks, because the series $\sum_{j=1}^{\infty} |\lambda_j|^{-4}$ is convergent. Let us denote $M = \sum |\lambda_j|^{-4}$. Then $\operatorname{diam} U_j < 2M$ for each j. Let $C_1' < C_1$. Denote $J = \{j \in \mathbf{N};\ \Lambda_{C_1',C_2} \cap U_j \neq \emptyset\}$. Then only a finite number of U_j-s could be not entirely included in Λ_{C_1,C_2}, i.e. $U_j \subset \Lambda_{C_1,C_2}$ for $j \geq j_0$, $j \in J$. Since (5.15) holds on ∂U_j and for $j \geq j_0$, $j \in J$, the cut-off resolvent $R_\chi(\lambda)$ is analytic in U_j, by the maximum principle we get that (5.15) holds in U_j for $j \geq j_0$, $j \in J$ with some other constant C''. Clearly, by choosing C'' sufficiently large, we can arrange (5.15) in the compact $\bar{\Lambda}_{C_1',C_2'} \cap \bigcup_{j<j_0} U_j$ as well. Therefore, (5.15) holds in the whole $\Lambda_{C_1',C_2'}$. □

Now, let us assume that there is only a finite number of resonances λ_j in Λ, i.e. if C_2 is sufficiently large, then $\mathcal{N}^{-1}(\lambda)$ is analytic in Λ. Consider $\Lambda_{a,b} = \{\lambda \in \mathbf{C};\ |\operatorname{Im}\lambda| \leq a\ln(\operatorname{Re}\lambda),\ \operatorname{Re}\lambda \geq b\}$. Choose $a < C_1$, $b > C_2$ so that $\Lambda_{a,b} \subset \Lambda$. Then $\partial\Lambda_{a,b}$ consists of the curves $l_\pm = \{\lambda \in \mathbf{C};\ \operatorname{Im}\lambda = \pm a\ln(\operatorname{Re}\lambda), \operatorname{Re}\lambda \geq b\}$ and the interval $\operatorname{Re}\lambda = b$, $|\operatorname{Im}\lambda| \leq a\ln b$. Let a, b be such that Proposition 5.1 holds. Then

$$\|\mathcal{N}^{-1}(\lambda)\|_{\mathcal{L}(H^{3/2}(\Gamma))} \leq \frac{C}{\ln(\operatorname{Re}\lambda)}, \qquad \lambda \in \partial\Lambda_{a,b}. \tag{5.16}$$

On the other hand, Proposition 5.2 implies that we have the following a priori estimate

$$\|\mathcal{N}^{-1}(\lambda)\|_{\mathcal{L}(H^{3/2}(\Gamma))} \leq C e^{C|\lambda|^4}, \qquad \lambda \in \Lambda_{a,b}, \tag{5.17}$$

under the assumption that $\mathcal{N}^{-1}(\lambda)$ is analytic in Λ. Indeed, (5.17) follows immediately from Proposition 5.2 and the relation (see [SV])

$$\mathcal{N}^{-1}(\lambda) = \gamma E - \gamma R_\chi(\lambda)(\Delta_e + \lambda^2)E,$$

where $\gamma f := f|_\Gamma$, $E : H^{1/2}(\Gamma) \to H^2(\Omega)$ is a fixed extension map from Γ to some small neighborhood of Γ in Ω, such that Ef satisfies the Neumann boundary conditions $\sum \boldsymbol{\sigma}_j(Ef)\nu_j = f$

on Γ. We are in position now to apply the Phragmén-Lindelöf principle (see [Ti]) in $\Lambda_{a,b}$ to the function $(\log\lambda)\mathcal{N}^{-1}(\lambda)$. Here $\log\lambda$ takes its principal branch $\ln\lambda$ for $\lambda\in\mathbf{R}_+$. We thus get by (5.16), (5.17)

$$\|\mathcal{N}^{-1}(\lambda)\|_{\mathcal{L}(H^{3/2}(\Gamma))}\le\frac{C}{\ln(\operatorname{Re}\lambda)},\qquad\lambda\in\Lambda_{a,b}. \tag{5.18}$$

The final step in our proof is to show that (5.18) leads to a contradiction for real λ. Assume in what follows that $\lambda\in\Lambda_{a,b}\cap\mathbf{R}$, i.e. $\lambda>C_2$. Denote the eigenvalues of $-c_R^2\Delta_\Gamma$ by λ_j^2, $j=1,2,\dots$ and the corresponding eigenfunctions by φ_j with $\|\varphi_j\|=1$. Recall that $\|\cdot\|$ stands for the norm in $H^{3/2}(\Gamma)$. Fix $\zeta^0\in\Sigma$ and let χ be supported in a small neighborhood U of ζ^0 in the elliptic region. Let us recall that the eigenvalues of $\sigma_p(N_e)$ in U are $\lambda a_1=\lambda(c_R^2|\eta|_x^2-1)a_1'$, λa_2, λa_3, where a_1', a_2, a_3 are smooth and do not vanish provided that U is sufficiently small. Let $\Pi(x,\eta)$, $(x,\eta)\in U$ be the projection onto the eigenspace corresponding to λa_1. Set

$$f_k(\,\cdot\,,\lambda_j)=\operatorname{Op}_{\lambda_j}(\chi\Pi)e_k\varphi_j, \tag{5.19}$$

$\{e_k\}_{k=1}^3$ being the standard base in $\mathbf{R}^3$. Denote $\Theta=\{\lambda_j\}_{j=1}^\infty$ and $f_k(x,\lambda)=f_k(x,\lambda_j)$, $\varphi(x,\lambda)=\varphi_j(x)$ for $\lambda\in\Theta$. Consider all ΨDO-s bellow as ΨDO-s with large parameter $\lambda\in\Theta$. Then

$$\mathcal{N}(\lambda)f_k=Ae_k\varphi, \tag{5.20}$$

where $A\in L^{0,1}_{0,0}(\Gamma)$, $\sigma_p(A)=\lambda(c_R^2|\eta|_x^2-1)a_1'\chi\Pi$. Since the principal symbol of $-c_R^2\Delta_\Gamma-\lambda^2$ is $\lambda^2(c_R^2|\eta|_x^2-1)$, we have

$$\mathcal{N}(\lambda)f_k=\lambda^{-1}\operatorname{Op}(\chi a_1'\Pi)\left(-c_R^2\Delta_\Gamma-\lambda^2\right)e_k\varphi+Be_k\varphi=Be_k\varphi, \tag{5.21}$$

where $B\in L^{0,0}_{0,0}(\Gamma)$. Thus

$$\|\mathcal{N}(\lambda)f_k\|\le C\qquad\text{for }k=1,2,3;\ \ \lambda\in\Theta. \tag{5.22}$$

According to (5.18), (5.19)

$$\|\operatorname{Op}(\chi\Pi)e_k\varphi\|\le\frac{C}{\ln\lambda}\qquad\text{for }k=1,2,3;\ \lambda\in\Theta. \tag{5.23}$$

Since the projection $\Pi(\zeta)$ is well defined and does not vanish near Σ, we have that $\sum|\Pi_{ij}|^2$ is elliptic in U provided that U is sufficiently close to Σ. Thus from (5.23) we deduce that

$$\|\operatorname{Op}(\chi'\chi'')\varphi\|\le\frac{C}{\ln\lambda}, \tag{5.24}$$

where $\chi'=\chi'(x)$, $\chi''=\chi''(\eta)$ and $\chi'(x)=1$, $\chi''(\eta)=1$ for (x,η) close to ζ^0, $\operatorname{supp}\chi'\chi''\subset\{\chi=1\}$. On the other hand, $(-c_R^2\Delta_\Gamma-\lambda^2)\varphi=0$ and $-c_R^2\Delta_\Gamma-\lambda^2$ is a ΨDO on Γ in $L^{2,2}_{0,0}(\Gamma)$ with principal symbol $\lambda^2(c_R^2|\eta|_x^2-1)$ elliptic outside Σ. Therefore, $\widetilde{\operatorname{WF}}(\varphi)\subset\Sigma$. Hence,

$$\|\operatorname{Op}(\chi'(1-\chi''))\varphi\|\le C_N\lambda^{-N},\qquad\forall N>0. \tag{5.25}$$

Combining (5.24) and (5.25) we get

$$\|\chi'\varphi\| \le \frac{C}{\ln\lambda}, \qquad \lambda \in \Theta,$$

for any cut-off function χ', such that $\chi' = 1$ near $x^0 = \pi_x(\zeta^0)$ and $\operatorname{supp}\chi'$ is sufficiently small. Since $\zeta^0 \in \Sigma$ was arbitrary, we get $\|\varphi\| \le C/\ln\lambda$ which contradicts the fact that $\|\varphi\| = 1$. The proof of Theorem 1.1 is complete. □

A Appendix

A.1 ΨDO-s and FIO-s with large parameter

We are going to construct a parametrix of the solution of the following problem

$$\begin{cases} (\Delta+\lambda^2)u &= 0 & \text{in } \Omega, \\ u &= f & \text{on } \Gamma, \\ u & — & \lambda\text{-outgoing} \end{cases} \tag{A.1}$$

as well as for the corresponding Neumann operator. Here λ is a large complex parameter and we will assume that

$$\lambda \in \Lambda := \{\lambda \in \mathbf{C};\ |\lambda_2| < C_1 \ln\lambda_1,\ \lambda_1 > C_2\}, \tag{A.2}$$

where $\lambda_1 = \operatorname{Re}\lambda$, $\lambda_2 = \operatorname{Im}\lambda$, $C_1 > 0$ is an arbitrary chosen constant, while C_2 is a large constant that will be specified latter. The definition of λ-outgoing function is the same as Definition 2.1 with the only difference that $R_0(\lambda)$ there has to be replaced by the free outgoing resolvent $S_0(\lambda)$ of the Laplacian in $\mathbf{R}^n$.

We will follow essentially Gérard [G] and Taylor [T2] with some modifications. We will deal with Pseudodifferential Operators (ΨDO-s) with large parameter λ. We refer to [G] (see also [D]) for more details about these operators and here we will give only some basic definitions and properties. Given an open set X in $\mathbf{R}^n$ denote by $\tilde{C}^\infty(X)$ the space of all functions $u(x,\lambda)$, $\lambda \in \Lambda$ such that $u(\,\cdot\,,\lambda) \in C^\infty(X)$ and $p(u(\,\cdot\,,\lambda)) = O(|\lambda|^{-\infty})$ for all seminorms p in $C^\infty(X)$. By a similar way we define $\tilde{C}^\infty(K)$, K being a compact, $\tilde{C}_0^\infty(X)$ and $\tilde{\mathcal{D}}'(X)$.

Given two open sets X, Y in $\mathbf{R}^n$, we set (see [G, Def. A.I.2]) for $m,k \in \mathbf{R}$, $\rho,\delta \in [0,1)$ the class $S^{m,k}_{\rho,\delta}(X\times Y)$ to be the set of all $a(x,y,\eta,\lambda) \in C^\infty(X\times Y\times\mathbf{R}^n)$, such that for any compact $K \subset\subset X\times Y$, all $\alpha,\beta,\gamma \in \mathbf{Z}^n$, $\lambda \in \Lambda$ we have

$$|\partial_x^\alpha \partial_y^\beta \partial_\eta^\gamma a| \le C_{\alpha,\beta,\gamma,K}|\lambda|^{k+\rho|\gamma|+\delta|\alpha+\beta|}(1+|\eta|)^{m-|\gamma|}. \tag{A.3}$$

If $X = Y$, we set $S^{m,k}_{\rho,\delta}(X) = S^{m,k}_{\rho,\delta}(X\times X)$. Given $a \in S^{m,k}_{\rho,\delta}(X\times Y)$, denote by $\operatorname{Op}(a)$ the operator

$$(\operatorname{Op}(a)u)(x,\lambda) = \left(\frac{\lambda}{2\pi}\right)^n \iint e^{i\lambda(x-y)\cdot\eta} a(x,y,\eta,\lambda)u(y,\lambda)\,dy\,d\eta. \tag{A.4}$$

If a has bounded support with respect to η, then $\mathrm{Op}(a)$ is well defined operator mapping $\tilde{C}_0^\infty(Y)$ into $\tilde{C}^\infty(X)$. On the other hand, when η could take arbitrary large values we will consider operators similar to those defined above with $\lambda_1 = \mathrm{Re}\,\lambda$ in the exponential. Denote $L^{m,k}_{\rho,\delta} = \mathrm{Op}\left(S^{m,k}_{\rho,\delta}\right)$. We say that $\mathrm{Op}(a)$ is properly supported if the distribution kernel of $\mathrm{Op}(a)$ is properly supported uniformly in λ. Then any ΨDO in $L^{m,k}_{\rho,\delta}$ can be represented as a properly supported ΨDO plus a neglectible operator (from $L^{-\infty,-\infty}_{\rho,\delta}$). It is convenient to set

$$\hat{u}(\eta,\lambda) = \int e^{-i\lambda\eta\cdot x} u(x,\lambda)\, dx. \tag{A.5}$$

Operators from $L^{m,k}_{\rho,\delta}$ have properties similar to those of the ordinary ΨDO-s (see for example [T2]). Given $a(x,y,\eta,\lambda) \in S^{m,k}_{\rho,\delta}$ with $\rho+\delta<1$, one can find a symbol $\sigma(A)$, where $A = \mathrm{Op}(a)$, depending only on x, η, λ such that A and $\mathrm{Op}(\sigma(A))$ differ by a neglectible operator (see [G, Pr. A.I.4]). By [G, Pr. A.I.5], if $A_j \in L^{m_j,k_j}_{\rho,\delta}(X)$, $j=1,2$ with $\rho+\delta<1$, then $A_1A_2 \in L^{m_1+m_2,k_1+k_2}_{\rho,\delta}(X)$ and

$$\sigma(A_1A_2) \sim \sigma(A_1)\circ\sigma(A_2) := \sum_{|\alpha|\geq 0} \frac{1}{\alpha!}\lambda^{-|\alpha|}\partial_\eta^\alpha\sigma(A_1)D_x^\alpha\sigma(A_2).$$

We note that the class of operators $L^{m,k}_{\rho,\delta}$ is the same considered by Gérard [G]. The only difference is that we allow $\lambda\in\Lambda$ if the support of the amplitude with respect to η is bounded, where Λ is the logarithmic domain defined in (A.2). Here and below we assume that X is always bounded, because we will work locally in small neighborhoods of boundary points. Note that the exponential in the definition of $\mathrm{Op}(a)$ is polynomially bounded in λ, if $\lambda\in\Lambda$. It is useful to note that operators of the form $\mathrm{Op}_\lambda(a)$ can be represented as ΨDO-s with large parameter λ_1 provided that $|\eta|$ is bounded on $\mathrm{supp}\,a$. Occasionally we will use the notations $\mathrm{Op}_\lambda(a)$ and $\mathrm{Op}_{\lambda_1}(a)$ in order to distinguish between ΨDO-s with large parameter λ and λ_1, respectively. To this end we write

$$\begin{aligned}\mathrm{Op}_\lambda(a)u &= \left(\frac{\lambda}{2\pi}\right)^n \iint e^{i\lambda(x-y)\cdot\eta} a(x,y,\eta,\lambda)u(y,\lambda)\,dy\,d\eta\\ &= \left(\frac{\lambda_1}{2\pi}\right)^n \iint e^{i\lambda_1(x-y)\cdot\eta} \tilde{a}(x,y,\eta,\lambda)u(y,\lambda)\,dy\,d\eta,\end{aligned}$$

where $\tilde{a}(x,y,\eta,\lambda) = (1+i\tan\arg\lambda)^n e^{-\lambda_2(x-y)\cdot\eta}a(x,y,\eta,\lambda)$. Assuming that $\lambda\in\Lambda$ we have $|e^{-\lambda_2(x-y)\cdot\eta}| < |\lambda|^N$ with some fixed N. Moreover, it is not hard to check that if $a\in S^{0,k}_{\rho,\delta}$, then $\tilde{a}$ is also an amplitude and $\tilde{a}\in S^{0,k+N}_{\rho+\varepsilon,\delta+\varepsilon}$ for any $\varepsilon>0$. The latter follows from the fact that $|\lambda_2|\leq C_\varepsilon|\lambda|^\varepsilon$ for $\lambda\in\Lambda$. We can consider here $\lambda_2/\ln\lambda_1\in[-C_1,C_1]$ (see (A.2)) as an additional parameter and then $\tilde{a}$ is continuous in this parameter in the Fréchet topology defined by the seminorms appearing in (A.3) (see also [T2]). Let us calculate the symbol a_0 (depending only on x, η, λ) of $\tilde{a}$. By [G, Pr. A.I.4, Pr. A.I.5], we get that in fact $a_0\in S^{0,k}_{\rho,\delta}$ and the principal symbol of $\mathrm{Op}_\lambda(a)$ considered as a ΨDO with large parameter λ_1 is $(1+i\tan\arg\lambda)^n\left(a(x,x,\eta,\lambda)+O(\ln|\lambda|/|\lambda|)\right)$. Therefore, if $A\in L^{0,k}_{\rho,\delta}$ with $\rho+\delta<1$, $\sigma(A)$ has bounded support in all variables, we have

$$\|A\|_{\mathcal{L}(L^2(X))} \leq C|\lambda|^k, \qquad \lambda\in\Lambda, \tag{A.6}$$

where C depends on A.

If η is unbounded on supp a, then $\mathrm{Op}(a)$ is well defined only for real λ, because for complex λ the integrand in (A.4) could be exponentially increasing. In this case, if $A \in L^{0,k}_{\rho,\delta}$, $\rho+\delta<1$, X is bounded and $|\sigma(A)(x,\eta,\lambda)| \le M$, we have

$$\|A\|_{\mathcal{L}(L^2(X))} \le M|\lambda|^k + C_N|\lambda|^{-N}, \qquad \lambda \in \mathbf{R} \tag{A.7}$$

for any $N \ge 0$ (see [G, Pr. A.I.6]).

We refer to [G, Def. A.I.10] for a definition of elliptic ΨDO-s elliptic symbols on $\hat{T}^*X = T^* \cup S^*X$ as well as for a definition of the wave front set $\widetilde{\mathrm{WF}}(f)$ of a distribution $f \in \tilde{\mathcal{D}}'(X)$ suggested by J. Sjöstrand. The following proposition about the invertibility of microlocally elliptic operators is used frequently in the paper.

Proposition A.1 *Let $A \in L^{0,k}_{\rho,\delta}(X)$ with $\rho+\delta<1$, X bounded, be elliptic in $\bar{U}$, where $U \subset \hat{T}^*X$ is an open set. We assume that either $\lambda \in \Lambda$ and U is bounded or $\lambda \in \mathbf{R}$. Let $\chi(x,\eta,\lambda) \in S^{0,0}_{0,0}$ be any amplitude with* supp $\chi \subset U$ *and let $\chi' \in C^\infty(T^*X)$ be any function such that $\chi' = 1$ on* supp χ. *Then for any $N>0$ and $s>0$ we have*

$$\|\mathrm{Op}(\chi)f\|_{H^s} \le C|\lambda|^{-k}\|\mathrm{Op}(\chi')Af\|_{H^s} + C_N|\lambda|^{-N}\|f\|_{H^s}.$$

Proof. Since $\sigma(A)$ is elliptic in U, there exists a symbol $b(x,\eta,\lambda) \in S^{0,-k}_{\rho,\delta}$, such that $b \circ \sigma(A) \sim 1$ in U. The proof of this assertion is the same as in the classical case (see e.g. [S]). Therefore, $\mathrm{Op}(\chi)\mathrm{Op}(b)Af = \mathrm{Op}(\chi)f + Rf$, where R has kernel in $\tilde{C}^\infty(X \times X)$. In order to complete the proof it is sufficient to observe that modulo neglectible operators

$$\mathrm{Op}(\chi)\mathrm{Op}(b)A = \mathrm{Op}(\chi)\mathrm{Op}(b)\mathrm{Op}(\chi')A.$$

□

Let us note that if U coincides with $\hat{T}^*X$, then the ellipticity of P implies that there exists $\tilde{P}$, such that $\tilde{P}P = I + R$, where $R \in L^{-\infty,-\infty}_{\rho,\delta}$, thus for large λ there exists the inverse operator $P^{-1} = (I+R)^{-1}\tilde{P}$. For $A \in L^{m,k}_{\rho,\rho}(X)$ with $\rho < 1/2$ one can define invariantly the principal symbol $\sigma_p(A) \in S^{m,k}_{\rho,\rho}(T^*X)/S^{m-1,k-(1-2\rho)}_{\rho,\rho}(T^*X)$.

We introduce also the Fourier Integral Operators (FIO-s) with large parameter, given by

$$(I_{a,\phi}u)(x,\lambda) = \left(\frac{\lambda}{2\pi}\right)^n \iint e^{i\lambda(\phi(x,\eta)-y\cdot\eta)}a(x,y,\eta,\lambda)u(y,\lambda)\,dy\,d\eta,$$

mapping $\tilde{C}^\infty_0(X)$ into $\tilde{C}^\infty(X)$. Here $a \in S^{m,k}_{\rho,\delta}(X)$, ϕ is a non-degenerate smooth phase function. We will deal with FIO-s for which η is bounded on supp a and $\lambda \in \Lambda$ or η is unbounded but $\lambda \in \mathbf{R}$. In both cases $I_{a,\phi}$ is well-defined. According to [G, Pr. A.I.9], if $\lambda \in \mathbf{R}$, $m=k=\rho=\delta=0$, $\mathrm{Im}\,\phi = 0$, then $\|I_{a,\phi}\|_{\mathcal{L}(L^2(X))}$ is uniformly bounded in λ. When η is bounded on supp a, then it is easy to get estimates of $\|I_{a,\phi}\|_{\mathcal{L}(L^2(X))}$ even for complex-valued ϕ.

We begin with construction of a parametrix to the solution of (A.1) following [G] and [T2]. Assume that $\Omega \subset \mathbf{R}^{n+1}$ is the exterior of a strictly convex body $\mathcal{O}$. The points in $\hat{T}^*\Gamma$ can be divided naturally into the following three regions

hyperbolic region $\{\zeta \in T^*\Gamma;\ \|\zeta\| < 1\}$,
glancing region $\{\zeta \in T^*\Gamma;\ \|\zeta\| = 1\}$,
elliptic region $\{\zeta \in \hat{T}^*\Gamma;\ \|\zeta\| > 1\}$.

Here $\|\cdot\|$ is the norm in $T^*\Gamma$. Let $\tilde{U}$ be a sufficiently small neighborhood of Γ and set $U = \tilde{U} \cap \Omega$. We will construct an operator $H(\lambda) : \tilde{C}^\infty(\Gamma) \to \tilde{C}^\infty(U)$, such that

$$\begin{cases} (\Delta + \lambda^2)Hf &= Kf \quad \text{in } U, \\ Hf &= f + Rf \quad \text{on } \Gamma, \end{cases}$$

where K, R have kernels in $\tilde{C}^\infty(U \times \Gamma)$ and $\tilde{C}^\infty(\Gamma \times \Gamma)$, respectively. Moreover, Hf will have some outgoing properties that will guarantee that Hf is a parametrix for the (λ-outgoing) solution to (A.1).

We will use the following notations. Given $x^0 \in \Gamma$, we choose $(z_1, \dots, z_{n+1})$ to be Euclidean coordinates such that $x = x^0$ corresponds to $z = 0$ and Γ is given locally by $z_1 = F(z_2, \dots, z_{n+1})$ with $F(0) = 0$, $\nabla F(0) = 0$. Then we set $x_1 = z_1 - F(z_2, \dots, z_{n+1})$, $x' = (z_2, \dots, z_{n+1})$. So, in these coordinates $x^0 = (0, 0)$, $x_1 > 0$ in Ω and the normal derivative $\partial/\partial\nu$ at $x^0 = (0, 0)$ is given by $\partial/\partial x_1$. Moreover, $x' = (z_2, \dots, z_{n+1})$ are local coordinates on Γ. In the sequel $\mathrm{Op}(\chi)$ will always denote the ΨDO with full symbol χ in the coordinates x'. Respectively, $|\eta|_x$ is the norm of the covector (x, η) written in the coordinates associated with x'.

A.2 The hyperbolic region

Fix $\zeta^0 \in T^*\Gamma$ with $\|\zeta^0\| < 1$. In the local coordinates defined above ζ^0 is given by $(0, \eta^0)$ with $|\eta^0| < 1$. Let $\chi \in C_0^\infty(T^*\Gamma)$ be a cut-off function supported in the hyperbolic region such that $\chi = 1$ near $(0, \eta^0)$. We will show that if $\operatorname{supp}\chi$ is sufficiently small, then there exists a FIO $H_h : \tilde{C}^\infty(\Gamma) \to \tilde{C}^\infty(U)$ with large parameter λ, such that

$$\begin{cases} (\Delta + \lambda^2)H_h f &= K_h f \quad \text{in } U, \\ H_h f &= \mathrm{Op}(\chi) f \quad \text{on } \Gamma, \end{cases} \tag{A.8}$$

$\lambda \in \Lambda$, where K_h has kernel in $\tilde{C}^\infty(U \times \Gamma)$ and H_h has the form

$$(H_h f)(x, \lambda) = \left(\frac{\lambda}{2\pi}\right)^n \iint e^{i\lambda(\psi(x,\eta) - y\cdot\eta)} a(x, y, \eta, \lambda) f(x, \lambda)\, dy\, d\eta.$$

The construction here goes along the same lines as that in [G] and the fact that $\lambda \in \Lambda$ does not lead to any complications. The phase function ψ solves the eikonal equations

$$\begin{cases} (\nabla\psi)^2 &= 1, \\ \psi|_\Gamma &= x \cdot \eta, \end{cases}$$

and $\frac{\partial\psi}{\partial\nu}(0, \eta^0) < 0$, where ν is the inner normal to $\Gamma = \partial\Omega$. The amplitude $a(x, y, \eta, \lambda)$ belongs to $S^{0,0}_{0,0}$ and has the asymptotic expansion

$$a \sim \sum_{j=0}^{\infty} a_j(x, y, \eta)\lambda^{-j},$$

where a_j solve the transport equations

$$\left\{\begin{array}{rcl} 2i\nabla\psi\cdot\nabla a_j + i\Delta\psi\cdot a_j &=& -\Delta a_{j-1}, \\ a_j|_\Gamma &=& \delta_{j,0}\chi(y,\eta). \end{array}\right.$$

According to [G, Corollary A.II.4] we have

$$\begin{aligned} \widetilde{\mathrm{WF}}'(H_h) \subset \{&(x,\xi,y,\eta)\in T^*(U\setminus\Gamma)\times T^*\Gamma;\ |\xi|_x = 1,\ (x,\xi) \text{ belongs} \\ &\text{to the outgoing ray issued from } (y,\eta)\in\operatorname{supp}\chi\}, \end{aligned} \tag{A.9}$$

which will help us in Section A.5 to deduce that H_h is a parametrix of the λ-*outgoing* solution to (2.1).

A.3 The glancing region

Here we will make some modifications in the scheme proposed by Gérard [G]. Let $\zeta^0 \in T^*\Gamma$, $\|\zeta^0\| = 1$, i.e. in local coordinates $\zeta^0 = (0,\eta^0)$ with $|\eta^0| = 1$. Let $\chi \in C_0^\infty(T^*\Gamma)$ be a cut-off function equal to 1 near $(0,\eta^0)$ and having its support in a small neighborhood of that point. We will construct an operator $H_g : \tilde{C}^\infty(\Gamma) \to \tilde{C}^\infty(U)$ such that

$$\left\{\begin{array}{rcll} (\Delta+\lambda^2)H_g f &=& K_g f & \text{in } U, \\ H_g f &=& \mathrm{Op}(\chi)f + R_g f & \text{on } \Gamma, \end{array}\right. \tag{A.10}$$

where K_g has kernel in $\tilde{C}^\infty(U\times\Gamma)$, R_g has kernel in $\tilde{C}^\infty(\Gamma\times\Gamma)$. The operator H_g has the form $H_g = \tilde{H}_g J^{-1}\mathrm{Op}(\chi)$, where J is an elliptic local FIO on Γ with large parameter $\lambda \in \Lambda$ with amplitude of class $S^{0,0}_{0,0}$ and

$$\begin{aligned} \tilde{H}_g w \;=\; \left(\frac{\lambda}{2\pi}\right)^n \int e^{i\lambda\theta(x,\eta)} \Bigg[& g_0(x,\eta,\lambda)\frac{\mathrm{Ai}_-(\lambda^{2/3}\rho(x,\eta))}{\mathrm{Ai}_-(\lambda^{2/3}\alpha)} \\ & + i\lambda^{-1/3} g_1(x,\eta,\lambda)\frac{\mathrm{Ai}'_-(\lambda^{2/3}\rho(x,\eta))}{\mathrm{Ai}_-(\lambda^{2/3}\alpha)}\Bigg]\hat{w}(\eta,\lambda)\,d\eta. \end{aligned}$$

Here $\alpha = |\eta| - 1$, $\lambda\in\Lambda$ and $\mathrm{Ai}_-(s) = \mathrm{Ai}(e^{-2\pi i/3}s)$, where Ai is the Airy function. Recall that (see (A.5))

$$\hat{w}(\eta,\lambda) = \int e^{-i\lambda y\cdot\eta} w(y,\lambda)\,dy.$$

We suppose that $\widetilde{\mathrm{WF}}(w)$ is contained in a small neighborhood of $\alpha = 0$. One can see that the construction in [G] goes without complications in the more general case $\lambda\in\Lambda$. In particular, $\lambda\in\Lambda$ implies that $\arg\lambda = O(\ln\lambda_1/\lambda_1)$ can be assumed arbitrary small if $C_2 \gg 1$ (see (A.2)), so $\lambda^{2/3}\rho$ and $\lambda^{2/3}\alpha$ are away from the zeros of Ai_- that lie on the line $\arg s = -\pi/3$ and all the estimates of $\mathrm{Ai}_-(\lambda^{2/3}\rho)$, $\mathrm{Ai}_-(\lambda^{2/3}\alpha)$, $\mathrm{Ai}'_-(\lambda^{2/3}\rho)$ used in [G] remain true. We will modify the parametrix a little bit in order to keep it closer in spirit to [T2] (see also [M]).

The phase functions ρ, θ solve the eikonal equations

$$\left\{\begin{array}{rcl} (\nabla\theta)^2 - \rho(\nabla\rho)^2 &=& 1, \\ \nabla\theta\cdot\nabla\rho &=& 0 \end{array}\right.$$

exactly in $\alpha \le 0$ and of order $O(\alpha^\infty)$ in $\alpha > 0$. Further, ρ, θ have the properties

$$\begin{cases} \left|\det \frac{\partial^2\theta}{\partial x' \partial \eta}\right| & \neq & 0 & \text{on } x_1 = 0, \\ \rho & = & \alpha + O(|\alpha|^\infty) & \text{on } x_1 = 0, \\ \rho & = & \alpha & \text{for } x_1 = 0,\ \alpha > 0, \\ \frac{\partial \rho}{\partial x_1} & < & 0 & \text{on } x_1 = 0. \end{cases} \tag{A.11}$$

The amplitudes g_0, g_1 solve the corresponding transport equations and

$$g_0(0, \eta^0, \lambda) \neq 0, \qquad g_1(0, \eta^0, \lambda) = 0.$$

Moreover, $g_1 = O(|\alpha|^\infty)$ together with its derivatives. Existence of such ρ, θ, g_1, g_0 follows directly from [T2].

It is shown in [G, pp. 114–124] that $\tilde{H}_g$ solves the Helmholtz equation up to an error $O(|\lambda|^{-\infty})$ for $\lambda \in \Lambda$. Let us see what kind of boundary conditions $\tilde{H}_g$ satisfies. Here we will follow [T2]. For $x_1 = 0$ we have

$$\tilde{H}_g w|_{x_1=0} = \left(\frac{\lambda}{2\pi}\right)^n \int e^{i\lambda\theta_0} \left[g_0 \frac{\mathrm{Ai}_-(\lambda^{2/3}\rho_0)}{\mathrm{Ai}_-(\lambda^{2/3}\alpha)} + i\lambda^{-1/3} g_1 \frac{\mathrm{Ai}'_-(\lambda^{2/3}\rho_0)}{\mathrm{Ai}_-(\lambda^{2/3}\alpha)} \right] \hat{w}\, d\eta. \tag{A.12}$$

Here $\theta_0 = \theta|_{x_1=0}$, $\rho_0 = \rho|_{x_1=0}$. Recall (A.11) that $\rho_0 = \alpha$ for $\alpha > 0$. Set $\varphi_0 = \theta_0 + \gamma$, where

$$\gamma = -\frac{2}{3}(-\rho_0)^{3/2} + \frac{2}{3}(-\alpha)^{3/2}.$$

Note that $\gamma = 0$ for $\alpha > 0$. Then (compare with [T2, p. 237])

$$\frac{\mathrm{Ai}_-(\lambda^{2/3}\rho_0)}{\mathrm{Ai}_-(\lambda^{2/3}\alpha)} = B(x', \eta, \lambda) e^{i\lambda\gamma(x', \eta)}.$$

We have $B = 1$, $\gamma = 0$ for $\alpha > 0$. Similarly to [T2, Lemma X.4.1] one can prove the following.

Lemma A.1 *$B(x', \eta, \lambda) \in S^{0,0}_{0,0}(\Gamma)$ near $\alpha = 0$.*

Proof. One can argue as in the proof of Lemma X.4.1 in [T2], but we will prove Lemma A.1 as a direct consequence of Taylor's lemma. If we compare our quantities ρ, η etc. with those in [T2] that we will denote by $\tilde{\rho}$, $\tilde{\xi}$, etc., we see that for real λ we have

$$\frac{\tilde{\rho}}{|\tilde{\xi}|} = -\rho, \qquad \frac{\tilde{\eta}}{|\tilde{\xi}|} = \alpha, \qquad \frac{\tilde{\psi}}{|\tilde{\xi}|} = \theta, \qquad \frac{\tilde{\theta}}{|\tilde{\xi}|} = \theta + \tilde{t}, \qquad |\tilde{\xi}| = \lambda.$$

So the symbol $\tilde{B}$ in [T2] is related to our symbol B by

$$B(x', \eta, \lambda) = \tilde{B}(x', \lambda\frac{\eta}{|\eta|}, \lambda(|\eta| - 1)).$$

From Taylor's lemma it follows that $\tilde{B} \in \tilde{S}^0$, $\tilde{S}^0$ being the standard class of symbols (see e.g. [T2]). Therefore,

$$|\partial_{x'}^{\alpha}\partial_{\eta}^{\beta}B(x',\eta,\lambda)| = |\partial_{x'}^{\alpha}\partial_{\eta}^{\beta}\tilde{B}(x',\lambda\frac{\eta}{|\eta|},\lambda(|\eta|-1))| \le C_{\alpha,\beta}.$$

This proves the lemma for $\lambda \in \mathbf{R}_+$. Assume that $\lambda \in \Lambda$, i.e. $\lambda = |\lambda|e^{i\arg\lambda}$, $\arg\lambda$ not necessarily zero. Then the desired estimate (even in a larger domain of the kind $|\arg\lambda| < \varepsilon$) follows from the fact that if we replace $\mathrm{Ai}_-(s)$ with $\mathrm{Ai}_-(e^{i\frac{2}{3}\arg\lambda}s)$, then all the arguments remain valid and the corresponding estimates are uniform in $\arg\lambda$. □

Following [T2] set also

$$\frac{\mathrm{Ai}'_-(\lambda^{2/3}\rho_0)}{\mathrm{Ai}'_-(\lambda^{2/3}\alpha)} = Ce^{i\lambda\gamma}.$$

Similarly, $C \in S^{0,0}_{0,0}$ for $x_1 = 0$ and $C = 1$ for $\alpha \ge 0$. Therefore, one can write down $\tilde{H}_g w$ in the form

$$\tilde{H}_g w|_{x_1=0} = \left(\frac{\lambda}{2\pi}\right)^n \int [g_0 B + ig_1 qC]\, e^{i\lambda\varphi_0}\hat{w}\, d\eta. \tag{A.13}$$

Here $q(\eta,\lambda) = \lambda^{-1/3}\frac{\mathrm{Ai}'_-}{\mathrm{Ai}_-}(\lambda^{2/3}\alpha)$. From the estimate

$$|\partial_\alpha^k q| \le C_k\left(|\lambda|^{-2/3} + \alpha\right)^{-k}$$

and the fact that $\mathrm{Ai}'_-/\mathrm{Ai}_- \in \tilde{S}^{1/2}_{1,0}$ for $|\arg s| < \pi/3 - \varepsilon$, $\varepsilon > 0$, it follows that

$$|\partial_\eta^\gamma q(\eta,\lambda)| \le C_\gamma\left(|\lambda|^{-2/3} + |\alpha|\right)^{-|\gamma|}. \tag{A.14}$$

Therefore,

$$q \in S^{0,0}_{2/3,0}. \tag{A.15}$$

Let us denote by Jw the right hand side of (A.12), i.e.

$$Jw = \left(\frac{\lambda}{2\pi}\right)^n \int [g_0 B + ig_1 qC]\, e^{i\lambda\varphi_0}\hat{w}\, d\eta. \tag{A.16}$$

Here g_0, g_1, B, C belong to $S^{0,0}_{0,0}$, while for q we have (A.15). Since $g_1 = O(|\alpha|^\infty)$, from (A.14) we get

$$g_1(x',\eta,\lambda)q(\eta,\lambda) \in S^{0,0}_{0,0}. \tag{A.17}$$

Therefore, J is a FIO with large parameter λ and amplitude of class $S^{0,0}_{0,0}$. The boundary condition $\tilde{H}_g w|_{x_1=0} = \mathrm{Op}(\chi)f$ is equivalent to

$$Jw = \mathrm{Op}(\chi)f. \tag{A.18}$$

It is easy to see that J is elliptic near $\alpha = 0$. Indeed, $g_1 q$ is small near $\alpha = 0$. On the other hand, for $\alpha = 0$ we have $B = 1$ and $|g_0| \ge c > 0$ near $\alpha = 0$, so J is elliptic. Moreover, J takes distributions with $\widetilde{\mathrm{WF}}(f)$ in a small neighborhood of $(0,\eta^0)$ to distributions Jf with $\widetilde{\mathrm{WF}}(Jf)$ supported near $\alpha = 0$. Thus there exists a FIO with large parameter λ (let us denote it by J^{-1}), such that $J^{-1}J - I$ has kernel in $\tilde{C}^\infty$ if applied to f with $\widetilde{\mathrm{WF}}(f)$ in a small neighborhood of $(0,\eta^0)$ and $JJ^{-1} - I$ has similar property acting on f with $\widetilde{\mathrm{WF}}(f)$ in a small neighborhood of $\alpha = 0$.

Remark. In contrast to [D], the large parameter λ here is a complex nimber. However, it can be seen that FIO-s with large parameter $\lambda \in \Lambda$ have properties similar to those with large real parameter considered in [D]. In particular, any elliptic FIO J with amplitude of class $S^{0,0}_{0,0}$ has asymptotic inverse J^{-1} in the same class. This can be seen directly by following the classical theory. For example, set

$$Ku = \left(\frac{\lambda}{2\pi}\right)^n \iint e^{i\lambda(y\cdot\eta - \varphi_0(y,\eta))} u(y,\lambda)\, dy\, d\eta$$

(here the integration is taken in a neighborhood of $\operatorname{supp}\chi$) and check that $P = JK$ is an elliptic ΨDO with large parameter λ. Now, set $J^{-1} = KP^{-1}$. The assertions about the wave front sets can be verified by integration by parts. Here it is important to note that the exponential in the integral above is bounded by $C|\lambda|^m$ with some fixed m.

Thus we proved that the solution to (A.10) is given by $H_g = \tilde{H}_g J^{-1}\mathrm{Op}(\chi)$. This completes the construction in the glancing region. It remains to show that (A.9) holds in the glancing region as well, i.e.

$$\begin{aligned} \widetilde{\mathrm{WF}}'(H_h) \subset \{&(x,\xi,y,\eta) \in T^*(U\setminus\Gamma)\times T^*\Gamma;\ |\xi|_x = 1,\ (x,\xi) \text{ belongs} \\ &\text{to the outgoing ray issued from } (y,\eta)\in \operatorname{supp}\chi\}, \end{aligned} \tag{A.19}$$

The proof of (A.19) is similar to that of Corollary A.II.8 in [G] and in particular (A.19) justifies the outgoing properties of the glancing parametrix (see section A.5). One considers three subregions $\alpha \le -C|\lambda|^{-\varepsilon}$, $|\alpha| \le C|\lambda|^{-\varepsilon}$ and $\alpha \ge C|\lambda|^{-\varepsilon}$. In the first two subregions the analysis is the same. In $\{\alpha \ge C|\lambda|^{-\varepsilon}\}$, $\tilde{H}_g J^{-1}$ reduces to an elliptic FIO with phase $\tilde{\theta}$ and amplitude $d(x,\eta,\lambda)$. We have $e^{i\lambda\tilde{\theta}} = e^{i\lambda_1\tilde{\theta}}e^{-\lambda_2\tilde{\theta}}$ and $|e^{-\lambda_2\tilde{\theta}}| \le \lambda_1^m$ for $\lambda\in\Lambda$ For $e^{i\lambda_1\tilde{\theta}}$ we have $\mathrm{Re}\,(i\lambda_1\tilde{\theta}) = -\lambda_1 \mathrm{Im}\,\tilde{\theta} \le -c\lambda_1^{1-\varepsilon/2}x_1$ (see [G, p. 127]), therefore $e^{i\lambda_1\tilde{\theta}}$ decays exponentially for $x_1 > 0$, so in $\{\alpha \ge C|\lambda|^{-\varepsilon}\}$ there is no contribution to $\widetilde{\mathrm{WF}}'(H_g)$. The rest of the proof of (A.19) is the same as in [G].

A.4 The elliptic region

Let $\zeta^0 \in T^*\Gamma$ with $\|\zeta^0\| > 1$. In the local coordinates considered above ζ^0 is given by $(0,\eta^0)$ with $|\eta^0| = 1+\varepsilon_0$, $\varepsilon_0 > 0$. Set $W = \{\eta;\ |\eta| > 1+\varepsilon_0/2\}$, let V be a small neighborhood of $x' = 0$ on Γ and let U be small neighborhood of $x_1 = 0$, $x' = 0$ in $\bar{\Omega}$. Let $\chi \in C^\infty(T^*\Gamma)$ be given locally by $\chi = \chi_1(x')\chi_2(\eta)$, where $\operatorname{supp}\chi_1 \subset V$, $\chi_2(\eta) = 1$ for $|\eta| > 1+3\varepsilon_0/4$, $\chi_2(\eta) = 0$ for $|\eta|_x < \varepsilon_0/2$. We will construct a FIO $H_e : \tilde{C}^\infty(V) \mapsto \tilde{C}^\infty(U)$ with large parameter λ_1 such that

$$\left\{ \begin{array}{rcl} (\Delta+\lambda^2)H_e f &=& K_e f, \\ H_e f|_\Gamma &=& \mathrm{Op}_{\lambda_1}(\chi) f, \end{array} \right. \tag{A.20}$$

where K_e has kernel in $\tilde{C}^\infty(U\times V)$, H_e has kernel in $\tilde{C}^\infty(U\setminus\Gamma\times V)$ and

$$\mathrm{Op}_{\lambda_1}(\chi) f = \left(\frac{\lambda_1}{2\pi}\right)^n \iint e^{i\lambda_1(x-y)\cdot\eta}\chi(x,\eta) f(y,\lambda_1)\, dy\, d\eta,$$

i.e. it is a ΨDO with large parameter $\lambda_1 = \operatorname{Re}\lambda$. The operator H_e is of the form

$$H_e f = \left(\frac{\lambda_1}{2\pi}\right)^n \iint e^{i\lambda_1(\varphi(x,\eta)-y\cdot\eta)} a(x,\eta,\lambda_1) f(y,\lambda_1)\, dy\, d\eta. \tag{A.21}$$

Here in fact φ and a depend also on $\arg\lambda$ and φ is complex valued.

Remark. Note that H_e here is a FIO with large parameter $\lambda_1 = \operatorname{Re}\lambda$, while in the hyperbolic region we took λ to be the large parameter. This choice of λ_1 in the exponential in H_e differs from [G]. The reason for this is that we found it difficult to interpret integrals like (A.21) with complex λ in the exponential. The problem is that $e^{i\lambda(\varphi(x,\eta)-y\cdot\eta)}$ could grow exponentially in η because $\operatorname{supp} a$ is no longer bounded in η. By the arguments in [G] (where λ belongs to some strip $|\operatorname{Im}\lambda| \le \delta$), we have (see also (A.24) below) $\operatorname{Im}\varphi \ge c_1 x_1(1+|\eta|)$, $|\operatorname{Re}\varphi| \le c_2(1+|\eta|)$, therefore $\operatorname{Re}(i\lambda(\varphi(x,\eta) - y\cdot\eta)) \le (c_2'\lambda_2 - c_1\lambda_1 x_1)(1+|\eta|)$ thus it is negative for any $x_1 > 0$ and for λ_1 sufficiently large. This guarantees that the corresponding integral is well defined. However, if $\operatorname{Im}\lambda = \delta > 0$, one gets $\operatorname{Re}(i\lambda(\varphi(x,\eta) - y\cdot\eta)) < 0$ only for $\lambda_1 > c\delta/x_1$, so as $x_1 \to 0$ we would have $\lambda_1 \to \infty$. It turns out that $\inf\{\lambda_1;\ \lambda \in \Lambda\}$ (and hence Λ) should depend on x_1. Therefore, it cannot be seen form these arguments how one can construct a parametrix in a small neighborhood of the boundary with λ belonging to some fixed set Λ that contains not only real numbers. All these problems do not exist when λ is real. That is why we consider $\lambda_1 = \operatorname{Re}\lambda$ as the large parameter in the elliptic region.

Applying $\Delta + \lambda^2$ to $H_e f$, we get

$$\begin{aligned}(\Delta+\lambda^2)H_e f &= \left(\frac{\lambda_1}{2\pi}\right)^n \iint e^{i\lambda_1(\varphi(x,\eta)-y\cdot\eta)}[\Delta a - 2i\lambda_1\nabla\varphi\cdot\nabla a \\ &\qquad - i\lambda_1 a\Delta\varphi + (\lambda^2 - \lambda_1^2(\nabla\varphi)^2)a] f(y)\, dy\, d\eta.\end{aligned}$$

Thus φ must satisfy the eikonal equation

$$\begin{cases} (\nabla\varphi)^2 &= \alpha^2, \\ \varphi|_\Gamma &= x\cdot\eta, \end{cases} \tag{A.22}$$

where $\alpha = \lambda/\lambda_1 = 1 + i\tan\arg\lambda$. If $\lambda \in \Lambda$ and C_2 in (A.2) is sufficiently large, then α is a complex parameter close to $\alpha = 1$. Moreover, for C_2 large enough (and therefore α sufficiently close to 1), there exists a complex valued function $\varphi = \varphi_\alpha$ satisfying (A.22) up to an error $O(x_1^\infty)$ with $\operatorname{Im}\varphi \ge 0$. In local coordinates (A.22) has the form

$$\begin{cases} \frac{\partial\varphi}{\partial x_1} &= p_\alpha(x, \frac{\partial\varphi}{\partial x'}), \\ \varphi|_{x_1=0} &= x'\cdot\eta, \end{cases} \tag{A.23}$$

where

$$p_\alpha(x,\eta) = \left(1+|\nabla F|^2\right)^{-1}\left\{\eta\cdot\nabla F + i\left[(1+|\nabla F|^2)(|\eta|^2-\alpha^2) - (\eta\cdot\nabla F)^2\right]^{1/2}\right\}.$$

Recall that F is that function for which Γ is given locally by $z_1 = F(z_2, \dots, z_{n+1})$. Then one can solve (A.23) of infinite order at $x_1 = 0$ as in [G] for all $\eta \in W$ provided that α is sufficiently close to $\alpha = 1$. Once we have $\varphi = \varphi_\alpha$ solving (A.22), we can solve the corresponding transport equations of infinite order at $x_1 = 0$ and the solution is $a(x, \eta, \lambda) = \sum_{j=0}^{\infty} a_j(x, \eta)\lambda_1^{-j}$ with a_j formal series in x_1 with coefficients in $\tilde{S}^{-j}(V \times W)$. In fact, a depends also on α and this dependence is continuous in the Fréchet topology given by the seminorms appearing in (A.3). Since φ has the same property, the operator H_e can be considered as a FIO with large parameter λ_1 continuously depending on the parameter α. The same will be true for the corresponding Neumann operator.

Now it is not hard to check that $H_e f$ is well defined and solves (A.20). According to the construction of φ (see [G]), one has $\operatorname{Im} \varphi \geq c x_1(1 + |\eta|)$ with $c > 0$ independent of α, thus

$$\operatorname{Re}\left(i\lambda_1(\varphi(x, \eta) - y \cdot \eta)\right) = -\lambda_1 \operatorname{Im} \varphi \leq -c\lambda_1 x_1(1 + |\eta|). \tag{A.24}$$

Therefore,

$$\left| e^{i\lambda_1(\varphi(x,\eta) - y\cdot\eta)} \right| \leq e^{-c\lambda_1 x_1 (1+|\eta|)}, \tag{A.25}$$

so the integrand in (A.21) is convergent for $x_1 > 0$. Using the inequality (see e.g. [T2, §VII.5])

$$\sup_{x_1 \geq 0} x_1^j e^{-c\lambda_1 x_1 (1+|\eta|)} \leq c_j (1 + |\eta|)^{-j} \lambda_1^{-j}, \tag{A.26}$$

one can easily show that the kernel of K_e is in $\tilde{C}^\infty(U \times V)$, i.e. $K_e = O(|\lambda|^{-\infty})$ uniformly in $x_1 \in [0, \varepsilon]$. Moreover, (A.25), (A.26) show that the kernel of H_e is in $\tilde{C}^\infty(U \setminus \Gamma \times V)$.

Finally, we note that if χ in (A.20) is compactly supported, i.e. if we work with η in a bounded set, then one can consider a parametrix H_e with $\lambda \in \Lambda$ in the exponential. Then the phase function will satisfy the usual eikonal equation $(\nabla\varphi)^2 = 1$.

A.5 Relationship between the parametrix and the exact solution

Having constructed the parametrix in the three regions, we will represent the exact solution in terms of the parametrix. First we note that the boundary operator $\operatorname{Op}_{\lambda_1}(\chi)$ in the elliptic region (see (A.20)) is a ΨDO with large parameter $\lambda_1 = \operatorname{Re} \lambda$, while the other two boundary operators (see (A.8), (A.10)) have λ as large parameter. We will modify the elliptic parametrix so that the boundary operator takes the form $\operatorname{Op}_\lambda(\chi')$.

Let H_e be the elliptic parametrix with $\chi = \chi_1(x)\chi_2(\eta)$ as before. Choose $\chi' = \chi_1'(x)\chi_2'(\eta)$, such that $\chi_j = 1$ on $\operatorname{supp} \chi_j'$. Let us define

$$\operatorname{Op}_\lambda(\chi') := \chi_1'(x) - \operatorname{Op}_\lambda(\chi_1'(1 - \chi_2')). \tag{A.27}$$

Then $H_e' := H_e \operatorname{Op}_\lambda(\chi')$ solves

$$\begin{cases} (\Delta + \lambda^2) H_e' f &= K_e' f, \\ H_e' f|_\Gamma &= \operatorname{Op}_\lambda(\chi') f + R_e' f, \end{cases}$$

where $K_e' = K_e \operatorname{Op}_\lambda(\chi')$ and R_e' has kernel in $\tilde{C}^\infty(V \times V)$. Indeed, one can easily check that

$$\operatorname{Op}_{\lambda_1}(\chi) \operatorname{Op}_\lambda(\chi') = \operatorname{Op}_\lambda(\chi') \tag{A.28}$$

modulo operators with kernels in $\tilde{C}^\infty(V \times V)$. To verify (A.28), one uses the definition (A.27) for $\mathrm{Op}_\lambda(\chi')$ given above and the fact that $\chi = 1$ on $\operatorname{supp} \chi'$.

Thus by using a pseudodifferential partition of unity, one can construct an operator $H(\lambda)$, such that

$$\begin{cases} (\Delta + \lambda^2) H(\lambda) f &= K(\lambda) f \qquad \text{in } U \\ H(\lambda) f|_\Gamma &= f + R(\lambda) f, \end{cases} \tag{A.29}$$

where the kernel of $K(\lambda)$ is in $\tilde{C}^\infty(U \times \Gamma)$, the kernel of $R(\lambda)$ is in $\tilde{C}^\infty(\Gamma \times \Gamma)$. Following [G], set

$$\tilde{H}(\lambda) = \chi H(\lambda) - S_0(\lambda)(\chi K(\lambda) + [\Delta, \chi] H(\lambda)), \tag{A.30}$$

where χ is a smooth cut-off function in $\bar{\Omega}$ with support in U, equal to 1 near Γ, $S_0(\lambda)$ is the free outgoing resolvent. Clearly, $\tilde{H}(\lambda) f$ is λ-outgoing for any f and any $\lambda \in \Lambda$, according to Definition 2.1. For $\tilde{H}(\lambda) f|_\Gamma$ we have

$$\tilde{H}(\lambda) f|_\Gamma = f + R(\lambda) f - S_0(\lambda)(\chi K(\lambda) + [\Delta, \chi] H(\lambda)) f|_\Gamma \,.$$

By (A.9), (A.19) one deduces (see [G, p. 136]) that the last term above defines a neglectible operator, i.e. an operator with kernel in $\tilde{C}^\infty(\Gamma \times \Gamma)$. Let us emphasize that this property is due to the right choice of the signs when solving the corresponding eikonal equations which determines the outgoing properties of $\widetilde{\mathrm{WF}}'(H)$ (see (A.9), (A.19)) and the fact that $S_0(\lambda)$ is the *outgoing* resolvent, hence $\widetilde{\mathrm{WF}}'(S_0(\lambda))$ has also outgoing properties (see [G, (A.II.24)]). Therefore, the singularities of $[\Delta, \chi] H(\lambda) f$ cannot go back to Γ under the action of $S_0(\lambda)$. We refer to [G] for more details.

If we denote by $u = \mathcal{H}(\lambda) f$ the exact solution of (A.1), then we get

$$\mathcal{H}(\lambda) = \tilde{H}(\lambda) \left(I + \tilde{R}(\lambda) \right)^{-1}, \tag{A.31}$$

where $\tilde{R}(\lambda)$ has kernel in $\tilde{C}^\infty(\Gamma \times \Gamma)$ and therefore $I + \tilde{R}(\lambda)$ is invertible for large $|\lambda|$. So (A.30) shows that the exact solution $\mathcal{H}(\lambda) f$ with $\widetilde{\mathrm{WF}}(f)$ belonging to the hyperbolic and the glancing region, respectively, coincides with the corresponding parametrix up to an error $O(|\lambda|^{-\infty})$ for $\lambda \in \Lambda$. Without loss of generality we can assume that the elliptic parametrix also has the form H_e (not $H_e \mathrm{Op}_\lambda(\chi)$), where H_e is the ΨDO with large parameter λ_1 constructed in the elliptic region. Indeed, given a cut-off function χ supported in the elliptic region as above, we know that $\mathcal{H}(\lambda)\mathrm{Op}_\lambda(\chi) = H_e(\lambda)\mathrm{Op}_\lambda(\chi)$ modulo neglectible operators. However, since $\mathrm{Op}_\lambda(\chi)\mathrm{Op}_{\lambda_1}(\chi_1) = \mathrm{Op}_{\lambda_1}(\chi_1)$ for any cut-off function χ_1, such that $\chi = 1$ on $\operatorname{supp} \chi_1$, we get that $\mathcal{H}(\lambda)\mathrm{Op}_{\lambda_1}(\chi_1) = H_e(\lambda)\mathrm{Op}_{\lambda_1}(\chi_1)$ modulo neglectible operators, i.e. we have the parametrix constructed above with another cut-off function with slightly shrunken support.

A.6 The Neumann operator

We proceed with a construction of a parametrix for the (outgoing) Neumann operator related to (A.1). Although we deal with somewhat different operators in the proof of the main result, we believe that the analysis of the Neumann operator for (A.1) is useful for better understanding the structure of the Neumann operator for the elasticity problem.

Given $f \in H^s(\Gamma)$, $s \geq 3/2$ denote by

$$N(\lambda) f = \left.\frac{\partial u}{\partial \nu}\right|_{\Gamma} \in H^{s-1}(\Gamma)$$

the normal derivative on Γ of the solution u to (A.1).

(i) *Hyperbolic region.* Let $\widetilde{\mathrm{WF}}(f)$ be supported in the hyperbolic region as above. Then using the hyperbolic parametrix H_h and (A.30), (A.31), one gets that up to a neglectible term $N(\lambda)f = N_h f$, where N_h is a local ΨDO with large parameter λ of class $L^{0,1}_{0,0}$. The principal symbol of that operator at $x' = 0$ is $i\lambda \partial\psi/\partial x_1|_{x_1=0}$ and writing this in invariant form, we get that the principal symbol reads

$$-i\lambda\sqrt{1 - |\eta|_x^2}.$$

(ii) *Elliptic region.* The analysis here is similar. Let $\widetilde{\mathrm{WF}}(f)$ be supported in the elliptic region. If $\widetilde{\mathrm{WF}}(f)$ is compact in η, then one can construct $H_e(\lambda)$ as a FIO with large parameter λ (not λ_1) and the principal symbol of the parametrix for the Neumann operator is

$$-\lambda\sqrt{|\eta|_x^2 - 1}.$$

If $\widetilde{\mathrm{WF}}(f)$ is not compact in η, then the elliptic parametrix is a FIO with large parameter λ_1. Therefore, the Neumann operator in this case is a ΨDO with large parameter λ_1 and principal symbol

$$-\lambda_1\sqrt{|\eta|_x^2 - \alpha^2}.$$

(iii) *Glancing region.* Here we follow [T2, §X.5]. Let $\widetilde{\mathrm{WF}}(f)$ be supported in a small neighborhood of a point ζ^0 in the glancing region. As shown above, the glancing parametrix has the form $\tilde{H}_g w$, where $w = J^{-1} f$. We have

$$\begin{aligned} \left.\frac{\partial}{\partial \nu}\right|_{\Gamma} \tilde{H}_g w &= \left(\frac{\lambda}{2\pi}\right)^n \int [\lambda \rho_\nu g_0 - \lambda\theta_\nu g_1 + i\partial_\nu g_1]\, C e^{i\lambda\varphi_0} q \hat{w}(\eta,\lambda)\, d\eta \\ &\quad + \left(\frac{\lambda}{2\pi}\right)^n \int [i\lambda\theta_\nu g_0 + i\lambda\rho\rho_\nu g_1 + \partial_\nu g_0]\, B e^{i\lambda\varphi_0} \hat{w}(\eta,\lambda)\, d\eta, \end{aligned} \tag{A.32}$$

where $\theta_\nu = \partial\theta/\partial\nu|_\Gamma$ etc. Let us first note that the terms containing g_1 are $O(|\alpha|^\infty)$. The construction of ρ guarantees that $\rho_\nu \neq 0$ for $|\alpha| \ll 1$, while $\theta_\nu = 0$ for $\alpha > 0$. Therefore, the first term in (A.32) defines an elliptic operator near $\alpha = 0$, while the second one has principal part that vanishes at $\alpha = 0$ as $|\lambda| \to \infty$. Let us rewrite (A.32) as

$$\left.\frac{\partial}{\partial \nu}\right|_{\Gamma} \tilde{H}_g w = (K_1 Q + K_2) w,$$

where $Q = \mathrm{Op}(q)$, $q \in S^{0,0}_{2/3,0}$, K_1, K_2 are FIO-s with large parameter λ with associated canonical transformation $\mathcal{J}$ (that of J). Let us set

$$A_1 = J^{-1} K_1, \qquad A_2 = J^{-1} K_2.$$

Then $A_1 \in L^{0,1}_{0,0}$, A_1 is elliptic near $\alpha = 0$; $A_2 \in L^{0,1}_{0,0}$, $\sigma_p(A_2) = 0$ at $\alpha = 0$; $Q \in L^{0,0}_{2/3,0}$. For $N_g := \frac{\partial}{\partial\nu}|_\Gamma \tilde{H}_g J^{-1}$ we get

$$N_g = J(A_1 Q + A_2)J^{-1} \tag{A.33}$$

(compare with [T2]).

It should be noted that unfortunately Q belongs to a class with $\rho = 2/3$ (this corresponds to $\rho = 1/3$ in the classical pseudodifferential calculus). Therefore, this does not enable us to conjugate directly $A_1Q + A_2$ with the FIO J and to claim that the result is again a ΨDO. This would be possible if $\rho < 1/2$. In fact, by using some special variants of Egorov's theorem [T2] and their generalizations to the calculus of ΨDO-s and FIO-s with large complex parameter, we could prove as in [T2] that locally N_g is a ΨDO and $N_g \in L^{0,1}_{2/3,2/3}$. Even this result would not allow us to interpret N_g as a ΨDO with large parameter on Γ globally (this requires $\rho = \delta < 1/2$). For our purposes however (A.33) is enough in order to construct an asymptotic inverse N_g^{-1} of N_g as done in [T2]. Note that $A_1Q + A_2 \in L^{0,1}_{2/3,0}$. It is useful to rewrite N_g as

$$N_g = J(A_1 + A_2Q^{-1})QJ^{-1}.$$

A priori $Q^{-1} \in L^{0,1/3}_{2/3,0}$, but from the fact that $\sigma_p(A_2) = 0$ for $\alpha = 0$ it is clear that $A_2Q^{-1} \in L^{0,1}_{2/3,0}$ and $\lambda^{-1}\sigma(A_2Q^{-1})$ is small near $\alpha = 0$. Therefore, $A_1 + A_2Q^{-1} \in L^{0,1}_{2/3,0}$ is elliptic near $\alpha = 0$. By the standard procedure we can construct asymptotic inverse $(A_1 + A_2Q^{-1})^{-1} \in L^{0,-1}_{2/3,0}$ (this requires $\rho + \delta < 1$, not $\rho < 1/2$). Therefore, N_g is hypoelliptic and modulo $O(|\lambda|^{-\infty})$ has inverse N_g^{-1} given by

$$N_g^{-1} = JQ^{-1}\left(A_1 + A_2Q^{-1}\right)^{-1} J^{-1}$$

with $Q^{-1}\left(A_1 + A_2Q^{-1}\right)^{-1} \in L^{0,-2/3}_{2/3,0}$.

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