MICROLOCAL ANALYSIS METHODS

PLAMEN STEFANOV

One of the fundamental ideas of classical analysis is a thorough study of functions near a point, i.e., locally. Microlocal analysis, loosely speaking, is analysis near points and directions, i.e., in the "phase space". We review here briefly the theory of pseudo-differential operators, and geometrical optics.

1. Wave front sets

The phase space in $\mathbf{R}^n$ is the cotangent bundle $T^*\mathbf{R}^n$ that can be identified with $\mathbf{R}^n \times \mathbf{R}^n$. Given a distribution $f \in \mathcal{D}'(\mathbf{R}^n)$, a fundamental object to study is the wave front set $\mathrm{WF}(f) \subset T^*\mathbf{R}^n \setminus 0$ viewed as the singularities of f, that we define below.

1.1. **Definition.** The basic idea goes back to the properties of the Fourier transform. If f is an integrable compactly supported function, one can tell whether f is smooth by looking at the behavior of $\hat f(\xi) = \int e^{-\mathrm{i}x\cdot\xi} f(x)\,\mathrm{d}x$ (that is smooth, even analytic) when $|\xi| \to \infty$. It is known that f is smooth if and only if for any N, $|\hat f(\xi)| \le C_N|\xi|^{-N}$ for some C_N. If we localize this requirement to a conic neighborhood V of some $\xi_0 \neq 0$ (V is conic if $\xi \in V \Rightarrow t\xi \in V, \forall t > 0$), then we can think of this as a smoothness in the cone V. To localize in the base x variable however, we first have to cut smoothly near a fixed x_0.

We say that $(x_0, \xi_0) \in \mathbf{R}^n \times (\mathbf{R}^n \setminus 0)$ is *not* in the wave front set $\mathrm{WF}(f)$ of $f \in \mathcal{D}'(\mathbf{R}^n)$ if there exists $\phi \in C_0^\infty(\mathbf{R}^n)$ with $\phi(x_0) \neq 0$ so that for any N, there exists C_N so that

$$|\widehat{\phi f}(\xi)| \le C_N|\xi|^{-N}$$

for ξ in some conic neighborhood of ξ_0. This definition is independent of the choice of ϕ. If $f \in \mathcal{D}'(\Omega)$ with some open $\Omega \subset \mathbf{R}^n$, to define $\mathrm{WF}(f) \subset \Omega \times (\mathbf{R}^n \setminus 0)$, we need to choose $\phi \in C_0^\infty(\Omega)$. Clearly, the wave front set is a closed conic subset of $\mathbf{R}^n \times (\mathbf{R}^n \setminus 0)$. Next, multiplication by a smooth function cannot enlarge the wave front set. The transformation law under

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Date: March 28, 2015.

First author partly supported by a NSF FRG Grant DMS-1301646.

This article first appeared as an appendix (pp. 310–320) to "Multi-wave methods via ultrasound" by Plamen Stefanov and Gunther Uhlmann, *Inverse Problems and Applications: Inside Out II*, edited by Gunther Uhlmann, Mathematical Sciences Research Institute Publications v.60, Cambridge University Press, New York, 2012, pp. 271-324.

coordinate changes is that of covectors making it natural to think of $\mathrm{WF}(f)$ as a subset of $T^*\mathbf{R}^n \setminus 0$, or $T^*\Omega \setminus 0$, respectively.

The wave front set $\mathrm{WF}(f)$ generalizes the notion $\mathrm{singsupp}(f)$ — the complement of the largest open set where f is smooth. The points (x,ξ) in $\mathrm{WF}(f)$ are referred to as *singularities* of f. Its projection onto the base is $\mathrm{singsupp}(f)$, i.e.,

$$\mathrm{singsupp}(f) = \{x;\ \exists \xi, (x,\xi) \in \mathrm{WF}(f)\}.$$

Examples.

(a) $\mathrm{WF}(\delta) = \{(0,\xi);\ \xi \neq 0\}$. In other words, the Dirac delta function is singular at $x = 0$, and in all directions there.

(b) Let $x = (x', x'')$, where $x' = (x_1, \dots, x_k)$, $x'' = (x_{k+1}, \dots, x_n)$ with some k. Then $\mathrm{WF}(\delta(x')) = \{(0, x'', \xi', 0), \xi' \neq 0\}$, where $\delta(x')$ is the Dirac delta function on the plane $x' = 0$, defined by $\langle \delta(x'), \phi\rangle = \int \phi(0,x'')\,\mathrm{d}x''$. In other words, $\mathrm{WF}(\delta(x'))$ consists of all (co)vectors $\neq 0$ with a base point on that plane, perpendicular to it.

(c) Let f be a piecewise smooth function that has a non-zero jump across some smooth surface S. Then $\mathrm{WF}(f)$ consists of all non-zero (co)vectors at points of S, normal to it. This follows from a change of variables that flattens S locally and reduces the problem to that for the Heaviside function multiplied by a smooth function.

(d) Let $f = \mathrm{pv}\frac{1}{x} - \pi\mathrm{i}\delta(x)$ in $\mathbf{R}$, where $\mathrm{pv}\frac{1}{x}$ is the regularized $1/x$ in the principal value sense. Then $\mathrm{WF}(f) = \{(0,\xi);\ \xi > 0\}$.

In example (d) we see a distribution with a wave front set that is not even in the ξ variable, i.e., not symmetric under the change $\xi \mapsto -\xi$. In fact, wave front sets do not have a special structure except for the requirement to be closed conic sets; given any such set, there is a distribution with a wave front set exactly that set. On the other hand, if f is real valued, then $\hat f$ is an even function; therefore $\mathrm{WF}(f)$ is even in ξ, as well.

Two distributions cannot be multiplied in general. However, if $\mathrm{WF}(f)$ and $\mathrm{WF}'(g)$ do not intersect, there is a "natural way" to define a product. Here, $\mathrm{WF}'(g) = \{(x,-\xi);\ (x,\xi) \in \mathrm{WF}(g)\}$.

2. Pseudodifferential Operators

2.1. **Definition.** We first define the symbol class $S^m(\Omega)$, $m \in \mathbf{R}$, as the set of all smooth functions $p(x,\xi)$, $(x,\xi) \in \Omega \times \mathbf{R}^n$, called symbols, satisfying the following symbol estimates: for any compact set $K \subset \Omega$, and any multi-indices α, β, there is a constant $C_{K,\alpha,\beta} > 0$ so that

$$|\partial_\xi^\alpha \partial_x^\beta p(x,\xi)| \le C_{K,\alpha,\beta}(1+|\xi|)^{m-|\alpha|}, \quad \forall (x,\xi) \in K \times \mathbf{R}^n. \tag{1}$$

More generally, one can define the class $S^m_{\rho,\delta}(\Omega)$ with $0 \le \rho,\ \delta \le 1$ by replacing $m - |\alpha|$ there by $m - \rho|\alpha| + \delta|\beta|$. Then $S^m(\Omega) = S^m_{1,0}(\Omega)$. Often, we omit Ω and simply write S^m. There are other classes in the literature, for example $\Omega = \mathbf{R}^n$, and (1) is required to hold for all $x \in \mathbf{R}^n$.

The estimates (1) do not provide any control of p when x approaches boundary points of Ω, or ∞.

Given $p \in S^m(\Omega)$, we define the pseudodifferential operator (ΨDO) with symbol p, denoted by $p(x, D)$, by

$$p(x, D)f = (2\pi)^{-n} \int e^{\mathrm{i}x\cdot\xi} p(x, \xi) \hat{f}(\xi)\, \mathrm{d}\xi, \quad f \in C_0^\infty(\Omega). \tag{2}$$

The definition is inspired by the following. If $P = \sum_{|\alpha|\leq m} a_\alpha(x) D^\alpha$ is a differential operator, where $D = -\mathrm{i}\partial$, then using the Fourier inversion formula we can write P as in (2) with a symbol $p = \sum_{|\alpha|\leq m} a_\alpha(x)\xi^\alpha$ that is a polynomial in ξ with x-dependent coefficients. The symbol class S^m allows for more general functions. The class of the pseudo-differential operators with symbols in S^m is denoted usually by Ψ^m. The operator P is called a ΨDO if it belongs to Ψ^m for some m. By definition, $S^{-\infty} = \cap_m S^m$, and $\Psi^{-\infty} = \cap_m \Psi^m$.

An important subclass is the set of the *classical symbols* that have an asymptotic expansion of the form

$$p(x, \xi) \sim \sum_{j=0}^{\infty} p_{m-j}(x, \xi), \tag{3}$$

where $m \in \mathbf{R}$, and p_{m-j} are smooth and positively homogeneous in ξ of order $m - j$ for $|\xi| > 1$, i.e., $p_{m-j}(x, \lambda\xi) = \lambda^{m-j} p_{m-j}(x, \xi)$ for $|\xi| > 1$, $\lambda > 1$; and the sign $\sim$ means that

$$p(x, \xi) - \sum_{j=0}^{N} p_{m-j}(x, \xi) \in S^{m-N-1}, \quad \forall N \geq 0. \tag{4}$$

Any ΨDO $p(x, D)$ is continuous from $C_0^\infty(\Omega)$ to $C^\infty(\Omega)$, and can be extended by duality as a continuous map from $\mathcal{E}'(\Omega)$ to $\mathcal{D}'(\Omega)$.

2.2. **Principal symbol.** The principal symbol of a ΨDO in $\Psi^m(\Omega)$ given by (2) is the equivalence class $S^m(\Omega)/S^{m-1}(\Omega)$, and any representative of it is called a principal symbol as well. In case of classical ΨDOs, the convention is to choose the principal symbol to be the first term p_m, that in particular is positively homogeneous in ξ.

2.3. **Smoothing Operators.** Those are operators than map continuously $\mathcal{E}'(\Omega)$ into $C^\infty(\Omega)$. They coincide with operators with smooth Schwartz kernels in $\Omega \times \Omega$. They can always be written as ΨDOs with symbols in $S^{-\infty}$, and vice versa — all operators in $\Psi^{-\infty}$ are smoothing. Smoothing operators are viewed in this calculus as negligible and ΨDOs are typically defined modulo smoothing operators, i.e., $A = B$ if and only if $A - B$ is smoothing. Smoothing operators are not "small".

2.4. **The pseudolocal property.** For any ΨDO P and any $f \in \mathcal{E}'(\Omega)$,

$$\text{singsupp}(Pf) \subset \text{singsupp}\, f. \tag{5}$$

In other words, a ΨDO cannot increase the singular support. This property is preserved if we replace singsupp by WF, see (13).

2.5. **Symbols defined by an asymptotic expansion.** In many applications, a symbol is defined by consecutively constructing symbols $p_j \in S^{m_j}$, $j = 0, 1, \dots$, where $m_j \searrow -\infty$, and setting

$$p(x,\xi) \sim \sum_j p_j(x,\xi). \tag{6}$$

The series on the right may not converge but we can make it convergent by using our freedom to modify each p_j for ξ in expanding compact sets without changing the large ξ behavior of each term. This extends the Borel idea of constructing a smooth function with prescribed derivatives at a fixed point. The asymptotic (6) then is understood in a sense similar to (4). This shows that there exists a symbol $p \in S^{m_0}$ satisfying (6). That symbol is not unique but the difference of two such symbols is always in $S^{-\infty}$.

2.6. **Amplitudes.** A seemingly larger class of ΨDOs is defined by

$$Af = (2\pi)^{-n} \int e^{\mathrm{i}(x-y)\cdot\xi} a(x,y,\xi) f(y)\,\mathrm{d}y\,\mathrm{d}\xi, \quad f \in C_0^\infty(\Omega), \tag{7}$$

where the amplitude a satisfies

$$|\partial_\xi^\alpha \partial_x^\beta \partial_y^\gamma a(x,y,\xi)| \le C_{K,\alpha,\beta,\gamma}(1+|\xi|)^{m-|\alpha|}, \quad \forall (x,y,\xi) \in K \times \mathbf{R}^n \tag{8}$$

for any compact set $K \subset \Omega \times \Omega$, and for any α, β, γ. In fact, any such A is a ΨDO with symbol $p(x,\xi)$ (independent of y) with the formal asymptotic expansion

$$p(x,\xi) \sim \sum_{\alpha \ge 0} D_\xi^\alpha \partial_y^\alpha a(x,x,\xi).$$

In particular, the principal symbol of that operator can be taken to be $a(x,x,\xi)$.

2.7. **Transpose and adjoint operators to a ΨDO.** The mapping properties of any ΨDO A indicate that it has a well defined transpose A', and a complex adjoint A^* with the same mapping properties. They satisfy

$$\langle Au, v\rangle = \langle u, A'v\rangle, \quad \langle Au, \bar v\rangle = \langle u, \overline{A^* v}\rangle, \quad \forall u, v \in C_0^\infty$$

where $\langle \cdot,\cdot\rangle$ is the pairing in distribution sense; and in this particular case just an integral of uv. In particular, $A^* u = \overline{A' \bar u}$, and if A maps L^2 to L^2 in a bounded way, then A^* is the adjoint of A in L^2 sense.

The transpose and the adjoint are ΨDOs in the same class with amplitudes $a(y,x,-\xi)$ and $\bar a(y,x,\xi)$, respectively; and symbols

$$\sum_{\alpha\ge 0} (-1)^{|\alpha|} \frac{1}{\alpha!} (\partial_\xi^\alpha D_x^\alpha p)(x,-\xi), \quad \sum_{\alpha \ge 0} \frac{1}{\alpha!} \partial_\xi^\alpha D_x^\alpha \bar p(x,\xi),$$

if $a(x,y,\xi)$ and $p(x,\xi)$ are the amplitude and/or the symbol of that ΨDO. In particular, the principal symbols are $p_0(x,-\xi)$ and $\bar{p}_0(x,\xi)$, respectively, where p_0 is (any representative of) the principal symbol.

2.8. **Composition of ΨDOs and ΨDOs with properly supported kernels.** Given two ΨDOs A and B, their composition may not be defined even if they are smoothing ones because each one maps C_0^∞ to C^∞ but may not preserve the compactness of the support. For example, if $A(x,y)$, and $B(x,y)$ are their Schwartz kernels, the candidate for the kernel of AB given by $\int A(x,z)B(z,y)\,\mathrm{d}z$ may be a divergent integral. On the the hand, for any ΨDO A, one can find a smoothing correction R, so that $A+R$ has properly supported kernel, i.e., the kernel of $A+R$, has a compact intersection with $K\times\Omega$ and $\Omega\times K$ for any compact $K\subset\Omega$. The proof of this uses the fact that the Schwartz kernel of a ΨDO is smooth away from the diagonal $\{x=y\}$ and one can always cut there in a smooth way to make the kernel properly supported at the price of a smoothing error. ΨDOs with properly supported kernels preserve $C_0^\infty(\Omega)$, and also $\mathcal{E}'(\Omega)$, and therefore can be composed in either of those spaces. Moreover, they map $C^\infty(\Omega)$ to itself, and can be extended from $\mathcal{D}'(\Omega)$ to itself. The property of the kernel to be properly supported is often assumed, and it is justified by considering each ΨDO as an equivalence class.

If $A\in\Psi^m(\Omega)$ and $B\in\Psi^k(\Omega)$ are properly supported ΨDOs with symbols a and b, respectively, then AB is again a ΨDO in $\Psi^{m+k}(\Omega)$ and its symbol is given by

$$\sum_{\alpha\geq 0}(-1)^{|\alpha|}\frac{1}{\alpha!}\partial_\xi^\alpha a(x,\xi)D_x^\alpha b(x,\xi).$$

In particular, the principal symbol can be taken to be ab.

2.9. **Change of variables and ΨDOs on manifolds.** Let Ω' be another domain, and let $\phi:\Omega\to\tilde{\Omega}$ be a diffeomorphism. For any $P\in\Psi^m(\Omega)$, $\tilde{P}f:=(P(f\circ\phi))\circ\phi^{-1}$ maps $C_0^\infty(\tilde{\Omega})$ into $C^\infty(\tilde{\Omega})$. It is a ΨDO in $\Psi^m(\tilde{\Omega})$ with principal symbol

$$p(\phi^{-1}(y),(\mathrm{d}\phi)'\eta) \tag{9}$$

where p is the symbol of P, $\mathrm{d}\phi$ is the Jacobi matrix $\{\partial\phi_i/\partial x_j\}$ evaluated at $x=\phi^{-1}(y)$, and $(\mathrm{d}\phi)'$ stands for the transpose of that matrix. We can also write $(\mathrm{d}\phi)'=((\mathrm{d}\phi^{-1})^{-1})'$. An asymptotic expansion for the whole symbol can be written down as well.

Relation (9) shows that the transformation law under coordinate changes is that of a covector. Therefore, the principal symbol is a correctly defined function on the cotangent bundle $T^*\Omega$. The full symbol is not invariantly defined there in general.

Let M be a smooth manifold, and $A:C_0^\infty(M)\to C^\infty(M)$ be a linear operator. We say that $A\in\Psi^m(M)$, if its kernel is smooth away from the diagonal in $M\times M$, and if in any coordinate chart (A,χ), where $\chi:U\to$

$\Omega \subset \mathbf{R}^n$, we have $(A(u\circ\chi))\circ\chi^{-1} \in \Psi^m(\Omega)$. As before, the principal symbol of A, defined in any local chart, is an invariantly defined function on T^*M.

2.10. **Mapping properties in Sobolev Spaces.** In $\mathbf{R}^n$, Sobolev spaces $H^s(\mathbf{R}^n)$, $s \in \mathbf{R}$, are defined as the completion of $\mathcal{S}'(\mathbf{R}^n)$ in the norm

$$\|f\|^2_{H^s(\mathbf{R}^n)} = \int (1+|\xi|^2)^s |\hat f(\xi)|^2 \, \mathrm{d}\xi.$$

When s is a non-negative integer, an equivalent norm is the square root of $\sum_{|\alpha|\le s} \int |\partial^\alpha f(x)|^2 \, \mathrm{d}x$. For such s, and a bounded domain Ω, one defines $H^s(\Omega)$ as the completion of $C^\infty(\bar\Omega)$ using the latter norm with the integral taken in Ω. Sobolev spaces in Ω for other real values of s are defined by different means, including duality or complex interpolation.

Sobolev spaces are also Hilbert spaces.

Any $P \in \Psi^m(\Omega)$ is a continuous map from $H^s_{\text{comp}}(\Omega)$ to $H^{s-m}_{\text{loc}}(\Omega)$. If the symbols estimates (1) are satisfied in the whole $\mathbf{R}^n \times \mathbf{R}^n$, then $P : H^s(\mathbf{R}^n) \to H^{s-m}(\mathbf{R}^n)$.

2.11. **Elliptic ΨDOs and their parametrices.** The operator $P \in \Psi^m(\Omega)$ with symbol p is called elliptic of order m, if for any compact $K \subset \Omega$, there exists constants $C > 0$ and $R > 0$ so that

$$C|\xi|^m \le |p(x,\xi)| \quad \text{for } x \in K, \text{ and } |\xi| > R. \tag{10}$$

Then the symbol p is called also elliptic of order m. It is enough to require the principal symbol only to be elliptic (of order m). For classical ΨDOs, see (3), the requirement can be written as $p_m(x,\xi) \ne 0$ for $\xi \ne 0$. A fundamental property of elliptic operators is that they have parametrices. In other words, given an elliptic ΨDO P of order m, there exists $Q \in \Psi^{-m}(\Omega)$, so that

$$QP - \mathrm{Id} \in \Psi^{-\infty}, \quad PQ - \mathrm{Id} \in \Psi^{-\infty}. \tag{11}$$

The proof of this is to construct a left parametrix first by choosing a symbol $q_0 = 1/p$, cut off near the possible zeros of p, that form a compact set any time when x is restricted to a compact set as well. The corresponding ΨDO Q_0 will then satisfy $Q_0P = \mathrm{Id} + R$, $R \in \Psi^{-1}$. Then we take a ΨDO E with asymptotic expansion $E \sim \mathrm{Id} - R + R^2 - R^3 + \dots$, that would be the formal Neumann series expansion of $(\mathrm{Id} + R)^{-1}$, if the latter existed. Then EQ_0 is a left parametrix that is also a right parametrix.

An important consequence is the following elliptic regularity statement. If P is elliptic (and properly supported), then

$$\operatorname{singsupp}(PF) = \operatorname{singsupp}(f), \quad \forall f \in \mathcal{D}'(\Omega), \tag{12}$$

compare to (5). In particular, $Pf \in C^\infty$ implies $f \in C^\infty$.

It is important to emphasize that elliptic ΨDOs are not necessarily invertible or even injective. For example, the Laplace-Beltrami operator $-\Delta_{S^{n-1}}$ on the sphere is elliptic, and then so is $-\Delta_{S^{n-1}} - z$ for every number z. The latter however so not injective for z an eigenvalue. On the other hand, on a compact manifold M, an elliptic $P \in \Psi^m(M)$ is "invertible" up to a compact error, because then $QP - \mathrm{Id} = K_1$, $PQ - \mathrm{Id} = K_2$, see (11) with $K_{1,2}$ compact operators. As a consequence, such an operator is Fredholm and in particular has a finitely dimensional kernel and cokernel.

3. ΨDOs and wave front sets

The microlocal version of the pseudo-local property is given by the following:

$$\mathrm{WF}(Pf) \subset \mathrm{WF}(f) \tag{13}$$

for any (properly supported) ΨDO P and $f \in \mathcal{D}'(\Omega)$. In other words, a ΨDO cannot increase the wave front set. If P is elliptic for some m, it follows from the existence of a parametrix that there is equality above, i.e., $\mathrm{WF}(Pf) = \mathrm{WF}(f)$, which is a refinement of (12).

We say that the ΨDO P is of order $-\infty$ in the open conic set $U \subset T^*\Omega\setminus 0$, if for any closed conic set $K \subset U$ with a compact projection on the the base "x-space", (1) is fulfilled for any m. The *essential support* $\mathrm{ES}(P)$, sometimes also called the *microsupport* of P, is defined as the smallest closed conic set on the complement of which the symbol p is of order $-\infty$. Then

$$\mathrm{WF}(Pf) \subset \mathrm{WF}(f) \cap \mathrm{ES}(P).$$

Let P have a homogeneous principal symbol p_m. The characteristic set $\mathrm{Char}\, P$ is defined by

$$\mathrm{Char}\, P = \{(x,\xi) \in T^*\Omega \setminus 0;\ p_m(x,\xi) = 0\}.$$

$\mathrm{Char}\, P$ can be defined also for general ΨDOs that may not have homogeneous principal symbols. For any ΨDO P, we have

$$\mathrm{WF}(f) \subset \mathrm{WF}(Pf) \cup \mathrm{Char}\, P, \quad \forall f \in \mathcal{E}'(\Omega). \tag{14}$$

P is called *microlocally elliptic* in the open conic set U, if (10) is satisfied in all compact subsets, similarly to the definition of $\mathrm{ES}(P)$ above. If it has a homogeneous principal symbol p_m, ellipticity is equivalent to $p_m \neq 0$ in U. If P is elliptic in U, then Pf and f have the same wave front set restricted to U, as follows from (14) and (13).

3.1. **The Hamilton flow and propagation of singularities.**

Let $P \in \Psi^m(M)$ be properly supported, where M is a smooth manifold, and suppose that P has a real homogeneous principal symbol p_m. The Hamiltonian vector field of p_m on $T^*M \setminus 0$ is defined by

$$H_{p_m} = \sum_{j=1}^{n} \left(\frac{\partial p_m}{\partial x_j}\frac{\partial}{\partial \xi_j} - \frac{\partial p_m}{\partial \xi_j}\frac{\partial}{\partial x_j} \right).$$

The integral curves of H_{p_m} are called *bicharacteristics* of P. Clearly, $H_{p_m} p_m = 0$, thus p_m is constant along each bicharacteristics. The bicharacteristics along which $p_m = 0$ are called *zero bicharacteristics*.

The Hörmander's theorem about propagation of singularities is one of the fundamental results in the theory. It states that if P is an operator as above, and $Pu = f$ with $u \in \mathcal{D}'(M)$, then

$$\mathrm{WF}(u) \setminus \mathrm{WF}(f) \subset \mathrm{Char}\, P,$$

and is invariant under the flow of H_{p_m}.

An important special case is the wave operator $P = \partial_t^2 - \Delta_g$, where Δ_g is the Laplace Beltrami operator associated with a Riemannian metric g. We may add lower order terms without changing the bicharacteristics. Let (τ, ξ) be the dual variables to (t, x). The principal symbol is $p_2 = -\tau^2 + |\xi|_g^2$, where $|\xi|_g^2 := \sum g^{ij}(x)\xi_i\xi_j$, and $(g^{ij}) = (g_{ij})^{-1}$. The bicharacteristics equations then are $\dot\tau = 0$, $\dot t = -2\tau$, $\dot x^j = 2\sum g^{ij}\xi_i$, $\dot\xi_j = -2\partial_{x^j}\sum g^{ij}(x)\xi_i\xi_j$, and they are null ones if $\tau^2 = |\xi|_g^2$. Here, $\dot x = \mathrm{d}x/\mathrm{d}s$, etc. The latter two equations are the Hamiltonian curves of $\tilde H := \sum g^{ij}(x)\xi_i\xi_j$ and they are known to coincide with the geodesics $(\gamma, \dot\gamma)$ on TM when identifying vectors and covectors by the metric. They lie on the energy surface $\tilde H = \mathrm{const}$. The first two equations imply that τ is a constant, positive or negative; and up to rescaling, one can choose the parameter along the geodesics to be t. That rescaling forces the speed along the geodesic to be 1. The null condition $\tau^2 = |\xi|_g^2$ defines two smooth surfaces away from $(\tau, \xi) = (0, 0)$: $\tau = \pm|\xi|_g$. This corresponds to geodesics starting from x in direction either ξ or $-\xi$. To summarize, for the homogeneous equation $Pu = 0$, we get that each singularity (x, ξ) of the initial conditions at $t = 0$ starts to propagate from x in direction either ξ or $-\xi$ or both (depending on the initial conditiosn) along the unit speed geodesic. In fact, we get this first for the singularities in $T^*(\mathbf{R}_t \times \mathbf{R}_x^n)$ first, but since they lie in $\mathrm{Char}\, P$, one can see that they project to $T^*\mathbf{R}_x^n$ as singularities again.

4. Geometrical Optics

Geometrical optics describes asymptotically the solutions of hyperbolic equations at large frequencies. It also provides a parametrix (a solution up to smooth terms) of the initial value problem for hyperbolic equations. The resulting operators are not ΨDOs anymore; they are actually examples of Fourier Integrals Operators. Geometrical Optics also studies the large frequency behavior of solutions that reflect from a smooth surface (obstacle scattering) including diffraction; reflect from an edge or a corner; reflect and refract from a surface where the speed jumps (transmission problems).

As an example, consider the acoustic equation

$$(\partial_t^2 - c^2(x)\Delta)u = 0, \quad (t, x) \in \mathbf{R}^n, \tag{15}$$

with initial conditions $u(0, x) = f_1(x)$, $u_t(0, x) = f_2$. It is enough to assume first that f_1 and f_2 are in C_0^∞, and extend the resulting solution operator to larger spaces later.

We are looking for a solution of the form

$$u(t,x) = (2\pi)^{-n} \sum_{\sigma=\pm} \int e^{\mathrm{i}\phi_\sigma(t,x,\xi)} \Big(a_{1,\sigma}(x,\xi,t)\hat{f}_1(\xi) + |\xi|^{-1} a_{2,\sigma}(x,\xi,t)\hat{f}_2(\xi)\Big)\mathrm{d}\xi, \tag{16}$$

modulo terms involving smoothing operators of f_1 and f_2. The reason to expect two terms is already clear by the propagation of singularities theorem, and is also justified by the eikonal equation below. Here the phase functions $\phi_\pm$ are positively homogeneous of order 1 in ξ. Next, we seek the amplitudes in the form

$$a_{j,\sigma} \sim \sum_{k=0}^{\infty} a_{j,\sigma}^{(k)}, \quad \sigma = \pm,\ j = 1, 2, \tag{17}$$

where $a_{j,\sigma}^{(k)}$ is homogeneous in ξ of degree $-k$ for large $|\xi|$. To construct such a solution, we plug (16) into (15) and try to kill all terms in the expansion in homogeneous (in ξ) terms.

Equating the terms of order 2 yields the *eikonal equation*

$$(\partial_t\phi)^2 - c^2(x)|\nabla_x\phi|^2 = 0. \tag{18}$$

Write $f_j = (2\pi)^{-n}\int e^{\mathrm{i}x\cdot\xi}\hat{f}_j(\xi)\,\mathrm{d}\xi$, $j = 1, 2$, to get the following initial conditions for $\phi_\pm$

$$\phi_\pm|_{t=0} = x\cdot\xi. \tag{19}$$

The eikonal equation can be solved by the method of characteristics. First, we determine $\partial_t\phi$ and $\nabla_x\phi$ for $t = 0$. We get $\partial_t\phi|_{t=0} = \mp c(x)|\xi|$, $\nabla_x\phi|_{t=0} = \xi$. This implies existence of two solutions $\phi_\pm$. If $c = 1$, we easily get $\phi_\pm = \mp|\xi|t + x\cdot\xi$. Let for any (z,ξ), $\gamma_{z,\xi}(s)$ be unit speed geodesic through (z,ξ). Then ϕ_+ is constant along the curve $(t, \gamma_{z,\xi}(t))$ that implies that $\phi_+ = z(x,\xi)\cdot\xi$ in any domain in which (t,z) can be chosen to be coordinates. Similarly, ϕ_- is constant along the curve $(t,\gamma_{z,-\xi}(t))$. In general, we cannot solve the eikonal equation globally, for all (t,x). Two geodesics $\gamma_{z,\xi}$ and $\gamma_{w,\xi}$ may intersect, for example, giving a non-unique value for $\phi_\pm$. We always have a solution however in a neighborhood of $t = 0$.

Equate now the order 1 terms in the expansion of $(\partial_t^2 - c^2\Delta)u$ to get that the principal terms of the amplitudes must solve the *transport equation*

$$\big((\partial_t\phi_\pm)\partial_t - c^2\nabla_x\phi_\pm\cdot\nabla_x + C_\pm\big)\, a_{j,\pm}^{(0)} = 0, \tag{20}$$

with

$$2C_\pm = (\partial_t^2 - c^2\Delta)\phi_\pm.$$

This is an ODE along the vector field $(\partial_t\phi_\pm, -c^2\nabla_x\phi)$, and the integral curves of it coincide with the curves $(t,\gamma_{z,\pm\xi})$. Given an initial condition at $t = 0$, it has a unique solution along the integral curves as long as ϕ is well defined.

Equating terms homogeneous in ξ of lower order we get transport equations for $a^{(k)}_{j,\sigma}$, $j=1,2,\dots$ with the same left-hand side as in (20) with a right-hand side determined by $a^{(k-1)}_{k,\sigma}$.

Taking into account the initial conditions, we get

$$a_{1,+}+a_{1,-}=1,\quad a_{2,+}+a_{2,-}=0\quad \text{for } t=0.$$

This is true in particular for the leading terms $a^{(0)}_{1,\pm}$ and $a^{(0)}_{2,\pm}$. Since $\partial_t\phi_\pm = \mp c(x)|\xi|$ for $t=0$, and $u_t=f_2$ for $t=0$, from the leading order term in the expansion of u_t we get

$$a^{(0)}_{1,+}=a^{(0)}_{1,-},\quad \mathrm{i}c(x)(a^{(0)}_{2,-}-a^{(0)}_{2,+})=1\quad \text{for } t=0.$$

Therefore,

$$a^{(0)}_{1,+}=a^{(0)}_{1,-}=\frac{1}{2},\quad a^{(0)}_{2,+}=-a^{(0)}_{2,-}=\frac{\mathrm{i}}{2c(x)}\quad \text{for } t=0. \tag{21}$$

Note that if $c=1$, then $\phi_\pm = x\cdot\xi\mp t|\xi|$, and $a_{1,+}=a_{1,-}=1/2$, $a_{2,+}=-a_{2,-}=\mathrm{i}/2$. Using those initial conditions, we solve the transport equations for $a^{(0)}_{1,\pm}$ and $a^{(0)}_{2,\pm}$. Similarly, we derive initial conditions for the lower order terms in (17) and solve the corresponding transport equations. Then we define $a_{j,\sigma}$ by (17) as a symbol.

The so constructed u in (16) is a solution only up to smoothing operators applied to (f_1,f_2). Using standard hyperbolic estimates, we show that adding such terms to u, we get an exact solution to (15). As mentions above, this construction may fail for t too large, depending on the speed. On the other hand, the solution operator $(f_1,f_2)\mapsto u$ makes sense as a global Fourier Integral Operator for which this construction is just one if its local representations.

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Department of Mathematics, Purdue University, West Lafayette, IN 47907