BOUNDARY AND LENS RIGIDITY, TENSOR TOMOGRAPHY AND ANALYTIC MICROLOCAL ANALYSIS

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Dedicated to Professor T. Kawai on the occasion of his 60th birthday

Abstract. The boundary rigidity problem consists of determining a compact, Riemannian manifold with boundary, up to isometry, by knowing the boundary distance function between boundary points. Lens rigidity consists of determining the manifold, by knowing the scattering relation which measures, besides the travel times, the point and direction of exit of a geodesic from the manifold if one knows its point and direction of entrance. Tensor tomography is the linearization of boundary rigidity and length rigidity. It consists of determining a symmetric tensor of order two from its integral along geodesics. In this paper we survey some recent results obtained on these problems using methods from microlocal analysis, in particular analytic microlocal analysis. Although we use the distribution version of analytic microlocal analysis, many of the ideas were based on the pioneer work of the Sato school of microlocal analysis of which Professor Kawai was a very important member.

1. Boundary Rigidity and Tensor Tomography

Let $(M, \partial M, g)$ be a compact Riemannian manifold with boundary. Denote by ρ_g the distance function in the metric g. The boundary rigidity problem consists of whether $\rho_g(x, y)$, known for all x, y on ∂M, determines the metric uniquely. It is clear that any isometry which is the identity at the boundary will give rise to the same distance functions on the boundary. Therefore, the natural question is whether this is the only obstruction to unique identifiability of the metric. The boundary distance function only takes into account the shortest paths and it is easy to find counterexamples where ρ_g does not carry any information about certain open subset of M, so one needs to pose some restrictions on the metric. One such condition is simplicity of the metric.

Definition 1. *We say that the Riemannian metric g is* simple *in M, if ∂M is strictly convex w.r.t. g, and for any $x \in M$, the exponential map $\exp_x : \exp_x^{-1}(M) \to M$ is a diffeomorphism.*

By strictly convex we mean convex (any two points are connected by a unique minimizing geodesic), and the second fundamental form on the boundary is positive.

Michel [18] conjectured that a *simple* metric g is uniquely determined, up to an action of a diffeomorphism fixing the boundary, by the boundary distance function $\rho_g(x, y)$ known for all x and y on ∂M.

This problem also arose in geophysics in an attempt to determine the inner structure of the Earth by measuring the travel times of seismic waves. It goes back to Herglotz [15] and

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Wiechert and Zoeppritz [38] who considered the case of a radial metric conformal to the Euclidean metric. Although the emphasis has been in the case that the medium is isotropic, the anisotropic case has been of interest in geophysics since the Earth is anisotropic. It has been found that even the inner core of the Earth exhibits anisotropic behavior [8].

Unique recovery of g (up to an action of a diffeomorphism fixing the boundary) from the boundary distance function is known for simple metrics conformal to each other [10], [6], [19], [20], [21], [4], for flat metrics [13], for locally symmetric spaces of negative curvature [5]. In two dimensions it was known for simple metrics with negative curvature [9] and [22], and recently it was shown in [24] for simple metrics with no restrictions on the curvature. In [29], the authors proved a local result for metrics in a small neighborhood of the Euclidean one. This result was used in [17] to prove a semiglobal solvability result assuming that one metric is close to the Euclidean and the other has bounded curvature. Burago and Ivanov have recently extended the latter result; they show that metrics close to the Euclidean metric are boundary rigid [7].

It is known [25], that a linearization of the boundary rigidity problem near a simple metric g is given by the following integral geometry problem: recover a symmetric tensor of order 2, which in any coordinates is given by $f = (f_{ij})$, by the geodesic X-ray transform

$$I_g f(\gamma) = \int f_{ij}(\gamma(t))\dot\gamma^i(t)\dot\gamma^j(t)\,\mathrm{d}t$$

known for all geodesics γ in M. It can be easily seen that $I_g dv = 0$ for any vector field v with $v|_{\partial M} = 0$, where dv denotes the symmetric differential

$$[dv]_{ij} = \frac{1}{2}\left(\nabla_i v_j + \nabla_j v_i\right), \tag{1.1}$$

and $\nabla_k v$ denote the covariant derivatives of the vector field v. This is the linear version of the fact that ρ_g does not change on $(\partial M)^2 := \partial M \times \partial M$ under an action of a diffeomorphism as above. The natural formulation of the linearized problem is therefore that $I_g f = 0$ implies $f = dv$ with v vanishing on the boundary. We will refer to this property as *s-injectivity* of I_g. More precisely, we have.

Definition 2. *We say that I_g is* s-injective *in M, if $I_g f = 0$ and $f \in L^2(M)$ imply $f = dv$ with some vector field $v \in H^1_0(M)$.*

Any symmetric tensor $f \in L^2(M)$ admits an orthogonal decomposition $f = f^s + dv$ into a *solenoidal* and *potential* parts with $v \in H^1_0(M)$, and f^s divergence free, i.e., $\delta f^s = 0$, where δ is the adjoint operator to $-d$ given by $[\delta f]_i = g^{jk}\nabla_k f_{ij}$ [25]. Therefore, I_g is s-injective, if it is injective on the space of solenoidal tensors.

The inversion of I_g is a problem of independent interest in integral geometry, also called *tensor tomography*. We first survey the recent results on this problem. S-injectivity, respectively injectivity for 1-tensors (1-forms) and functions is known, see [25] for references. S-injectivity of I_g was proved in [23] for metrics with negative curvature, in [25] for metrics with small curvature and in [27] for Riemannian surfaces with no focal points. A conditional and non-sharp stability estimate for metrics with small curvature is also established in [25]. In [30], we proved stability estimates for s-injective metrics (see (1.5) below) and sharp estimates about the recovery of a 1-form $f = f_j dx^j$ and a function f from the associated $I_g f$

which is defined by

$$I_g f(\gamma) = \int f_i(\gamma(t))\dot\gamma^i(t))\ dt.$$

The stability estimates proven in [30] were used to prove local uniqueness for the boundary rigidity problem near any simple metric g with s-injective I_g.

Similarly to [36], we say that f is analytic in the set K (not necessarily open), if it is real analytic in some neighborhood of K.

The results that follow in this section are based on [32]. The first main result we discuss is about s-injectivity for simple analytic metrics.

Theorem 1. *Let g be a simple, real analytic metric in M. Then I_g is s-injective.*

Sketch of the proof. Note that a simple metric g in M can be extended to a simple metric in some M_1 with $M \Subset M_1$. A simple manifold is diffeomorphic to a (strictly convex) domain $\Omega \subset \mathbf{R}^n$ with the Euclidean coordinates x in a neighborhood of Ω and a metric $g(x)$ there. For this reason, it is enough to prove the results of this section for domains Ω in $\mathbf{R}^n$ provided with a Riemannian metric g.

The proof of Theorem 1 is based on the following. For smooth metrics, the normal operator $N_g = I_g^* I_g$ is a pseudodifferential operator with a non-trivial null space which is given by

$$(N_g f)_{kl}(x) = \frac{2}{\sqrt{\det g}} \int \frac{f^{ij}(y)}{\rho_g(x,y)^{n-1}} \frac{\partial \rho_g}{\partial y^i}\frac{\partial \rho_g}{\partial y^j}\frac{\partial \rho_g}{\partial x^k}\frac{\partial \rho_g}{\partial x^l} \det \frac{\partial^2(\rho_g^2/2)}{\partial x \partial y}\,\mathrm{d}y, \quad x \in \Omega. \tag{1.2}$$

In the case that the metric g is real-analytic, N_g is an analytic pseudodifferential operator with a non-trivial kernel. We construct an analytic parametrix, using the analytic pseudodifferential calculus in [36], that allows us to reconstruct the solenoidal part of a tensor field from its geodesic X-ray transform, up to a term that is analytic near Ω. If $I_g f = 0$, we show that for some v vanishing on $\partial\Omega$, $\tilde f := f - dv$ must be flat at $\partial\Omega$ and analytic in $\bar\Omega$, hence $\tilde f = 0$. This is similar to the known argument that an analytic elliptic pseudodifferential operator resolves the analytic singularities, hence cannot have compactly supported functions in its kernel. In our case we have a non-trivial kernel, and complications due to the presence of a boundary, in particular a lost of one derivative. For more details see [32]. □

As shown in [30], the s-injectivity of I_g for analytic simple g implies a stability estimate for I_g. In next theorem we show something more, namely that we have a stability estimate for g in a neighborhood of each analytic metric, which leads to stability estimates for generic metrics.

As above, let $M_1 \Supset M$ be a compact manifold which is a neighborhood of M and g extends as a simple metric there. We always assume that our tensors are extended as zero outside M, which may create jumps at ∂M. Define the *normal operator* $N_g = I_g^* I_g$, where I_g^* denotes the operator adjoint to I_g with respect to an appropriate measure. We showed in [30] that N_g is a pseudo-differential operator in M_1 of order -1.

We introduce the norm $\|\cdot\|_{\tilde H^2(M_1)}$ of $N_g f$ in $M_1 \supset M$ in the following way. Choose $\chi \in C_0^\infty$ equal to 1 near ∂M and supported in a small neighborhood of ∂M and let $\chi = \sum_{j=1}^J \chi_j$ be a partition of χ such that for each j, on $\operatorname{supp}\chi_j$ we have coordinates (x'_j, x^n_j), with x^n_j a normal coordinate. Set

$$\|f\|^2_{\tilde H^1} = \int \sum_{j=1}^J \chi_j \left(\sum_{i=1}^{n-1} |\partial_{x^i_j} f|^2 + |x^n_j \partial_{x^n_j} f|^2 + |f|^2 \right) \mathrm{d}x, \tag{1.3}$$

$$\|N_g f\|_{\tilde H^2(M_1)} = \sum_{i=1}^n \|\partial_{x^i} N_g f\|_{\tilde H^1} + \|N_g f\|_{H^1(M_1)}. \tag{1.4}$$

In other words, in addition to derivatives up to order 1, $\|N_g f\|_{\tilde H^2(M_1)}$ includes also second derivatives near ∂M but they are realized as first derivatives of $\nabla N_g f$ tangent to ∂M.

The reason to use the $\tilde H^2(M_1)$ norm, instead of the stronger $H^2(M_1)$ one, is that this allows us to work with $f \in H^1(M)$, not only with $f \in H^1_0(M)$, since for such f, extended as 0 outside M, we still have that $N_g f \in \tilde H^2(M_1)$, see [30]. On the other hand, $f \in H^1(M)$ implies $N_g f \in \tilde H^2(M_1)$ despite the possible jump of f at ∂M.

Our stability estimate for the linearized inverse problem is as follows:

Theorem 2. *There exists k_0 such that for each $k \geq k_0$, the set $\mathcal{G}^k(M)$ of simple $C^k(M)$ metrics in M for which I_g is s-injective is open and dense in the $C^k(M)$ topology. Moreover, for any $g \in \mathcal{G}^k(M)$,*

$$\|f^s\|_{L^2(M)} \leq C \|N_g f\|_{\tilde H^2(M_1)}, \quad \forall f \in H^1(M), \tag{1.5}$$

with a constant $C > 0$ that can be chosen locally uniform in $\mathcal{G}^k(M)$ in the $C^k(M)$ topology.

Of course, $\mathcal{G}^k(M)$ includes all real analytic simple metrics in M, according to Theorem 1.

Sketch of the proof. The proof of the basic estimate (1.5) is based on the following ideas. For g of finite smoothness, one can still construct a parametrix Q_g of N_g as above that allows us to reconstruct f^s from $N_g f$ up to smoothing operator terms. This is done in a way similar to that in [30] in two steps: first we invert N_g modulo smoothing operators in a neighborhood M_1 of M, and that gives us $f^s_{M_1}$, i.e., the solenoidal projection of f but associated to the manifold M_1. Next, we compare $f^s_{M_1}$ and f^s and show that one can get the latter from the former by an operator that loses one derivative. This is the same construction as in the proof of Theorem 1 above but the metric is only C^k, $k \gg 1$.

After applying the parametrix Q_g, the equation for recovering f^s from $N_g f$ is reduced to solving the Fredholm equation

$$(\mathcal{S}_g + K_g) f = Q_g N_g f, \quad f \in \mathcal{S}_g L^2(M) \tag{1.6}$$

where $\mathcal{S}_g$ is the projection to solenoidal tensors, similarly we denote by $\mathcal{P}_g$ the projection onto potential tensors. Here, K_g is a compact operator on $\mathcal{S}_g L^2(M)$. We can write this as an equation in the whole $L^2(M)$ by adding $\mathcal{P}_g f$ to both sides above to get

$$(I + K_g) f = (Q_g N_g + \mathcal{P}_g) f. \tag{1.7}$$

Then the solenoidal projection of the solution of (1.7) solves (1.6). A finite rank modification of K_g above can guarantee that $I + K_g$ has a trivial kernel, and therefore is invertible, if and only if N_g is s-injective. The problem then reduces to that of invertibility of $I + K_g$. The operators above depend continuously on $g \in C^k$, $k \gg 1$. Since for g analytic, $I + K_g$ is invertible by Theorem 1, it would still be invertible in a neighborhood of any analytic g, and estimate (1.5) is true with a locally uniform constant. Analytic (simple) metrics are dense in the set of all simple metrics, and this completes the sketch of the proof of Theorem 2. For more details see [32]. □

The analysis of I_g can also be carried out for symmetric tensors of any order, see e.g., [25] and [26]. Since we are motivated by the boundary rigidity problem, and to simplify the exposition, we study only tensors of order 2.

Theorem 2 and especially estimate (1.5) allow us to prove the following local generic uniqueness result for the non-linear boundary rigidity problem.

Theorem 3. *Let k_0 and $\mathcal{G}^k(M)$ be as in Theorem 2. There exists $k \geq k_0$, such that for any $g_0 \in \mathcal{G}^k$, there is $\varepsilon > 0$, such that for any two metrics g_1, g_2 with $\|g_m - g_0\|_{C^k(M)} \leq \varepsilon$, $m = 1, 2$, we have the following:*

$$\rho_{g_1} = \rho_{g_2} \text{ on } (\partial M)^2 \quad \text{implies } g_2 = \psi_* g_1 \tag{1.8}$$

with some $C^{k+1}(M)$-diffeomorphism $\psi : M \to M$ fixing the boundary.

Sketch of the proof. We prove Theorem 3 by linearizing and using Theorem 2, and especially (1.5), see also [30]. This requires first to pass to special semigeodesic coordinates related to each metric in which $g_{in} = \delta_{in}$, $\forall i$. We denote the corresponding pull-backs by g_1, g_2 again. Then we show that if g_1 and g_2 have the same distance on the boundary, then $g_1 = g_2$ on the boundary with all derivatives. As a result, for $f := g_1 - g_2$ we get that $f \in C_0^l(\bar\Omega)$ with $l \gg 1$, if $k \gg 1$; and $f_{in} = 0$, $\forall i$. Then we linearize to get

$$\|N_{g_1} f\|_{L^\infty(\Omega_1)} \leq C\|f\|_{C^1}^2,$$

where $\Omega_1 \supset \bar\Omega$ is as above. Combine this with (1.5) and interpolation estimates, to get $\forall \mu < 1$,

$$\|f^s\|_{L^2} \leq C\|f\|_{L^2}^{1+\mu}.$$

One can show that tensors satisfying $f_{in} = 0$ also satisfy $\|f\|_{L^2} \leq C\|f^s\|_{H^2}$, and using this, and interpolation again, we get

$$\|f\|_{L^2} \leq C\|f\|_{L^2}^{1+\mu'}, \quad \mu' > 0.$$

This implies $f = 0$ for $\|f\| \ll 1$. Note that the condition $f \in C_0^l(\bar\Omega)$ is used to make sure that f, extended as zero in $\Omega_1 \setminus \Omega$, is in $H_0^l(\Omega)$, and then use this fact in the interpolation estimates. Again, for more details see [32]. □

Finally, in [32] it is proven a conditional stability estimate of Hölder type. A similar estimate near the Euclidean metric was proven in [37] based on the approach in [29].

Theorem 4. *Let k_0 and $\mathcal{G}^k(M)$ be as in Theorem 2. Then for any $\mu < 1$, there exits $k \geq k_0$ such that for any $g_0 \in \mathcal{G}^k$, there is an $\varepsilon_0 > 0$ and $C > 0$ with the property that that for any two metrics g_1, g_2 with $\|g_m - g_0\|_{C(M)} \leq \varepsilon_0$, and $\|g_m\|_{C^k(M)} \leq A$, $m = 1, 2$, with some $A > 0$, we have the following stability estimate*

$$\|g_2 - \psi_* g_1\|_{C^2(M)} \leq C(A)\|\rho_{g_1} - \rho_{g_2}\|_{C(\partial M \times \partial M)}^\mu$$

with some diffeomorphism $\psi : M \to M$ fixing the boundary.

Sketch of the proof. To prove Theorem 4, we basically follow the uniqueness proof sketched above by showing that each step is stable. The analysis is more delicate near pairs of points too close to each other. An important ingredient of the proof is stability at the boundary, that is also of independent interest:

Theorem 5. *Let g_0 and g_1 be two simple metrics in Ω, and $\Gamma \subset\subset \Gamma' \subset \partial\Omega$ be two sufficiently small open subsets of the boundary. Then for some diffeomorphism ψ fixing the boundary,*

$$\left\|\partial_{x^n}^k(\psi_* g_1 - g_0)\right\|_{C^m(\bar{\Gamma})} \leq C_{k,m}\left\|\rho_{g_1}^2 - \rho_{g_0}^2\right\|_{C^{m+2k+2}\left(\overline{\Gamma'\times\Gamma'}\right)},$$

where $C_{k,m}$ depends only on Ω and on a upper bound of g_0, g_1 in $C^{m+2k+5}(\bar{\Omega})$.

Theorem 4 can be used to obtain stability near generic simple metrics for the inverse problem of recovering g from the hyperbolic *Dirichlet-to-Neumann map* Λ_g. It is known that g can be recovered uniquely from Λ_g, up to a diffeomorphism as above, see e.g. [3]. This result however relies on a unique continuation theorem by Tataru [35] and it is unlikely to provide Hölder type of stability estimate as above. By using the fact that ρ_g is related to the leading singularities in the kernel of Λ_g, we proved a Hölder stability estimate under the assumptions above, relating g and Λ_g. We refer to [31] for details. □

2. Lens rigidity for a class of non-simple manifolds

Suppose we have a Riemannian metric in Euclidean space which is the Euclidean metric outside a compact set. The inverse scattering problem for metrics is to determine the Riemannian metric by measuring the scattering operator (see [14]). A similar obstruction to the boundary rigidity problem occurs in this case with the diffeomorphism ψ equal to the identity outside a compact set. It was proven in [14] that from the wave front set of the scattering operator, one can determine, under some conditions on the metric including non-trapping, the *scattering relation* on the boundary of a large ball. This uses high frequency information of the scattering operator. In the semiclassical setting Alexandrova [1], [2] has shown for a large class of operators that the scattering operator associated to potential and metric perturbations of the Euclidean Laplacian is a semiclassical Fourier integral operator that quantizes the scattering relation. The scattering relation maps the point and direction of a geodesic entering the manifold to the point and direction of exit of the geodesic. As mentioned in the previous section, the boundary rigidity problem only takes into account the shortest paths. For non-simple manifolds in particular, if we have conjugate points or the boundary is not strictly convex, we need to look at the behavior of all the geodesics and the scattering relation encodes this information. We proceed to define in more detail the scattering relation and the lens rigidity problem and state our results. We note that we also consider the case of incomplete data, that is when we don't have information about all the geodesics entering the manifold. More details can be found in [33], [34].

Denote by $SM = \{(x,\xi)\in TM;\ |\xi|=1\}$ the unit sphere bundle and set

$$\partial_\pm SM = \{(x,\xi)\in\partial SM;\ \pm\langle\nu,\xi\rangle<0\}\,, \tag{2.1}$$

where ν is the unit interior normal, $\langle\cdot,\cdot\rangle$ and stands for the inner product. The scattering relation

$$\Sigma : \partial_- SM \to \overline{\partial_+ SM} \tag{2.2}$$

is defined by $\Sigma(x,\xi) = (y,\eta) = \Phi^\ell(x,\xi)$, where Φ^t is the geodesic flow, and $\ell > 0$ is the first moment, at which the unit speed geodesic through (x,ξ) hits ∂M again. If such an ℓ does not exists, we formally set $\ell = \infty$ and we call the corresponding initial condition and the corresponding geodesic *trapping*. This defines also $\ell(x,\xi)$ as a function $\ell : \partial_- SM \to [0,\infty]$. Note that Σ and ℓ are not necessarily continuous.

It is convenient to think of Σ and ℓ as defined on the whole ∂SM with $\Sigma = \mathrm{Id}$ and $\ell = 0$ on $\overline{\partial_+ SM}$.

We parametrize the scattering relation in a way that makes it independent of pulling it back by diffeomorphisms fixing ∂M pointwise. Let $\kappa_\pm : \partial_\pm SM \to B(\partial M)$ be the orthogonal projection onto the (open) unit ball tangent bundle that extends continuously to the closure of $\partial_\pm SM$. Then $\kappa_\pm$ are homeomorphisms, and we set

$$\sigma = \kappa_+ \circ \Sigma \circ \kappa_-^{-1} : \overline{B(\partial M)} \longrightarrow \overline{B(\partial M)}. \tag{2.3}$$

According to our convention, $\sigma = \mathrm{Id}$ on $\partial(\overline{B(\partial M)}) = S(\partial M)$. We equip $\overline{B(\partial M)}$ with the relative topology induced by $T(\partial M)$, where neighborhoods of boundary points (those in $S(\partial M)$) are given by half-neighborhoods, i.e., by neighborhoods in $T(\partial M)$ intersected with $\overline{B(\partial M)}$.

It is possible to define σ in a way that does not require knowledge of $g|_{T(\partial M)}$ by thinking of any boundary vector ξ as characterized by its angle with ∂M and the direction of its tangential projection. Let $\mathcal{D}$ be an open subset of $\overline{B(\partial M)}$. A priori, the latter depends on $g|_{T(\partial M)}$. By the remark above, we can think of it as independent of $g|_{T(\partial M)}$ however.

The lens rigidity question we study is the following:

Given M and $\mathcal{D}$, do σ and ℓ, restricted to $\mathcal{D}$, determine g uniquely, up to a pull back of a diffeomorphism that is identity on ∂M?

The answer to this question, even when $\mathcal{D} = B(\partial M)$, is negative, see [12]. The known counter-examples are trapping manifolds.

The boundary rigidity problem and the lens rigidity one are equivalent for simple metrics.

2.1. Main assumptions.

Definition 3. *We say that $\mathcal{D}$ is **complete** for the metric g, if for any $(z,\zeta) \in T^*M$ there exists a maximal in M, finite length unit speed geodesic $\gamma : [0,l] \to M$ through z, normal to ζ, such that*

$$\{(\gamma(t),\dot\gamma(t));\ 0 \le t \le l\} \cap S(\partial M) \subset \mathcal{D}, \tag{2.4}$$

$$\text{there are no conjugate points on } \gamma. \tag{2.5}$$

*We call the C^k metric g **regular**, if a complete set $\mathcal{D}$ exists, i.e., if $\overline{B(\partial M)}$ is complete.*

If $z \in \partial M$ and ζ is conormal to ∂M, then γ may reduce to one point.

Topological Condition (T): Any path in M connecting two boundary points is homotopic to a polygon $c_1 \cup \gamma_1 \cup c_2 \cup \gamma_2 \cup \cdots \cup \gamma_k \cup c_{k+1}$ with the properties that for any j,

(i) c_j is a path on ∂M;

(ii) $\gamma_j : [0, l_j] \to M$ is a geodesic lying in $M^{\rm int}$ with the exception of its endpoints and is transversal to ∂M at both ends; moreover, $\kappa_-(\gamma_j(0), \dot\gamma_j(0)) \in \mathcal{D}$;

Notice that (T) is an open condition w.r.t. g, i.e., it is preserved under small C^2 perturbations of g. To define the $C^K(M)$ norm below in a unique way, we choose and fix a finite atlas on M.

2.2. **Results about the linear problem.** We refer to [33] for more details about the results in this subsection. It turns out that a linearization of the lens rigidity problem is again the problem of s-injectivity of the ray transform I. Here and below we sometimes drop the subscript g. Given $\mathcal{D}$ as above, we denote by $I_{\mathcal{D}}$ (or $I_{g,\mathcal{D}}$) the ray transform I restricted to the maximal geodesics issued from $(x,\xi) \in \kappa_-^{-1}(\mathcal{D})$.

The first result of this subsection generalizes Theorem 1.

Theorem 6. *Let g be an analytic, regular metric on M. Let $\mathcal{D}$ be complete and open. Then $I_{\mathcal{D}}$ is s-injective.*

Sketch of the proof. Since we know integrals over a subset of geodesics only, this creates difficulties with cut-offs in the phase variable that cannot be analytic. For this reason, the proof of Theorem 6 is different than that of Theorem 1.

Let g be an analytic regular metrics in M, and let $M_1 \Supset M$ be the manifold where g is extended analytically. There is an analytic atlas in M, and ∂M can be assumed to be analytic, too. In other words, now $(M, \partial M, g)$ is a real analytic manifold with boundary. We denote by $\mathcal{A}(M)$ (respectively $\mathcal{A}(M_1)$) the set of analytic functions on M (respectively M_1). Next, $f^s_{M_1}$ denotes the solenoidal part of the tensor f, extended as zero to M_1, in the manifold M_1.

The main step is to show that $I_{\mathcal{D}} f = 0$ implies $f^s \in \mathcal{A}(M)$. In order to do that one shows that $f^s_{M_1} \in \mathcal{A}(M_1)$. Let us first notice, that in $M_1 \setminus M$, $f^s_{M_1} = -dv_{M_1}$, where v_{M_1} satisfies $\delta dv_{M_1} = 0$ in $M_1 \setminus M$, $v|_{\partial M_1} = 0$ since $f = 0$ in $M_1 \setminus M$. Therefore, v_{M_1} is analytic up to ∂M_1. Therefore, we only need to show that $f^s_{M_1}$ is analytic in the interior of M_1. Below, $\mathrm{WF}_{\mathrm{A}}(f)$ stands for the analytic wave front set of f, see [28, 36].

The crucial point is the following microlocal analytic regularity result.

Proposition 1. *Let γ_0 be a fixed maximal geodesic in M with endpoints on ∂M, without conjugate points, and let $I_g f(\gamma) = 0$ for $\gamma \in \mathrm{neigh}(\gamma_0)$. Let g be analytic in $\mathrm{neigh}(\gamma_0)$. Then*

$$N^*\gamma_0 \cap \mathrm{WF}_{\mathrm{A}}(f^s_{M_1}) = \emptyset. \tag{2.6}$$

Sketch of the proof. Set $f = f^s_{M_1}$. Let U_ε be a tubular neighborhood of γ_0, and $x = (x', x^n)$ be semigeodesic coordinates in it such that $x' = 0$ on γ_0. Fix $x_0 \in \gamma_0 \cap M$. We can assume that $x_0 = 0$ and $g_{ij}(0) = \delta_{ij}$. Then we can assume that $U_\varepsilon = \{-l_1 - \varepsilon < x^n < l_2 + \varepsilon,\ |x'| < \varepsilon\}$ with the part of γ_0 corresponding to $x^n \not\in [-l_1, l_2]$ outside M.

Fix $\xi^0 = ((\xi^0)', 0)$ with $\xi^0_n = 0$. We will show that

$$(0, \xi^0) \not\in \mathrm{WF}_{\mathrm{A}}(f). \tag{2.7}$$

We choose a local chart for the geodesics close to γ_0. Set first $Z = \{x^n = 0;\ |x'| < 7\varepsilon/8\}$, and denote the x' variable on Z by z'. Then z', θ' (with $|\theta'| \ll 1$) are local coordinates in $\mathrm{neigh}(\gamma_0)$ determined by $(z', \theta') \to \gamma_{(z',0),(\theta',1)}$ where the latter denotes the geodesic through the point $(z',0)$ in the direction $(\theta',1)$. Let $\chi_N(z')$ be a smooth cut-off function equal to 1 for $|z'|\le 3\varepsilon/4$ and supported in Z, also satisfying $|\partial^\alpha \chi_N|\le (CN)^{|\alpha|}$, $|\alpha|\le N$. Set $\theta=(\theta',1)$, $|\theta'|\ll 1$, and multiply

$$If\left(\gamma_{(z',0),\theta}\right)=0$$

by $\chi_N(z')e^{\mathrm{i}\lambda z'\cdot\xi'}$, where $\lambda>0$, ξ' is in a complex neighborhood of $(\xi^0)'$, and integrate w.r.t. z' to get

$$\iint e^{\mathrm{i}\lambda z'\cdot\xi'}\chi_N(z')f_{ij}\left(\gamma_{(z',0),\theta}(t)\right)\dot\gamma^i_{(z',0),\theta}(t)\dot\gamma^j_{(z',0),\theta}(t)\,\mathrm{d}t\,\mathrm{d}z'=0. \tag{2.8}$$

Set $x=\gamma_{(z',0),\theta}(t)$. If $\theta'=0$, we have $x=(z',t)$. By a perturbation argument, for θ' fixed and small enough, (t,z') are analytic local coordinates, depending analytically on θ'. In particular, $x=(z'+t\theta',t)+O(|\theta'|)$ but this expansion is not enough for the analysis below. Performing a change of variables in (2.8), we get

$$\int e^{\mathrm{i}\lambda z'(x,\theta')\cdot\xi'}a_N(x,\theta')f_{ij}(x)b^i(x,\theta')b^j(x,\theta')\,\mathrm{d}x=0 \tag{2.9}$$

for $|\theta'|\ll 1$, $\forall\lambda$, $\forall\xi'$, where, for $|\theta'|\ll 1$, the function $(x,\theta')\mapsto a_N$ is positive for x in a neighborhood of γ_0, vanishing for $x\notin U_\varepsilon$, and satisfies the same estimate as χ_N. The vector field b is analytic, and $b(0,\theta')=\theta$, $a_N(0,\theta')=1$.

To clarify the approach, note that if g is Euclidean in neigh(γ_0), then (2.9) reduces to

$$\int e^{\mathrm{i}\lambda(\xi',-\theta'\cdot\xi')\cdot x}\chi f_{ij}(x)\theta^i\theta^j\,\mathrm{d}x=0,$$

where $\chi=\chi(x'-x^n\theta')$. Then $\xi=(\xi',-\theta'\cdot\xi')$ is perpendicular to $\theta=(\theta',1)$. This implies that

$$\int e^{\mathrm{i}\lambda\xi\cdot x}\chi f_{ij}(x)\theta^i(\xi)\theta^j(\xi)\,\mathrm{d}x=0 \tag{2.10}$$

for any function $\theta(\xi)$ defined near ξ^0, such that $\theta(\xi)\cdot\xi=0$. This has been noticed and used before if g is close to the Euclidean metric (with $\chi=1$), see e.g., [29]. We will assume that $\theta(\xi)$ is analytic. A simple argument (see e.g. [25, 29]) shows that a constant symmetric tensor f_{ij} is uniquely determined by the numbers $f_{ij}\theta^i\theta^j$ for finitely many θ's (actually, for $N'=(n+1)n/2$ θ's); and in any open set on the unit sphere, there are such θ's. On the other hand, f is solenoidal. To simplify the argument, assume for a moment that f vanishes on ∂M. Then $\xi^i\hat f_{ij}(\xi)=0$. Therefore, combining this with (2.10), we need to choose $N=n(n-1)/2$ vectors $\theta(\xi)$, perpendicular to ξ, that would uniquely determine the tensor $\hat f$ on the plane perpendicular to ξ. To this end, it is enough to know that this choice can be made for $\xi=\xi^0$, then it would be true for $\xi\in$ neigh(ξ^0). This way, $\xi^i\hat f_{ij}(\xi)=0$ and the N equations (2.10) with the so chosen $\theta_p(\xi)$, $p=1,\dots,N$, form a system with a tensor-valued symbol elliptic near $\xi=\xi^0$. The C^∞ ΨDO calculus easily implies the statement of the lemma in the C^∞ category, and the complex stationary phase method below, or the analytic ΨDO calculus in [36] with appropriate cut-offs in ξ, implies the lemma in this special case (g locally Euclidean).

The general case is considered in [33], and is based on an application of a complex stationary phase argument [28] to (2.9) as in [16]. □

Proposition 1 makes it possible to prove that $f^s \in \mathcal{A}(M)$. We combine this with a boundary determination theorem for tensors, a linear version of Theorem 10 below, to conclude that then $f = 0$. $\square$

Next, we formulate a stability estimate in the spirit of Theorem 2. We need first to parametrize (a complete subset of) the geodesics issued from $\mathcal{D}$ in a different way that would make them a manifold. The parametrization provided by $\mathcal{D}$ is inconvenient near the directions tangent to ∂M.

Let H_m be a finite collection of smooth hypersurfaces in M_1^{int}. Let $\mathcal{H}_m$ be an open subset of $\{(z,\theta) \in SM_1;\ z \in H_m, \theta \notin T_z H_m\}$, and let $\pm l_m^{\pm}(z,\theta) \geq 0$ be two continuous functions. Let $\Gamma(\mathcal{H}_m)$ be the set of geodesics

$$\Gamma(\mathcal{H}_m) = \left\{\gamma_{z,\theta}(t);\ l_m^-(z,\theta) \leq t \leq l_m^+(z,\theta),\ (z,\theta) \in \mathcal{H}_m\right\}, \tag{2.11}$$

that, depending on the context, is considered either as a family of curves, or as a point set. We also assume that each $\gamma \in \Gamma(\mathcal{H}_m)$ is a simple geodesic (no conjugate points).

If g is simple, then one can take a single $H = \partial M_1$ with $l^- = 0$ and an appropriate $l^+(z,\theta)$. If g is regular only, and Γ is any complete set of geodesics, then any small enough neighborhood of a simple geodesic in Γ has the properties listed in the paragraph above and by a compactness argument one can choose a finite complete set of such $\Gamma(\mathcal{H}_m)$'s, that is included in the original Γ.

Given $\mathcal{H} = \{\mathcal{H}_m\}$ as above, we consider an open set $\mathcal{H}' = \{\mathcal{H}'_m\}$, such that $\mathcal{H}'_m \Subset \mathcal{H}_m$, and let $\Gamma(\mathcal{H}'_m)$ be the associated set of geodesics defined as in (2.11), with the same $l_m^{\pm}$. Set $\Gamma(\mathcal{H}) = \cup\Gamma(\mathcal{H}_m)$, $\Gamma(\mathcal{H}') = \cup\Gamma(\mathcal{H}'_m)$.

The restriction $\gamma \in \Gamma(\mathcal{H}'_m) \subset \Gamma(\mathcal{H}_m)$ can be modeled by introducing a weight function α_m in $\mathcal{H}_m$, such that $\alpha_m = 1$ on $\mathcal{H}'_m$, and $\alpha_m = 0$ otherwise. More generally, we allow α_m to be smooth but still supported in $\mathcal{H}_m$. We then write $\alpha = \{\alpha_m\}$, and we say that $\alpha \in C^k(\mathcal{H})$, if $\alpha_m \in C^k(\mathcal{H}_m)$, $\forall m$.

We consider $I_{\alpha_m} = \alpha_m I$, or more precisely, in the coordinates $(z,\theta) \in \mathcal{H}_m$,

$$I_{\alpha_m} f = \alpha_m(z,\theta) \int_0^{l_m(z,\theta)} \left\langle f(\gamma_{z,\theta}), \dot\gamma_{z,\theta}^2 \right\rangle \mathrm{d}t, \quad (z,\theta) \in \mathcal{H}_m. \tag{2.12}$$

Next, we set

$$I_\alpha = \{I_{\alpha_m}\}, \quad N_{\alpha_m} = I_{\alpha_m}^* I_{\alpha_m} = I^*|\alpha_m|^2 I, \quad N_\alpha = \sum N_{\alpha_m}, \tag{2.13}$$

where the adjoint is taken w.r.t. the measure $\mathrm{d}\mu := |\langle \nu(z), \theta\rangle|\, \mathrm{d}S_z\, \mathrm{d}\theta$ on $\mathcal{H}_m$, $\mathrm{d}S_z\, \mathrm{d}\theta$ being the induced measure on SM, and $\nu(z)$ being a unit normal to H_m.

S-injectivity of N_α is equivalent to s-injectivity for I_α, which in turn is equivalent to s-injectivity of I restricted to $\operatorname{supp}\alpha$.

Theorem 7.
(a) Let $g = g_0 \in C^k$, $k \gg 1$ be regular, and let $\mathcal{H}' \Subset \mathcal{H}$ be as above with $\Gamma(\mathcal{H}')$ complete. Fix $\alpha = \{\alpha_m\} \in C^\infty$ with $\mathcal{H}'_m \subset \operatorname{supp}\alpha_m \subset \mathcal{H}_m$. Then if I_α is s-injective, we have

$$\|f^s\|_{L^2(M)} \leq C\|N_\alpha f\|_{\tilde H^2(M_1)}. \tag{2.14}$$

(b) Assume that $\alpha = \alpha_g$ in (a) depends on $g \in C^k$, so that $C^k(M_1) \ni g \to C^l(\mathcal{H}) \ni \alpha_g$ is continuous with $l \gg 1$, $k \gg 1$. Assume that $I_{g_0,\alpha_{g_0}}$ is s-injective. Then estimate (2.14) remains true for g in a small enough neighborhood of g_0 in $C^k(M_1)$ with a uniform constant $C > 0$.

The theorem above allows us to formulate a generic result:

Theorem 8. *Let $\mathcal{G} \subset C^k(M)$ be an open set of regular Riemannian metrics on M such that (T) is satisfied for each one of them. Let the set $\mathcal{D}' \subset \partial SM$ be open and complete for each $g \in \mathcal{G}$. Then there exists an open and dense subset $\mathcal{G}_s$ of $\mathcal{G}$ such that $I_{g,\mathcal{D}'}$ is s-injective for any $g \in \mathcal{G}_s$.*

Of course, the set $\mathcal{G}_s$ includes all real analytic metrics in $\mathcal{G}$.

Corollary 1. *Let $\mathcal{R}(M)$ be the set of all regular C^k metrics on M satisfying (T) equipped with the $C^k(M_1)$ topology. Then for $k \gg 1$, the subset of metrics for which the X-ray transform I over all simple geodesics through all points in M is s-injective, is open and dense in $\mathcal{R}(M)$.*

2.3. **Results about the non-linear lens rigidity problem.** Using the results above, we prove the following about the lens rigidity problem on manifolds satisfying the assumptions in Section 2.1. More details can be found in [34].

Theorem 9 below says, loosely speaking, that for the classes of manifolds and metrics we study, the uniqueness question for the non-linear lens rigidity problem can be answered locally by linearization. This is a non-trivial implicit function type of theorem however because our success heavily depends on the a priori stability estimate that the s-injectivity of $I_{\mathcal{D}}$ implies; see Theorem 7; and the latter is based on the hypoelliptic properties of $I_{\mathcal{D}}$. We work with two metrics g and $\hat{g}$; and will denote objects related to $\hat{g}$ by $\hat{\sigma}$, $\hat{\ell}$, etc.

Theorem 9. *Let (M, g_0) satisfy the topological assumption (T), with $g_0 \in C^k(M)$ a regular Riemannian metric with $k \gg 1$. Let $\mathcal{D}$ be open and complete for g_0, and assume that there exists $\mathcal{D}' \Subset \mathcal{D}$ so that $I_{g_0,\mathcal{D}'}$ is s-injective. Then there exists $\varepsilon > 0$, such that for any two metrics g, $\hat{g}$ satisfying*

$$\|g - g_0\|_{C^k(M)} + \|\hat{g} - g_0\|_{C^k(M)} \leq \varepsilon, \tag{2.15}$$

the relations

$$\sigma = \hat{\sigma}, \quad \ell = \hat{\ell} \quad \text{on } \mathcal{D}$$

imply that there is a C^{k+1} diffeomorphism $\psi : M \to M$ fixing the boundary such that

$$\hat{g} = \psi^* g.$$

By Theorem 8, the requirement that $I_{g_0,\mathcal{D}'}$ is s-injective is a generic one for g_0. Therefore, Theorems 9 and 8 combined imply that there is local uniqueness, up to isometry, near a generic set of regular metrics.

Corollary 2. *Let $\mathcal{D}' \Subset \mathcal{D}$, $\mathcal{G}$, $\mathcal{G}_s$ be as in Theorem 8. Then the conclusion of Theorem 9 holds for any $g_0 \in \mathcal{G}_s$.*

2.4. **Boundary determination of the jet of g.** The first step of the proof of Theorem 9 is to determine all derivatives of g on ∂M. The following theorem is interesting by itself. Notice that g below does not need to be analytic or generic.

Theorem 10. *Let (M, g) be a compact Riemannian manifold with boundary. Let $(x_0, \xi_0) \in S(\partial M)$ be such that the maximal geodesic γ_{x_0,ξ_0} through it is of finite length, and assume that x_0 is not conjugate to any point in $\gamma_{x_0,\xi_0} \cap \partial M$. If σ and ℓ are known on some neighborhood of (x_0, ξ_0), then the jet of g at x_0 in boundary normal coordinates is determined uniquely.*

Sketch of the proof of Theorem 10. To make the arguments below more transparent, assume that the geodesic γ_0 issued from (x_0, ξ_0) hits ∂M for the first time transversally at $\gamma_0(l_0) = y_0$, $l_0 > 0$. Then y_0 is the only point on ∂M reachable from (x_0, ξ_0), and x_0, y_0 are not conjugate points on γ_0 by assumption. Assume also that γ_0 is tangent of finite order at x_0. Then there is a half neighborhood V of x_0 on ∂M visible from y_0. The latter is not always true if γ_0 is tangent to ∂M of infinite order at x_0.

Choose local boundary normal coordinates near x_0 and y_0, and let g_0 be the Euclidean metric in each of them w.r.t. to the so chosen coordinates. We can then consider a representation of Σ, denoted by $\Sigma^\sharp$ below, defined locally on $\mathbf{R}^{n-1} \times S^{n-1}$, with values on another copy of the same space. If $(x, \theta) \in \mathbf{R}^{n-1} \times S^{n-1}$, then the associated vector at $x \in \partial M$ is $\xi = \theta/|\theta|_g$; and $\Sigma^\sharp(x, \theta) = \Sigma(x, \xi)$. The same applies to the second component of $\Sigma^\sharp(x, \theta)$. Namely, if $(y, \eta) = \Sigma(x, \xi)$, then we set $\omega = \eta/|\eta|_{g_0}$, then $\Sigma^\sharp : (x, \theta) \mapsto (y, \omega)$. Similarly, we set $\ell^\sharp(x, \theta) = \ell(x, \xi)$. Let also θ_0 and ω_0 correspond to ξ_0 and η_0, respectively, where $\Sigma(x_0, \xi_0) = (y_0, \eta_0)$.

Set $\tau(x) := \tau(x, y_0)$, where τ is the smooth travel time function localized near $x = x_0$ such that $\tau(x_0, y_0) = l_0$. Then τ is well defined in a small neighborhood of x_0 by the implicit function theorem and the assumption that x_0 and y_0 are not conjugate on γ_0. In the normal boundary coordinates $x = (x', x^n)$ near x_0, $g_{in} = \delta_{in}$, $\forall i$. Since x_0 and y_0 are not conjugate, for $\eta \in S_{y_0}M$ close enough to η_0, the map $\eta \mapsto x \in \partial M$ is a local diffeomorphism as long as the geodesic connecting x and y_0 is not tangent to ∂M at x. Moreover, that map is known, being the inverse of Σ. Similarly, the map $S^{n-1} \ni \omega \mapsto x$ is a local diffeomorphism and is also known. Then we know $(x, -\theta) = \Sigma^\sharp(y_0, -\omega)$, and we know $\ell^\sharp(y_0, -\omega) = \ell^\sharp(x, \theta) = \tau(x)$. Then we can recover $\operatorname{grad}'\tau = -\theta'/|\theta|_g$, where the prime stands for tangential projection as usual. Taking the limit $\omega \to \omega_0$, we recover $|\theta_0|_g^2 = g_{\alpha\beta}\theta_0^\alpha\theta_0^\beta$. We use again the fact that a symmetric $n \times n$ tensor f_{ij} can be recovered by knowledge of $f_{ij}p_k^i p_k^j$ for $N = n(n+1)/2$ "generic" vectors p_k, $k = 1, \dots, N$; and such N vectors exist in any open set on S^{n-1}, see e.g. [34]. Thus choosing appropriate $n(n-1)/2$ perturbations of θ_0's, we recover $g(x_0)$. Thus, we recover g in a neighborhood of x_0 as well; we can assume that V covers that neighborhood.

Note that we know all tangential derivatives of g in $V \ni x_0$. Then τ solves the eikonal equation

$$g^{\alpha\beta}\tau_{x^\alpha}\tau_{x^\beta} + \tau_{x^n}^2 = 1. \tag{2.16}$$

Next, in V, we know τ_{x^α}, $\alpha \leq n-1$, we know g, therefore by (2.16), we get $\tau_{x^n}^2$. It is easy to see that $\tau_{x^n} \leq 0$ on the visible part, so we recover τ_{x^n} there. We therefore know the tangential derivatives of τ_{x^n} on ∂M near x_0.

Differentiate (2.16) w.r.t. x^n at $x = x_0$ to get

$$\left[\frac{dg^{\alpha\beta}}{dx^n}\tau_{x^\alpha}\tau_{x^\beta} + 2g^{\alpha\beta}\tau_{x^\alpha x^n}\tau_{x^\beta} + 2\tau_{x^n x^n}\tau_{x^n}\right]\Bigg|_{x=x_0} = 0. \tag{2.17}$$

Since γ_0 is tangent to ∂M at x_0, we have $\tau_{x^n}(x_0) = 0$ by (2.16). The third term in the r.h.s. of (2.17) therefore vanishes. Therefore the only unknown term in (2.17) is $\gamma^{\alpha\beta} := dg^{\alpha\beta}/dx^n$ at $x = x_0$. Since $\tau_{x^\alpha}(x_0) = -\xi_0$, using the fact that $\operatorname{grad}\tau(x_0) = -\xi_0$ again, we get that we have to determine $\gamma^{\alpha\beta}$ from $\gamma_{\alpha\beta}\xi_0^\alpha\xi_0^\beta$. This is possible if as above, we repeat the construction and replace ξ_0 by a finite number of vectors, close enough to ξ_0. So we get an explicit formula for $\partial g/\partial x^i|_{\partial M}$ in fact.

Next, for $x \in V$ but not on ∂V, we can recover $\tau_{x^n x^n}(x)$ by (2.17) because $\tau_{x^n}(x) < 0$. By continuity, we recover $\tau_{x^n x^n}(x_0)$, therefore we know $\tau_{x^n x^n}$ near x_0, and all tangential derivatives of the latter.

We differentiate (2.17) w.r.t. x^n again, and as above, recover $d^2g/d(x^n)^2|_{\partial M}$ near x_0. Then we recover $d^3\tau/d(x^n)^3$, etc.

In the general case, we repeat those arguments with ξ_0 replaced by $\xi_0 + \varepsilon\nu$, where ν is the interior unit normal, and take the limit $\varepsilon \to 0$. □

Sketch of the proof of Theorem 9. We first find suitable metric $\hat{g}_1$ isometric to $\hat{g}$, and then we show that $\hat{g}_1 = g$. First, we can always assume that g and $\hat{g}$ have the same boundary normal coordinates near ∂M. By [11], there is a metric h isometric to $\hat{g}$ so that h is solenoidal w.r.t. g. Moreover, $h = \hat{g} + O(\varepsilon)$. By a standard argument, by a diffeomorphism that identifies normal coordinates near ∂M for h and g, and is identity away from some neighborhood of the boundary, we find a third $\hat{g}_1$ isometric to h (and therefore to $\hat{g}$), so that $\hat{g}_1 = \hat{g}$ near ∂M, and $\hat{g}_1 = h$ away from some neighborhood of ∂M (and there is a region that $\hat{g}_1$ is neither). Then $\hat{g}_1 - h$ is as small as $g - h$, more precisely,

$$\|\hat{g}_1 - h\|_{C^{k-3}} \le C\|g - h\|_{C^{k-1}}, \quad k \gg 1. \tag{2.18}$$

Set

$$f = h - g, \quad \tilde{f} = \hat{g}_1 - g. \tag{2.19}$$

Estimate (2.18) implies

$$\|\tilde{f} - f\|_{C^{l-3}} \le C\|f\|_{C^{l-1}}, \quad \forall l \le k. \tag{2.20}$$

By (2.15), (2.20),

$$\|f\|_{C^{k-1}} \le C\varepsilon, \quad \|\tilde{f}\|_{C^{k-3}} \le C\varepsilon. \tag{2.21}$$

By Theorem 10,

$$\partial^\alpha \tilde{f} = 0 \quad \text{on } \partial M \text{ for } |\alpha| \le k - 5. \tag{2.22}$$

It is known [25] that $2dv$ is the linearization of $\psi_\tau^* g$ at $\tau = 0$, where ψ_τ is a smooth family of diffeomorphisms, and $v = d\psi_\tau/d\tau$ at $\tau = 0$. Next proposition is therefore a version of Taylor's expansion:

Proposition 2. *Let $\hat g$ and g be in C^k, $k\ge 2$ and isometric, i.e.,*

$$\hat g = \psi^* g$$

for some diffeomorphism ψ fixing ∂M. Set $f=\hat g - g$. Then there exists v vanishing on ∂M, so that

$$f = 2dv + f_2,$$

and for g belonging to any bounded set U in C^k, there exists $C(U)>0$, such that

$$\|f_2\|_{C^{k-2}} \le C(U)\|\psi - \mathrm{Id}\|^2_{C^{k-1}}, \quad \|v\|_{C^{k-1}} \le C(U)\|\psi - \mathrm{Id}\|_{C^{k-1}}.$$

We will sketch now the rest of the proof of Theorem 9. We apply Proposition 2 to h and $\hat g_1$ to get

$$\tilde f = f + 2dv + f_2, \quad \|f_2\|_{C^{l-3}} \le C\|f\|^2_{C^{l-1}}, \quad \forall l\le k. \tag{2.23}$$

In other words, $\tilde f^s = f$ up to $O(\|f\|^2)$.

We can assume that g is extended smoothly on $M_1 \Supset M$. Next, with g extended as above, we extend $\hat g_1$ so that $\hat g_1 = g$ outside M. This can be done in a smooth way by Theorem 10.

The next step is to reparametrize the scattering relation. We show that one can extend the maximal geodesics of g, respectively $\hat g_1$, outside M (where $g=\hat g_1$), and since the two metrics have the same scattering relation and travel times, they will still have the same scattering relation and travel times if we locally push ∂M a bit outside M. Then we can arrange that the new pieces of ∂M are transversal to the geodesics close to a fixed one, which provides a smooth parametrization. By a compactness argument, one can do this near finitely many geodesics issued from point on $\mathcal{D}$, and still have a complete set. This puts as in the situation of Theorem 7, where the set of geodesics is parametrized by $\alpha=\{\alpha_j\}$.

Next, we linearize the energy functional near each geodesic (in our set of data) related to g. Using the assumption that g and $\hat g_1$ have the same scattering relation and travel times, we deduct

$$\|N_{\alpha_j}\tilde f\|_{L^\infty} \le C\|\tilde f\|^2_{C^1}, \quad \forall j. \tag{2.24}$$

Using interpolation inequalities, and the fact that the extension of $\tilde f$ outside M is smooth enough across ∂M as a consequence of the boundary recovery, we get by (2.24), and (2.20),

$$\|N_\alpha \tilde f\|_{\tilde H^2(M_1)} \le C\|\tilde f\|^{3/2}_{C^3} \le C'\|f\|^{3/2}_{C^5}. \tag{2.25}$$

Since $I_{g_0,\mathcal{D}'}$ is s-injective, so is N_α, related to g_0, by the support properties of α. Now, since g is close enough to g_0 with s-injective N_α by (2.15), N_α (the one related to g) is s-injective as well by Theorem 7. Therefore, by (2.25) and (2.14),

$$\|f^s\|_{L^2(M)} \le C\|N_\alpha \tilde f\|_{\tilde H^2} \le C'\|f\|^{3/2}_{C^5}. \tag{2.26}$$

A decisive moment of the proof is that by Proposition 2, see (2.23), $\tilde f^s = f + f_2^s$, the latter being the solenoidal projection of f_2. Therefore,

$$\|\tilde f^s\|_{L^2(M)} \ge \|f\|_{L^2(M)} - C\|f\|^2_{C^2}.$$

Together with (2.26), this yields

$$\|f\|_{L^2(M)} \le C\left(\|f\|^2_{C^2} + \|f\|^{3/2}_{C^5}\right) \le C'\|f\|^{3/2}_{C^5}$$

because the C^5 norm of f is uniformly bounded when $\varepsilon \leq 1$. Using interpolation again, we easily deduct $\|f\|_{L^2(M)} \geq 1/C$ if $f \neq 0$. This contradicts (2.21) if $\varepsilon \ll 1$.

Now, $f = 0$ implies $h = g$, therefore, g and $\hat{g}$ are isometric.

This concluses the sketch proof of Theorem 9. □

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