LOCAL LENS RIGIDITY WITH INCOMPLETE DATA FOR A CLASS OF NON-SIMPLE RIEMANNIAN MANIFOLDS

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Abstract. Let σ be the scattering relation on a compact Riemannian manifold M with non-necessarily convex boundary, that maps initial points of geodesic rays on the boundary and initial directions to the outgoing point on the boundary and the outgoing direction. Let ℓ be the length of that geodesic ray. We study the question of whether the metric g is uniquely determined, up to an isometry, by knowledge of σ and ℓ restricted on some subset $\mathcal{D}$. We allow possible conjugate points but we assume that the conormal bundle of the geodesics issued from $\mathcal{D}$ covers T^*M; and that those geodesics have no conjugate points. Under an additional topological assumption, we prove that σ and ℓ restricted to $\mathcal{D}$ uniquely recover an isometric copy of g locally near generic metrics, and in particular, near real analytic ones.

1. Introduction and main results

Let (M,g) be a compact Riemannian manifold with boundary. Let Φ^t be the geodesic flow on TM, where for each $(x,\xi)\in M$, $t\mapsto\Phi^t(x,\xi)$ is defined over its maximal interval containing $t=0$, in particular this interval is allowed to be the zero point only. Let SM be the unit tangent bundle. Then ∂SM represents all elements in SM with a base point on ∂M.

Denote

$$\partial_\pm SM = \{(x,\xi)\in\partial SM;\ \pm\langle\nu,\xi\rangle<0\}\,, \tag{1}$$

where ν is the unit interior normal, $\langle\cdot,\cdot\rangle$ and stands for the inner product. The *scattering relation*

$$\Sigma:\partial_-SM\to\overline{\partial_+SM} \tag{2}$$

is defined by $\Sigma(x,\xi)=(y,\eta)=\Phi^{\mathcal{L}}(x,\xi)$, where $\mathcal{L}>0$ is the first moment, at which the (unit speed) geodesic through (x,ξ) hits ∂M again. If such an $\mathcal{L}$ does not exist, we formally set $\mathcal{L}=\infty$ and we call the corresponding geodesic *trapped*. This defines also $\mathcal{L}(x,\xi)$ as a function $\mathcal{L}:\partial_-SM\to[0,\infty]$. Note that Σ and $\mathcal{L}$ are not necessarily continuous.

It is convenient to think of Σ and $\mathcal{L}$ as defined on the whole ∂SM with $\Sigma=\mathrm{Id}$ and $\mathcal{L}=0$ on $\overline{\partial_+SM}$.

We parametrize the scattering relation in a way that makes it independent of pulling it back by diffeomorphisms fixing ∂M pointwise. Let $\kappa_\pm:\partial_\pm SM\to B(\partial M)$ be the orthogonal projection onto the (open) unit ball tangent bundle that extends continuously to the closure of $\partial_\pm SM$. Then $\kappa_\pm$ are homeomorphisms, and we set

$$\sigma=\kappa_+\circ\Sigma\circ\kappa_-^{-1}:\overline{B(\partial M)}\longrightarrow\overline{B(\partial M)},\quad \ell=\mathcal{L}\circ\kappa_-^{-1}:\overline{B(\partial M)}\longrightarrow[0,\infty]. \tag{3}$$

According to our convention, $\sigma=\mathrm{Id}$, $\ell=0$ on $\partial(\overline{B(\partial M)})=S(\partial M)$. We equip $\overline{B(\partial M)}$ with the relative topology induced by $T(\partial M)$, where neighborhoods of boundary points (those in $S(\partial M)$) are given by half-neighborhoods.

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Let $\mathcal{D}$ be an open subset of $\overline{B(\partial M)}$. The *lens rigidity* question we study in this paper is the following:

Given M and $g|_{T(\partial M)}$, do σ and ℓ, restricted to $\mathcal{D}$, determine g uniquely, up to a pull back of a diffeomorphism that is identity on ∂M?

More generally, one can ask whether one can determine the topology of M as well. One motivation for the lens rigidity problem is the study of the inverse scattering problem for metric perturbations of the Laplacian. Suppose that we are in Euclidean space equipped with a Riemannian metric which is Euclidean outside a compact set. The inverse problem is to determine the Riemannian metric from the scattering operator, which is a Fourier integral operator, if the metric is non-trapping (see [Gu]). It was proven in [Gu] that from the wave front set of the scattering operator, one can determine, under some conditions on the metric including non-trapping, the scattering relation on the boundary of a large ball. This uses high frequency information of the scattering operator. In the semiclassical setting, Alexandrova has shown that the scattering operator associated to potential and metric perturbations of the Euclidean Laplacian is a semiclassical Fourier integral operator that quantizes the scattering relation [A1], [A2]. The scattering relation is also encoded in the hyperbolic Dirichlet to Neumann map on ∂M. Lens rigidity is also considered in [PoR] in the study of the AdS/CFT duality and holography, namely the idea that the "bulk" space-time can be captured by conformal field theory on a "holographic screen". The lens rigidity problem appears also naturally when considering rigidity questions in Riemannian geometry [C1, C2].

The lens rigidity problem is also closely related to the *boundary rigidity* problem. Denote by ρ_g the distance function in the metric g. The boundary rigidity problem consists of whether $\rho_g(x, y)$, known for all x, y on ∂M, determines the metric uniquely. It is clear that any isometry which is the identity at the boundary will give rise to the same distance functions on the boundary. Therefore, the natural question is whether this is the only obstruction to unique identifiability of the metric. The boundary distance function only takes into account the shortest paths and it is easy to find counterexamples where ρ_g does not carry any information about certain open subset of M, so one needs to pose some restrictions on the metric. One such condition is simplicity of the metric.

Definition 1. *We say that the Riemannian metric g is* simple *in M, if ∂M is strictly convex w.r.t. g, and for any $x \in M$, the exponential map $\exp_x : \exp_x^{-1}(M) \to M$ is a diffeomorphism.*

The manifold (M, g) is called *boundary rigid* if one can determine the metric (and more generally, the topology) from the boundary distance function up to an isometry which is the identity at the boundary. It is a conjecture of Michel [Mi] that the simple manifolds are boundary rigid. This has been proved recently in two dimensions [PU], for subdomains of Euclidean space [Gr] or for metrics close to Euclidean [BI], or symmetric spaces of negative curvature [BCG]. It was shown in [SU3] that metrics a priori close to a metric in a generic set, which includes real-analytic metrics, are boundary rigid. For other local results see [CDS], [E], [LSU], [SU1]. The lens rigidity problem is equivalent to the boundary rigidity problem if the manifold is simple [Mi].

Of course there is more information in the lens rigidity problem if the manifold is not simple. Even so, the answer to the lens rigidity problem, even when $\mathcal{D} = B(\partial M)$, is negative, as shown by the examples in [CK]. Note that in these examples the manifold is trapping, that is, there are geodesics of infinite length. The natural conjecture is that the scattering relation for non-trapping manifolds determines the metric uniquely up to the natural obstruction ([U]). There are very few results about this problem when the manifold is not simple. Croke has shown that if a manifold is lens rigid, a finite quotient of it is also lens rigid [C2].

We redefine σ in a way that removes the need to know g on $T(\partial M)$. Denote by $T^0(\partial M)$ the tangent bundle $T(\partial M)$ considered as a conic set, i.e., vectors with the same direction in $T(\partial M)$ are identified. For any metric $g|_{T(\partial M)}$, $T^0(\partial M)\setminus 0$ is isomorphic to the unit tangent bundle $S(\partial M)$ (in the metric g) but has the advantage to be independent of the choice of g. Given $0\neq\xi'\in\overline{B_x(\partial M)}$, we set

$$\lambda = |\xi'|_g\in[0,1],\quad \theta=\xi'/|\xi'|_g\in T^0_x(\partial M), \tag{4}$$

i.e., λ and θ are polar coordinates of ξ'. If $\xi'=0$, then θ is undefined. If $\xi'=\kappa_\pm(\xi)$, knowing λ, θ is equivalent to knowing the angle that ξ makes with the boundary, and the direction of the tangential projection of the same vector. Given two metrics g and $\hat g$ on M, and $(x,\xi')\in\overline{B(\partial M)}$, $(x,\hat\xi')\in\overline{\hat B(\partial M)}$, where $\hat B(\partial M)$ is related to $\hat g$, we say that $\xi'\equiv\hat\xi'$ iff $|\xi'|_g=|\hat\xi'|_{\hat g}$, and $\xi'=s\hat\xi'$ for some $s>0$. In other words, we require that ξ' and $\hat\xi'$ have the same polar coordinates (4). Note that this induces a homeomorphism $\overline{B(\partial M)}\mapsto\overline{\hat B(\partial M)}$ given by $\xi'\mapsto|\xi'|_g\xi'/|\xi'|_{\hat g}$ if $\xi'\neq0$, $0\mapsto0$. With that identification of $B(\partial M)$ for different metrics, it makes sense to study σ restricted to the same set $\mathcal D$ for a family of metrics, and in particular, an a priori knowledge of g on $T(\partial M)$ is not needed to define $\mathcal D$ and σ on it. If $\sigma(x,\xi')=(y,\eta')$, we just think of ξ' and η' as expressed in the polar coordinates (4). Also, the notion of $\mathcal D$ being open is independent of g.

Our first result is that we can recover the full jet of the metric under some non-degenerate assumptions. We refer to section 3 for the definition of boundary normal coordinates.

Theorem 1. *Let (M,g) be a compact Riemannian manifold with boundary. Let $(x_0,\xi_0)\in S(\partial M)$ be such that the maximal geodesic γ_0 through it is of finite length, and assume that x_0 is not conjugate to any point in $\gamma_0\cap\partial M$. If σ and ℓ are known on some neighborhood of (x_0,ξ_0), then the jet of g at x_0 in boundary normal coordinates is determined uniquely.*

Until now, this was known for simple metrics only [LSU]. The proof in [LSU] is non-constructive and relies heavily on the convexity of the boundary, using geodesics converging to a point. When (M,g) is simple, knowledge of x, y, ℓ (the graph of the boundary distance function) determines ξ, η uniquely [Mi]. The proof of Theorem 1 shows in particular that this can be greatly extended. As a corollary we extend the result of [LSU] on the determination of the jet of the metric from the boundary distance function for simple manifolds to any manifold with non-conjugate points without the convexity assumption on the boundary.

A linearization of the boundary rigidity problem and the lens rigidity problem, see section 4.4, is the following integral geometry problem. Given a family of geodesics Γ with endpoints on ∂M, we define the ray transform

$$I_\Gamma f(\gamma)=\int\langle f(\gamma(t)),\dot\gamma^2(t)\rangle\,\mathrm{d}t,\quad \gamma\in\Gamma, \tag{5}$$

of symmetric 2-tensor fields f (playing the role of the variation of the metric g), where $\langle f,\theta^2\rangle$ is the action of f on the vector θ. Locally, $\langle f,\theta^2\rangle=f_{ij}\theta^i\theta^j$. We will omit the subscript Γ to denote an integral over a chosen geodesic, or over all geodesics. Any such f can be decomposed orthogonally into a potential part dv and a solenoidal one f^s (see section 4), and I vanishes on potential tensors. The linearized boundary rigidity or lens rigidity problem then is the following: can we recover uniquely the solenoidal projection f^s of f from its ray transform? If so, we call I_Γ *s-injective.*

S-injectivity of I was proved for metrics with negative curvature in [PS], for metrics with a specific a priori upper bound on the curvature in [Sh1, Sh2, D, Pe], and for simple Riemannian surfaces [Sh3], see also [ShU]. A conditional and non-sharp stability estimate for metrics with small curvature is also established in [Sh1]. In [SU2], we proved stability estimates for s-injective simple metrics, see (30); and sharp estimates about the recovery of a 1-form $f=f_jdx^j$ and a function f from the associated If. Recently, a sharp stability estimate for 2-tensors was proved in [S]. These stability estimates were used in [SU2] to prove local uniqueness for the boundary rigidity problem near any simple metric g with an s-injective I. In [SU3], we showed that the simple metrics g for which I is s-injective is generic, and applied this to the boundary rigidity problem. We note that in all the above mentioned results the metric has no conjugate points. On the other hand, in [SU4] we proved generic s-injectivity for a class of non-simple manifolds described below. Within that class, the boundary is not necessarily strictly convex, we might have conjugate points on the metric, the manifold might be trapping, and we have partial or incomplete information, i.e., we do not know the scattering relation or the ray transform for all sets of geodesics.

Given $(x,\xi)\in\mathcal{D}$, let $\gamma_{\kappa_-^{-1}(x,\xi)}$ denote the geodesic issued from $\kappa_-^{-1}(x,\xi)$ with endpoint $\pi(\sigma(x,\xi))$, where π is the natural projection onto the base point. With some abuse of notation, we define

$$I_{\mathcal{D}}(x,\xi) = I(\gamma_{\kappa_-^{-1}(x,\xi)}),\quad (x,\xi)\in\mathcal{D}.$$

Definition 2. *We say that $\mathcal{D}$ is **complete** for the metric g, if for any $(z,\zeta)\in T^*M$ there exists a maximal in M, finite length unit speed geodesic $\gamma:[0,l]\to M$ through z, normal to ζ, such that*

$$\{(\gamma(t),\dot\gamma(t));\ 0\le t\le l\}\cap S(\partial M)\subset\mathcal{D}, \tag{6}$$

$$\text{there are no conjugate points on } \gamma. \tag{7}$$

*We call the C^k metric g **regular**, if a complete set $\mathcal{D}$ exists, i.e., if $\overline{B(\partial M)}$ is complete.*

If $z\in\partial M$ and ζ is conormal to ∂M, then γ may reduce to one point. Since (6) includes points where γ is tangent to ∂M, and $\sigma=\mathrm{Id}$, $\ell=0$ there, knowing σ and ℓ on them provides no information about the metric g. On the other hand, we require below that $\mathcal{D}$ is open, so the purpose of (6) is to make sure that we know σ near such tangent points.

Definition 3. *We say that (M,g) satisfies the **Topological Condition (T)** if any path in M connecting two boundary points is homotopic to a polygon $c_1\cup\gamma_1\cup c_2\cup\gamma_2\cup\dots\cup\gamma_k\cup c_{k+1}$ with the properties that for any j,*

(i) c_j is a path on ∂M;

(ii) $\gamma_j:[0,l_j]\to M$ is a geodesic lying in $M^{\rm int}$ with the exception of its endpoints and is transversal to ∂M at both ends; moreover, $\kappa_-(\gamma_j(0),\dot\gamma_j(0))\in\mathcal{D}$;

Notice that (T) is not only topological assumption because it depends on g. It is an open condition w.r.t. g, i.e., it is preserved under small C^2 perturbations of g.

We showed in [SU4] that if $\mathcal{D}$ is complete, then $I_{\mathcal{D}}f$ recovers the singularities of f^s. Next, see also Theorem 3 below, under the same conditions, and assuming (T) as well, $I_{\mathcal{D}}$ is s-injective for real-analytic metrics, and if $k\gg 2$, also for generic metrics.

To define the $C^k(M)$ norm in a unique way, and to make sense of real analytic g's, we choose and fix a finite real analytic atlas on M.

Clearly, any simple manifold satisfies the regularity condition and condition (T). Even in this case our results improve the known ones because the data, given by the set $\mathcal{D}$, can be a subset of all possible points and directions on the boundary. Examples of non-simple manifolds satisfying the regularity condition and condition (T) can be constructed as follows. Let $(M',\partial M')$ be a simple compact Riemannian manifold with boundary with $\dim M'\ge 2$, and let M'' be a compact Riemannian manifold with or without boundary. Let M be a small enough C^2 perturbation of $M'\times M''$. Then M is regular. Indeed, for $M'\times M''$, as a complete set, one can choose any neighborhood of the geodesics above fixed points of M''. Since our conditions remain true under small enough perturbations, this proves our claim. In particular, on any manifold, a small enough neighborhood of a finite length geodesics segment that may have conjugate points (a "perturbed cylinder") falls into this class. Also, the interior of a "perturbed torus": a small enough neighborhood of a periodic geodesic on a Riemannian manifold, is in this class. This shows that there are manifolds satisfying our assumptions that are trapping. For more details, we refer to [SU4].

Theorem 2 below says, loosely speaking, that for the classes of manifolds and metrics we study, the uniqueness question for the non-linear lens rigidity problem can be answered locally by linearization. This is a non-trivial local injectivity type of theorem however because our success heavily depends on the a priori stability estimate that the s-injectivity of $I_{\mathcal{D}}$ implies, and the latter is based on certain hypoelliptic properties of $I_{\mathcal{D}}$, as shown in [SU4], see (30). We work with two metrics g and $\hat g$; and will denote objects related to $\hat g$ by $\hat\sigma$, $\hat\ell$, etc. Note that (T) is not assumed in the next theorem.

Theorem 2. *Let $g_0 \in C^k(M)$ be a regular Riemannian metric on M with $k \gg 2$ depending on $\dim(M)$ only. Let $\mathcal{D}$ be open and complete for g_0, and assume that there exists $\mathcal{D}' \Subset \mathcal{D}$ so that $I_{g_0,\mathcal{D}'}$ is s-injective. Then there exists $\varepsilon > 0$, such that for any two metrics g, $\hat g$ satisfying*

$$\|g - g_0\|_{C^k(M)} + \|\hat g - g_0\|_{C^k(M)} \leq \varepsilon, \tag{8}$$

the relations

$$\sigma = \hat\sigma, \quad \ell = \hat\ell \quad \text{on } \mathcal{D}$$

imply that there is a C^{k+1} diffeomorphism $\psi : M \to M$ fixing the boundary such that

$$\hat g = \psi^* g.$$

Next theorem is a version of [SU4, Theorem 3]. It states that the requirement that $I_{g_0,\mathcal{D}'}$ is s-injective is a generic one for g_0.

Theorem 3. *Let $\mathcal{G} \subset C^k(M)$, with $k \gg 2$ depending on $\dim(M)$ only, be an open set of regular Riemannian metrics on M such that (T) is satisfied for each one of them. Let the set $\mathcal{D}' \subset \overline{B(\partial M)}$ be open and complete for each $g \in \mathcal{G}$. Then there exists an open and dense subset $\mathcal{G}_s$ of $\mathcal{G}$ such that $I_{g,\mathcal{D}'}$ is s-injective for any $g \in \mathcal{G}_s$.*

Theorems 2 and 3 combined imply that there is local uniqueness, up to isometry, near a generic set of regular metrics.

Corollary 1. *Let $\mathcal{D}'$, $\mathcal{G}$, $\mathcal{G}_s$ be as in Theorem 3, and let $\mathcal{D} \Supset \mathcal{D}'$ be open and complete for any $g \in \mathcal{G}$. Then the conclusion of Theorem 2 holds for any $g_0 \in \mathcal{G}_s$.*

Remark 1. Condition (T) in Theorem 3, and Corollary 1 in some cases can be replaced by the assumption that (M, g) can be extended to $(\tilde M, \tilde g)$ that satisfies (T). One such case is if $(\tilde M, \tilde g)$ is a simple manifold, and we study σ, ℓ on its maximal domain, i.e., $\mathcal{D} = \overline{B(\partial M)}$. In particular, we get local generic lens rigidity for subdomains of simple manifolds when $\mathcal{D}$ is maximal. See section 5 for more details.

2. Preliminaries

We allow the geodesics to have segments on ∂M, then they are called geodesics if they satisfy the geodesic equation in a half-neighborhood of each boundary point. Such segments are included in determining the maximal interval, where $\Phi^t(x,\xi)$ is defined. If $(\tilde M, \tilde g)$ is any extension of (M, g) (a Riemannian manifold of the same dimension of which M is a submanifold), then any geodesic in $\tilde M$ restricted to M is a geodesic in M. For the maximal geodesics in M we have the following property that indicates that the property of γ to be maximal in M does not change under such extensions.

Lemma 1. *Let $\gamma(t)$, $0\le t\le l$, $0\le l<\infty$, be a maximal geodesic in M. Let $(\tilde M,\tilde g)$ be any $C^{1,1}$ extension of (M,g). Then there is no interval $I\supset[0,l]$ strictly larger than $[0,l]$, such that γ can be extended as a geodesic in $\tilde M$ for $t\in I$, and $\{\gamma(t);\ t\in I\}\subset M$.*

Proof. Suppose that there is such I. Without loss of generality, we may assume that $I\supset[0,l+\delta]$, $\delta>0$. Then γ solves the geodesic equation for $t\in[0,l+\delta]$, with the Christoffel symbols Γ^k_{ij} depending on g and its first derivatives restricted to M. Since $\tilde g$ is continuous and has continuous first derivatives across ∂M, their restriction to M depends on g only. Therefore, γ is a geodesic in M for $t\in I$, and this is a contradiction. □

In [SU4] we studied geodesics originating from points outside M given some extension $(\tilde M,\tilde g)$. We connect the notion of $\mathcal D$ being open with the analysis in [SU4] by the following.

Lemma 2. *Let $\tilde M$ be an extension of M, and let $\tilde g$ be a $C^{1,1}$ extension of g on $\tilde M$. Let $\gamma_0:[0,l]\mapsto\tilde M^{\rm int}$ be a unit speed geodesic with endpoints in $\tilde M^{\rm int}\setminus M$ such that*

$$\big\{(\gamma_0(t),\dot\gamma_0(t));\ t\in[0,l]\big\}\cap\overline{\partial_-SM}\subset\kappa_-^{-1}(\mathcal D). \tag{9}$$

Then there exists a neighborhood W of $(x_0,\xi_0)=(\gamma_0(0),\dot\gamma_0(0))$ such that any geodesic γ with initial conditions in W and the same interval of definition $t\in[0,l]$ still satisfies (9).

Moreover, if $\tilde g$ belongs to a class of extensions satisfying $\|\tilde g\|_{C^{1,1}}\le A$ with some $A>0$, then W can be chosen independently of $\tilde g$.

Proof. Let E_0 be the l.h.s. of (9). Then E_0 is compact in $S\tilde M$. Given any neighborhood U of E_0 (in $S\tilde M$), the set $\partial SM\setminus U$ is compact, therefore there exists a neighborhood W of (x_0,ξ_0) such that the geodesic flow through W for $0\le t\le l$ will miss that set, therefore, its only common points with $\overline{\partial_-SM}$ must be in U. To construct U, we first choose a neighborhood $U_{x,\xi}$ in $S\tilde M$ of each point $(x,\xi)\in\kappa_-^{-1}(\mathcal D)$ so that $U_{x,\xi}\cap\overline{\partial_-SM}\subset\kappa_-^{-1}(\mathcal D)$. This is easy to do in local coordinates because $\mathcal D$ is open. Then we set $U=\cup_{(x,\xi)\in\kappa_-^{-1}(\mathcal D)}U_{x,\xi}$.

To prove the second part, we use the theorem of continuous dependence of solutions of an ODE, over a fixed interval, on the initial conditions and on the coefficients of the ODE. As long the Lipschitz constant related to the generator of the geodesic flow is uniformly bounded, we can choose W uniformly w.r.t. $\tilde g$. □

Lemma 3. *Let (M,g), $(\tilde M,\tilde g)$, γ_0 be as in Lemma 2. Let H be a hypersurface through $x_0=\gamma_0(0)$ transversal to γ_0, and set $\mathcal H=SM\cap\pi^{-1}(H)$. Then there exists a small enough neighborhood U of (x_0,ξ_0) in $\mathcal H$, so that the geodesics issued from U with interval of definition $t\in[0,l]$ satisfy (9) and are transversal to ∂M at all common points with it except for a closed set of initial conditions in U of measure zero.*

Proof. We only need to prove the statement about the measure zero set. We study the geodesic flow on SM issued from U. The corresponding geodesic is tangent to ∂M at some point x if the corresponding integral curve in the phase space is tangent to ∂SM at $\pi^{-1}(x)$. By Sard's theorem, this happens only on a closed set of measure zero. □

Those lemmas will be used to reformulate the results in [SU4] in the situation in the paper. In [SU4], we extended the geodesics slightly outside M and parametrized them by initial points and directions on surfaces H_m transversal to them, see section 4.3, instead by points and directions on ∂M. This can be done, because when studying I_Γ, the metric g is known and can be extended outside M in a known way. The reason for doing this was to prevent working with a parametrization, where the geodesics can be tangent to the surface, which is the case with ∂M. We can still do this even for the lens rigidity problem, using the boundary recovery result in section 3 below, see Proposition 2. We prefer however to parametrize the scattering relation by points on ∂M and corresponding directions. This does not preserve the smooth structure of the previous parametrization but the lemmas above show that it preserves the topology, at least.

3. Recovery of the jet of g on ∂M.

In this section, we prove Theorem 1. We start with some remarks and preliminaries. We first recall the definition of boundary normal coordinates (x', x^n), $x^n \geq 0$, $x' \in \mathbf{R}^{n-1}$ near a boundary point, also called semigeodesic coordinates. Given $p \in M$ in a small enough neighborhood of some $p_0 \in \partial M$, we let x^n to be the distance to ∂M, and x' to be the local coordinates of the closest point on ∂M to p in any fixed in advance local coordinates on ∂M. In those coordinates, $g_{in} = \delta_{in}$, $\forall i$, the lines $x' =$ const. are geodesics, and $\Gamma^n_{in} = \Gamma^i_{nn} = 0$, $\forall i$.

If ∂M is convex in a neighborhood of some point x_0, then it is known that the jet of g near x_0 in boundary normal coordinates is uniquely determined by studying the "short" geodesics connecting x_0 and y, and then letting $y \to x_0$, see [LSU]. This argument does not provide explicit recovery however. A constructive approach is proposed in [UW]. If ∂M is strictly convex near x_0, then there is also a conditional Lipschitz stability estimate, see [SU3]. In Theorem 3 we show that we can do this also without the convexity assumption, or more generally, without assuming existence of "short geodesics" issued from x_0 converging to a point by studying "long geodesics" that converge to a geodesic tangential to the boundary at x_0. A non-conjugacy condition is imposed. Note that there is no generic assumption. This part of the proof is constructive but the one relying on "short geodesics" ($l_0^* = 0$ below) is not, although perhaps it can be made constructive using the argument sin [UW].

Let $(\tilde M, \tilde g)$ be a smooth extension of (M, g). Let $x_0 \in M$, $y_0 \in M$ be endpoints of a geodesic $\gamma_0 : [0,1] \to\in M$, and assume that x_0, y_0, are not conjugate points on γ_0. Then there exist neighborhoods $U \ni x_0$, $V \ni y_0$ in $\tilde M$, such that for $x \in U$, the exponential map neigh$(\dot\gamma_0(0)) \ni \xi \mapsto y = \exp_x \xi \in V$ is a local diffeomorphism, therefore it has a smooth inverse $V \ni y \mapsto \xi := \exp_x^{-1} y \in T_x\tilde M$ such that $\exp^{-1}_{x_0} y_0 = \dot\gamma_0(0)$. The geodesic $\gamma_{x,\xi} : [0,1] \to \tilde M$ then connects x and y and is unique among the geodesics issued from x in directions close enough to ξ. One can also define a travel time function $\tau(x,y)$ on $U \times V$, smooth for $x \neq y$, by $\tau(x,y) = |\exp_x^{-1} y|_g$ with $\tau(x_0,y_0) = |\dot\gamma(0)|_g$. If there are no conjugate points on γ_0, then τ locally minimizes the distance but in general, this is no longer true. On the other hand, τ is always a critical value of the length functional and of the energy functional.

In the situation described in the paragraph above, we will call $y \in V$ *visible* from $x \in U$, if $\gamma_{x,\xi} \subset M$. In the next theorem, given $(x,\xi) \in TM$, we call $y \in M$ *reachable* from (x,ξ), if there exists $s \geq 0$, such that $\gamma_{x,\xi}(s) \in M$ for $t \in [0,s]$, and $\gamma_{x,\xi}(s) = y$.

The so defined function τ solves the eikonal equation $|\text{grad}_x \tau|^2_g = 1$ in U despite the possible existence of pairs of conjugate points not in $U \times V$. Indeed, clearly, $d\tau(\gamma_{x,\xi}(t), y)/dt|_{t=0} = -|\xi|_g$. Next, by the Gauss lemma, the x-derivatives of $\tau(x,y)$ in directions perpendicular to ξ vanish. Therefore, $\langle \xi, \text{grad}_x \tau\rangle = -|\xi|_g$, and the two vectors are parallel, therefore, $\text{grad}_x \tau = -\xi/|\xi|_g$, and $|\text{grad}_y \tau|^2_g = 1$. Also, if $\eta = \text{grad}_y \tau$, then for some $l \geq 0$, $\Phi^l(x,\xi) = (y,\eta)$. In particular, $\eta' = \text{grad}'_y \tau$, where the prime stands for a tangential projection. If σ is locally known, then integrating the known $\eta' = \eta'(y)$ along a curve on ∂M connecting y and y_0 recovers uniquely $\tau(x_0,y)$ up to a constant.

Assume further that γ_0 is transversal to ∂M at both ends and does not touch ∂M elsewhere. Then

$$\sigma(x, -\text{grad}'_x \tau(x,y)) = (y, \text{grad}'_y(x,y)), \quad \ell(x, -\text{grad}'_x \tau(x,y)) = \tau(x,y), \tag{10}$$

Therefore, at least in this non-degenerate situation, knowledge of $\tau(x,y)$ recovers uniquely σ, ℓ locally.

The travel (arrival) times are widely used in the applied literature. Assume that g on $T(\partial M)$ is known and fixed. Fix x, y on ∂M. We call the number $\tau\ge 0$ a *travel time* between x and y, if there exists a geodesic of length τ connecting x and y. Then we have a map that associates to any $(x,y)\in\partial M\times\partial M$ a subset of $[0,\infty]$. In the situation above, $\tau(x,y)$ is one of the possible travel times between x and y.

Proof of Theorem 1. We will show first that the assumptions about (x_0,ξ_0) in the theorem are preserved under a small perturbation of that point in $S(\partial M)$. Let U be a neighborhood of (x_0,ξ_0) in $T\tilde M$. Since $\gamma_0\cap\partial M$ consists of points that are not conjugate to x_0, there is an open set $W\supset\gamma_0\cap\partial M$ in $\tilde M$ that stays away from the points conjugate to x along the geodesics $t\mapsto \exp_x t\xi$, $0\le t\le l_0$ if $(x,\xi)\in U$ and if U is small enough. Here l_0 is the length of γ_0. On the other hand, the possible common points of those geodesics with ∂M must be in W, if U is small enough, as in the proof of Lemma 2. By Lemma 1, there exists $s>1$ so that $(x_0,s\xi_0)\in U$, and $\exp_{x_0}(s\xi_0)\notin M$. The geodesics issued from $(x,\xi)\in U$ close enough to $(x_0,s\xi_0)$, and the same time interval $[0,l_0]$ of definition, still have endpoints outside M, therefore they are longer than the maximal segment in M. On the other hand, we showed that they meet ∂M at points that are not conjugate to x_0. We can now replace $s\xi_0$ by ξ_0 and rescale the corresponding geodesic. This shows that the assumption of the theorem is preserved in a small neighborhood of (x_0,ξ_0) in $T\tilde M$, not necessarily in $S(\partial M)$. We now restrict that neighborhood to $S(\partial M)$. Let, in fixed boundary normal coordinates, $X\times\Xi\subset S(\partial M)$ be such a neighborhood of (x_0,ξ_0). In the beginning, (x_0,ξ_0) is as in the theorem but later, we will repeat the arguments with (x_0,ξ_0) an arbitrary point in $X\times\Xi\subset S(\partial M)$.

Let $\xi_\varepsilon=\xi_0+\varepsilon\nu$, $0<\varepsilon\ll 1$, where ν is the interior unit normal at x_0. In the coordinate system above, $\nu=(0,\dots,0,1)$. Let γ_ε be the geodesic issued from (x_0,ξ_ε) until it hits ∂M for the first time. By a compactness argument, we can choose a sequence $\varepsilon_j\to 0$ such that $y_{\varepsilon_j}\to y_0^*$ that can be different from y_0 but we still have that $y_0^*\in\partial M$ and y_0^* is reachable from (x_0,ξ_0). Then $l_{\varepsilon_j}\to l_0^*$ with some $l_0^*\ge 0$. If $y_0^*\neq x_0$, then x_0 and y_0^* are not conjugate points on γ_0 by assumption, and the function $\tau(x,y)$ is then well-defined and smooth near (x_0,y_0^*) satisfying the eikonal equation. Next, γ_{ε_j} connects x_0 and y_{ε_j}, and only the endpoints are not in M^{int}. The advantage now is that γ_{ε_j} hits ∂M at x_0 transversely, its interior lies in the interior of M but it might be tangent to ∂M at y_{ε_j}.

For a fixed j, by [Sh2] (see the proof of Lemma 2.3 there) if U is a small enough neighborhood of x_0 on ∂M, then U can be expressed as the disjoint union $U^+\cup H\cup U^-$, where H is a hypersurface on ∂M through x_0, and at least one of the half-neighborhoods $U^\pm$ is visible from y_{ε_j}, let us say that this is U^+. If γ_{ε_j} is transversal to ∂M at y_{ε_j}, then even better, the whole U is visible from y_{ε_j} if U is small enough. If $n=2$, then H reduces to the point x_0 and the modifications are obvious. We set $\tau(x)=\tau(x,y_{\varepsilon_j})$, where τ is the localized travel time function discussed above. Note that τ depends on (x_0,ξ_0) that we will vary later in the proof, and on j.

We will show first that we can recover $g|_{T(\partial M)}$ near x_0. Recall that without knowledge of $g|_{T(\partial M)}$, σ and ℓ are considered as parametrized by polar coordinates, see (4). Choose local boundary normal coordinates near x_0 and y_{ε_j}. In boundary normal coordinates, σ, ℓ determine uniquely Σ, $\mathcal{L}$ in a trivial way. In particular, ξ_ε is uniquely characterized by the direction of ξ_0 and by the relation $\lambda=(1+\varepsilon^2)^{-1/2}$. We use below the notation $(y,\eta)=\Sigma(x,\xi)$.

Since x_0 and y_{ε_j} are not conjugate by the first paragraph of this proof, for $\eta\in S_{y_{\varepsilon_j}}M$ close enough to η_0, the map $\eta\mapsto x\in\partial M$ is a local diffeomorphism as long as the geodesic connecting x and y_{ε_j} is not tangent to ∂M at x. That condition is fulfilled for $x=x_0$ by construction, and is therefore true on U^+ if the latter is small enough. Moreover, that map is known (with η parametrized by polar coordinates, as above) for $x \in U^+$, if U^+ is small enough, because it is determined by the inverse of Σ near (x_0, ξ_0). Then we know $(x, -\xi) = \Sigma(y_{\varepsilon_j}, -\eta)$ in polar coordinates, and we know $\mathcal{L}(y_{\varepsilon_j}, -\eta) = \mathcal{L}(x, \xi) = \tau(x)$ on U^+. Take an one-sided derivative of τ at x_0, from U^+, to recover $\operatorname{grad}' \tau(x_0) = -\xi'_{\varepsilon_j}/|\xi_{\varepsilon_j}|_g = -(1+\varepsilon_j^2)^{-1/2}\xi_0$. Take the limit $\varepsilon_j \to 0$ to recover ξ_0. Since ξ_0 is unit in the unknown metric $g|_{T(\partial M)}$, this allows us to recover the quadratic form $g(x_0)$ restricted to vectors in the direction of ξ_0.

We use now the fact that a symmetric $n \times n$ tensor f_{ij} can be recovered by knowledge of $f_{ij}v_k^i v_k^j$ for $N = n(n+1)/2$ "generic" vectors v_k, $k = 1, \dots, N$; and such N vectors exist in any open set on the sphere, see e.g. [SU4]. We apply this to the tensor $g(x_0)|_{T_{x_0}(\partial M)}$ in any fixed coordinates. Thus choosing appropriate $n(n-1)/2$ perturbations of θ_0's, we recover $g(x_0)$. Next, we vary x_0 in X to recover g in X as well.

Let g' be another metric with the same σ, ℓ in $X \times \Xi$; then $g' = g$ in X. One can replace g' by a diffeomorphic metric that we call g' again, so that g' and g have the same boundary normal coordinates (x', x^n) near $x_0 = (x'_0, 0)$. Set $f = g - g'$. Then $f_{in} = 0$, $\forall i$. We will show that

$$\partial_{x^n}^k f_{ij}\big|_X = 0, \quad \forall k. \tag{11}$$

We just showed that (11) holds for $k = 0$. We will show next that it holds for $k = 1$.

Note that we know now all tangential derivatives of g in some neighborhood of x_0 on ∂M. We showed above that $\tau(x)$, defined as above and depending also on y_{ε_j}, solves the eikonal equation

$$g^{\alpha\beta}\tau_{x^\alpha}\tau_{x^\beta} + \tau_{x^n}^2 = 1. \tag{12}$$

Next, in U^+, we know τ_{x^α}, $\alpha \le n-1$, we know g, therefore by (12), we get $\tau_{x^n}^2$. We recover $\tau_{x^n}^2$ on U^+, and therefore $\tau_{x^n}^2(x_0)$ by continuity. Since $\tau_{x^n}(x_0) = -\varepsilon_j/|\xi_{\varepsilon_j}|_g$, and therefore, $\tau_{x^n} < 0$ near x_0, we can take a negative square root to recover τ_{x^n} on $U^+ \cup H$, and the tangential derivatives of τ_{x^n}.

Differentiate (12) w.r.t. x^n at $x = x_0$ to get

$$\left[\frac{\partial g^{\alpha\beta}}{\partial x^n}\tau_{x^\alpha}\tau_{x^\beta} + 2g^{\alpha\beta}\tau_{x^\alpha x^n}\tau_{x^\beta} + 2\tau_{x^n x^n}\tau_{x^n}\right]\bigg|_{x=x_0} = 0. \tag{13}$$

Therefore,

$$G_{\alpha\beta}(x_0)\xi_0^\alpha\xi_0^\beta/|\xi_{\varepsilon_j}|_g^2 = \left(-2g^{\alpha\beta}\tau_{x^\alpha x^n}\tau_{x^\beta} + 2\tau_{x^n x^n}\varepsilon_j/|\xi_{\varepsilon_j}|_g\right)\bigg|_{x=x_0},$$

where $G^{\alpha\beta} = \partial g^{\alpha\beta}/\partial x^n$. Assume first that $y_0^* \ne x_0$. Then τ is smooth near (x_0, y_0^*), $\tau_{x^n x^n}(x_0) = \tau_{x^n x^n}(x_0, y_{\varepsilon_j})$ remains bounded and even has a limit, as $j \to \infty$. Similarly the other terms above have a limit. Therefore, the second term on the r.h.s. above tends to zero, as $j \to \infty$, and we recover $G_{\alpha\beta}(x_0)\xi_0^\alpha\xi_0^\beta$, and therefore $(\partial g_{\alpha\beta}/\partial x^n)(x_0)\xi_0^\alpha\xi_0^\beta$.

We show now that $(\partial g_{\alpha\beta}/\partial x^n)(x_0)\xi_0^\alpha\xi_0^\beta$ is uniquely determined if $y_0^* = x_0$ as well. This happens, for example, if ∂M is convex at x_0, as in [LSU] but not only in that case. The arguments there still work in our case, and below we follow [LSU, Theorem 2.1]. The argument that follows is not constructive, and we show that $(\partial f_{\alpha\beta}/\partial x^n)(x_0)\xi_0^\alpha\xi_0^\beta = 0$. Assume that this is not true. Without loss of generality, we may assume that $(\partial f_{\alpha\beta}/\partial x^n)(x_0)\xi_0^\alpha\xi_0^\beta > 0$. Since $g' = g$ on X, the Taylor expansion of g', g w.r.t. x^n shows that

$$f_{\alpha\beta}(x)\xi^\alpha\xi^\beta > 0 \tag{14}$$

in some neighborhood of (x_0, ξ_0) in M, excluding (x, ξ) for which $x^n = 0$, i.e., excluding the boundary points. Let γ'_{ε_j} be as above but related to g'. Since $\sigma' = \sigma$, $\ell' = \ell$ near (x_0, ξ_0), γ'_{ε_j} has the same first encounter point with ∂M, and the same length, as γ_{ε_j}. Parametrize both γ_{ε_j} and γ'_{ε_j} by $t \in [0,1]$. Then as in [LSU], since the geodesics minimize the energy functional, and since γ_{ε_j} and γ'_{ε_j} have the same length, we get $If(\gamma_{\varepsilon_j}) \leq 0$ for $j \gg 1$. On the other hand, integrating (14) over γ_{ε_j} which interior lies in the interior of M, we get a contradiction.

To recover $\partial g/\partial x^n$ at x_0, we need to perturb ξ_0. As we showed above, (x_0, ξ) satisfies the assumptions of the theorem, if $\xi \in \Xi$. We can therefore recover $G_{\alpha\beta}(x_0)\xi^\alpha\xi^\beta$ for such ξ, which recovers $G_{\alpha\beta}(x_0)$, and therefore $\partial g/\partial x^n$ at $x = x_0$.

Since we can apply this argument to any $(x_0, \xi_0) \in X \times \Xi$, we get (11) for $k = 1$.

To recover the higher order derivatives, we proceed in the same way. Assume that we have proved (11) for k replaced by $0, \dots, k-1$. Let $(x_0, \xi_0) \in X \times \Xi$. For each j, we recover consecutively $\partial^m_{x^n}\tau$, $m = 2, \dots, k$ on $U_+ \cup H$ by (12), differentiated $m-1$ times. Then we can also recover the tangential derivatives of $\partial^m_{x^n}\tau$ there. If for the corresponding y_0^* we have $y_0^* \neq x_0$, to recover $\partial^k g/\partial(x^n)^k$, we differentiate (12) k times, and solve for $\partial^k g^{\alpha\beta}/\partial(x^n)^k$ at $x = x_0$. The only unknown term in the r.h.s. will be $\partial^{k+1}\tau/\partial(x^n)^{k+1}$ at $x = x_0$ but it will be multiplied by $\tau_{x^n}(x_0)$ that equals $-\varepsilon_j/|\xi_{\varepsilon_j}|_g$. Then taking the limit $j \to \infty$ will recover $\partial^k g_{\alpha\beta}/\partial(x^n)^k\xi^\alpha\xi^\beta$ at (x_0, ξ_0) as above. If $y_0^* = x_0$, assume that $\partial^k f_{\alpha\beta}/\partial(x^n)^k\xi^\alpha\xi^\beta \neq 0$ at (x_0, ξ_0). Without loss of generality we may assume that it is positive. We write the Taylor expansion of f w.r.t. x^n at $x^n = 0$, using (11) that holds for $0, \dots, k-1$. Then we see that (14) holds as before near (x_0, ξ_0) excluding the boundary points. The we get a contradiction as before. We now vary ξ_0 first, and then x_0, to prove (11). $\square$

Remark 2. If g has a finite smoothness $g \in C^k(M)$, then the proof above implies that we can recover $\partial^\alpha g|_{\partial M}$ for $|\alpha| \leq k-2$ in boundary normal coordinates.

4. Local interior rigidity; Proof of Theorem 2

Given a symmetric 2-tensor $f = f_{ij}$, the *divergence* of f is an 1-tensor δf defined by

$$[\delta f]_i = g^{jk}\nabla_k f_{ij}$$

in any local coordinates, where ∇ is the covariant derivative of the tensor f. Given an 1-tensor (a vector field or an 1-form that we identify through the metric) v, we denote by dv the 2-tensor called symmetric differential of v:

$$[dv]_{ij} = \frac{1}{2}\left(\nabla_i v_j + \nabla_j v_i\right).$$

Operators d and $-\delta$ are formally adjoint to each other in $L^2(M)$. It is easy to see that for each smooth v with $v = 0$ on ∂M, we have $I(dv)(\gamma) = 0$ for any geodesic γ with endpoints on ∂M. This follows from the identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle v(\gamma(t)), \dot\gamma(t)\rangle = \langle dv(\gamma(t)), \dot\gamma^2(t)\rangle. \tag{15}$$

It is known (see [Sh1] and (16) below) that for g smooth enough, each symmetric tensor $f \in L^2(M)$ admits unique orthogonal decomposition $f = f^s + dv$ into a *solenoidal* tensor $\mathcal{S}f := f^s$ and a *potential* tensor $\mathcal{P}f := dv$, such that both terms are in $L^2(M)$, f^s is solenoidal, i.e., $\delta f^s = 0$ in M, and $v \in H^1_0(M)$ (i.e., $v = 0$ on ∂M). In order to construct this decomposition, introduce the operator $\Delta^s = \delta d$ acting on vector fields. This operator is elliptic in M, the Dirichlet problem satisfies the Lopatinskii condition, and has a trivial kernel and cokernel. Denote by Δ^s_D the Dirichlet realization of Δ^s in M. Then

$$v = \left(\Delta^s_D\right)^{-1}\delta f, \quad f^s = f - d\left(\Delta^s_D\right)^{-1}\delta f. \tag{16}$$

Therefore, we have

$$\mathcal{P} = d\,(\Delta_D^s)^{-1}\,\delta, \quad \mathcal{S} = \mathrm{Id} - \mathcal{P},$$

and for any $g \in C^1(M)$, the maps

$$(\Delta_D^s)^{-1} : H^{-1}(M) \to H_0^1(M), \quad \mathcal{P}, \mathcal{S} : L^2(M) \longrightarrow L^2(M) \tag{17}$$

are bounded and depend continuously on g, see [SU3, Lemma 1] that easily generalizes for manifolds. This admits the following easy generalization: for $s = 0, 1, \dots$, the resolvent above also continuously maps H^{s-1} into $H^{s+1} \cap H_0^1$, similarly, $\mathcal{P}$ and $\mathcal{S}$ are bounded in H^s, if $g \in C^k$, $k \gg 2$ (depending on s). Moreover those operators depend continuously on g.

Notice that even when f is smooth and $f = 0$ on ∂M, then f^s does not need to vanish on ∂M. In particular, f^s, extended as 0 to $\tilde{M}$, may not be solenoidal anymore. To stress on the dependence on the manifold, when needed, we will use the notation v_M and f_M^s as well.

Operators $\mathcal{S}$ and $\mathcal{P}$ are orthogonal projectors. The problem about the s-injectivity of I, restricted to a subset of geodesics, then can be posed as follows: if $If = 0$ on those geodesics, show that $f^s = 0$, in other words, show injectivity on the subspace $\mathcal{S}L^2$ of solenoidal tensors.

4.1. **Shift of a small perturbation of a metric to a solenoidal one, after** [CDS]**.** The following lemma is a slight generalization of [CDS, Lemma 2.2]. The reason we include it is that we need to track the dependence of the constants on g. Note first that g is solenoidal w.r.t. itself, and let $\hat{g}$ be close enough to it. The lemma below show that we can replace $\hat{g}$ by an isometric copy h so that h is solenoidal w.r.t. g. Then $f = h - g$ is a solenoidal tensor w.r.t. g that is convenient when linearizing near g.

Let $k \geq 2$, and $\alpha \in (0, 1)$ be fixed. Let τ_M denote the vector bundle on M, let τ_M' be the covector bundle, and let $S^2\tau_M'$ stand for all symmetric tensor fields of type $(0, 2)$. For a small enough neighborhood Ω of zero in $C_0^{k,\alpha}(\tau_M)$, the map

$$e_v : M \to M, \quad e_v(x) = \exp_x v(x) \tag{18}$$

is well defined for all $v \in \Omega$. Moreover, $e_v \in \mathrm{Diff}_0^{k,\alpha}(M)$, if Ω is small enough. We define $C^{k,\alpha}$ through a fixed choice of a finite atlas on M, and in particular, independently of the metric. Instead of trying to find a diffeomorphism ψ, we will look for a vector field v, and then $\psi = e_v$.

Lemma 4. *Let (M, g_0) be a compact Riemannian manifold with boundary, let $k \geq 2$ be an integer, and let $\alpha \in (0, 1)$. Let $\Omega \subset C_0^{k,\alpha}(\tau_M)$ be a neighborhood of zero such that (18) is well defined. Then there exists a neighborhood $G \subset C^{k,\alpha}(S^2\tau_M')$ of zero and a continuous map $\beta : G \to \Omega$ such that $\beta(0) = 0$ and the tensor field $\left(e_{\beta(f)}\right)^*(g_0 + f)$ is solenoidal w.r.t. g_0.*

There exist m, and $\delta > 0$, so that the conclusion above holds for g_0 replaced by any metric $g \in U := \{g;\ \|g - g_0\|_{C^m(S^2\tau_M')} < \delta\}$ with $G = \{f;\ \|f\|_{C^{k,\alpha}(S^2\tau_M')} < \delta\}$. Moreover, $\|\beta(f)\|_{C^{k,\alpha}(\tau_M)} \leq C\|f\|_{C^{k,\alpha}(S^2\tau_M')}$ for such f and g.

Proof. The proof in [CDS] is based on an application of the implicit function theorem in Banach spaces, see e.g., [AMR, Theorem 2.5.7]. Consider the map

$$F : \Omega \times C^{k,\alpha}(S^2\tau_M') \to C^{k-2,\alpha}(\tau_M')$$

given by

$$F(v, f) = \delta(e_v^*(g_0 + f)).$$

We want to solve $F(v, f) = 0$ for v near $f = 0$ so that $v = 0$ when $f = 0$. It is shown then that F_v, F_f exist (in Gâteaux sense), and are continuous (therefore they are Fréchet differentiable). Then it is shown that

$$F_v(0,0) = 2\delta d, \tag{19}$$

where the latter is considered as an operator from $C_0^{k,\alpha}(\tau_M)$ to $C^{k-2,\alpha}(\tau_M')$. Since δd is an isomorphism, the implicit function theorem yields the result.

To prove the second statement of the lemma, we recall the implicit function theorem in the form presented in [AMR, Theorem 2.5.7]. Let $F : X\times Y \to Z$ be a continuous map , where X, Y are Banach spaces. Let F be C^r, $r\ge 1$ near some (x_0, y_0), and assume that $F_y(x_0, y_0) : Y \to Z$ is an isomorphism. Then the equation $F(x, y) = z$ has unique C^r solution $y = y(x, z)$ for (x, z) near $(x_0, f(x_0, y_0))$. To apply this to our setting, set $x = (f, g)$, $y = v$; $x_0 = (0, g_0)$, and

$$X = C^m(S^2\tau_M')\times C^{k,\alpha}(S^2\tau_M'),\quad Y = C_0^{k,\alpha}(\tau_M),\quad Z = C^{k-2,\alpha}(S^2\tau_M').$$

Note that we are not using covariant derivatives in the spaces above, and they are independent of g. Let F be as above but now g is not fixed. We choose $m = m(k, r)$ so that $F \in C^r$ (see, e.g., [FSU, Lemma 3]), i.e., that the derivative $D^r_{f,g}F$ exist and is continuous. Then $F_y(x_0, y_0) = 2\delta_{g_0}d_{g_0}$ by (19), and it is an isomorphism. Now the statement of the theorem about m, U, and G follows from [AMR, Theorem 2.5.7] by setting $z = 0$. Thus we get $v = \beta(f, g)$ that is C^r in both variables. For the purpose of the lemma, choosing $r = 1$ is enough. Since $\beta\in C^1$, it is Lipshitz on $G \times U$, that proves the last estimate in the lemma. □

Remark 3. Notice that under the assumptions of the lemma, $\left(e_{\beta(f)}\right)^*(g_0 + f)$ is a metric again for $\delta \ll 1$, and belongs to $C^{k-1,\alpha}$. Moreover, it is $O\big(\|f\|_{C^{k,\alpha}(S^2\tau_M')}\big)$ close to g in that norm.

We start with the proof of Theorem 2. We will split the proof into several steps. Let g_0, g, $\hat g$, k and ε be as in Theorem 2. Many of the steps below hold for $\varepsilon \ll 1$, and $k \gg 1$, and in each of the finite many steps we may need to decrease ε or increase k.

4.2. **Choosing a suitable metric isometric to $\hat g$.** Any two metrics such that one of them is a pull-back of the other under a diffeomorphism fixing the boundary pointwise, will be called below isometric. Such a diffeomorphism is necessarily C^{k+1} if the metrics are C^k, see e.g., [SU3], and the norm of its derivatives are controlled by those of the two metrics, see [SU3, Lemma 6].

We first find a metric isometric to $\hat g$, that we denote by $\hat g$ again, so that the boundary normal coordinates related to g and $\hat g$ coincide in some neighborhood of the boundary, see e.g., the beginning of the proof of Theorem 2.1 in [LSU]. We can still assume that (8) holds because it is preserved with k replaced by $k-2$ and ε replaced by $C\varepsilon$.

By Lemma 4, and the remark above, if $\varepsilon \ll 1$, since g and $\hat g$ satisfy (8), if $k \ge 3$, there is $h \in C^k$ isometric to $\hat g$ so that h is solenoidal w.r.t. g. Moreover,

$$\|h - \hat g\|_{C^{k-2}} \le C\varepsilon, \tag{20}$$

and we can replace $k-2$ by $k-1$ if we work in the $C^{k,\alpha}$ spaces. By a standard argument, by a diffeomorphism that identifies normal coordinates near ∂M for h and g, and is identity away from some neighborhood of the boundary, we find a third $\hat g_1$ isometric to h (and therefore to $\hat g$), so that $\hat g_1 = \hat g$ near ∂M, and $\hat g_1 = h$ away from some neighborhood of ∂M (and there is a region that $\hat g_1$ is neither). Then $\hat g_1 - h$ is as small as $g - h$, more precisely,

$$\|\hat g_1 - h\|_{C^{l-3}} \le C\|g - h\|_{C^{l-1}},\quad l \le k. \tag{21}$$

This follows from the fact that $\hat g_1 = \phi^* h$, with a diffeomorphism ϕ that is identity on the boundary, and

$$\|\phi - \mathrm{Id}\|_{C^{l-2}} \le C\|\hat g - h\|_{C^{l-1}}. \tag{22}$$

Set

$$f = h - g, \quad \tilde f = \hat g_1 - g. \tag{23}$$

We aim to show that $f = \tilde f = 0$. Estimate (21) implies

$$\|\tilde f - f\|_{C^{l-3}} \le C\|f\|_{C^{l-1}}, \quad \forall l \le k. \tag{24}$$

By (8), (20) and (24),

$$\|f\|_{C^{k-2}} \le C\varepsilon, \quad \|\tilde f\|_{C^{k-4}} \le C\varepsilon. \tag{25}$$

By Theorem 1, and the remark after it,

$$\partial^\alpha \tilde f = 0 \quad \text{on } \partial M \text{ for } |\alpha| \le k-6. \tag{26}$$

We have now two isometric copies of $\hat g$: the first one is h that has the advantage of being solenoidal w.r.t. g; and the second one $\hat g_1$ that has the same jet as g on ∂M. We need both properties below to show that $g = h$, i.e., $f = 0$ (or $g = \hat g_1$, i.e., $\tilde f = 0$) but so far we cannot prove that $h = \hat g_1$. The next proposition shows that h and $\hat g_1$ are equal up to $O(\|f\|^2) = O(\|\tilde f\|^2)$.

Proposition 1. *Let $\hat g$ and g be in C^k, $k \ge 2$ and isometric, i.e.,*

$$\hat g = \psi^* g$$

for some diffeomorphism ψ fixing ∂M pointwise. Set $f = \hat g - g$. Then there exists v vanishing on ∂M, so that

$$f = 2dv + f_2,$$

and for g belonging to any bounded set U in C^k, there exists $C(U) > 0$, such that

$$\|f_2\|_{C^{k-2}} \le C(U)\|\psi - \mathrm{Id}\|^2_{C^{k-1}}, \quad \|v\|_{C^{k-1}} \le C(U)\|\psi - \mathrm{Id}\|_{C^{k-1}}.$$

Proof. Extend g to $\tilde M$ in such a way that the C^k norm of the extension is bounded by $C\|g\|_{C^k(M)}$. Set $v(x) = \exp_x^{-1}(\psi(x))$ that is a well defined vector field in $C^1(M)$ if ψ is close enough to identity in C^1 (it is enough to prove the theorem in this case only), and $v = 0$ on ∂M. Set $\psi_\tau(x) = \exp_x(\tau v(x))$, $0 \le \tau \le 1$. Let $g^\tau = \psi_\tau^* g$. Under the smallness condition above, v is small enough in C^1, and therefore ψ_τ is close enough to identity in the $C^1(\tilde M)$ norm. Therefore, $\psi_\tau : M \to \psi_\tau(M) \subset \tilde M$ is a diffeomorphism. Next, ψ_τ fixes ∂M pointwise, therefore, $\psi_\tau(M) = M$.

The Taylor formula implies

$$\hat g = g + \frac{d}{d\tau}\Big|_{\tau=0} g^\tau + h = g + 2dv + h,$$

where

$$|h| \le \frac12 \max_{\tau\in[0,1]} \left|\frac{d^2 g^\tau}{d\tau^2}\right|,$$

and $2dv$ is the linearization of g^τ at $\tau = 0$, see [Sh1]. To estimate h, write

$$g^\tau_{ij} = g_{kl} \circ \psi_\tau \frac{\partial \psi_\tau^k}{\partial x^i}\frac{\partial \psi_\tau^l}{\partial x^j},$$

and differentiate twice w.r.t. τ. Notice that

$$\left|\frac{\partial^2 \psi_\tau}{\partial \tau^2}\right| \le C\|v\|^2_{L^\infty}, \quad \left|\frac{\partial^2 \nabla\psi_\tau}{\partial \tau^2}\right| \le C\|v\|^2_{C^1}.$$

This yields the stated estimate for f_2 for $k = 2$. The estimates for $k > 2$ go along similar lines by expressing the remainder h in its Lagrange form, and estimating the derivatives of h. □

We apply Proposition 1 to h and $\hat g_1$ to get by (22),

$$\tilde f = f + 2dv + f_2, \quad \|f_2\|_{C^{l-3}} \le C\|f\|^2_{C^{l-1}}, \quad \forall l \le k. \tag{27}$$

In other words, $\tilde f^s = f$ up to $O(\|f\|^2)$.

Next, with g extended as above, we extend $\hat g_1$ so that $\hat g_1 = g$ outside M. Then $g \in C^k$ and $\hat g_1 \in C^{k-6}$ by (26).

4.3. **Reparametrizing the scattering relation.** We proceed with some preliminary work that would allow us to apply [SU4, Theorem 2]. Assume first that the underlying metric is fixed to be g_0. Let $(\tilde M, \tilde g_0)$ be a C^k extension as above. In [SU4], the geodesics are extended to $\tilde M \setminus M$, parametrized by initial points, and corresponding directions on a finite collection $\{H_m\}$ of smooth connected hypersurfaces in $\tilde M$, having additional properties as explained below. Given two complete $\mathcal{D}' \Subset \mathcal{D}$, we will construct such a family issued from a set $\mathcal{D}''$ with $\mathcal{D}' \Subset \mathcal{D}'' \Subset \mathcal{D}$, that is also complete.

For any $(z_0, \zeta_0) \in T^*M$, including the case where $z_0 \in \partial M$, there is a maximal geodesic γ_0 through z_0 normal to ζ_0, satisfying the conditions of Definition 2. Let us assume that γ_0 is parametrized by $t \in [l^-, l^+]$, $\pm l^\pm \ge 0$, and $\gamma(0) = z$, $\dot\gamma_0(0) = \xi_0$, with ζ_0 conormal to ξ_0. Since γ_0 is maximal in M, by Lemma 1, for any $\delta_1 > 0$ there exists $\delta \in (0, \delta_1)$ so that the extension $\tilde\gamma_0$ of γ to $\tilde M$ corresponding to $t \in [l^- - \delta, l^+ + \delta]$ is well defined and has endpoints in $\tilde M^{\text{int}} \setminus M$. By (6),

$$\left\{(\gamma_0(t), \dot\gamma_0(t));\ l^- \le t \le l^+\right\} \cap S(\partial M) \subset \mathcal{D}.$$

The extension $(\tilde\gamma_0, \dot{\tilde\gamma}_0)$ may have additional point on ∂M corresponding to $l^- - \delta \le t \le l^-$, and $l^+ \le t \le l^+ + \delta$. However, we choose $\delta_1 \ll 1$ so that they still belong to $\mathcal{D}$, and this is possible to do because $\mathcal{D}$ is open. Now, by Lemma 2, any geodesic that is obtained by a small enough perturbation (z, ξ) of the initial conditions (z_0, ξ_0) of γ_0 at $t = 0$, with the same interval $t \in [l^- - \delta, l^+ + \delta]$, will satisfy condition (9). Condition (7) will also be satisfied by a perturbation argument, if δ is small enough as well. Now we can perturb $\tilde g_0$ in the $C^2(\tilde M)$ topology to ensure the same property. Note that if $\|g - g_0\|_{C^2(M)} \le \varepsilon$, one can choose an extension $\tilde g$ of g to $\tilde M$ so that $\|\tilde g - \tilde g_0\|_{C^2(M)} \le C\varepsilon$ with $\tilde g_0$ a fixed extension of g_0 as above.

To summarize, we proved the following.

Lemma 5. *Under the conditions of Theorem 2, for any $(z_0, \zeta_0) \in S^*M$, there exists a geodesic in the metric $\tilde g_0$ through z_0 normal to ζ_0 with endpoints in $\tilde M^{\text{int}} \setminus M$ so that conditions (7), (9) are satisfied.*

Moreover, if that geodesic has initial conditions (z_0, ξ_0) at $t = t_0$ for some $t_0 \in [0, l]$, and an interval of definition $0 \le t \le l$, properties (7) and (9) remain true under small enough perturbations of (z_0, ξ_0), and $\tilde g_0$ in $C^2(\tilde M)$.

Let us assume now that the underlying metric is g as in the theorem, with $\varepsilon \ll 1$. Since $\overline{\mathcal{D}'}$ is compact, there are finitely many geodesics

$$\left\{\gamma_m(t);\ l_m^- - \delta_m \le t \le l_m^+ + \delta_m\right\},$$

with the following properties. If $\dot\gamma_m(0) = \xi_m \in S_{z_m}M$, then for any m there exists neighborhoods $U'_m \Subset U_m$ of (z_m, ξ_m) in SM, such that if Γ_m, Γ'_m is the set of geodesic with initial conditions in U_m, respectively U'_m, and the same interval of definition as γ_m, then for any $(x, \zeta) \in T^*M$ there is a geodesic $\gamma \in \cup\Gamma'_m$ so that (7), (9) are satisfied with $\mathcal{D}$ replaced by $\mathcal{D}'$; and all geodesics in $\cup\Gamma_m$ satisfy (7), (9) as well. Moreover,

$$\cup\Gamma'_m \supset \kappa_-^{-1}(\overline{\mathcal{D}'}), \tag{28}$$

where Γ_m' is regarded as a point set in the phase space $S\tilde{M}$ consisting of the points on all integral curves.

We will parametrize Γ_m, Γ_m', with initial points outside M. Choose a family of finitely many small enough smooth hypersurfaces $\{H_m\}$ in $\tilde{M}^{\rm int}\setminus M$, each one transversal to γ_m. Without loss of generality, we can assume that all geodesics in Γ_m can be extended in $\tilde{M}^{\rm int}\setminus M$ so that they intersect H_m once and are transversal to H_m as well. We still denote the set of the extended geodesics by Γ_m and Γ_m', respectively. Let $\mathcal{H}_m$ be the open subset of $\{(x,\xi)\in SM;\ x\in H_m,\ \xi\not\in T_xH_m\}$ formed by those (x,ξ) that coincide with the left endpoint and the corresponding direction of some geodesic in Γ_m. Then their endpoints belong to $\tilde{M}^{\rm int}\setminus M$ again, and their length is a smooth function $l_m(x,\xi)>0$ (actually, we may even assume that l_m is constant, if U_m is small enough).

We have

$$\Gamma_m=\Gamma(\mathcal{H}_m)=\left\{\gamma_{x,\xi}(t);\ 0\le t\le l_m(x,\xi),\ (x,\xi)\in\mathcal{H}_m\right\}, \tag{29}$$

where $\gamma_{z,\xi}$, as usual, is the geodesic with initial conditions (x,ξ). We define $\mathcal{H}_m'$ in a similar way, related to U_m'. We consider also the geodesics in the metric $\hat{g}$ defined as in (29). Then (7), (9) hold for those geodesics, too, provided that $\varepsilon\ll 1$.

Combining the arguments above with Lemma 3, we have the following.

Proposition 2. *Let g and $\hat{g}_1$ be as in the end of Section 4.2. Then, if $\varepsilon\ll 1$,*

$$\Phi^{l_m(x,\xi)}(x,\xi)=\hat{\Phi}^{l_m(x,\xi)}(x,\xi),\quad \forall (x,\xi)\in\mathcal{H}_m,\ \forall m,$$

where $\hat{\Phi}$ is related to $\hat{g}_1$. Moreover, $\cup\Gamma_m$ satisfies (7), (9), and $\cup\Gamma_m'$ is complete in the sense that $N^*(\cup\Gamma_m')\supset T^*M$, and satisfies (28); similarly $\cup\hat{\Gamma}_m$ and $\cup\hat{\Gamma}_m'$ have the same properties.

In other words, informally speaking, we pushed the boundary, where the scattering relation is defined, to a collection of hypersurfaces outside M, so that the corresponding geodesics are always transversal to them, and the endpoints are away from ∂M.

Proof of Proposition 2. The proof is straightforward, if the geodesic issued from (x,ξ), for $0\le l\le l_m(x,\xi)$, always intersects ∂M transversally. Observe that $\kappa_\pm=\hat{\kappa}_\pm$ because g and $\hat{g}$ have the same normals on ∂M, therefore $\sigma=\hat{\sigma}$ implies $\Sigma=\hat{\Sigma}$. The points $(x,\xi)\in\mathcal{H}_m$ where this transversality does not hold is a closed set of measure zero by Lemma 3, and for such points, one can approximate with points outside this set, and to use the continuity of Φ^t. $\square$

4.4. **Linearization near g.** Now we are in the situation of [SU4], see Theorem 2 there. Choose smooth functions α_m supported in $\mathcal{H}_m$ and equal to 1 on $\mathcal{H}_m'$. Set $\alpha=\{\alpha_m\}$ and $I_{\alpha_m}=\alpha_m I$, more precisely,

$$I_{\alpha_m}f(z,\xi)=\alpha_m(x,\xi)\int_0^{l_m(x,\xi)}\langle f(\gamma_{z,\xi}),\dot{\gamma}_{z,\xi}^2\rangle\,\mathrm{d}t,\quad (z,\xi)\in\mathcal{H}_m.$$

Also set

$$I_\alpha=\{I_{\alpha_m}\},\quad N_{\alpha_m}=I_{\alpha_m}^*I_{\alpha_m},\quad N_\alpha=\sum N_{\alpha_m},$$

where the adjoint is taken w.r.t. the measure $|\langle\nu,\xi\rangle|\mathrm{d}\Sigma_{2n-2}$, where $\mathrm{d}\Sigma_{2n-2}$ is the induced measure on ∂SM by the volume form, and ν is a unit normal to H_m.

In [SU4, Theorem 2] we showed that if for a fixed g, I_α is s-injective, then we have the following a priori estimate

$$\|f^s\|_{L^2(M)}\le C\|N_\alpha f\|_{\tilde{H}^2(\tilde{M})}, \tag{30}$$

with a suitable norm $\|\cdot\|_{\tilde H^2(\tilde M)}$ so that $H^2(\tilde M)\subset \tilde H^2(\tilde M)\subset H^1(\tilde M)$. Moreover, (30) remains true under small C^k perturbations of g with a constant C that can be choose uniformly. Note that (T) is not needed in [SU4, Theorem 2].

Fix m and $(x,\xi)\in\mathcal{H}_m$. Let $\{\gamma_{x,\xi}(t),\, 0\le t\le l_m(x,\xi)\}$, and $\{\hat\gamma_{x,\xi}(t),\, 0\le t\le l_m(x,\xi)\}$ be the geodesic issued from (x,ξ) related to g and $\hat g_1$, respectively. By Proposition 2, their endpoints and directions coincide. We reparametrize $\gamma_{x,\xi}$, $\hat\gamma_{x,\xi}$ so that $t\in[0,1]$; then they have the same speeds $|\dot\gamma_{x,\xi}(t)|=|\dot{\hat\gamma}_{x,\xi}(t)|=l_m(x,\xi)$.

Define the following variation of $\gamma_{x,\xi}$, where exp is related to g:

$$c_\tau(t)=\exp_{\gamma_{x,\xi}(t)}\left(\tau v(t)\right),\quad v(t)=\exp^{-1}_{\gamma_{x,\xi}(t)}\left(\hat\gamma_{x,\xi}(t)\right), \tag{31}$$

where $0\le t\le 1$, $0\le\tau\le1$. Then $c_0=\gamma_{x,\xi}$, $c_1=\hat\gamma_{x,\xi}$. Set $g^\tau=g+\tau(\hat g_1-g)$. Let

$$E(\tau)=\int_0^1\langle\dot c_\tau(t),\dot c_\tau(t)\rangle_{g_\tau}\,\mathrm{d}t, \tag{32}$$

where, in local coordinates, $\langle\dot c_\tau(t),\dot c_\tau(t)\rangle_{g^\tau}=g^\tau_{ij}(c_\tau)\dot c^i_\tau\dot c^j_\tau$. Apply Taylor's formula

$$E(1)=E(0)+E'(0)+\int_0^1(1-\tau)E''(\tau)\,\mathrm{d}\tau$$

to get

$$E'(0)=-\int_0^1(1-\tau)E''(\tau)\,\mathrm{d}\tau \tag{33}$$

because $E(0)=E(1)=l_m^2(x,\xi)$. Write

$$\psi(\tau,s)=\int_0^1\langle\dot c_\tau(t),\dot c_\tau(t)\rangle_{g^s}\,\mathrm{d}t=\int_0^1 g^s_{ij}(c_\tau)\dot c^j_\tau\dot c^i_\tau\,\mathrm{d}t,$$

where the second integrand is written in local coordinates. Then $E(\tau)=\psi(\tau,\tau)$. For E' we get

$$E'(\tau)=\psi_\tau(\tau,\tau)+\psi_s(\tau,\tau). \tag{34}$$

Since $c_0=\gamma_{x,\xi}$ is a critical curve for the energy functional, we get $\psi_\tau(0,0)=0$, therefore,

$$E'(0)=\int_0^1\langle\tilde f,\dot\gamma^2_{x,\xi}\rangle\,\mathrm{d}t,$$

recall (23). Together with (33) this yields

$$I\tilde f(\gamma_{x,\xi})=-\int_0^1(1-\tau)E''(\tau)\,\mathrm{d}\tau. \tag{35}$$

To estimate the r.h.s. above, note that

$$E''(\tau)=\psi_{\tau\tau}(\tau,\tau)+2\psi_{\tau s}(\tau,\tau) \tag{36}$$

because $\psi_{ss}=0$. Note that

$$\left|\frac{\partial c_\tau(t)}{\partial\tau}\right|+\left|\frac{\partial\dot c_\tau(t)}{\partial\tau}\right|\le C\big(|v(t)|+|\dot v(t)|\big), \tag{37}$$

and

$$\left|\frac{\partial^2 c_\tau(t)}{\partial\tau^2}\right|+\left|\frac{\partial^2\dot c_\tau(t)}{\partial\tau^2}\right|\le C\big(|v(t)|+|\dot v(t)|\big)^2. \tag{38}$$

We have that $|\exp_x^{-1} y| \le C|x-y|$ for $|x-y| \ll 1$, where the norm is in any fixed coordinate chart, and $C>0$ depends on an upper bound of g in C^k, $k \gg 2$. All constants below will have the same property. This and (31) imply

$$|v(t)| \le C\,|\hat\gamma_{x,\xi}(t) - \gamma_{x,\xi}(t)| \le C'\|\tilde f\|_{C^1}. \tag{39}$$

Since in fixed coordinates, $D_x \exp_x^{-1} y + D_y \exp_x^{-1} y = 0$ when $x=y$, we have

$$\left|D_x \exp_x^{-1} y + D_y \exp_x^{-1} y\right| \le C|x-y|.$$

This allows us to estimate $\dot v(t)$, see (31), as follows

$$|\dot v(t)| \le C\left(\left|\hat\gamma_{x,\xi}(t) - \gamma_{x,\xi}(t)\right| + \left|\dot{\hat\gamma}_{x,\xi}(t) - \dot\gamma_{x,\xi}(t)\right|\right) \le C'\|\tilde f\|_{C^1}. \tag{40}$$

By (35), (36), (37), (38), (39), (40),

$$|I\tilde f(\gamma_{x,\xi})| \le C'\|\tilde f\|^2_{C^1}.$$

This is the same estimate that was used in the linearization argument in [SU2, SU3] and goes back to [E], but now proven in this more general situation. Therefore,

$$\|I_{\alpha_j}\tilde f\|_{L^\infty} \le C\|\tilde f\|^2_{C^1}. \tag{41}$$

That implies the same for $\|N_{\alpha_j}\tilde f\|_{L^\infty}$, see e.g., [SU3], therefore,

$$\|N_{\alpha_j}\tilde f\|_{L^\infty} \le C\|\tilde f\|^2_{C^1}, \quad \forall j. \tag{42}$$

4.5. **End of the proof of Theorem 2.** We will use interpolation to estimate $\|N_{\alpha_j}\tilde f\|_{\tilde H^2(\tilde M)}$ through some power of $\|N_{\alpha_j}\tilde f\|_{L^\infty}$. Since N_{α_j} is a ΨDO of order -1 and $\tilde f$, extended as 0 outside M, is smooth enough if $k \gg 2$ by (26), we get

$$\|N_{\alpha_j}\tilde f\|^2_{\tilde H^2(\tilde M)} \le \|N_{\alpha_j}\tilde f\|^2_{H^2(\tilde M)} \le C\|\tilde f\|_{H^3(M)}\|N_{\alpha_j}\tilde f\|_{L^2(\tilde M)}. \tag{43}$$

Combine (42) and (43) to get

$$\|N_\alpha \tilde f\|^2_{\tilde H^2(\tilde M)} \le C\|\tilde f\|_{C^3}\|\tilde f\|^2_{C^1} \le C'\|f\|^3_{C^5}, \tag{44}$$

by (24).

Since $I_{g_0,\mathcal{D}'}$ is s-injective, so is N_α, related to g_0, by the support properties of α. Now, since N_α by (8) is s-injective for $g=g_0$, we get from [SU4, Theorem 2] that N_α (the one related to g) is s-injective as well provided that

$$\|g-g_0\|_{C^k(M)} \le \varepsilon_0 \tag{45}$$

with some $k \gg 1$ and $\varepsilon_0 > 0$. Moreover, (30) is true. Assume now that both (45) and (8) are satisfied. The geodesics issued from $\operatorname{supp}\alpha$ form a complete set by the second statement of Proposition 2, therefore, by (44) and (30),

$$\|\tilde f^s\|_{L^2(M)} \le C\|N_\alpha \tilde f\|_{\tilde H^2} \le C'\|f\|^{3/2}_{C^5}. \tag{46}$$

By (27), $\tilde f^s = f + f_2^s$, therefore,

$$\|\tilde f^s\|_{L^2(M)} \ge \|f\|_{L^2(M)} - \|f_2^s\|_{L^2(M)} \ge \|f\|_{L^2(M)} - C\|f\|^2_{C^2}.$$

Together with (46), this yields

$$\|f\|_{L^2(M)} \le C\left(\|f\|^2_{C^2} + \|f\|^{3/2}_{C^5}\right) \le C'\|f\|^{3/2}_{C^5}$$

because the C^5 norm of f is uniformly bounded when $\varepsilon \le 1$. We can use now interpolation estimates in C^k, see [Tri], and Sobolev embedding estimates to get $\|f\|_{C^5} \le C\|f\|_{C^0}^{\mu} \le C'\|f\|_{H^{n/2+1}}^{\mu}$ with any $\mu \in (0,1)$ as long as $k = k(\mu) \gg 1$ in (25). Next, interpolation estimates in Sobolev spaces imply $\|f\|_{H^{n/2+1}} \le C\|f\|_{L^2}^{\mu}$, so in the end, we get

$$\|f\|_{L^2(M)} \le C\|f\|_{L^2(M)}^{3\mu^2/2}.$$

Choose $2/3 < \mu^2 < 1$ with a corresponding $k \gg 2$, to get $\|f\|_{L^2(M)} \ge 1/C$ if $f \ne 0$. Choose $\varepsilon \ll 1$ to get a contradiction with (25). This proves Theorem 3 for ε replaced by $\min(\varepsilon_0, \varepsilon)$.

Now, $f = 0$ implies $h = g$, therefore, g and $\hat g$ are isometric.

This concludes the proof of Theorem 2.

5. Proof of Theorem 3 and Corollary 1

Proof of Theorem 3. As we mentioned in the Introduction, Theorem 3 is a reformulation of [SU4, Theorem 3], as we show below. Notice first, that the s-injectivity of $I_{g,\mathcal{D}''}$ for some $\mathcal{D}'' \subset \mathcal{D}'$ implies s-injectivity of $I_{g,\mathcal{D}'}$. One can always assume that M is equipped with a finite analytic atlas. Note that the assumption that $\mathcal{D}'$ is open implies that the corresponding set of geodesic is open in the sense of [SU4] by Lemma 2. Let $\mathcal{D}'' \Subset \mathcal{D}'$ be such that $\mathcal{D}''$ is still complete and open. It can be constructed as in Section 4.3. As in Section 4.3 again, with $\mathcal{D}''$ and $\mathcal{D}'$ playing the roles of $\mathcal{D}'$ and $\mathcal{D}$ there, respectively, we construct N_α such that all geodesics through $\operatorname{supp}\alpha$ cover $\kappa_-^{-1}(\mathcal{D}'')$ and are contained in the interior of $\kappa_-^{-1}(\mathcal{D}')$. Then N_α is s-injective for each analytic $g_0 \in \mathcal{G}$ by [SU4, Theorem 1]. It is still s-injective under a small enough C^k, $k \gg 2$, perturbation of $g \in C^k(M)$ by [SU4, Theorem 2]. Note that [SU4, Theorem 2] requires that the perturbation must be considered in $C^k(\tilde M)$ but one can use extensions of g near a fixed g_0 so that their norms in $C^k(\tilde M)$ are bounded by $\|g\|_{C^k(M)}$ with a fixed C. Using the fact that $\mathcal{D}'' \Subset \mathcal{D}'$, Lemma 2, and the support properties of α, we deduce that $I_{g,\mathcal{D}'}$ is s-injective for g close enough in $C^k(M)$ to a fixed analytic $g_0 \in \mathcal{G}$. We can therefore build $\mathcal{G}_s$ as a small enough neighborhood of the analytic metrics in $\mathcal{G}$, that form a dense set.

This completes the proof of Theorem 3. □

Corollary 1 is an immediate consequence of Theorem 2 and Theorem 3.

Proof of Remark 1. Condition (T) is not needed for [SU4, Theorem 2], see also (30) but it is used in the proof of Theorems 2 (s-injectivity for analytic metrics) and Theorem 3 (generic s-injectivity) there. Assume now that (T) in Theorem 3 of this paper is replaced by the assumption that $(\tilde M, \tilde g)$ is simple as in the remark. In the proof above, in this situation, we need to show that N_α is s-injective for a dense set of metrics in $\mathcal{G}$. Fix $g_0 \in \mathcal{G}$. It can be extended as $\tilde g_0$ to $\tilde M$ so that $(\tilde M, \tilde g)$ is simple. Given $\varepsilon > 0$ we can find a real analytic $\tilde g$ so that $\|\tilde g - \tilde g_0\|_{C^k(\tilde M)} \le \varepsilon$ for k fixed. Then the ray transform I related to $\tilde g$ over all maximal geodesics in $\tilde M$ is s-injective, see [SU3, SU4]. Let now $N_{g,\alpha} f = 0$ in M, where $g = \tilde g|_M$, and the subscript g in $N_{g,\alpha}$ indicates that this is the normal operator related to g. Then we get after integration by parts that $I_g f(\gamma) = 0$ for all maximal geodesics in M. Let $\tilde f$ be the extension of f as zero outside M. Then $I_{\tilde g} \tilde f(\gamma) = 0$ for all maximal geodesics in $\tilde M$. Therefore, $\tilde f = d\tilde v$ in $\tilde M$, $\tilde v \in H_0^1(M)$. Since $\tilde f = 0$ in $\tilde M \setminus M$, on can see that the same holds for $\tilde v$ as well, see also [Sh2]. Therefore, f is potential, thus I_g is s-injective, and so is $N_{g,\alpha}$, see [SU4, Lemma 2]. □

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