Local Uniqueness for the Fixed Energy Fixed Angle Inverse Problem in Obstacle Scattering

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Abstract
We prove local uniqueness for the inverse problem in obstacle scattering at a fixed energy and fixed incident angle.

We consider the inverse problem of determining a sound-soft obstacle in $\mathbb{R}^n$, $n \geq 2$, from its scattering amplitude at a fixed incident direction $\theta \in S^{n-1}$ and a fixed energy $k > 0$. This is a formally determined inverse problem, since the data depends on the same number of variables, $n - 1$, as does the object we want to recover.

The purpose of this note is to give a simple proof of local uniqueness for this problem. Roughly speaking, we show that if two domains are close to a given obstacle, in a precise sense described below, and have the same scattering amplitude at a fixed angle and fixed energy then they must be the same. Previously, it was shown in [CS] that local uniqueness holds for small obstacles. The Fréchet derivative of the nonlinear map from the domain to the scattering amplitude at fixed energy and angle was computed in [P], and one can easily show that it is injective. However, this does not imply a local result, since we cannot directly apply the implicit function theorem.

The proof of our result follows by using the arguments of Schiffer’s well-known proof, presented in [LP], of uniqueness when given all incident directions and the Poincaré inequality.

By obstacles, we mean compact subsets of $\mathbb{R}^n$ with $C^2$ boundary and connected complement. The scattering amplitude $A_\mathcal{O}(k, \theta, \omega)$ related to an obstacle $\mathcal{O}$ is defined as follows. For $k > 0$, $\theta \in S^{n-1}$, we define the scattering solution $u(x, \theta, k)$ as the solution to the boundary value problem (see e.g., [CK])

\[
\begin{cases}
(-\Delta - k^2)u = 0, & \text{in } \mathbb{R}^n \setminus \mathcal{O}, \\
u|_{\partial \mathcal{O}} = 0,
\end{cases}
\]

such that $u = e^{ik\theta \cdot x} + v$, with $v$ satisfying the Sommerfeld outgoing condition at infinity: $(\partial/\partial r - ik)v = O(r^{-(n+1)/2})$, as $r = |x| \to \infty$. Then

\[
v(x, \theta, k) = e^{ik\theta \cdot x} + \frac{e^{ikr}}{r^{(n-1)/2}} A_\mathcal{O}\left(k, \theta, \frac{x}{r}\right) + O\left(\frac{1}{r^{(n+1)/2}}\right), \quad \text{as } r = |x| \to \infty.
\]

The function $A_\mathcal{O}(k, \theta, \omega)$ is the scattering amplitude related to $\mathcal{O}$.

It is known that Schiffer’s proof implies uniqueness if $A_\mathcal{O}$ is known for all $\omega$, fixed $k_0 > 0$, and $N$ incident directions $\theta$; or for all $\omega$, fixed $\theta_0$, and $N$ frequencies $k \leq k_0$, where $N$ is greater than the number of the

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Dirichlet eigenvalues $k^2 \leq k_0^2$ of the Laplacian in a ball containing the obstacles. In particular, as mentioned above, this implies uniqueness at a fixed $\theta_0$ and a fixed $k_0$ for all obstacles contained in a ball with sufficiently small radius $R$. In the 3D case, the condition is given by $k_0 R < \pi$. We refer to [CK], [I], [KK], [CS] for details and references.

In what follows, $\omega_n$ is the volume of the unit ball in $\mathbb{R}^n$ (not to be confused with the outgoing direction $\omega$); in particular, $\omega_3 = 4\pi/3$.

Our main result is the following.

**Theorem 1** Fix $k_0 > 0$, $\theta_0 \in S^{n-1}$. Let $\mathcal{O}_- \subset \mathcal{O}_+$ be two obstacles and assume that $\operatorname{Vol}(\mathcal{O}_+ \setminus \mathcal{O}_-) < \omega_n k_0^{-n}$. Let $\mathcal{O}_- \subset \mathcal{O}_j \subset \mathcal{O}_+$, $j = 1, 2$ be two other obstacles and assume that $A_{\mathcal{O}_1}(k_0, \theta_0, \omega) = A_{\mathcal{O}_2}(k_0, \theta_0, \omega)$. Then $\mathcal{O}_1 = \mathcal{O}_2$.

In particular, for any fixed obstacle $\mathcal{O}$, and fixed $k_0 > 0$, $\theta_0$, any small enough perturbation of the boundary gives an obstacle with different scattering amplitude.

More precisely, there exists $\varepsilon = \varepsilon(\mathcal{O}, k_0, \theta_0) > 0$ such that if $\partial \mathcal{O}_1$ is given in boundary normal coordinates $(x', x_n) \in \partial \mathcal{O} \times (-\delta, \delta)$ by $x_n = f(x')$ with $|f(x')| \leq \varepsilon$, $\forall x'$, then $A_{\mathcal{O}_1}(k_0, \theta_0, \omega) = A_{\mathcal{O}_0}(k_0, \theta_0, \omega)$ implies $\mathcal{O}_1 = \mathcal{O}$. We would like to emphasize that this is different from the uniqueness for obstacles with small diameters mentioned above.

In Theorem 1 and Proposition 1 below, we do not impose smallness assumptions on $k_0$ or on the diameters of the obstacles. We prove unconditional local uniqueness at fixed $k_0$, $\theta_0$ near any obstacle.

Let $\mathcal{O}_{\text{ext}}$ be the connected unbounded component of $\mathbb{R}^n \setminus (\mathcal{O}_1 \cup \mathcal{O}_2)$. Set $\Omega_{\text{int}} = \mathbb{R}^n \setminus \overline{\mathcal{O}_{\text{ext}}}$. Then $\Omega_{\text{int}} \supset \mathcal{O}_1 \cup \mathcal{O}_2$. Note that $\Omega_{\text{int}}$ is an open set that contains the interior of $\mathcal{O}_1 \cup \mathcal{O}_2$ as well as all components of $\mathbb{R}^n \setminus (\mathcal{O}_1 \cup \mathcal{O}_2)$ disconnected from infinity.

Theorem 1 follows from the following.

**Proposition 1** Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two obstacles. Assume that for the corresponding scattering amplitudes we have

$$A_{\mathcal{O}_1}(k_0, \theta_0, \omega) = A_{\mathcal{O}_2}(k_0, \theta_0, \omega)$$

for a fixed $\theta_0 \in S^{n-1}$, fixed $k_0 > 0$ and all $\omega \in S^{n-1}$. If

$$\operatorname{Vol}(\Omega_{\text{int}} \setminus \mathcal{O}_i) < \omega_n k_0^{-n}, \quad i = 1, 2,$$

(1)

then $\mathcal{O}_1 = \mathcal{O}_2$.

Our argument is based on an estimate of the first eigenvalue of the Dirichlet Laplacian in a bounded domain.

**Lemma 1** Let $k^2$ be a Dirichlet eigenvalue of $-\Delta$ in the bounded domain $G$. Then

$$\omega_n \leq k^n \operatorname{Vol}(G).$$

**Proof.** We use the Poincaré inequality in the form presented in [GT]:

$$\|u\| \leq \left(\frac{\operatorname{Vol}(G)}{\omega_n}\right)^{1/n} \|\nabla u\|, \quad \text{for any } u \in H^1_0(G). \quad (2)$$

Let $u$ be a normalized eigenfunction corresponding to $k^2$. Then $\|\nabla u\| = k$ and $u \in H^1_0(G)$. Applying (2), we get

$$1 \leq k \left(\frac{\operatorname{Vol}(G)}{\omega_n}\right)^{1/n},$$

which implies the lemma. \qed
Proof of Proposition 1. The proof is a combination of Schiffer’s idea and Lemma 1.

Let \( u_j(x, \theta, k) \) be the scattering solution related to \( \mathcal{O}_j \), \( j = 1, 2 \). By a well-known argument based on Rellich’s lemma, \( A_{\mathcal{O}_1}(k_0, \theta_0, \omega) = A_{\mathcal{O}_2}(k_0, \theta_0, \omega) \) implies that \( u_1(x, \theta_0, k_0) = u_2(x, \theta_0, k_0) \) for all \( x \) outside a ball containing \( \mathcal{O}_1 \cup \mathcal{O}_2 \). We know that \( u_1 \) and \( u_2 \) solve

\[
\begin{align*}
(-\Delta - k_0^2)u_j &= 0, \quad \text{in } \mathbb{R}^n \setminus \mathcal{O}_j, \\
u_j|_{\partial \mathcal{O}_j} &= 0.
\end{align*}
\]

Then by analytic continuation, \( u_1 = u_2 \) on \( \partial \Omega_{\text{ext}} \).

Suppose that \( \mathcal{O}_1 \neq \mathcal{O}_2 \). Then for \( j = 1 \) or \( j = 2 \), \( \Omega_{\text{int}} \setminus \mathcal{O}_j \) is an open nonempty set. Suppose that this happens for \( j = 1 \). Let \( G \) be any connected component of \( \Omega_{\text{int}} \setminus \mathcal{O}_1 \). Then \( u_1 = 0 \) on \( \partial G \), and therefore \( u_1|_G \in H^1_0(G) \). Since \( \partial G \) may not be smooth, the latter needs some justification. This was done in [CK] by approximating \( u_1 \) by a sequence \( u_{1,n} \in C^\infty_0(G) \); see [CK, Theorem 5.1 and Lemma 3.8]. Therefore, \( u_1 \) solves the problem

\[
\begin{align*}
(-\Delta - k_0^2)u_1 &= 0, \quad \text{in } G, \\
u_1|_G &\in H^1_0(G).
\end{align*}
\]

Moreover, \( u_1 \) is not identically equal to zero in \( G \), because it is a real analytic function in the domain \( \mathbb{R}^n \setminus \mathcal{O}_1 \) not vanishing for large \( x \). Thus \( k_0^2 \) is a Dirichlet eigenvalue of the Laplacian in \( G \). By Lemma 1, \( \omega_1 k_0^{-n} \leq \text{Vol}(G) \leq \text{Vol}(\Omega_{\text{int}} \setminus \mathcal{O}_1) \). This contradicts our assumption (1), which proves the proposition. Note that in (1), we can actually replace \( \Omega_{\text{int}} \setminus \mathcal{O}_j \) by the biggest (in terms of volume) connected component of this set.

Proof of Theorem 1. We claim that the open sets \( \Omega_{\text{int}} \setminus \mathcal{O}_j \) are included in \( \mathcal{O}_+ \setminus \mathcal{O}_- \).

To prove that, note that \( \Omega_{\text{int}} \setminus \mathcal{O}_1 \), for example, is a union of the interior of \( \mathcal{O}_2 \setminus \mathcal{O}_1 \) and all bounded components of \( \mathbb{R}^n \setminus (\mathcal{O}_1 \cup \mathcal{O}_2) \). We only need to show that any such component is in \( \mathcal{O}_+ \setminus \mathcal{O}_- \). Assume that there is a point \( x_0 \) in such a component with \( x_0 \notin \mathcal{O}_+ \setminus \mathcal{O}_- \). Clearly, \( x_0 \notin \mathcal{O}_+ \). Then we can connect \( x_0 \) and infinity with a continuous curve lying outside \( \mathcal{O}_+ \), because \( \mathcal{O}_+ \) is an obstacle. This curve is in \( \mathbb{R}^n \setminus (\mathcal{O}_1 \cup \mathcal{O}_2) \), and this contradicts the assumption that \( x_0 \) is in a bounded component of this set. This proves the inclusion, and the theorem now follows from Proposition 1. \( \square \)

References


