Local Uniqueness for the Fixed Energy Fixed Angle Inverse Problem in Obstacle Scattering

Plamen Stefanov*
Department of Mathematics
Purdue University
West Lafayette, IN 47907

Gunther Uhlmann†
Department of Mathematics
University of Washington
Seattle, WA 98195

Abstract

We prove local uniqueness for the inverse problem in obstacle scattering at a fixed energy and fixed incident angle.

We consider the inverse problem of determining a sound-soft obstacle in $\mathbf{R}^n$, $n \geq 2$, from its scattering amplitude at a fixed incident direction $\theta \in S^{n-1}$ and a fixed energy $k > 0$. This is a formally determined inverse problem, since the data depends on the same number of variables, $n-1$, as does the object we want to recover.

The purpose of this note is to give a simple proof of local uniqueness for this problem. Roughly speaking, we show that if two domains are close to a given obstacle, in a precise sense described below, and have the same scattering amplitude at a fixed angle and fixed energy then they must be the same. Previously, it was shown in [CS] that local uniqueness holds for small obstacles. The Fréchet derivative of the nonlinear map from the domain to the scattering amplitude at fixed energy and angle was computed in [P], and one can easily show that it is injective. However, this does not imply a local result, since we cannot directly apply the implicit function theorem.

The proof of our result follows by using the arguments of Schiffer's well-known proof, presented in [LP], of uniqueness when given all incident directions and the Poincaré inequality.

By obstacles, we mean compact subsets of $\mathbf{R}^n$ with C^2 boundary and connected complement. The scattering amplitude $A_{\mathcal{O}}(k,\theta,\omega)$ related to an obstacle $\mathcal{O}$ is defined as follows. For $k > 0$, $\theta \in S^{n-1}$, we define the scattering solution $u(x,\theta,k)$ as the solution to the boundary value problem (see e.g., [CK])

$$\left\{ \begin{array}{rcl} (-\Delta - k^2)u & = & 0, \quad \text{in } \mathbf{R}^n \setminus \mathcal{O}, \\ u|_{\partial\mathcal{O}} & = & 0, \end{array} \right.$$

such that $u = e^{ik\theta\cdot x} + v$, with v satisfying the Sommerfeld outgoing condition at infinity: $(\partial/\partial r - ik)v = O(r^{-(n+1)/2})$, as $r = |x| \to \infty$. Then

$$v(x,\theta,k) = e^{ik\theta\cdot x} + \frac{e^{ikr}}{r^{(n-1)/2}} A_{\mathcal{O}}\Big(k,\theta,\frac{x}{r}\Big) + O\left(\frac{1}{r^{(n+1)/2}}\right), \quad \text{as } r = |x| \to \infty.$$

The function $A_{\mathcal{O}}(k,\theta,\omega)$ is the scattering amplitude related to $\mathcal{O}$.

It is known that Schiffer's proof implies uniqueness if $A_{\mathcal{O}}$ is known for all ω, fixed $k_0 > 0$, and N incident directions θ; or for all ω, fixed θ_0, and N frequencies $k \leq k_0$, where N is greater than the number of the

*Partly supported by NSF Grant DMS-0196440 and MSRI

†Partly supported by NSF Grant DMS-007048 and a John Simon Guggenheim fellowship. Both authors would like to thank the hospitality of the Mathematical Sciences Research Institute where part of this work was done.

Dirichlet eigenvalues $k^2 \le k_0^2$ of the Laplacian in a ball containing the obstacles. In particular, as mentioned above, this implies uniqueness at a fixed θ_0 and a fixed k_0 for all obstacles contained in a ball with sufficiently small radius R. In the 3D case, the condition is given by $k_0 R < \pi$. We refer to [CK], [I], [KK], [CS] for details and references.

In what follows, ω_n is the volume of the unit ball in $\mathbf{R}^n$ (not to be cofused with the outgoing direction ω); in particular, $\omega_3 = 4\pi/3$.

Our main result is the following.

Theorem 1 *Fix $k_0 > 0$, $\theta_0 \in S^{n-1}$. Let $\mathcal{O}_- \subset \mathcal{O}_+$ be two obstacles and assume that $\mathrm{Vol}(\mathcal{O}_+ \backslash \mathcal{O}_-) < \omega_n k_0^{-n}$. Let $\mathcal{O}_- \subset \mathcal{O}_j \subset \mathcal{O}_+$, $j = 1, 2$ be two other obstacles and assume that $A_{\mathcal{O}_1}(k_0, \theta_0, \omega) = A_{\mathcal{O}_2}(k_0, \theta_0, \omega)$. Then $\mathcal{O}_1 = \mathcal{O}_2$.*

In particular, for any fixed obstacle $\mathcal{O}$, and fixed $k_0 > 0$, θ_0, any small enough perturbation of the boundary gives an obstacle with different scattering amplitude.

More precisely, there exists $\varepsilon = \varepsilon(\mathcal{O}, k_0, \theta_0) > 0$ such that if $\partial\mathcal{O}_1$ is given in boundary normal coordinates $(x', x_n) \in \partial\mathcal{O} \times (-\delta, \delta)$ by $x_n = f(x')$ with $|f(x')| \le \varepsilon$, $\forall x'$, then $A_{\mathcal{O}_1}(k_0, \theta_0, \omega) = A_{\mathcal{O}}(k_0, \theta_0, \omega)$ implies $\mathcal{O}_1 = \mathcal{O}$. We would like to emphasize that this is different from the uniqueness for obstacles with small diameters mentioned above.

In Theorem 1 and Proposition 1 below, we do not impose smallness assumptions on k_0 or on the diameters of the obstacles. We prove unconditional local uniqueness at fixed k_0, θ_0 near any obstacle.

Let Ω_{ext} be the connected unbounded component of $\mathbf{R}^n \setminus (\mathcal{O}_1 \cup \mathcal{O}_2)$. Set $\Omega_{\text{int}} = \mathbf{R}^n \setminus \overline{\Omega}_{\text{ext}}$. Then $\overline{\Omega}_{\text{int}} \supset \mathcal{O}_1 \cup \mathcal{O}_2$. Note that Ω_{int} is an open set that contains the interior of $\mathcal{O}_1 \cup \mathcal{O}_2$ as well as all components of $\mathbf{R}^n \setminus (\mathcal{O}_1 \cup \mathcal{O}_2)$ disconnected from infinity.

Theorem 1 follows from the following.

Proposition 1 *Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two obstacles. Assume that for the corresponding scattering amplitudes we have*

$$A_{\mathcal{O}_1}(k_0, \theta_0, \omega) = A_{\mathcal{O}_2}(k_0, \theta_0, \omega)$$

for a fixed $\theta_0 \in S^{n-1}$, fixed $k_0 > 0$ and all $\omega \in S^{n-1}$. If

$$\mathrm{Vol}(\Omega_{\text{int}} \setminus \mathcal{O}_i) < \omega_n k_0^{-n}, \quad i = 1, 2, \tag{1}$$

then $\mathcal{O}_1 = \mathcal{O}_2$.

Our argument is based on an estimate of the first eigenvalue of the Dirichlet Laplacian in a bounded domain.

Lemma 1 *Let k^2 be a Dirichlet eigenvalue of $-\Delta$ in the bounded domain G. Then*

$$\omega_n \le k^n \mathrm{Vol}(G).$$

Proof. We use the Poincaré inequality in the form presented in [GT]:

$$\|u\| \le \left(\frac{\mathrm{Vol}(G)}{\omega_n}\right)^{1/n} \|\nabla u\|, \quad \text{for any } u \in H_0^1(G). \tag{2}$$

Let u be a normalized eigenfunction corresponding to k^2. Then $\|\nabla u\| = k$ and $u \in H_0^1(G)$. Applying (2), we get

$$1 \le k \left(\frac{\mathrm{Vol}(G)}{\omega_n}\right)^{1/n},$$

which implies the lemma. $\square$

Proof of Proposition 1. The proof is a combination of Schiffer's idea and Lemma 1.

Let $u_j(x,\theta,k)$ be the scattering solution related to $\mathcal{O}_j$, $j=1,2$. By a well-known argument based on Rellich's lemma, $A_{\mathcal{O}_1}(k_0,\theta_0,\omega)=A_{\mathcal{O}_2}(k_0,\theta_0,\omega)$ implies that $u_1(x,\theta_0,k_0)=u_2(x,\theta_0,k_0)$ for all x outside a ball containing $\mathcal{O}_1\cup\mathcal{O}_2$. We know that u_1 and u_2 solve

$$\left\{\begin{array}{rcl}(-\Delta-k_0^2)u_j &=& 0, \quad \text{in } \mathbf{R}^n\setminus\mathcal{O}_j,\\ u_j|_{\partial\mathcal{O}_j} &=& 0.\end{array}\right.$$

Then by analytic continuation, $u_1=u_2$ on $\partial\Omega_{\text{ext}}$.

Suppose that $\mathcal{O}_1\neq\mathcal{O}_2$. Then for $j=1$ or $j=2$, $\Omega_{\text{int}}\setminus\mathcal{O}_j$ is an open nonempty set. Suppose that this happens for $j=1$. Let G be any connected component of $\Omega_{\text{int}}\setminus\mathcal{O}_1$. Then $u_1=0$ on ∂G, and therefore $u_1|_G\in H_0^1(G)$. Since ∂G may not be smooth, the latter needs some justification. This was done in [CK] by approximating u_1 by a sequence $u_{1,n}\in C_0^\infty(G)$; see [CK, Theorem 5.1 and Lemma 3.8]. Therefore, u_1 solves the problem

$$\left\{\begin{array}{rcl}(-\Delta-k_0^2)u_1 &=& 0, \quad \text{in } G,\\ u_1|_G &\in& H_0^1(G).\end{array}\right.$$

Moreover, u_1 is not identically equal to zero in G, because it is a real analytic function in the domain $\mathbf{R}^n\setminus\mathcal{O}_1$ not vanishing for large x. Thus k_0^2 is a Dirichlet eigenvalue of the Laplacian in G. By Lemma 1, $\omega_n k_0^{-n}\le \text{Vol}(G)\le\text{Vol}(\Omega_{\text{int}}\setminus\mathcal{O}_1)$. This contradicts our assumption (1), which proves the proposition. Note that in (1), we can actually replace $\Omega_{\text{int}}\setminus\mathcal{O}_j$ by the biggest (in terms of volume) connected component of this set. $\square$

Proof of Theorem 1. We claim that the open sets $\Omega_{\text{int}}\setminus\mathcal{O}_j$ are included in $\mathcal{O}_+\setminus\mathcal{O}_-$.

To prove that, note that $\Omega_{\text{int}}\setminus\mathcal{O}_1$, for example, is a union of the interior of $\mathcal{O}_2\setminus\mathcal{O}_1$ and all bounded components of $\mathbf{R}^n\setminus(\mathcal{O}_1\cup\mathcal{O}_2)$. We only need to show that any such component is in $\mathcal{O}_+\setminus\mathcal{O}_-$. Assume that there is a point x_0 in such a component with $x_0\notin\mathcal{O}_+\setminus\mathcal{O}_-$. Clearly, $x_0\notin\mathcal{O}_+$. Then we can connect x_0 and infinity with a continuous curve lying outside $\mathcal{O}_+$, because $\mathcal{O}_+$ is an obstacle. This curve is in $\mathbf{R}^n\setminus(\mathcal{O}_1\cup\mathcal{O}_2)$, and this contradicts the assumption that x_0 is in a bounded component of this set. This proves the inclusion, and the theorem now follows from Proposition 1. $\square$

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