Estimates on the residue of the scattering amplitude

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1 Introduction

In this paper we study of the residue of the scattering amplitude for resonances near the real axis. We are motivated by the following interesting result by Lahmar-Benbernou and Martinez [Be], [BeM]. Consider the semiclassical scattering amplitude related to the Schrödinger equation in the case of a "well in an island". They studied a simple resonance $z_0(h)$ exponentially close to the real axis, corresponding to a simple eigenvalue of the harmonic oscillator that approximates the Hamiltonian near the bottom of the well. Under some additional assumptions they showed that $z_0(h)$ that is also a pole of the scattering amplitude $A(\omega,\theta,z,h)$, has residue satisfying the estimate

$$A^{\rm res}(\omega,\theta,h) = O(h^N)|\mathrm{Im}\, z_0(h)| \tag{1}$$

with some fixed N. Therefore, in the decomposition of A into a singular and holomorphic part around $z_0(h)$

$$A(\omega,\theta,z,h) = \frac{A^{\rm res}(\omega,\theta,h)}{z-z_0(h)} + A^{\rm hol}(\omega,\theta,z,h), \tag{2}$$

for z real and near $\mathrm{Re}\, z_0(h)$, the growth of the singular term $(z-z_0(h))^{-1}$ is exactly compensated by the decay of the residue up to a polynomial factor. The papers [Be] and [BeM], among the other results there, give a detailed analysis of this polynomial term $O(h^N)$ under additional assumptions, it turns out that for some directions (ω,θ) it is $O(h^\infty)$, while for some other directions it has full asymptotic expansion with a non-vanishing principal term.

The main purpose of this paper is to show that (1) holds in much more general situations, namely, for any black-box compactly supported perturbation of the Laplacian, provided that $z_0(h)$ is an isolated resonance close to the real axis. We give two proofs of this under two slightly different sets of assumptions. First, we assume that the resonances cannot approach the real axis more than exponentially fast, i.e., that

$$e^{-C/h} \le -\mathrm{Im}\, z(h) \tag{3}$$

for any resonance $z(h)$ with $E_1 \le \mathrm{Re}\, z(h) \le E_2$, where $0 < E_1 < E_2$ are fixed. In this paper by C we will denote different positive constants. This is connected to the estimate $\|\mathbf{1}_{|x|\le R} R(z,h)\mathbf{1}_{|x|\le R}\| = O(e^{C/h})$, $R>0$, $E_1 \le z \le E_2$. The bound (3) has been proved by Burq [B1], [B2] for very general systems and also by Vodev [V2], [V3] and Cardoso–Vodev [CV]. In Theorem 2 we prove (1) under the assumption (3).

In Theorem 1, we give a simpler proof of (1) assuming in addition that

$$\|\mathbf{1}_{R_1\le|x|\le R_2} R(z,h)\mathbf{1}_{R_1\le|x|\le R_2}\| = O(h^{-1}), \quad 0<E_1\le z\le E_2, \tag{4}$$

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where $R(z,h)$ is the outgoing meromorphic extension of the resolvent defined below, and $1 \ll R_1 < R_2$. Estimate (4) is proven in [B2], [CV] for a large number of long-range systems.

Most of the important scattering systems are covered by the assumptions in [B2], [CV] however, and for them both (3) and (4) hold. The proof of Theorem 2 under the assumption (3) only is a bit longer than that of Theorem 1 but it also provides some insight into the structure of the scattering amplitude and its connection with the singular part of the so-called scattering solution (known also as a distorted harmonic wave, or scattered wave) near a resonance with $-\mathrm{Im}\, z(h) = O(h^\infty)$. As a byproduct, we get in section 7 estimate on the residue of the scattering solution.

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2 Preliminaries

We work in the general framework of *black-box scattering* proposed by Sjöstrand and Zworski [SjZ] (see also [Sj], [TZ1]). We consider only compactly supported perturbations of the semiclassical Schrödinger operator $-h^2\Delta$. Let $\mathcal{H}$ be a complex Hilbert space of the form

$$\mathcal{H} = \mathcal{H}_{R_0} \oplus L^2(\mathbf{R}^n \setminus B(0,R_0)),$$

where $R_0 > 0$ is fixed and $B(0,R_0)$ is the ball centered at the origin with radius R_0. We consider a family of self-adjoint unbounded operators $P(h)$ in $\mathcal{H}$ with common domain $\mathcal{D}$, whose projection onto $L^2(\mathbf{R}^n \setminus B(0,R_0))$ is $H^2(\mathbf{R}^n \setminus B(0,R_0))$. In what follows we will denote by $\mathbf{1}_{B(0,R_0)}$ the orthogonal projector onto $\mathcal{H}_{R_0}$. We will also denote the same projector by $\mathbf{1}_{|x|\le R_0}$, and will use the notation $\mathcal{H}_R$ for the space $\mathcal{H}_{R_0} \oplus L^2(B(R,0) \setminus B(R_0,0))$, where $R > R_0$. We assume that

$$\mathbf{1}_{B(0,R_0)} \left(P(h)+i\right)^{-1} : \mathcal{H} \to \mathcal{H}$$

is compact. Outside $\mathcal{H}_{R_0}$, $P(h)$ is assumed to coincide with the semiclassical Schrödinger operator, i.e.,

$$\mathbf{1}_{\mathbf{R}^n\setminus B(0,R_0)} P(h) u = -h^2 \Delta \left(u|_{\mathbf{R}^n \setminus B(0,R_0)} \right).$$

Finally, we assume that $P(h) > -C_0$, $C_0 > 0$. Under those assumptions, one can define (the semi-classical) resonances $\mathcal{R}(P(h))$ of $P(h)$ in a conic neighborhood of the real axis by the method of complex scaling (see [SjZ], [Sj]). Resonances are also poles of the meromorphic continuation of the resolvent $(P(h)-z)^{-1} : \mathcal{H}_{\mathrm{comp}} \to \mathcal{H}_{\mathrm{loc}}$ from $\mathrm{Im}\, z > 0$ into a conic neighborhood of the real line. We will denote the so continued resolvent by $R(z,h)$. In classical scattering, we consider P as above independent of h by formally assuming that $h = 1$. Then P has classical resonances $\mathcal{R}(P)$ defined as the poles of the meromorphic continuation of the resolvent $(P-\lambda^2)^{-1} : \mathcal{H}_{\mathrm{comp}} \to \mathcal{H}_{\mathrm{loc}}$ from $\mathrm{Im}\,\lambda > 0$ to a neighborhood of the real line. For such P we then set $P(h) = h^2 P$ and define resonances $z(h)$ as above. Then the semi-classical resonances and the classical ones are related by $\lambda^2 = h^{-2} z$.

As in [SjZ], [Sj], we construct a reference selfadjoint operator $P^\#(h)$ from $P(h)$ on $\mathcal{H}^\# = \mathcal{H}_{R_0} \oplus L^2(M \setminus B(0,R_0))$, where $M = (\mathbf{R}/R\mathbf{Z})^n$ for some $R \gg R_0$. Then for the number of eigenvalues of $P^\#$ in a given interval $[-\lambda,\lambda]$, we assume

$$\#\{z \in \mathrm{Spec} P^\#(h);\ -\lambda \le z \le \lambda;\ \} \le C(\lambda/h^2)^{n^\#/2}, \quad \lambda \ge 1,$$

with some $n^\# \ge n$. This implies (see [SjZ] and [Sj]) that

$$\#\{z \in \mathcal{R}(P(h));\ 0 < a_0 \le \mathrm{Re}\, z \le b_0;\ 0 \le -\mathrm{Im}\, z \le c_0\} \le C(a_0,b_0,c_0) h^{-n^\#}, \tag{5}$$

$$\#\{\lambda \in \mathcal{R}(P);\ 1 \le |\lambda| \le r;\ 0 \le -\mathrm{Im}\,\lambda \le 1\} \le C r^{n^\#}, \quad r > 1. \tag{6}$$

Polynomial estimates of this type have been proved also in [M1], [Z1], [SjZ], [V1], [Sj].

We will denote by C various positive constants that may change from line to line.

3 The scattering amplitude

We introduce here the scattering amplitude in the black-box setting. Our definition is equivalent to that given in [PZ], [Z4], for example, but is somewhat closer to the classical one and we include it in order to keep the exposition self-contained.

Choose a smooth cut-off function χ_1 such that $\chi_1 = 0$ on $B(0, R_0 + 1)$, and $\chi_1 = 1$ outside $B(0, R + 2)$. For any $\theta \in S^{n-1}$, and any $z > 0$, we are looking for a solution $\psi(x, \theta, z, h)$ to the problem $(P(h) - z)\psi = 0$, $\psi \in \mathcal{D}_{\rm loc}(P(h))$ such that

$$\psi = \chi_1 e^{i\sqrt{z}\theta\cdot x/h} + \psi_{\rm sc}, \tag{7}$$

with $\psi_{\rm sc}$ satisfying the Sommerfeld outgoing condition at infinity: $(\partial/\partial r - i\sqrt{z}/h)\psi_{\rm sc} = O(r^{-(n+1)/2})$, as $r = |x| \to \infty$. Then

$$\psi(x, \theta, z, h) = e^{i\sqrt{z}\theta\cdot x/h} + \frac{e^{i\sqrt{z}r/h}}{r^{(n-1)/2}} A\Big(\frac{x}{r}, \theta, z, h\Big) + O\left(\frac{1}{r^{(n+1)/2}}\right), \quad \text{as } r = |x| \to \infty. \tag{8}$$

The function $A(\omega, \theta, z, h)$ is the scattering amplitude related to $P(h)$. In order to justify this definition, we will show that $\psi_{\rm sc}$ is well defined and the limit above exists.

Before proceeding, we will recall the definition for outgoing solution in the case that z is not necessarily real that we will need later. In short, "outgoing" function is a function equal for large x to $R_0(z, h)f$ for some compactly supported f. Here $R_0(z, h) : \mathcal{H}_{\rm comp} \to \mathcal{H}_{\rm loc}$ is the outgoing free resolvent, i.e., the analytic continuation of $R_0(z, h) = (-h^2\Delta - z)^{-1}$ from a neighborhood of the positive real axis in the upper half-plane into the lower half-plane in $\mathbf{C}$. The extension from the lower to the upper half plane is called incoming.

Definition 1 *Given $z \in \mathbf{C}$, we say that the function u is z-outgoing (or simply, outgoing, if z is understood from the context), if there exists $a > 0$ and $f \in \mathcal{H}_{\rm comp}$ such that $u|_{|x|>a} = R_0(z, h)f|_{|x|>a}$.*

Similarly one defines incoming functions.

Proposition 1 ([St1], see also Lemma 1 in [Z4])

(a) For any $f \in \mathcal{H}_{\rm comp}$ and any z not a resonance, the function $u = R(z, h)f$ is z-outgoing. Moreover, if χ is a smooth cut-off function such that $\chi = 1$ for $|x| > a$, and $\chi = 0$ in a neighborhood of $B(0, R_0)$ and supp f, *then we have $R(z, h)f|_{|x|>a} = -R_0(z, h)[h^2\Delta, \chi]R(z, h)f|_{|x|>a}$.*

(b) Let $u \in \mathcal{D}_{\rm loc}(P(h))$, $(P(h) - z)u = f \in \mathcal{H}_{comp}$, z is not a resonance, and u is z-outgoing. Then $u = R(z, h)f$.

Proof. First, consider (a). Let $\chi \in C^\infty$ be such that $\chi = 0$ near $B(0, R_0)$, and $\chi = 1$ for large $|x|$. Then $(-h^2\Delta - z)\chi u = -[h^2\Delta, \chi]u + \chi f$ is compactly supported. Since $u \in \mathcal{H}$ for $\operatorname{Im} z > 0$, we have $\chi u = R_0(z, h)(-h^2\Delta - z)\chi u$ there. Both sides of this equality are meromorphic in z in a neighborhood of the real axis, therefore by analytic continuation it also hold in the lower half-plane. In particular,

$$\chi R(z, h)f = R_0(z, h)\left(-[h^2\Delta, \chi]R(z, h)f + \chi f\right).$$

This proves that u is outgoing. If χ is as in the proposition, then $\chi f = 0$ and this proves the second statement.

Next, consider (b). Choose a smooth partition of unity $\chi_1 + \chi_2 = 1$, where $\chi_1 = 1$ for $|x| < a$ and χ_1 has compact support. Here a is such that $u|_{|x|>a} = R_0(z, h)f|_{|x|>a}$ with some $f \in \mathcal{H}_{\rm comp}$. Set

$$v(w) = \chi_1 u + \chi_2 R_0(w, h)f.$$

Observe first that $v(z) = u$. Next, $(P(h) - w)v(w) = g(w)$ with $g(w) = (P(h) - w)\chi_1 u - [h^2\Delta, \chi_2]R_0(w, h)f + \chi_2 f$ is compactly supported. Therefore, for $\operatorname{Im} w > 0$, $v(w)$ solves the problem $(P(h) - w)v(w) = g(w) \in \mathcal{H}$, also, $v(w) \in \mathcal{H}$, thus $v(w) = R(w, h)g(w)$ there. By meromorphic continuation through the real line, this is true for $\operatorname{Im} z < 0$ as well, and in particular for $w = z$, thus $u = v(z) = R(z, h)g(z) = R(z, h)f$. $\square$

The scattering solution $\psi_{\rm sc}$ can be constructed as follows. Apply $P(h)-z$ to $\psi_{\rm sc}$ to get

$$(P(h)-z)\psi_{\rm sc} = -(P(h)-z)\chi_1 e^{i\sqrt{z}\theta\cdot x/h} = [h^2\Delta,\chi_1]e^{i\sqrt{z}\theta\cdot x/h}. \tag{9}$$

Then, since $\psi_{\rm sc}$ is outgoing, by Proposition 1(b),

$$\psi_{\rm sc}(x,\theta,z,h) = R(z,h)[h^2\Delta,\chi_1]e^{i\sqrt{z}\theta\cdot x/h}. \tag{10}$$

Choose a smooth function χ_2 with $\chi_2=1$ for large x and $\chi_2=0$ on $\operatorname{supp}\chi_1$. Then, by Proposition 1(a),

$$\chi_2\psi_{\rm sc}(x,\theta,z,h) = -R_0(z,h)[h^2\Delta,\chi_2]R(z,h)[h^2\Delta,\chi_1]e^{i\sqrt{z}\theta\cdot x/h},$$

To take the asymptotic as $x=r\omega$, $r=|x|\to\infty$, we recall the asymptotic formula for $R_0(z,h)f$, where f has compact support, see [M2, section 1.7] (note that in [M2], we have $\lambda = h^{-1}\sqrt{z}$ and we have to take complex conjugate since the resonances there are in the upper half-plane)

$$[R_0(z,h)f](r\omega) = \frac{e^{i\sqrt{z}r/h}}{r^{\frac{n-1}{2}}}\left(v_\infty(\omega,z,h)+O\Big(\frac{1}{r}\Big)\right), \tag{11}$$

where

$$v_\infty = h^{-2}\frac{i}{2}(2\pi)^{-\frac{n+1}{2}}h^{-\frac{n-3}{2}}z^{\frac{n-3}{4}}e^{-i\pi\frac{n-1}{4}}\hat f(h^{-1}\sqrt{z}\omega). \tag{12}$$

The function v_∞ is called in the applied literature the far-field pattern of the outgoing solution v to $(-h^2\Delta - z)v=0$ for large x (which always can be expressed as $v=R_0(z,h)f$ for large x). In our case, v_∞ is just the scattering amplitude, if $v=\psi_{\rm sc}$. Thus we get

$$A(\omega,\theta,z,h) = \frac{1}{2}e^{-i\pi\frac{n-3}{4}}(2\pi h)^{-\frac{n+1}{2}}z^{\frac{n-3}{4}}\int e^{-i\sqrt{z}\omega\cdot x/h}[h^2\Delta,\chi_2]R(z,h)[h^2\Delta,\chi_1]e^{i\sqrt{z}\theta\cdot\bullet/h}dx. \tag{13}$$

It is clear from this formula, that the scattering amplitude A can be extended meromorphically everywhere, where the resolvent admits continuation as well. In particular, resonances are poles of A as well.

As in [Z4], [PZ], introduce the operators

$$[\mathbf{E}_\pm(z,h)f](\omega) = \int e^{\pm i\sqrt{z}\omega\cdot x/h}f(x)\,dx = \hat f(\mp h^{-1}\sqrt{z}\omega), \quad \omega\in S^{n-1},$$

and we will apply $\mathbf{E}_\pm(z,h)$ only to functions f with compact support. Let ${}^t\mathbf{E}_\pm(z,h)$ be the transpose operators defined as operator with Schwartz kernels ${}^tE(x,\omega)=E(\omega,x)$. Then viewing the scattering amplitude as an operator $\mathbf{A}(z,h)$ on $L^2(S^{n-1})$ with kernel $A(\omega,\theta,z,h)$, we recover the formula for A in [PZ] modulo normalizing factors:

$$\mathbf{A}(z,h) = \frac{1}{2}e^{-i\pi\frac{n-3}{4}}(2\pi h)^{-\frac{n+1}{2}}z^{\frac{n-3}{4}}\mathbf{E}_-(z,h)[h^2\Delta,\chi_2]R(z,h)[h^2\Delta,\chi_1]\,{}^t\mathbf{E}_+(z,h). \tag{14}$$

We will recall another formula in classical scattering. Let v be an outgoing solution of the Helmholtz equation $(-h^2\Delta - z)v=0$ outside $B(0,R_0+1)$. Of course, the situation we have in mind is the solution $v=\psi_{\rm sc}$ related to ψ as in (8). Using the Green's formula and the outgoing condition, we get for any $R>R_0+1$,

$$v(x,\theta,z,h) = h^2\int_{|x|=R}\left(R_0(x-y,z,h)\frac{\partial v(y,\theta,z,h)}{\partial r_y} - \frac{\partial R_0(x-y,z,h)}{\partial r_y}v(y,\theta,z,h)\right)dS_y, \quad |x|>R,$$

where $R_0(x-y,z,h)$ denoted the kernel of $R_0(z,h)$, and $r_y=|y|$. Take the asymptotic (11), (12) again (for z positive) as $x=r\omega\to\infty$ to get

$$\begin{aligned} v_\infty(w) &= \frac{1}{2}e^{-i\pi\frac{n-3}{4}}(2\pi h)^{-\frac{n+1}{2}}z^{\frac{n-3}{4}}h^2\int_{|y|=R}\left(e^{-i\sqrt{z}\omega\cdot y/h}\frac{\partial v(y)}{\partial r_y} - \frac{\partial e^{-i\sqrt{z}\omega\cdot y/h}}{\partial r_y}v(y)\right)dS_y \\ &= \frac{1}{2}e^{-i\pi\frac{n-3}{4}}(2\pi h)^{-\frac{n+1}{2}}z^{\frac{n-3}{4}}h\int_{|y|=R}e^{-i\sqrt{z}\omega\cdot y/h}\left(h\frac{\partial v(y)}{\partial r_y}+i\sqrt{z}\omega\cdot\frac{y}{|y|}v(y)\right)dS_y. \end{aligned} \tag{15}$$

Formula (15) relates the near field $v(x)$ to the far field $v_\infty(\omega)$ via the Cauchy data of v on any closed surface outside the black box. In our case, this is the sphere $|x| = R$. We can extend this map $v(x) \mapsto v_\infty(\omega)$ by analyticity for z not necessarily real and positive, even though then the limit as $|x| \to \infty$ is problematic, since $v(x)$ grows exponentially fast for $\operatorname{Im} z < 0$. With the aid of (12), we get the following representation for $\mathbf{E}_\pm[h^2\Delta, \chi]$ acting on outgoing solutions: for each outgoing v as above,

$$\mathbf{E}_\mp[h^2\Delta, \chi]v = h \int_{|y|=R} e^{\mp i\sqrt{z}\omega\cdot y/h} \left(h\frac{\partial v(y)}{\partial r_y} \pm i\sqrt{z}\omega \cdot \frac{y}{|y|} v(y) \right) dS_y. \tag{16}$$

In view of this, one can regard (14) as taking two consecutive limits of the Schwartz kernel $R(x, y, z, h)$ of the resolvent $R(z, h)$ as $x = r\omega$, $r \to \infty$ first, and $y = -r\theta$, $r \to \infty$, second (or in reverse order) and this gives A modulo multiplication factors. Each limit can be replaced by the integral (16), and this makes sense for complex z as well. This generalizes a well known fact in scattering theory to the black-box setting.

4 Estimate on the residue of the scattering amplitude

Let A^{res}, A^{hol} be as in (2). In this section we prove the following.

Theorem 1 *Fix $0 < E_1 < E_2$ and let $z_0(h)$ be a simple resonance with $0 < -\operatorname{Im} z_0(h) \le h^{\frac{3n^\#+5}{2}}$, $E_1 \le \operatorname{Re} z_0(h) \le E_2$ such that there is no other resonance in*

$$\Omega(h) = \left\{ z \in \mathbf{C};\ |\operatorname{Re} z - \operatorname{Re} z_0(h)| \le h^{-\frac{3n^\#+4}{2}} |\operatorname{Im} z_0(h)|,\ 0 \le -\operatorname{Im} z \le h^{-n^\#-2} |\operatorname{Im} z_0(h)| \right\}. \tag{17}$$

Assume (3) and (4). Then

$$|A^{\mathrm{res}}(\omega, \theta, h)| \le Ch^{-\frac{n-1}{2}} |\operatorname{Im} z_0(h)|, \quad |A^{\mathrm{hol}}(\omega, \theta, z, h)| \le Ch^{-\frac{n-1}{2}} \quad \text{in } \tilde\Omega(h),$$

where $\tilde\Omega = \left\{ z \in \mathbf{C};\ |\operatorname{Re} z - \operatorname{Re} z_0(h)| \le \frac{1}{2} h^{-\frac{3n^\#+4}{2}} |\operatorname{Im} z_0(h)|,\ 0 \le -\operatorname{Im} z \le 2|\operatorname{Im} z_0(h)| \right\}$.

Proof of Theorem 1. For R_1, R_2 such that $R_0 < R_1 < R_2$ denote

$$R_{R_1,R_2}(z, h) = \mathbf{1}_{R_1 \le |x| \le R_2} R(z, h) \mathbf{1}_{R_1 \le |x| \le R_2}.$$

By (4), for some R_1, R_2 we have $\|R_{R_1,R_2}(z, h)\| = O(h^{-1})$ for $z \in [E_1, E_2]$. Set

$$G(z, h) = \frac{z - z_0(h)}{z - \bar z_0(h)} R_{R_1,R_2}(z, h).$$

Then $G(z, h)$ is holomorphic in $\Omega(h)$ and satisfies

$$\|G(z, h)\| = O(h^{-1}) \quad \text{for } z \in [E_1, E_2] \tag{18}$$

as well. On the other hand, we claim that $G(z, h)$ satisfies the following a priori estimate:

$$\|G(z, h)\| \le Ce^{Ch^{-n^\#-1}} \quad \text{in } \Omega_{3/4}(h) := \tfrac{3}{4}(\Omega(h) - \operatorname{Re} z_0(h)) + \operatorname{Re} z_0(h). \tag{19}$$

This estimate is a direct consequence of the exponential a priori estimate of the resolvent due to M. Zworski [Z2], see [TZ1], [TZ2] for a proof in this generality:

$$\|R_\chi(z, h)\|_{\mathcal{H}\to\mathcal{H}} \le e^{C_\Omega h^{-n^\#} \log(1/g(h))} \quad \text{for } z \in \Omega(h),\ |z - z_j| \ge g(h),\ \forall z_j \in \mathcal{R}(P(h)),\ 0 < g(h) \ll 1, \tag{20}$$

where χ is any compactly supported function and $R_\chi(z,h) = \chi R(z,h)\chi$. This implies a similar estimate for $G(z,h)$ with $g(h) = e^{-Ch^{-1}}$, in $\Omega_{3/4}(h)$, $C \gg 1$, since this choice of $g(h)$ guarantees by (3) that all resonances are at least at a distance $g(h)$ from $\partial\Omega_{3/4}(h)$. Indeed, this is true for $z_0(h)$, which is inside $\Omega_{3/4}(h)$, and it is true for all other resonances that are outside $\Omega(h)$. Therefore, $G(z,h)$ satisfies (19) on the boundary of $\Omega_{3/4}(h)$, and since it is holomorphic inside, it also satisfies the same estimate inside $\Omega_{3/4}(h)$. This proves (19).

We need the semi-classical maximum principle in the form presented in [St2, Lemma 1]:

Lemma 1 *Fix $n^\# > 0$. Let $0 < h < 1$ and $a(h) \le b(h)$. Suppose that $F(z,h)$ is a holomorphic function of z defined in a neighborhood of*

$$\Omega(h) = [a(h) - 5w(h), b(h) + 5w(h)] + i[-S_-(h), S_+(h)h^{-n^\# - \varepsilon}],$$

where $0 \le S_-(h) \le S_+(h) \le w(h)h^{3n^\#/2+2\varepsilon}$, $S_+(h) > 0$ and $\varepsilon > 0$. If $F(z,h)$ satisfies

$$|F(z,h)| \le Ae^{Ah^{-n^\#}\log(1/h)} \quad \text{on } \Omega(h), \tag{21}$$

$$|F(z,h)| \le M(h) \quad \text{on } [a(h) - 5w(h), b(h) + 5w(h)] - iS_-(h) \tag{22}$$

with $M(h) \ge 1/C$, as $h \to 0$, then there exists $h_1 = h_1(S_-, S_+, A, \varepsilon) > 0$ such that

$$|F(z,h)| \le 2e^3 M(h), \quad \forall z \in \tilde\Omega := [a(h) - w(h), b(h) + w(h)] + i[-S_-(h), S_+(h)]$$

for $h \le h_1$.

Lemma 1 is the same as [St2, Lemma 1] with three exceptions. First, we allow here $S_-(h) = 0$, while in [St2] it is required that $S_-(h) > 0$. Second, we dropped the requirement that $\omega(h) = o(1)$. Finally, we replaced the assumption $M(h) \to 0$ by $M(h) \ge 1/C$. It can be seen from the proof that those requirements are not needed. By Lemma 1, we get from (18), (19),

$$\|G(z,h)\| \le Ch^{-1}, \quad \text{in } \tilde\Omega(h), \tag{23}$$

where $\tilde\Omega(h)$ is as in Theorem 1. Then, from

$$R_{R_1,R_2}(z,h) = \frac{z - \bar z_0(h)}{z - z_0(h)} G(z,h).$$

we get that

$$R_{R_1,R_2}(z,h) = \frac{R^{\text{res}}_{R_1,R_2}(h)}{z - z_0(h)} + R^{\text{hol}}_{R_1,R_2}(z,h) \tag{24}$$

with

$$\|R^{\text{res}}_{R_1,R_2}(h)\| \le Ch^{-1}|\operatorname{Im} z_0(h)|, \quad \|R^{\text{hol}}_{R_1,R_2}(z,h)\| \le Ch^{-1} \quad \text{in } \tilde\Omega(h). \tag{25}$$

After proving the first of the above estimates, in order to prove the second one, we apply the maximum principle to R^{hol} in $\tilde\Omega(h)$. By a standard argument, we can get the same estimates for the $L^2 \to H^2$ norm of the operators above in the annulus $R_1 \le |x| \le R_2$ and therefore, for the $L^2 \to H^1$ norm. Here, as usual, $\|f\|_{H^s} = \sum_{|\alpha|\le s} \|(hD)^\alpha f\|$. Then the estimates above and (13), (14) imply

$$|A^{\text{res}}(\omega,\theta,h)| \le Ch^{-\frac{n-1}{2}}|\operatorname{Im} z_0(h)|, \quad |A^{\text{hol}}(\omega,\theta,z,h)| \le Ch^{-\frac{n-1}{2}} \quad \text{in } \tilde\Omega(h).$$

In the derivation of those estimates we used the fact that the integration in (13) is performed over a compact region and in this case, $|\int g\,dx| \le C\|g\|$. This proves Theorem 1. □

5 An absorption estimate and estimates on the resonant states

The following is a slight improvement over Proposition 2.2 in [B1].

Proposition 2 *For any $\gamma > 0$ there exists $R_1 > 0$ such that for any $R > R_1$ the following estimate holds*

$$|\lambda| \int_{|x|=2R} \left(|u|^2 + |\lambda^{-1}\nabla u|^2\right) dS_x \le \pm C \int_{|x|=2R} \operatorname{Im}\left(\bar{u}\partial_r u\right) dS_x + Ce^{-\gamma|\lambda|} \int_{|x|=R} \left(|u|^2 + |\lambda^{-1}\nabla u|^2\right) dS_x$$

for any λ with $|\operatorname{Im}\lambda| \le 1$, $\operatorname{Re}\lambda \gg 1$ and for any outgoing/incoming solution u of the equation $(-\Delta - \lambda^2)u = 0$ outside $B(0, R_1)$. The positive sign above corresponds to an outgoing solution, the negative — to incoming solution.

The difference between the proposition above and the corresponding result in [B1] is that we show that one can choose the constant γ large enough if R is large enough. We will show below that this is in fact implicit in Burq's proof. Our proof below also indicates that we can replace above R by R_1 and $2R$ by $R_2 > R_1$, and then one can see that the constant γ is bounded from below by $C_1 R_1$, if $R_2/R_1 \ge C_2$.

In [B1], outgoing solutions correspond to incoming in our paper. For this reason, below we work with incoming solutions and it is clear that all estimates translate in a natural way for outgoing solutions as well. As in [B1], we develop an incoming solution u in spherical harmonics for $|x| > R_1$:

$$u = \sum_\nu u_\nu(r) Y_\nu(\omega), \quad x = r\omega,\ r > 0,\ \omega \in S^{n-1},$$

where ν runs over the eigenvalues of the Laplacian on S^{n-1}, and Y_ν is an orthonormal basis of eigenfunctions. The functions u_ν are expressed in terms of the Hankel functions of type 2, i.e.,

$$u_\nu(r) = \alpha_\nu r^{1-n/2} h_\mu(\lambda r), \quad \mu := \sqrt{\nu^2 + (n/2-1)^2},$$

where h_μ (usually denoted by $H_\mu^{(2)}$) admits the following asymptotic at infinity $h_\mu(r) \sim Cr^{-1/2}e^{-ir}$. The lack of term involving $H_\mu^{(1)}$ is due to the incoming condition imposed on u. A well known fact is that the asymptotics for $h_\mu(\lambda r)$ have different behavior in the hyperbolic region $\mu/(\operatorname{Re}\lambda R) < 1$, the glancing region $\mu/(\operatorname{Re}\lambda R) \sim 1$, and the elliptic region $\mu/(\operatorname{Re}\lambda R) > 1$ (see, e.g., [O], [StV1]).

The following lemma is similar to Lemma 2.5 in [B1].

Lemma 2 *There exists $\gamma > 0$, such that for any $R > 1$ there exist $C > 0$, $\lambda_0 > 0$, with the property that for $\sqrt{2} \le \mu/(\operatorname{Re}\lambda R)$, $\operatorname{Re}\lambda \ge \lambda_0$, $|\operatorname{Im}\lambda| \le 1$, we have*

$$\begin{aligned} |h_\mu(2\lambda R)| &\le Ce^{-\gamma\mu}|h_\mu(\lambda R)|, \\ |h'_\mu(2\lambda R)| &\le Ce^{-\gamma\mu}|h_\mu(\lambda R)|. \end{aligned}$$

Proof. Following [B1], we start with the following representation of the Hankel functions h_μ

$$h_\mu(r) = \int_{-\infty}^{+\infty - i\pi} e^{r\sinh t - \mu t} dt.$$

Using it, it is shown in [B1] that

$$|h_\mu(\lambda R)| \ge \frac{C}{\mu} e^{\mu(\tanh t_1 - t_1)} - C, \tag{26}$$

where $t_1 = -\operatorname{arccosh}(\mu/(\operatorname{Re}\lambda R))$ is the negative critical point of the function $e^{\lambda r \sinh t - \mu t}$. Under our assumption, $-t_1 \ge \operatorname{arccosh}\sqrt{2}$. Therefore, for $\mu \gg 1$, the term $-C$ above is absorbed by the exponential term. On the other hand,

$$|h_\mu(2\lambda R)| \le \begin{cases} Ce^{\mu(\tanh t_2 - t_2)}, & \text{if } 2\operatorname{Re}\lambda R \le \mu, \text{ where } t_2 = -\operatorname{arccosh}\left(\dfrac{\mu}{2\operatorname{Re}\lambda R}\right), \\ C, & \text{otherwise.} \end{cases} \tag{27}$$

Note that the first line above corresponds to the elliptic region, while the second one is in the hyperbolic region. The proof of the first estimate in the lemma is based on (26), (27). Before combining those two estimates, we will estimate the difference of the phase functions there. Set

$$g(s) = \tanh t - t, \quad t = -\mathrm{arccosh}\Big(\frac{\mu}{\mathrm{Re}\,\lambda R}\Big).$$

A straightforward calculation shows that

$$g(s) = \mathrm{arccosh}(s) - \frac{\sqrt{s^2-1}}{s}.$$

We are interested in the difference of g at $s_1 = \mu/(\mathrm{Re}\,\lambda R) \geq \sqrt{2}$ and $s_2 = \mu/(2\mathrm{Re}\,\lambda R) = s_1/2 \geq \sqrt{2}/2$. We claim that there exists a constant $c_0 > 0$ such that

$$\begin{array}{ll} c_0 \leq g(s) - g(s/2), & \text{for } 2 \leq s, \\ c_0 \leq g(s) - g(1), & \text{for } \sqrt{2} \leq s \leq 2, \end{array} \tag{28}$$

(observe that $g(1) = 0$). Since $g'(s) = s^{-2}\sqrt{s^2-1} = s^{-1}(1+o(s))$, as $s \to \infty$, the first inequality above is clearly true for large s, and by compactness argument it also holds for all $2 \leq s$. The second one also holds with $c_0 = g(\sqrt{2})$. From (28) we get

$$0 < c_0 < g(\mu/(\mathrm{Re}\,\lambda R)) - g(\max\{1, \mu/(2\mathrm{Re}\,\lambda R)\}).$$

With the aid of (26) and (27) we get

$$\begin{aligned} |h_\mu(2\lambda R)| &\leq Ce^{\mu g(\max\{1,\mu/(2\mathrm{Re}\,\lambda R\}))} \leq Ce^{-c_0\mu} e^{\mu g(\mu/(\mathrm{Re}\,\lambda R))} \\ &\leq C\mu e^{-c_0\mu}|h_\mu(\lambda R)| \leq Ce^{-c_0\mu/2}|h_\mu(\lambda R)|. \end{aligned}$$

The proof for h'_μ is similar. □

By Lemma 2, we get

$$|u_\nu(2R)| + |u'_\nu(2R)| \leq Ce^{-\gamma\nu}|u_\nu(R)| \tag{29}$$

for $\nu/(\mathrm{Re}\,\lambda R) \geq \sqrt{2}$ (which implies $\mu/(\mathrm{Re}\,\lambda R) \geq \sqrt{2}$) and $|\lambda| \gg 1$, where $\gamma > 0$ can be chosen independently of R, if R is large enough. Therefore,

$$|u_\nu(2R)| + |u'_\nu(2R)| \leq Ce^{-\gamma'|\lambda|R}|u_\nu(R)|, \quad \text{for } \nu/(\mathrm{Re}\,\lambda R) \geq \sqrt{2}. \tag{30}$$

Note that the condition above implies that the right-hand side is in the elliptic region, but at a positive distance from it, while the left-hand side might be in the hyperbolic one.

With the aid of (29) and (30) we complete the proof of Proposition 2 as in [B1]. To this end, write the integral $-\int_{|x|=2R} \mathrm{Im}\,(\bar{u}\partial_r u) dS_x$ in terms of the spherical harmonics decomposition, separate into a partial sum in the hyperbolic (on the sphere $|x| = 2R$) region $\nu/(\mathrm{Re}\,\lambda 2R) \leq \sqrt{2}/2$, and another sum over the indexes belonging to the complement $\sqrt{2} \leq \nu/(\mathrm{Re}\,\lambda R)$, which include the whole elliptic region, the glancing one, and a part of the hyperbolic one: $\sqrt{2}/2 \leq \nu/(\mathrm{Re}\,\lambda 2R) \leq 1+\varepsilon$. The first sum contributes to the term in the left-hand side of the estimate in Proposition 2, while the second one can be estimated using (29) and (30) and gives the exponentially small remainder. The latter argument is based on the observation that the contribution to the terms in the region $\sqrt{2}/2 \leq \nu/(\mathrm{Re}\,\lambda 2R) \leq 1+\varepsilon$ on the boundary of $B(0,2R)$ is coming from rays that do not reach $B(0,R)$, and therefore belong to the elliptic region of the ball $B(0,R)$. Similar analysis using studying the parametrix of the corresponding Neumann operator on a convex boundary surrounded the obstacle was carried out in [St3]. We refer to [B1] for more details. □

Using Proposition 2, we can estimate the resonant states outside $B(0,R)$, $R \gg 1$ in terms of the imaginary part of the resonance. The proposition below is an improved version of an argument in [St3], and the use of Burq's type of absorption estimates as those in Proposition 2 for proving the estimate below was suggested by M.Zworski [Z3].

Proposition 3 *Assume (3). Let $z(h)$ be a resonance with $E_1 \le \operatorname{Re} z(h) \le E_2$, $0 < -\operatorname{Im} z(h) \le Ch$, and $u(h)$ be an outgoing resonant state such that $(P(h) - z(h))u(h) = 0$. Then for $R \gg 1$ and $0 < h \ll 1$,*

$$\int_{|x|=2R} \left(|u|^2 + |h\nabla u|^2\right) dS_x \le -Ch^{-1} \operatorname{Im} z(h) \int_{|x|\le 2R} |u|^2 dx.$$

Proof. Note first that after standard scaling, Proposition 2 implies

$$\int_{|x|=2R} \left(|u|^2 + |h\nabla u|^2\right) dS_x \le C \int_{|x|=2R} \operatorname{Im}\left(\bar{u} h \partial_r u\right) dS_x + Ce^{-\gamma/h} \int_{|x|=R} \left(|u|^2 + |h\nabla u|^2\right) dS_x, \tag{31}$$

where $\gamma > 0$ can be chosen arbitrary large, if $R \gg 1$. The exponentially decaying term above can be estimated by $Ce^{-\gamma'/h} \int_{R/2\le|x|\le 2R} |u|^2 dx$ after applying the trace theorem, a local elliptic estimate for $-h^2\Delta$, and using the fact that u solves $(P - z)u = 0$.

Apply Green's formula for black boxes in $B(0, 2R)$, we obtain

$$-2 \operatorname{Im} z(h) \int_{|x|\le 2R} |u|^2 dx = 2h \int_{|x|=2R} \operatorname{Im}\left(\bar{u} h \partial_r u\right) dS_x, \tag{32}$$

therefore,

$$\int_{|x|=2R} \left(|u|^2 + |h\nabla u|^2\right) dS_x \le -Ch^{-1} \operatorname{Im} z(h) \|u\|^2_{L^2(B(0,2R))} + Ce^{-\gamma/h} \int_{R/2\le|x|\le 2R} |u|^2 dx. \tag{33}$$

By our assumption (3), $e^{-C_0/h} \le -\operatorname{Im} z(h)$ with $C_0 > 0$ depending on $P(h)$. Choose $R \gg 1$, so that $R/2 > R_0$ and $\gamma > C_0$. Then the exponential term above is absorbed by the one containing $\operatorname{Im} z(h)$ and this completes the proof. □

6 The singular part of the scattering amplitude

Assume that $z_0(h)$ is a simple pole of $R(z, h)$ in the domain $\Omega(h)$. Then

$$R(z, h) = \frac{R^{\text{res}}(h)}{z - z_0(h)} + R^{\text{hol}}(z, h),$$

where $R^{\text{hol}}(z, h)$ is holomorphic in $\Omega(h)$, and $R^{\text{res}}(h)$ is the residue. The simplicity of the pole $z_0(h)$ implies that $R^{\text{res}}(h)$ is a rank one operator. Therefore, $R^{\text{res}}(h)f = (f, u_-)_{\mathcal{H}} u_+$ with some $u_\pm(h) \in \mathcal{H}_{\text{loc}}$. It is easy to check that $(P(h) - z_0(h))R^{\text{res}}(h) = 0$, therefore $(P(h) - z_0(h))u_+(h) = 0$. It is also easy to see that $u_+(h)$ is outgoing. Indeed, by the Cauchy integral formula, for any $f \in \mathcal{H}_{\text{comp}}$,

$$R^{\text{res}}(h)f = \frac{1}{2\pi i} \oint_{|z - z_0(h)|\ll 1} R(z, h) f \, dz,$$

with positively oriented contour. According to Proposition 1, for $R \gg 1$ and z not a resonance, we have $\mathbf{1}_{\{R<|x|\}} R(z, h) f = \mathbf{1}_{\{R<|x|\}} R_0(z, h) g(z)$, where $g \in \mathcal{H}_{\text{comp}}$ is given by $g = -[h^2\Delta, \chi] R(z, h) f$ with suitable cut-off function χ. Clearly, there exists $f \in \mathcal{H}_{\text{comp}}$ such that $(f, u_-)_{\mathcal{H}} \ne 0$. We thus get that

$$\mathbf{1}_{\{R<|x|\}} u_+ = C \frac{1}{2\pi i} \oint_{|z - z_0(h)|\ll 1} \mathbf{1}_{\{R<|x|\}} R_0(z, h) g(z) \, dz = C \mathbf{1}_{\{R<|x|\}} R_0(z_0(h), h) \frac{1}{2\pi i} \oint_{|z - z_0(h)|\ll 1} g(z) \, dz,$$

therefore, u_+ is outgoing. On the other hand, by studying the incoming resolvent $R(\bar{z}, h)^*$, we derive that u_- is incoming. In what follows, consider for simplicity the case when $P(h)$ is real, i.e., when $\mathcal{H}_{R_0}$ is a function space and $\overline{P(h)u} = P(h)\bar{u}$. Then $R^{\rm res}(h)$ is preserved under the change $(u_-, u_+) \mapsto (\bar{\lambda}^{-1}u_-, \lambda u_+)$, $0 \neq \lambda \in \mathbf{C}$, and if we choose $|\lambda| = 1$, then we have $u_- = \bar{u}_+$.

Therefore, $u_+(h)$ is a resonant state, and we have

$$R(z,h) = \frac{u_+(h) \otimes u_+(h)}{z - z_0(h)} + R^{\rm hol}(z,h). \tag{34}$$

Using (13), (14), we get that the scattering amplitude has similar form near the pole z_0:

$$A(\omega,\theta,z,h) = \frac{1}{2} e^{-i\pi\frac{n-3}{4}} (2\pi h)^{-\frac{n+1}{2}} z_0(h)^{\frac{n-3}{4}} \frac{a(\omega,h)a(-\theta,h)}{z - z_0(h)} + A^{\rm hol}(\omega,\theta,z,h), \tag{35}$$

where $a(\omega,h)$ is the "far field pattern" of $u_+(h)$ given by the right-hand side of (16), i.e.,

$$a(\omega,h) = h\int_{|y|=R} e^{-i\sqrt{z_0(h)}\omega\cdot y/h}\left(h\frac{\partial u_+(y)}{\partial r_y} + i\sqrt{z_0(h)}\omega\cdot\frac{y}{|y|}u_+(y)\right)dS_y.$$

Let us apply Proposition 3 to the integral above with $R > R_0$ large enough. We get

$$|a(\omega,h)| \leq Ch^{1/2}\sqrt{-\mathrm{Im}\, z_0(h)}\; \|u_+(h)\|_R, \tag{36}$$

where $\|\cdot\|_R$ is the norm in $\mathcal{H}_R$. It remains to estimate $\|u_+\|_R$. Here we are going to use essentially the assumption that $z_0(h)$ is an isolated pole. By (34), $\|u_+(h)\|^2_{\mathcal{H}_R}$ is just the norm of the residue $R^{\rm res}(h)$ at $z = z_0(h)$ restricted to $\mathcal{H}_R$. We will use the semi-classical maximum principle again. Set

$$G(z,h) = \frac{z - z_0(h)}{z - \bar{z}_0(h) + 2\mathrm{Im}\, z_0(h)} R_\chi(z,h),$$

where χ is a compactly supported cut-off function equal to 1 near $B(0,R)$. Then $\|G(z,h)\| \leq C/|\mathrm{Im}\, z_0(h)|$ on the line $\mathrm{Im}\, z = -\mathrm{Im}\, z_0(h)$. Applying Lemma 1 to $G(z,h)$ in $\Omega(h)$, we get

$$\|G(z,h)\| \leq C/|\mathrm{Im}\, z_0(h)| \quad \text{in } \tilde{\Omega}(h),$$

and this implies

$$\|R^{\rm res}_\chi\| \leq C \quad \Longrightarrow \quad \|u_+\|_R \leq C.$$

By (36),

$$|a(\omega,h)| \leq Ch^{1/2}\sqrt{-\mathrm{Im}\, z_0(h)},$$

and by (35),

$$|A^{\rm res}| \leq Ch^{-\frac{n-1}{2}}|\mathrm{Im}\, z_0(h)|.$$

Similarly to the proof of Theorem 1, we estimate the holomorphic part of A as well. We have therefore proved the following.

Theorem 2 *Theorem 1 remains true, if we drop the assumption (4).*

7 Estimates on the scattering solution

In this section we prove estimates on the scattering solution $\psi(x,\theta,z,h)$ defined by (8) for real z in terms of the distance

$$\mathrm{dist}\{z, \mathcal{R}(P(h))\} = |z - z_0(h)|$$

from z to the closest resonance $z_0(h)$. The main result in this section is the following.

Theorem 3 *Let $0 < E_1 < E_2$ and assume that (3) holds for $E_1' \le \operatorname{Re} z \le E_2'$ with some $0 < E_1' < E_1$, $E_2 < E_2'$. Then for $h \ll 1$, $\varepsilon > 0$,*

$$|\psi(x,\theta,z,h)| \le \frac{Ch^{-N}}{\sqrt{d(z,h)}} \quad \text{for all } x \in B(0,R_0),\ \theta \in S^{n-1},\ E_1 \le z \le E_2$$

with $N = 3n^{\#}/4 - 1/4 + \varepsilon$, where $d(z,h) = \min(\operatorname{dist}\{z, \mathcal{R}(P(h))\}, 1)$.

Note that for some non-trapping systems, for example in the case of the classical wave equation with variable sound speed, one can construct an asymptotic expansion of ψ by using geometric optics, under the additional assumption of no caustics in a compact set. Then $\psi = O(1)$. In the same case of variable speed, M. Taylor [T] proved the optimal estimate $\psi = O(|\lambda|)$ in L^∞ (which translates to $\psi = O(h^{-1})$ in the semiclassical formalism) for non-trapping wave speeds that may generate caustics in a fixed compact. Theorem 3 also proves a polynomial estimate in the non-trapping case but this example shows that the polynomial factor is not sharp at least in the case studied in [T].

The proof of Theorem 3 is based on [St2, Proposition 2]. We will formulate here a semiclassical version of this proposition.

Proposition 4 *Let χ be the multiplication with a compactly supported function. Then under the assumptions of Theorem 3, $\forall \epsilon > 0$, for $h \ll 1$,*

$$\|\chi R(z,h)\chi\| \le \frac{h^{-\frac{3}{2}n^{\#}-\frac{3}{2}-\varepsilon}}{d(z,h)} \quad \text{for } E_1 \le z \le E_2.$$

Proof. For any $z_0 \in [E_1, E_2]$, the disk $B(z_0, d(z_0,h))$ is free of resonances. In particular, there are no resonances in

$$\Omega(h) = \Big[z_0 - \frac{1}{2}d, z_0 + \frac{1}{2}d\Big] + i\Big[-dh^{(n^{\#}+1)/2+\varepsilon}, 0\Big],$$

where $d = d(z_0,h)$. We will apply the semiclassical maximum principle to the cut-off resolvent in $\Omega(h)$. For this reason, note first that that the closest resonance to $\partial\Omega(h)$ is at distance at least $g(h) = \frac{1}{2}(1+o(h))d$ and by (3), $d \ge e^{-C/h}$. By (20), this choice of $g(h)$ implies the exponential estimate (21) for the cut-off resolvent with $n^{\#}$ replaced by $n^{\#}+1$. Set $S_-(h) = S_+(h) = dh^{3(n^{\#}+1)/2+3\varepsilon}$, $10\omega(h) = d$. All requirement of Lemma 1 are now satisfied with $M(h) = 1/S_-(h)$ and we therefore get the estimate in the proposition for $h \ll 1$. Observe that the constants in Lemma 1 depend on the a priori exponential estimate only (which in turn depend on C in (3)) and is independent of the choice of z_0. □

Proof of Theorem 3. We will use Proposition 4 above. Let $f \in \mathcal{H}_R$ with $R > R_0$ (i.e., $f = 0$ for $|x| \ge R$). Set $g = R(z,h)f$, where we assume that z is real. Applying Green's formula, we get similarly to (32),

$$h\int_{|x|=2R} \operatorname{Im}(\bar{g}h\partial_r g)\, dS_r \le \|g\|_{2R}\|f\|,$$

where $\|g\|_R$ is the norm of g restricted to $\mathcal{H}_R$. Apply Proposition 2 with R large enough to get

$$\int_{|x|=2R} \left(|g|^2 + |h\nabla g|^2\right) dS_x \le C\|g\|_{2R}\|f\|. \tag{37}$$

By Proposition 4,

$$\|g\|_{2R} = \|R(z,h)f\|_{2R} \le \frac{Ch^{-N}}{d(z,h)}\|f\|, \tag{38}$$

with $N = 3n^{\#}/2 + 3/2 + \varepsilon$, $\varepsilon > 0$. Therefore,

$$\int_{|x|=2R} \left(|g|^2 + |h\nabla g|^2\right) dS_x \le \frac{Ch^{-N}}{d(z,h)}\|f\|^2. \tag{39}$$

We have therefore proved that

$$\left\| \mathbf{1}_{R_2 \le |x| \le R_3} R(z,h) \mathbf{1}_{|x|\le R_1} \right\| \le \frac{Ch^{-N/2}}{\sqrt{d(z,h)}}, \tag{40}$$

where $R_0 \le R_1 \ll R_2 < R_3$. A similar estimate holds for ∇R. In order to complete the proof of the theorem, it is enough to apply (10) and (7) and we obtain the required estimate with N replaced by $N/2-1$. □

Remark. Estimate (40) can be considered as an intermediate estimate between Burq's bound (4) that in Proposition 4.

Finally, we present a quasi-example showing that in some cases, the estimate in Theorem 3 should be optimal up to the polynomial factor. We call it quasi-example, because it is about a non-compactly supported perturbation of the Laplacian, while in proving our results we consider compactly supported perturbations only.

Consider the following situation: Let $0 < E_1 < E_2$ be fixed and assume that $z_0(h)$ is a simple resonance with $E_1 \le \operatorname{Re} z_0(h) \le E_2$ and that $-\operatorname{Im} z_0(h) = O(h^\infty)$. Assume also that $z_0(h)$ is isolated in the following sense: $\operatorname{dist}\{z_0(h), \mathcal{R}(P(h))\} \ge h^M$ for some $M > 0$. Then $R(z,h)$ has the form (34) in $|z - z_0(h)| < h^{-M}$. Next, applying the semi-classical maximum principle to $(z - z_0(h))\chi R(z,h)\chi$ and using essentially the fact that $z_0(h)$ is isolated, we get as in [TZ2],

$$\|\chi R^{\text{res}}(h)\chi\| + \|\chi R^{\text{hol}}(z,h)\chi\| = O(h^{-N}), \quad z \in [\operatorname{Re} z_0(h) - h^M/2, \operatorname{Re} z_0(h) + h^M/2], \tag{41}$$

where χ is a cut-off function with compact support, and $N > 0$ depends on M. Let us use the representation (10) of ψ_{sc} combined with (34). We assume that z is real and belongs to the interval specified in (41). By the estimate (41) above, the holomorphic part $R^{\text{hol}}(z,h)$ contributes to polynomially bounded term $\psi_{\text{sc}}^{\text{hol}}$ when estimating ψ_{sc} in a compact set, so it is enough to study the contribution of the singular term $(z - z_0(h))^{-1} u_+(h) \otimes u_+(h)$ in (34) to ψ_{sc} in (10). We therefore get

$$\psi_{\text{sc}}^{\text{res}}(x,\theta,h) = u_+(x,h) \int u_+(y,h)[h^2\Delta, \chi_1] e^{i\sqrt{z}\theta\cdot y/h}\, dy = u_+(x,h) \left(\mathbf{E}_+(z_0(h),h)[h^2\Delta,\chi_1]u_+ \right)(\theta)$$

Note that by (15), (16), $\mathbf{E}_+[h^2\Delta,\chi_1]u_+$ above is, up to multiplication factors, the far field pattern of the resonant state u_+. Using the notation in section 6, we get that $\left(\mathbf{E}_+(z_0(h),h)[h^2\Delta,\chi_1]u_+\right)(\theta) = a(\theta,h)$, thus

$$\psi_{\text{sc}}^{\text{res}}(x,\theta,h) = u_+(x,h) a(\theta,h). \tag{42}$$

We need an estimate of $\|u_+\|_R$ from below. The simplest thing to do is to work with the complex scalled operator P_{θ_0} at this point, see e.g., [SjZ]. Here the angle θ_0 of complex scalling is choosen small enough and fixed. Then $z_0(h)$ is an eigenvalue of the non self-adjoint operator P_{θ_0} and the residue $R_{\theta_0}^{\text{res}}(z,h)$ of the resolvent $R_{\theta_0}(z,h) = (P_{\theta_0} - z)^{-1}$ at $z_0(h)$ is a (non-orthogonal) projector. Therefore, $\|R_{\theta_0}^{\text{res}}(z,h)\| \ge 1$. The representation (34) is valid for the scaled resolvent as well, with the same polynomial estimate on the holomorphic part and with $u_+(x,h)$ replaced by its scaled version $u_{\theta_0}(x,h)$. Therefore, $\|u_{\theta_0}(x,h)\| \ge 1$. By Theorem 3.1 in [St4] and the remark after it, $\|u(x,h)\|_R \ge 1 - O(h^\infty)$ for $R \gg 1$. Therefore, by (42),

$$\frac{1}{2}|a(\theta,h)| \le \|\psi^{\text{res}}(\bullet,\theta,h)\|_R. \tag{43}$$

Assume now that the estimate in Theorem 3 can be improved in this particular case, in the sense that

$$\psi(x,\theta,z,h) = \frac{O(h^\infty)}{\sqrt{\operatorname{dist}\{z, \mathcal{R}(P(h))\}}} \quad \text{for all } x \in B(0,R_0),\ \theta \in S^{n-1},\ z \in [\operatorname{Re} z_0 - h^M/2, \operatorname{Re} z_0 + h^M/2]. \tag{44}$$

Apply the semi-classical maximum proncipule to ψ again as in (23), (24), (25). Then we get

$$\|\psi^{\rm res}\|_R = \|\psi^{\rm res}_{\rm sc}\|_R = O(h^\infty)\sqrt{-\operatorname{Im} z_0(h)}$$

This, combined with (43) gives

$$a(\theta, h) = O(h^\infty)\sqrt{-\operatorname{Im} z_0(h)}, \quad \forall\theta.$$

By (35),

$$A^{\rm res}(\omega, \theta, h) = O(h^\infty)|\operatorname{Im} z_0(h)|, \quad \forall\omega, \theta. \tag{45}$$

Now, consider the situation studied by Lamahr-Benbernou and Martinez [BeM]. Let $P(h) = -h^2\Delta + V(x)$, where $V(x) = O(1+|x|)^{-\tau}$, $\tau > 0$ for $|x| \to \infty$, V is real analytic and extends holomorphically in a conic neighborhood of $\mathbf{R}^n$ in $\mathbf{C}^n$. Assume that $V(x)$ has a non-degenerate local minimum $\lambda_0 = V(x_0)$ at x_0. This is known as a well in the island $V(x) \geq \lambda_0$. Then under some additional assumptions one can show that for some potential like that, there is unique simple resonance $z_0(h)$ in $[\lambda_0 - \delta h, \lambda_0 + \delta h] - i[0, \delta]$ for some $\delta > 0$. It is known [HSj] that $-\operatorname{Im} z_0(h) = f(h)e^{-2S_0/h}$, where $f(h)$ is an elliptic symbol of finite order, and S_0 is the Agmon distance from x_0 to the boundary of the "island". Then for some directions (ω, θ), Lahmar-Benbernou and Martinez [BeM] showed that

$$A^{\rm res}(\omega, \theta, h) = O(h^N)e^{-2S_0/h} \tag{46}$$

with a finite N, while for some other directions it may happen that $N = \infty$. Estimate (46) is optimal under additional assumptions in the sense that for some ω, θ, the factor $O(h^N)$ above admits a full asymptotic expansion with non-zero first term. Thus (45) cannot hold for all ω, θ. As we mentioned above, this is not a real counterexample to (44), because we managed to prove (45) for compactly supported perturbations of the Laplacian, while the example in [BeM] is about non-compactly supported potential.

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