SHARP STABILITY ESTIMATE FOR THE GEODESIC RAY TRANSFORM

YERNAT M. ASSYLBEKOV AND PLAMEN STEFANOV

Abstract. We prove a sharp $L^2 \to H^{1/2}$ stability estimate for the geodesic X-ray transform of tensor fields of order 0, 1 and 2 on a simple Riemannian manifold.

1. Introduction

Let (M, g) be a smooth compact n-dimensional Riemannian manifold with boundary ∂M. We assume that (M, g) is *simple*, meaning that ∂M is strictly convex and that any two points on ∂M are joined by a unique minimizing geodesic. The weighted *geodesic ray transform* $I_{m,\kappa} f$ of a smooth covariant symmetric m-tensor field f on M is given by

$$I_{m,\kappa} f(\gamma) := \int \kappa(\gamma(t), \dot\gamma(t)) f_{i_1\dots i_m}(\gamma(t)) \dot\gamma^{i_1}(t) \cdots \dot\gamma^{i_m}(t)\, \mathrm{d}t \tag{1.1}$$

where κ is a smooth weight, γ runs over the set Γ of all unit speed geodesics connecting boundary points, and the integrand, written in local coordinates, is invariantly defined.

When $\kappa = 1$, we drop the κ subscript and simply write I_m. It is well known and can be checked easily that for every ϕ regular enough with $\phi = 0$ on ∂M, we have $\mathrm{d}\phi \in \operatorname{Ker} I_1$. Similarly, for every regular enough covector field v of order $m-1$ vanishing at ∂M, we have $\mathrm{d}^s v \in \operatorname{Ker} I_m$, where d^s is the symmetrized covariant differential. Those differentials are called potential fields. Many works have studied injectivity of those transforms up to potential fields and stability estimates.

In the present paper, the bundle of symmetric covariant m-tensors on M will be denoted by S^m_M. If F is a notation for a function space (H^s, C^∞, L^p, etc.), then we denote by $F(M; S^m_M)$ the corresponding space of sections of S^m_M. Note that $S^0_M = \mathbb{C}$ and in this case we simply write $F(M)$ instead of $F(M; S^0_M)$.

The goal of this paper is to prove sharp $L^2(M; S^m_M) \to H^{1/2}_\Gamma$ stability estimates for those transforms when $m = 0, 1, 2$. The Sobolev exponents 0 and 1/2 are natural in view of the properties of I_m as a Fourier Integral Operator. To prove a sharp estimate, we define a suitable realization of H^s_Γ. Before stating the main results, we will review the known estimates first.

If $g = e$ is Euclidean, a natural parameterization of the lines in $\mathbb{R}^n$ is by points on $\Sigma := \{(z, \theta) \in \mathbb{R}^n \times S^{n-1} | \ z \cdot \theta = 0\}$ by $\ell_{z,\theta} = \{x + t\theta, t \in \mathbb{R}\}$. One defines the Sobolev spaces $\bar H^s(\Sigma)$ by using derivatives w.r.t. z only. Then, with I^e_0 being the Euclidean X-ray transform on functions,

$$\|f\|_{H^s(\mathbb{R}^n)}/C \le \|I^e_0 f\|_{\bar H^{s+1/2}(\Sigma)} \le C\|f\|_{H^s(\mathbb{R}^n)}, \tag{1.2}$$

for every $f \in C^\infty_0(\Omega)$ with $\Omega \subset \mathbb{R}^n$ a smooth bounded domain, see [24, Theorem II.5.1] with $C = C(s, n, \Omega)$ (the constant on the right depends on n only). This implies the estimate for every $f \in H^s_{\bar\Omega}$, see the discussion of the Sobolev spaces in Section 3.

Date: June 2, 2018.

The work of the first author was partially supported by AMS-Simons Travel Grant.

Second author partly supported by NSF Grant DMS-1600327.

Estimate (1.2) was recently proved by Boman and Sharafutdinov [4] for symmetric tensor fields of every order m for $s = 0$ and f replaced by the solenoidal part f^s of f, which is the projection of f to the orthogonal complement of its kernel in L^2:

$$\|f^s\|_{L^2(\Omega;S^m_\Omega)}/C \le \|I^e_m f\|_{H^{1/2}(\Sigma)} \le C\|f^s\|_{L^2(\Omega;S^m_\Omega)}, \tag{1.3}$$

where I^e_m is the Euclidean ray transform of tensor fields of order m supported in $\bar\Omega$.

In the Riemannian case, injectivity of I_m up to potential fields (called s-injectivity) has been studied extensively, see, e.g., [13, 15, 20, 21, 22, 27, 31, 30, 32, 38, 33, 35, 40, 39, 45]. The first proofs of injectivity/s-injectivity of I_0 and I_1 for simple metrics in [23, 3, 1] provide a stability estimate with a loss of an $1/2$ derivative. The ray transform there is parameterized by endpoins of geodesics. Another estimate with a loss of an $1/2$ derivative follows from Sharafutdinov's estimate in [29] for I_m, see (1.5) below. Stability estimates in terms of the normal operator $N_m = I^*_m I_m$ are established in [33]:

$$\|f^s\|_{L^2(M;S^m_M)}/C \le \|N_m f\|_{H^1(M_1;S^m_{M_1})} \le C\|f^s\|_{L^2(M;S^m_M)}, \quad \forall f \in L^2(M;S^m_M), \quad m = 0,1, \tag{1.4}$$

where $M_1 \Supset M$ is some extension of M with g extended to M_1 as a simple metric. When $m = 0$, $f^s = f$ above. In [12], this estimate was extended to the weighted transform $I_{0,\kappa}$, with κ never vanishing, under the assumption that the latter is injective, and even to more general geodesic-like families of curves without conjugate points. An analogous estimate for the weighted version of I_1, assuming injectivity, is proved in [14]. Those estimates are based on the fact that N_m is a ΨDO of order -1 elliptic on solenoidal tensor fields (or just elliptic for $m = 0$) and injective. The need to have M_1 there comes from the fact that the standard ΨDO calculus is not suited for studying operators on domains with boundary. On the other hand, ΨDOs satisfying the transmission condition can be used for such problems. In [21], it is showed that N_0 does not satisfy the transmission condition but satisfies a certain modified version of it. Then one can replace M_1 by M in (1.4) for $m = 0$ at the expense of replacing H^1 by a certain Hörmander type of space. It is also shown in [21] that (1.4) does not hold with $M_1 = M$ because when M is the unit disk in $\mathbb{R}^2$, for the function $f_0 = (1 - |x|^2)^{-1/2}$ we have $I_{e,0} f_0 =$ const., but $f_0 \notin L^2$. A sharp stability estimate for $I_{0,\kappa} : H^{-1/2}(M) \to L^2_\mu(\partial_+ SM)$ on the orthogonal complement on the kernel is established in [15]; see next section for the Sobolev norms we use.

The case $m \ge 2$ is harder and the $m = 2$ one contains all the difficulties already. S-injectivity is known under an a priori upper bound of the curvature [31] and also for an open dense set of simple metrics, including real analytic ones [35] (and for a class of non-simple metrics, see [37]). It was shown in [25] that I_m is s-injectivity on arbitrary simple surfaces for all $m \ge 2$. Under the curvature condition, Sharafutdinov [31] proved the stability estimate

$$\|f\|_{L^2(M;S^m_M)} \le C\left(\|I_m f\|_{H^1(\partial_+ SM)} + m(m-1)\|I_m f\|_{L^2(\partial_+ SM)} \left\|j_\nu f|_{\partial M}\right\|_{L^2(\partial M;S^{m-1}_M)}\right), \tag{1.5}$$

$\forall f \in H^1(M;S^m_M)$, where $j_\nu f$ equals f "shortened" by the unit normal ν and the spaces above are introduced in the next section. This estimate is of conditional type since f appears on the r.h.s. as well, and not sharp since one would expect $I_m f$ to be in some form of an $H^{1/2}$ norm, as in (1.2). In terms of the normal operator, a non-sharp stability estimate for I_2 was established in [35]; and in [32], the second author proved the sharp stability estimate (1.4) for $m = 2$:

$$\|f^s\|_{L^2(M;S^2_M)}/C \le \|N_2 f\|_{H^1(M_1;S^2_{M_1})} \le C\|f^s\|_{L^2(M;S^2_M)}, \quad \forall f \in L^2(S^2_M). \tag{1.6}$$

The new ingredient in [32] was to use the Korn inequality estimating $\|v\|_{H^1(M;S^1_M)}$ in terms of $\|v\|_{L^2(M;S^1_M)} + \|\mathrm{d}^s v\|_{L^2(M;S^2_M)}$.

The main result of this paper is a sharp estimate of the kind (1.2), (1.3) (where g is Euclidean) for simple metrics and $m = 0, 1, 2$. Our starting point are the estimates (1.4) for $m = 0, 1$ and (1.6) for $m = 2$. First, we define $H_\Gamma^{1/2}$ appropriately in an intrinsic for M way, i.e., without extending (M, g). Clearly, it cannot be $H^{1/2}(\Gamma)$ with some of the classical definitions because the function f_0 defined above would not satisfy the estimate, see Section 6. We parameterize the maximal unit geodesics in M in some neighborhood of the boundary by a point z on each one maximizing the distance to ∂M and a unit direction θ at that point, see also Figure 1. One can view this as taking the the strictly convex foliation $\mathrm{dist}(\cdot, \partial M) = p$, $0 \le p \ll 1$ first and then taking geodesics tangent to each such hypersurface. For this reason, we call it the foliation parameterization. One can extend it smoothly to geodesic in $M_1 \Supset M$, with g extended as a simple metric there, in a natural way. Then we define $H_\Gamma^{1/2}$ as the subspace of $H_0^{1/2}(\Gamma_1)$ (where Γ_1 is as Γ but related to M_1), consisting of functions supported in Γ. We refer to section 3.1 for more details. The resulting space is independent of the extension (M_1, g). In section 6, we show that in the Euclidean case, this is equivalent to the parameterization of lines by Σ as in (1.2).

Our main results is the following.

Theorem 1.1. *Let (M, g) be a simple manifold.*

(a) If $I_{0,\kappa}$ is injective, then for all $f \in L^2(M)$,

$$\|f\|_{L^2(M)}/C \le \|I_{0,\kappa} f\|_{H_\Gamma^{1/2}} \le C\|f\|_{L^2(M)}.$$

(b) For all $f \in L^2(M; S^1_M)$,

$$\|f^s\|_{L^2(M;S^1_M)}/C \le \|I_1 f\|_{H_\Gamma^{1/2}} \le C\|f^s\|_{L^2(M;S^1_M)}.$$

(c) If I_2 is s-injective, then for all $f \in L^2(M; S^2_M)$,

$$\|f^s\|_{L^2(M;S^2_M)}/C \le \|I_2 f\|_{H_\Gamma^{1/2}} \le C\|f^s\|_{L^2(M;S^2_M)}.$$

Note that if κ is constant, or more generally related to an attenuation depending on the position only, then $I_{0,\kappa}$ is injective [28], and I_1 is injective, too [1]. Injectivity and stability of $I_{1,\kappa}$ has been studied in [14] and the estimate there implies an estimate of the type above which we do not formulate. Conditions for injectivity of I_2 can be found in [31, 29, 36, 35, 40].

Acknowledgments. The authors thank Gabriel Paternain and François Monard for the discussion about the results in [21] and for their helpful comments.

2. Preliminaries

Consider a simple manifold (M, g). Let $SM := \{(x, v) \in TM : |v|_{g(x)} = 1\}$ be its unit sphere bundle and $\partial_\pm SM$ be the set of inward/outward unit vectors on ∂M,

$$\partial_\pm SM := \{(x, v) \in SM : x \in \partial M \text{ and } \pm \langle v, \nu(x)\rangle_{g(x)} \ge 0\},$$

where ν is the inward unit normal to ∂M. By $d\Sigma^{2n-1}$ we denote the Liouville volume form on SM and by $d\Sigma^{2n-2}$ its induced volume form on $\partial_\pm SM$. Following [29], we work with the Sobolev spaces $H^s(SM)$, $H^s_0(SM)$, $H^{-s}(SM)$ and $H^s(\partial_\pm SM)$, $H^s_0(\partial_\pm SM)$, $H^{-s}(\partial_\pm SM)$, for $s \ge 0$, w.r.t. the measures $d\Sigma^{2n-1}$ and $d\Sigma^{2n-2}$, respectively, defined in a standard way.

2.1. **Weighted Sobolev spaces on $\partial_\pm SM$.** In a similar way, we define the weighted Sobolev spaces on $H^s_\mu(\partial_+SM)$, $s\in\mathbb{R}$. More precisely, for $k\geq 0$ integer, $H^k_\mu(\partial_+SM)$ is the H^k-Sobolev space on ∂_+SM w.r.t. the measure $d\mu(x,v) := \langle v,\nu(x)\rangle_{g(x)}\, d\Sigma^{2n-2}(x,v)$. For arbitrary $s\geq 0$, $H^s_\mu(\partial_+SM)$ is defined via interpolation, $H^s_{\mu,0}(\partial_+SM)$ is the completion of $C_0^\infty((\partial_+SM)^{\rm int})$ in $H^s_\mu(\partial_+SM)$, and $H^{-s}_\mu(SM) := (H^s_{\mu,0}(\partial_+SM))^*$.

We will need the following lemma which is an analog of [18, Theorem 3.40(i)].

Lemma 2.1. *For all $0\leq s\leq 1/2$, we have $H^s_{\mu,0}(\partial_+SM) = H^s_\mu(\partial_+SM)$.*

Proof. Suppose that $w\in(H^s_\mu(\partial_+SM))^*$ with $\langle w,\varphi\rangle = 0$ for all $\varphi\in C^\infty((\partial_+SM)^{\rm int})$. The inclusion $(H^s_\mu(\partial_+SM))^*\subset(H^s(\partial_+SM))^*$, which follows from $H^s(\partial_+SM)\subset H^s_\mu(\partial_+SM)$, says that w belongs to $(H^s(\partial_+SM))^*$. Also, the hypothesis $0\leq s\leq 1/2$ and [18, Theorem 3.40(i)] imply $(H^s(\partial_+SM))^* = H^{-s}(\partial_+SM)$. Therefore, w is in fact in $H^{-s}(\partial_+SM)$ and satisfies $\langle w,\varphi\rangle = 0$ for all $\varphi\in C^\infty((\partial_+SM)^{\rm int})$. This implies $w=0$ which means that $C_0^\infty((\partial_+SM)^{\rm int})$ is dense in $H^s_\mu(\partial_+SM)$. Hence, $H^s_{\mu,0}(\partial_+SM) = H^s_\mu(\partial_+SM)$ as desired. ☐

2.2. **Geodesics and scattering relation.** One way to parameterize the geodesics going from ∂M into M is by the set ∂_+SM. More precisely, for $(x,v)\in\partial_+SM)$, we write $\gamma_{x,v}(t)$, $0\leq t\leq\tau(x,v)$, for the unique geodesic with $x=\gamma_{x,v}(0)$ and $v=\dot\gamma_{x,v}(0)$. Here and in what follows, we set $\tau(x,v) := \max\{t : \gamma_{x,v}(s)\in M \text{ for all } 0\leq s\leq t\}$ for $(x,v)\in SM$, i.e. the first positive time when $\gamma_{x,v}$ exits M. The next result holds by [29, Lemma 4.1.1]:

Lemma 2.2. *If (M,g) is simple, then the following function is smooth on $\partial(SM)$*

$$\tau_-(x,v) = \begin{cases}\tau(x,v), & (x,v)\in\partial_+SM,\\ \tau(x,-v), & (x,v)\in\partial_-SM.\end{cases}$$

In particular, $\tau:\partial_+SM\to\mathbb{R}$ is smooth.

The *scattering relation* α_g maps the point and direction of entrance of a geodesic to the point and direction of exit of that geodesic. In other words,

$$\alpha_g : \partial_+SM\ni(x,v)\mapsto\big(\gamma_{x,v}(\tau(x,v)),\dot\gamma_{x,v}(\tau(x,v))\big)\in\partial_-SM.$$

According to Lemma 2.2, α_g is a diffeomorphism from ∂_+SM to ∂_-SM.

2.3. **The weighted geodesic ray transform and its adjoint.** We write S^m_M for the bundle of symmetric covariant m-tensors on M. Let κ be a smooth function on SM. Then the weighted geodesic ray transform $I_{m,\kappa}f$ of $f\in C^\infty(M;S^m_M)$ in (1.1) can be expressed as

$$I_{m,\kappa}f(x,v) = \int_0^{\tau(x,v)}\kappa(\gamma_{x,v}(t),\dot\gamma_{x,v}(t))\,f_{i_1\dots i_m}(\gamma_{x,v}(t))\,\dot\gamma^{i_1}_{x,v}(t)\cdots\dot\gamma^{i_m}_{x,v}(t)\,\mathrm{d}t, \qquad (x,v)\in\partial_+SM.$$

Using Santaló formula [8, Lemma A.8], one can see that $I_{m,\kappa}$ is bounded from $L^2(M;S^m_M)$ to $L^2_\mu(\partial_+SM)$. By similar arguments as in [29, Section 4.2], $I_{m,\kappa}$ can be extended to a bounded operator $H^s(M;S^m_M)\to H^s(\partial_+SM)$ for all $s\geq 0$. We show in Section 4 that in fact $I_{m,\kappa}$ is bounded from $H^s(M;S^m_M)$ to $H^{s+1/2}_\mu(\partial_+SM)$ for all $s\geq -1/2$.

Consider the adjoint operator $I^*_{m,\kappa} : L^2_\mu(\partial_+SM) \to L^2(M; S^m_M)$. Then again by Santaló's formula [8, Lemma A.8],

$$
\begin{aligned}
(I_{m,\kappa} f, w)_{L^2_\mu(\partial_+SM)} &= \int_{\partial_+SM} \int_0^{\tau(x,v)} \kappa(\gamma_{x,v}, \dot\gamma_{x,v}) f_{i_1\dots i_m}(\gamma_{x,v})\, \dot\gamma^{i_1}_{x,v} \cdots \dot\gamma^{i_m}_{x,v}\, \overline{w(x,v)}\, dt\, d\mu(x,v) \\
&= \int_{SM} \kappa(x,v) f_{i_1\dots i_m}(x) v^{i_1} \cdots v^{i_m}\, \overline{w_\psi(x,v)}\, d\Sigma^{2n-1},
\end{aligned}
$$

where w_ψ is the function on SM that is constant along geodesics and $w_\psi|_{\partial_+SM} = w$. Hence, we have

$$
I^*_{m,\kappa} w(x) = \int_{S_xM} v^{i_1} \cdots v^{i_m}\, \overline{\kappa(x,v)}\, w_\psi(x,v)\, d\sigma_x(v),
$$

where $d\sigma_x(v)$ is the measure on S_xM such that $d\sigma_x(v) d\operatorname{Vol}_g(x) = d\Sigma^{2n-1}(x,v)$

3. Folation parametrization and the corresponding Sobolev spaces

3.1. **Parameterizations of the geodesic manifold.** There are three main parameterizations of the set Γ of the maximal directed unit speed geodesics on a simple manifold (M,g). We include geodesics generating to a point corresponding to initial directions tangent to ∂M to make Γ compact; we call that set $\partial\Gamma$. We recall those three parameterizations below, and we include our foliation one for completeness. Note that the first three are global and their correctness is guaranteed by the simplicity assumption.

∂_+SM parameterization: by initial points and incoming directions. Each $\gamma \in \Gamma$ is parameterized by an initial point $x \in \partial M$ and initial unit direction v at x, i.e., by $(x,v) \in \partial_+SM$. We write $\gamma = \gamma_{x,v}(t)$, $0 \le t \le \tau(x,v)$, where the latter is the lenght of the maximal geodesic issued from (x,v).

$B(\partial M)$ parametrization: by initial points and tangential projections of incoming directions. Each $\gamma \in \Gamma$ is parameterized by an initial point $x \in \partial M$ and the orthogonal tangential projection v' of its initial unit direction v at x, i.e., by $(x,v') \in B(\partial M)$, where B stands for the unit ball bundle. We write somewhat incorrectly $\gamma = \gamma_{x,v'}$.

$\partial M \times \partial M$ parametrization: by initial and end points. Each $\gamma \in \Gamma$ is parameterized by its endpoints x and y on ∂M. If we think of γ as a directed geodesic, then the direction is from x to y. We use the notation $\gamma = \gamma_{[x,y]}$.

foliation parametrization: Near $\partial\Gamma$, let z be the point where the maximum of $\operatorname{dist}(\gamma, \partial M)$ is attained, and let $\theta \in SM$ be the direction at z. We use the notation $\gamma = \gamma(\cdot, z, \theta)$. Away from $\partial\Gamma$, we can use any of the other parameterizations. We give more details below.

Identifying Γ with the corresponding set of parameters, each one of them being a manifold, introduces a natural manifold structure on it. While those differential structures are different (near $\partial\Gamma$), the first two ones are homeomorphic. In the ∂_+SM and in the $B(\partial M)$ parameterizations, Γ is a compact manifold with boundary $\partial\Gamma$. The boundary in the first one can be removed by allowing geodesics to propagate backwards. In the $\partial M \times \partial M$ one, Γ is a compact manifold without a boundary; then $\partial\Gamma$ is an incorrect notation and it represents the diagonal. If we project the unit sphere bundle to the unit ball one in the standard way $v \mapsto v'$, the resulting map is not a diffeomorphism up to the boundary, i.e., at v tangent to ∂M. The foliation parameterization makes Γ a manifold with a boundary $\partial\Gamma$ as well but it allows a natural smooth extension of Γ to a smooth manifold of geodesic Γ_1 on an extended $M_1 \Supset M$, as we show below. In section 6, we compare those four parameterizations for the unit circle in $\mathbb{R}^2$.

We describe the foliation parameterization in more detail now. Fix a point $q \in \partial M$ and assume that ∂M is strictly convex at q w.r.t. g. Let ∂M_1 be as above. We work in boundary normal coordinates near q in which $q = 0$ and x^n is the signed distance to ∂M, non-negative in M. We can always assume that ∂M_1 is given locally by $x^n = -\delta$ with some $0 < \delta \ll 1$. Let Γ_1 be a small neighborhood of the geodesics tangent to ∂M at q extended until they hit ∂M_1. Note that this includes geodesic segments which may lie outside of M. We will choose a parameterization of Γ_1 in the following way. First, since any geodesic $\gamma \in \Gamma_1$ hits ∂M_1 transversally at both ends when $\delta \ll 1$, we can parameterize γ by its initial point $y' \in \partial M_1$ and incoming unit directions w or their projections w' on $T_{y'}\partial M_1$. Denote this geodesic by $\gamma_{y',w}$. The foliation parameterization of γ is by (z, θ), where $z = (z', z^n)$ is the point maximizing the signed distance form γ to ∂M (regardless of whether γ is entirely outside M or hits ∂M), and by the direction θ at z which must be tangent to the hypersurface $x^n = z^n$. In Figure 1 on the left, we illustrate this on an almost Euclidean looking example (which is more intuitive) and in the right, we do this in boundary normal coordinates. We call the corresponding geodesic $\gamma(\cdot, z, \theta)$.

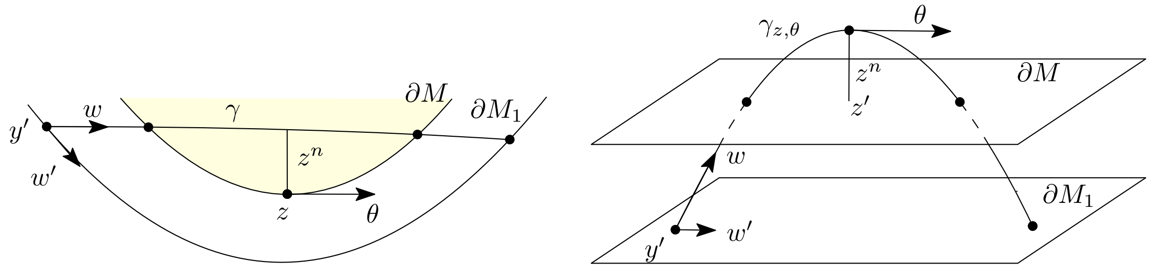

FIGURE 1. The foliation parameterization

Another way to describe the foliation parameterization, which explains it name, is to think of the hyperplanes $\Sigma_p := \{x^n = p\}$, $|p| \ll 1$, as a strictly convex foliation near q. Then $\gamma_{z,\theta}$ is the geodesic through $z \in \Sigma_{z^n}$ tangent to it with unit direction $\theta \in S_z\Sigma_{z^n}$. This defines a natural measure on the set of (z, θ) which we may identify with Γ_1 given by $\mathrm{d}\,\mathrm{Vol}_z\ \mathrm{d}\mu_\theta$, where $\mathrm{d}\mu_\theta$ is the natural measure on $S_z\Sigma_p$. Then (z, θ) belongs locally to the foliation $T\Sigma_p$, $|p| \ll 1$ with $p = z^n$, $(z', \theta) \in T\Sigma_p$.

Let us compare the ∂_+SM parameterization by $(y', w) \in \partial_+SM_1$ to the $B(\partial M)$ one by (y', w'). They are related by a diffeomorphism which becomes singular when w is tangent to ∂M_1. Such almost tangent geodesics however do not hit M; therefore when parameterizing If with $\operatorname{supp} f \subseteq M$, those two parameterizations are diffeomorphic to each other.

Proposition 3.1. *Assume that ∂M is strictly convex at q. Then the map $(y', w) \mapsto (z, \theta)$ is a local diffeomorphism.*

Proof. Let $\tau(y', w)$ be the travel time of the unit speed geodesic issued from $(y', w) \in \partial_+SM_1$ to z, i.e., τ maximizes $\gamma^n_{y',w}(\tau)$ locally. Then τ is a critical point, i.e., $\dot\gamma^n_{y',w}(\tau) = 0$. Let $\gamma_{y'_0,w_0}$ be a geodesic tangent to ∂M at $q = \gamma_{y'_0,w_0}(\tau_0)$ with some τ_0. To solve $\dot\gamma^n_{y',w}(\tau) = 0$ for (y'_0, w_0) near (y', w), we apply the Implicit Function Theorem. Since $\ddot\gamma^n_{y'_0,w_0}(\tau_0) = -\Gamma^n_{ij}(q)\dot\gamma^i_{y'_0,w_0}(\tau_0)\dot\gamma^j_{y'_0,w_0}(\tau_0)$ and the latter equals twice the second fundamental form at q, we get a unique smooth $\tau(y', w)$ with $\tau(y'_0, w_0) = \tau_0$.

Since $z = \gamma_{y',w}(\tau(y', w))$ and $\theta = \gamma'_{y',w}(\tau(y', w))$ (the prime stands for the projection onto the first $n - 1$ coordinates in boundary normal coordinates), we get that $(y', w) \mapsto (z, \theta)$ is smooth.

To verify that the inverse map $(z,\theta)\mapsto(y',w)$ is smooth, it is enough to show that the travel time $t(z,\theta)$ at which $\gamma_{z,\theta}(t)$ reaches $\partial M_1=\{z^n=-\delta\}$ is a smooth function as well. This follows easily from the fact that geodesics tangent to ∂M hit ∂M_1 transversely when $\delta\ll 1$. □

3.2. **Sobolev spaces.** We recall that for $s\ge 0$, there are several "natural" ways to define a Sobolev space when $\Omega\subset\mathbb{R}^n$ is a domain with a smooth boundary: $H^s(\Omega)$ is the restriction of distributions in $H^s(\mathbb{R}^n)$ to Ω; next, $H^s_0(\Omega)$ is the completion of $C_0^\infty(\Omega)$ in $H^s(\Omega)$; and $H^s_{\bar\Omega}$ is the space of all $u\in H^s(\mathbb{R}^n)$ supported in $\bar\Omega$, also equal to the completion of $C_0^\infty(\Omega)$ in $H^s(\mathbb{R}^n)$, also the space of all f which, extended as zero outside $\bar\Omega$, belong to $H^s(\mathbb{R}^n)$. We have $H^s_{\bar\Omega}=H^s_0(\Omega)$ for s not a half-integer, and $H^s_{\bar\Omega}\subset H^s_0(\Omega)$ in general; and $H^s_0(\Omega)=H^s(\Omega)$ for $0\le s\le 1/2$. Those definitions extend naturally to manifolds with boundary. We refer to [18] for more details.

In the (z,θ) coordinates, Γ is given by $z^n\ge 0$. For u supported in Γ, we define the Sobolev space H^s_Γ as $H^s_{\bar\Omega}$ above. In particular, when s is a non-negative integer, identifying θ locally with some parameterization in $\mathbb{R}^{n-1}$, we have locally

$$\|u\|^2_{H^s_\Gamma}=\int_{(\mathbb{R}^{n-1})^2}\sum_{|\alpha|\le s}|\partial^\alpha_{z,\theta}u|^2dz\,d\theta.$$

This norm is not invariantly defined but under changes of variables, it transforms into equivalent norms. Note that u above is considered as a function defined on Γ_1 but supported in Γ, as in the definition of $H^s_{\bar\Omega}$ above.

We make this definition global now. Without changing the notation, let Γ_1 be the manifold of all geodesics with endpoints on ∂M_1, and let Γ be those intersecting M. We can choose an open cover of Γ consisting of neighborhoods of geodesics tangent to M as above, plus an open set Γ_0 of geodesics passing through interior points only, and having a positive lower bound of the angle they make with ∂M. In the latter, we take the classical H^s norm w.r.t. the parameterization (y',w), for example. In the former neighborhood, we use the norms H^s_Γ defined above. Then using a partition of unity, we extend the norm H^s_Γ to functions defined in Γ_1 and supported in Γ. This defines a Hilbert space which we call H^s_Γ again.

On the other hand, we have the space $H^s_\Gamma(\partial_+SM_1)$ of distributions on Γ defined through the parameterization of Γ given by $(y',w)\in\partial_+SM_1$ in a similar way: we define the H^s norm for functions supported in the interior of Γ_1 first (the behavior near the boundary of Γ_1 corresponding to w tangent to ∂M_1 does not matter in what follows), and then define $H^s_\Gamma(\partial_+SM_1)$ as the subspace of those $u\in H^s(\partial_+SM_1)$ which are supported in Γ.

Proposition 3.1 then implies the following.

Proposition 3.2. *The Hilbert spaces $H^s_\Gamma(\partial_+SM_1)$ and H^s_Γ are topologically equivalent.*

4. Mapping properties of $I_{m,\kappa}$ and $I^*_{m,\kappa}$

The main purpose of this section is to study the mapping properties of $I_{m,\kappa}$, $I^*_{m,\kappa}$ and the *normal operator* $N_{m,\kappa}:=I^*_{m,\kappa}I_{m,\kappa}$.We start with the latter one.

Proposition 4.1. *Suppose (M,g) is a simple manifold, $\kappa\in C^\infty(SM)$, and $m\ge 0$. Then $N_{m,\kappa}:H^s(M;S^m_M)\to H^{s+1}(M;S^m_M)$ is bounded for all $s\le 0$.*

Proof. Our arguments mainly follow [15]. As before, we embed (M,g) into the interior of a simple manifold (M_1,g) (the metric on M_1 is an extension of the metic on M). We also extend κ smoothly to SM_1 and keep the same notation for the extension. We denote by $I^{M_1}_{m,\kappa}$ the geodesic ray transform on M_1. Take an open set $M_{1/2}$ such that $M\Subset M_{1/2}\Subset M_1^{\text{int}}$. Choose an arbitrary $\chi\in C_0^\infty(M_{1/2})$

such that $\chi \equiv 1$ on M. Since $N_{m,\kappa}^{M_1} := (I_{m,\kappa}^{M_1})^* I_{m,\kappa}^{M_1}$ is a pseudo-differential operator on M_1^{int} of order -1 (see [12, 32]), using [17, Theorem 18.1.13] we have boundedness of

$$m_\chi \circ N_{m,\kappa}^{M_1} \circ m_\chi : H^s(M_1; S_{M_1}^m) \to H^{s+1}(M_1; S_{M_1}^m),$$

where m_χ is the multiplication by χ operator. Suppose now that $f \in H^s(S_M^m)$. Then one can see that

$$N_{m,\kappa} f = r_M \circ m_\chi \circ N_{m,\kappa}^{M_1} \circ m_\chi \circ \mathscr{E}_{M_1} f,$$

where r_M is the restriction operator from M_1 to M and $\mathscr{E}_{M_1}$ is the operator which extends all tensors on M to $M_1 \setminus M$ by zero. Note that $r_M : H^s(M_1; S_{M_1}^m) \to H^s(M; S_M^m)$ is bounded for all s and $\mathscr{E}_{M_1} : H^s(M; S_M^m) \to H^s(M_1; S_{M_1}^m)$ is bounded for all $s \le 0$. Combining all of these,

$$\begin{aligned}\|N_{m,\kappa} f\|_{H^{s+1}(M;S_M^m)} &= \|r_M \circ m_\chi \circ N_{m,\kappa}^{M_1} \circ m_\chi \circ \mathscr{E}_{M_1} f\|_{H^{s+1}(M;S_M^m)} \\ &\le C\|m_\chi \circ N_{m,\kappa}^{M_1} \circ m_\chi \circ \mathscr{E}_{M_1} f\|_{H^{s+1}(M_1;S_{M_1}^m)} \le C\|\mathscr{E}_{M_1} f\|_{H^s(M_1;S_{M_1}^m)} \le C\|f\|_{H^s(M;S_M^m)}.\end{aligned}$$

The proof is thus complete. ☐

Using this result, we now prove boundedness of $I_{m,\kappa}$ between appropriate spaces. Note that we only need $s = 0$ and M in $L^2(M)$ replaced by $M_0 \Subset M$ for our main result.

Proposition 4.2. *Suppose (M, g) is a simple manifold, $\kappa \in C^\infty(SM)$, and $m \ge 0$. Then $I_{m,\kappa} : H^s(M; S_M^m) \to H_\mu^{s+1/2}(\partial_+ SM)$ is bounded for all $s \ge -1/2$.*

Proof. We start with the case $s = -1/2$. Take arbitrary $f \in C^\infty(M)$. Then by Proposition 4.1 (with $s = -1/2$),

$$\|I_{m,\kappa} f\|_{L_\mu^2(\partial_+ SM)}^2 = (N_{m,\kappa} f, f)_{L^2(M;S_M^m)} \le \|N_{m,\kappa} f\|_{H^{1/2}(M;S_M^m)} \|f\|_{H^{-1/2}(M;S_M^m)} \le C\|f\|_{H^{-1/2}(M;S_M^m)}^2.$$

We used here the fact that $H_0^{1/2}(M; S_M^m) = H^{1/2}(M; S_M^m)$, which follows from [18, Theorem 3.40(i)]. By a density argument, this implies that $I_{m,\kappa} : H^{-1/2}(M; S_M^m) \to L_\mu^2(\partial_+ SM)$ is bounded.

Next, we assume that $s \ge 3/2$ is a half-integer. Let $U \subset \partial_+ SM$ be a domain with a local coordinate system $(y^1, \dots, y^{2n-2})$. Then it suffices to prove the estimate

$$\|\varphi I_{m,\kappa} f\|_{H_\mu^{s+1/2}(U)} \le C\|f\|_{H^s(M;S_M^m)}$$

for $f \in C^\infty(M; S_M^m)$ and $\varphi \in C^\infty(\partial_+ SM)$ with $\mathrm{supp}(\varphi) \subset U$. Following [26], we write $\ell_m f(x, v) := f_{i_1 \dots i_m} v^{i_1} \cdots v^{i_m}$. Then it is not difficult to see that $\ell_m : H^s(M; S_M^m) \to H^s(SM)$ is bounded. By a direct calculation, similar to those performed in [29] to prove (4.2.13), we obtain

$$\begin{aligned}\partial_y^\alpha(\varphi(y) I_{m,\kappa} f(y)) = &\sum_{|\beta|+|\gamma|=|\alpha|} \partial_y^\gamma \varphi(y) \sum_{|\sigma| \le |\beta|} I_{m+|\sigma|,\kappa_\sigma} P_\sigma f(y) \\ &+ \sum_{\substack{|\beta|+|\gamma|+|\delta|=|\alpha| \\ |\delta|<|\alpha|}} C_{\beta\gamma\delta}^\alpha \partial_y^\beta \varphi(y) \partial_y^\gamma \tau(y) \partial_y^\delta \big((\ell_m f \circ \alpha_g)(y)\big),\end{aligned}$$

where κ_σ are smooth weights, P_σ are differential operators of order $|\sigma|$ on M, and $C_{\beta\gamma\delta}^\alpha$ are constants. According to Lemma 2.2, the functions $\partial_y^\gamma \tau$ are locally bounded. Using the fact that the scattering relation $\alpha_g : \partial_+ SM \to \partial_- SM$ is a diffeomorphism and the boundedness of $I_{m,\kappa} : H^{-1/2}(M; S_M^m) \to$

$L^2_\mu(\partial_+SM)$, we show

$$\begin{aligned}\|\varphi I_{m,\kappa}f\|_{H^{s+1/2}_\mu(U)} &\le C\sum_{|\sigma|\le s+1/2}\|I_{m+|\sigma|,\kappa_\sigma}P_\sigma f\|_{L^2_\mu(U)} + C\|\ell_m f\|_{H^{s-1/2}(\partial_-SM)}\\ &\le C\sum_{|\sigma|\le s+1/2}\|P_\sigma f\|_{H^{-1/2}(M;S^{m+|\sigma|}_M)} + C\|\ell_m f\|_{H^s(SM)} \le C\|f\|_{H^s(M;S^m_M)}.\end{aligned}$$

In the last step we also used the boundedness of ℓ_m. By compactness of ∂_+SM, this shows that $I_{m,\kappa}: H^s(M;S^m_M)\to H^{s+1/2}_\mu(\partial_+SM)$ is bounded for all $s\ge 3/2$ half-integers. The case of general $s\ge -1/2$ follows by interpolation. □

For a compact subset K of $(\partial_+SM)^{\mathrm{int}}$, set $H^s_K(\partial_+SM) := \{w\in H^s(\partial_+SM) : \mathrm{supp}(w)\subseteq K\}$.

Proposition 4.3. *Suppose that (M,g) is simple, $\kappa\in C^\infty(SM)$, and $m\ge 0$.*

(a) If $-1/2\le s\le 0$, then $I^*_{m,\kappa}: H^s_\mu(\partial_+SM)\to H^{s+1/2}(M;S^m_M)$ is bounded.

(b) If K is a compact subset of $(\partial_+SM)^{\mathrm{int}}$, then $I^*_{m,\kappa}: H^s_K(\partial_+SM)\to H^{s+1/2}(M;S^m_M)$ is bounded for all $s\ge -1/2$.

Proof. (a) We start with the case $s=-1/2$. According to Proposition 4.2, $I_{m,\kappa}: L^2(M;S^m_M)\to H^{1/2}_\mu(\partial_+SM)$ is bounded. This, together with Lemma 2.1, gives boundedness of $I^*_{m,\kappa}: H^{-1/2}_\mu(\partial_+SM)\to L^2(M;S^m_M)$. The case $s=0$ follows by taking the dual of $I_{m,\kappa}: H^{-1/2}(M;S^m_M)\to L^2_\mu(\partial_+SM)$ from Proposition 4.2. The case of general $-1/2\le s\le 0$ follows by interpolation.

(b) Suppose $w\in C^\infty(\partial_+SM)$ with $\mathrm{supp}(w)\subseteq K$ and $s\ge 1/2$ is a half-integer. Let $V\subset M$ be a domain with a local coordinate system $(x^1,\dots,x^n)$. Then it suffices to prove the estimate

$$\|\varphi I^*_{m,\kappa}w\|_{H^{s+1/2}(V;S^m_M)}\le C\|w\|_{H^s(\partial_+SM)}$$

for $\varphi\in C^\infty(M)$ with $\mathrm{supp}(\varphi)\subset V$. For $x\in M$, let $\Theta_x: S_xM\to S^{n-1}$ be a diffeomorphism smoothly depending on x. Since

$$I^*_{m,\kappa}w(x)=\int_{S_xM} v^{i_1}\cdots v^{i_m}\,\overline{\kappa(x,v)}w_\psi(x,v)\,d\sigma_x(v),$$

by a direction calculation,

$$\begin{aligned}&\partial^\alpha_x(\varphi(x)I^*_{m,\kappa}w(x)) = \partial^\alpha_x\Big(\varphi(x)\int_{S_xM} v^{i_1}\cdots v^{i_m}\,\overline{\kappa(x,v)}w_\psi(x,v)\,d\sigma_x(v)\Big)\\ &=\partial^\alpha_x\Big(\varphi(x)\int_{S^{n-1}}\overline{\kappa(x,\Theta_x^{-1}(\theta))}\,w_\psi(x,\Theta_x^{-1}(\theta))|J_{\Theta_x^{-1}}(\theta)|\,d\theta\Big)\\ &=\sum_{|\beta|+|\gamma|+|\sigma|+|\delta|=|\alpha|}\partial^\beta_x\varphi(x)\int_{S^{n-1}}\partial^\gamma_x\overline{\kappa(x,\Theta_x^{-1}(\theta))}\,\partial^\sigma_x w_\psi(x,\Theta_x^{-1}(\theta))\partial^\delta_x\big(v^{i_1}\cdots v^{i_m}\,|J_{\Theta_x^{-1}}(\theta)|\big)\,d\theta\\ &=\sum_{|\beta|+|\gamma|+|\sigma|+|\delta|=|\alpha|}\partial^\beta_x\varphi(x)\int_{S_xM}\partial^\gamma_x\overline{\kappa(x,v)}\,\partial^\sigma_x w_\psi(x,v)\partial^\delta_x\big(v^{i_1}\cdots v^{i_m}|J_{\Theta_x^{-1}}(\Theta_x(v))|\big)|J_{\Theta_x}(x,v)|d\sigma_x(v).\end{aligned}$$

Using chain rule, it is not difficult to see that $\partial^\sigma_x w_\psi=\sum_{|\rho|\le|\sigma|}c_\rho\,(P_\rho w)_\psi$, where P_ρ are differential operators of order $|\rho|$ on ∂_+SM and c_ρ are functions on SM. The latter ones involve sums and multiplications of derivatives of components of $\gamma_{x,v}(-\tau(x,-v))$ and $\dot\gamma_{x,v}(-\tau(x,-v))$. Since we assume that $\operatorname{supp}(w) \subseteq K$, we involve only smooth derivatives of τ. Hence, each c_ρ is smooth on SM. Using this expression of $\partial_x^\sigma w_\psi$ and part (a), we can write

$$\begin{aligned}\|\varphi I^*_{m,\kappa} w\|_{H^{s+1/2}(V;S^m_M)} &\le C \sum_{|\sigma|\le s+1/2} \|I^*_{0,\kappa_\sigma} P_\sigma w\|_{L^2(V)} \\ &\le C \sum_{|\sigma|\le s+1/2} \|P_\sigma w\|_{H_\mu^{-1/2}(\partial_+SM)} \le C\|w\|_{H^s_\mu(\partial_+SM)},\end{aligned}$$

where $\kappa_\sigma \in C^\infty(SM)$. This together with compactness of M imply boundedness of $I^*_{m,\kappa}$: $H^s_\mu(\partial_+SM) \to H^{s+1/2}(M;S^m_M)$ for all $s \ge 3/2$ half-integers. For general $s \ge -1/2$, apply interpolation. □

5. Proof of the main theorem

Proof. The starting point are the stability estimates (1.4) for $m = 0, 1$ and (1.6) for $m = 2$, the latter due to the second author [32, Theorem 1], valid for all symmetric 2-tensor field $f \in L^2(M; S^2_M)$. First we will replace $\|N^{M_1} f\|_{H^1(M_1)}$ in the first inequality in (1.4), respectively (1.6), by $\|If\|_{H^{1/2}_\Gamma}$ with the corresponding ray transform I. We will take $m = 2$ below and the proof is the same for $m = 0, 1$.

By Proposition 4.2, applied for the extension of $f \in L^2(M; S^2_M)$ by zero to $M_1 \setminus M$, we have $I_2^{M_1} f \in H^{1/2}_\mu(\partial_+SM)$. We also have that $\operatorname{supp}(I_2^{M_1} f)$ is contained in the compact subset Γ of $(\partial_+SM_1)^{\text{int}}$ which depends only on M (of course also on a fixed extended manifold M_1). Using Proposition 4.3 to the middle term of (1.6), we obtain

$$\|f^s\|_{L^2(M;S^2_M)} \le C\|I_2^{M_1} f\|_{H^{1/2}(\partial_+SM_1)}, \quad f \in L^2(M; S^2_M). \tag{5.1}$$

Note that the norm on the right can be replaced by the $H^{1/2}$ norm in any of the discussed parameterizations (with respect to any measure) since $I_2^{M_1} f$ is supported in the compact set Γ of the geodesics having common points with M (and in particular away from geodesic tangent to ∂M_1). Then Proposition 3.2 implies that we can replace the $H^{1/2}(\partial_+SM_1)$ norm by the $H^{1/2}_\Gamma$ one in (5.1). This completes the proof of the first inequality in the theorem.

To prove the second one, note that by Proposition 4.2, $I_2^{M_1} : L^2(M; S^2_M) \to H^{1/2}(\partial_+SM_1)$ is continuous. As above, the latter norm is topologically equivalent to the $H^{1/2}_\Gamma$ one on functions supported in Γ. □

6. Example: the unit disk in $\mathbb{R}^2$

Let M be the unit disk $(x^1)^2 + (x^2)^2 \le 1$ in $\mathbb{R}^2$. If (x^1, x^2) are the Euclidean coordinates of z, then for $z^n = z^2$, we have $z^2 = 1 - |x|$ and z' can be identified with the polar angle of x. Given z, the direction θ is a vector through it normal to it, and therefore, it is uniquely determined by z up to its orientation. Recall that in the general case, θ belongs to S^{n-2} locally. Therefore, the (z', z^n, θ) coordinates we introduced earlier are essentially the polar coordinates (r, ω) of z with r transformed by a simple linear transformation:

$$z^2 = 1 - r, \quad z' = \boldsymbol{\omega} := (\cos\omega, \sin\omega).$$

Then $l_{z,\theta} = \{x|\ \boldsymbol{\omega}\cdot x = r\}$ which is the typical Radon transform parameterization of lines in the plane by points in $Z := \{(\theta, z)|\ z \perp \theta\}$ which can be identified with TS^{n-1}, with the natural measure there. The Sobolev spaces H^s we defined then coincide with the standardly defined Sobolev spaces $H^s(Z)$. [include a remark here why the usual definition includes z-derivatives only]

The $\partial_+ SM$ parameterization of the same lines is given by initial points $x \in S^1$ and initial directions $v \in S^1$. If we parameterize them by their polar angles ω_x and ω_v, respectively, the lines in this parameterization are given by $l_{x,v} = \{\boldsymbol{\omega_x} + s\boldsymbol{\omega_v}\}$, see Figure 2. An elementary calculation

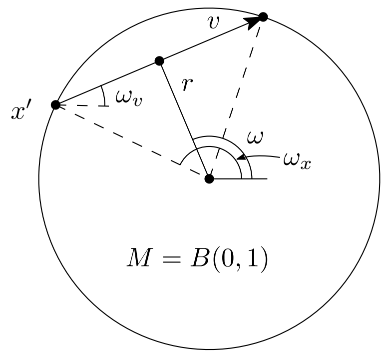

FIGURE 2. Relation between the $\partial_+ SM$ parameterization and the foliation one

shows (with a specific choice of the orientation along the lines) that the two coordinate systems (r,ω) and (ω_x,ω_v) are related by

$$\omega_x = \omega + \arccos r, \quad \omega_v = \omega - \pi/2$$

for $r \le 1$; respectively,

$$r = \sin(\omega_x - \omega_v), \quad \omega = \omega_v + \pi/2.$$

Set $\lambda := \langle v, \nu(x)\rangle_{g(x)} = (1-r^2)^{\frac{1}{2}}$. This is a boundary defining function of $\partial\Gamma$ in the classical parameterizations; and in the foliation one, λ^2 is such a function. The corresponding Jacobians are given by

$$\frac{\partial(\omega_x,\omega_v)}{\partial(r,\omega)} = \begin{pmatrix} -1/\lambda & 1 \\ 0 & 1 \end{pmatrix}, \qquad \frac{\partial(r,\omega)}{\partial(\omega_x,\omega_v)} = \begin{pmatrix} -\lambda & \lambda \\ 0 & 1 \end{pmatrix}.$$

where we used the fact that $\cos(\omega_x - \omega_v) = -\lambda$. We see that the first one has one entry which blows up at $r=1$ at a rate $1/\lambda$, and that entry also equals the determinant; while the second one is smooth but it has a determinant λ vanishing of order one at $\lambda = 0$. The latter corresponds to v tangent to the boundary S^1. In this parameterization, the boundary measure is $\lambda\, d\omega_x\, d\omega_v$ while in the foliation one, it is $dr\, d\omega$. The factor λ is the determinant of the second Jacobian above but the derivatives after the change of the variables will be affected by the singularity. More precisely,

$$\partial_{\omega_x,\omega_v} = \begin{pmatrix} -1/\lambda & 1 \\ 0 & 1 \end{pmatrix} \partial_{r,\omega}, \quad \partial_{r,\omega} = \begin{pmatrix} -\lambda & \lambda \\ 0 & 1 \end{pmatrix} \partial_{\omega_x,\omega_v}.$$

Therefore, none of the norms $\|\cdot\|_{\bar H^s}$ and $\|\cdot\|_{H^s_\lambda(\partial_+SM)}$ is stronger (or weaker) than the other for $s \ge 0$ integer unless $s = 0$ when $\|\cdot\|_{L^2_\lambda(\partial_+SM)} \le C\|\cdot\|_{\bar L^2}$. Formally, one gets the same inequality for $s \le 1/2$ because then the factor λ in the measure seems to compensate for the singularity but we will not pursue this further.

In the $B(\partial M)$ parameterization, we replace v by its tangential projection v'. In our case, an elementary calculations shows that $v' = r$. The $B(\partial M)$ variables are therefore (ω_x, r). The corresponding Jacobians are given by

$$\frac{\partial(\omega_x,r)}{\partial(r,\omega)} = \begin{pmatrix} -1/\lambda & 1 \\ 1 & 0 \end{pmatrix}, \qquad \frac{\partial(r,\omega)}{\partial(\omega_x,r)} = \begin{pmatrix} 0 & 1 \\ 1 & 1/\lambda \end{pmatrix}.$$

Both Jacobians have determinants equal to -1 but they have singular entries at $r = 1$. The boundary measure in this case is $d\omega_x\,dr$. Similarly, we get that none of the norms $\|\cdot\|_{\tilde{H}^s}$ and $\|\cdot\|_{H^s(B(\partial M))}$ is stronger (or weaker) than the other one unless $s = 0$ when they are equivalent.

In the $\partial M \times \partial M$ parameterization, if ω_y is the polar angle of x (similarly to ω_y), then $\omega_y = \omega - \arccos r$ and the Jacobians are given by

$$\frac{\partial(\omega_x,\omega_y)}{\partial(r,\omega)} = \begin{pmatrix} -1/\lambda & 1 \\ 1/\lambda & 1 \end{pmatrix}, \qquad \frac{\partial(r,\omega)}{\partial(\omega_x,\omega_y)} = \frac{1}{2}\begin{pmatrix} -\lambda & \lambda \\ 1 & 1 \end{pmatrix}.$$

where we used the fact that $\frac{1}{2}\sin\frac{\omega_x-\omega_y}{2} = \lambda$ one can just take the inverse of the left-hand matrix to get the right-hand one. As we can see, the first Jacobian has singular entries and the second one has vanishing ones at $r = 1$, and a vanishing determinant there, as well. The boundary measure here is $d\omega_x\,d\omega_y$. As above,

$$\partial_{\omega_x,\omega_y} = \begin{pmatrix} 1/\lambda & 1 \\ -1/\lambda & 1 \end{pmatrix}\partial_{r,\omega}, \quad \partial_{r,\omega} = \begin{pmatrix} -\lambda & \lambda \\ 1 & 1 \end{pmatrix}\partial_{\omega_x,\omega_y}.$$

Again, we see that none of the norms $\|\cdot\|_{\tilde{H}^s}$ and $\|\cdot\|_{H^s(\partial M\times\partial M)}$ is stronger or weaker than the other one unless $s = 0$ when they are equivalent.

Therefore, none of the three classical parameterizations is diffeomorphic to the foliation one. The first two are also isometric to the foliation one; the last one is not but the determinant of the Jacobian cannot compensate for the singular factor in the Jacobian even in the definition of H^1. We want to emphasize that here we are studying equivalence or not of norms of functions on Γ not restricted to the range of the ray transform. Also, the we establish the non-equivalence of the norms for s integer only. On the other hand, for the function $f_0(x) = (1-|x|^2)^{-1/2}$, we have that $I_0 f_0$, which is constant, belongs to $H^1(\partial_+SM)$, $H^1(BM)$, $H^1_\mu(\partial_+SM)$, $H^1(\partial M\times\partial M)$ because it belongs to the L^2 versions of those spaces and it has a zero differential. Therefore, the estimate in Theorem 1.1(a) cannot hold with $H^{1/2}_\Gamma$ replaced by the $H^{1/2}$ version of some of those spaces since $f_0 \notin L^2(M)$.

References

[1] Y. E. Anikonov and V. G. Romanov. On uniqueness of determination of a form of first degree by its integrals along geodesics. *J. Inverse Ill-Posed Probl.*, 5(6):487–490 (1998), 1997.

[2] Y. M. Assylbekov, F. Monard, and G. Uhlmann. Inversion formulas and range characterizations for the attenuated geodesic ray transform. *Journal de Mathématiques Pures et Appliquées*, 111:161–190, March 2018.

[3] I. N. Bernstein and M. L. Gerver. A problem of integral geometry for a family of geodesics and an inverse kinematic seismics problem. *Dokl. Akad. Nauk SSSR*, 243(2):302–305, 1978.

[4] J. Boman and V. Sharafutdinov. Stability estimates in tensor tomography. *preprint*, 2017.

[5] W. Chen and J. Jost. A Riemannian version of Korn's inequality. *Calculus of Variations and Partial Differential Equations*, **14**(4):517–530, 2002.

[6] P. G. Ciarlet and P. Ciarlet Jr. Another approach to linearized elasticity and korn's inequality. *Comptes Rendus Mathematique*, **339**(4):307–312, 2004.

[7] P. G. Ciarlet and L. Gratie. Another approach to linear shell theory and a new proof of Korn's inequality on a surface. *Comptes Rendus Mathematique*, **340**(6):471–478, 2005.

[8] N. S. Dairbekov, G. P. Paternain, P. Stefanov, and G. Uhlmann. The boundary rigidity problem in the presence of a magnetic field. *Advances in Mathematics*, **216**(2):535–609, 2007.

[9] N. S. Dairbekov and V. A. Sharafutdinov. On conformal Killing symmetric tensor fields on Riemannian manifolds. *Siberian Advances in Mathematics*, **21**(1):1–41, 2011.

[10] N. S. Dairbekov and G. Uhlmann. Reconstructing the metric and magnetic field from the scattering relation. *Inverse problems & Imaging*, **4**(3):397, 2010.

[11] G. Duvaut and J. L. Lions. *Inequalities in mechanics and physics.* Springer, 1976.

[12] B. Frigyik, P. Stefanov, and G. Uhlmann. The X-ray transform for a generic family of curves and weights. *J. Geom. Anal.*, 18(1):89–108, 2008.
[13] S. Holman, F. Monard, and P. Stefanov. The attenuated geodesic x-ray transform. *Inverse Problems*, 34(6):064003, 2018.
[14] S. Holman and P. Stefanov. The weighted Doppler transform. *Inverse Probl. Imaging*, 4(1):111–130, 2010.
[15] S. Holman and G. Uhlmann. On the microlocal analysis of the geodesic X-ray transform with conjugate points. *J. Differential Geom.*, 108(3):459–494, 2018.
[16] C. O. Horgan. Korn's inequalities and their applications in continuum mechanics. *SIAM review*, **37**(4):491–511, 1995.
[17] L. Hörmander. *The analysis of linear partial differential operators. III*, volume 274. Springer-Verlag, Berlin, 1985. Pseudodifferential operators.
[18] W. McLean. *Strongly elliptic systems and boundary integral equations*. Cambridge University Press, Cambridge, 2000.
[19] W. C. H. McLean. *Strongly elliptic systems and boundary integral equations*. Cambridge University Press, 2000.
[20] F. Monard. Numerical implementation of geodesic X-ray transforms and their inversion. *SIAM J. Imaging Sci.*, 7(2):1335–1357, 2014.
[21] F. Monard, R. Nicki, and G. Paternain. Efficient nonparametric Bayesian inference for X-ray transforms. *arXiv:1708.06332*, 2018.
[22] F. Monard, P. Stefanov, and G. Uhlmann. The geodesic ray transform on Riemannian surfaces with conjugate points. *Communications in Mathematical Physics*, pages 1–23, 2015.
[23] R. G. Muhometov. The reconstruction problem of a two-dimensional Riemannian metric, and integral geometry. *Dokl. Akad. Nauk SSSR*, 232(1):32–35, 1977.
[24] F. Natterer. *The mathematics of computerized tomography*. B. G. Teubner, Stuttgart, 1986.
[25] G. P. Paternain, M. Salo, and G. Uhlmann. Tensor tomography on surfaces. *Inventiones Mathematicae*, **193**(1):229–247, 2013.
[26] G. P. Paternain and H. Zhou. Invariant distributions and the geodesic ray transform. *Analysis & PDE*, **9**(8):1903–1930, 2016.
[27] L. Pestov and G. Uhlmann. On characterization of the range and inversion formulas for the geodesic X-ray transform. *Int. Math. Res. Not.*, (80):4331–4347, 2004.
[28] M. Salo and G. Uhlmann. The attenuated ray transform on simple surfaces. *J. Differential Geom.*, 88(1):161–187, 2011.
[29] V. A. Sharafutdinov. *Integral geometry of tensor fields*. Inverse and Ill-posed Problems Series. VSP, Utrecht, 1994.
[30] V. A. Sharafutdinov. A finiteness theorem for the ray transform on a Riemannian manifold. *Dokl. Akad. Nauk*, 355(2):167–169, 1997.
[31] V. A. Sharafutdinov. Ray transform on Riemannian manifolds, lecture notes, UW–Seattle. available at: http://www.ima.umn.edu/talks/workshops/7-16-27.2001/sharafutdinov/, 1999.
[32] P. Stefanov. A sharp stability estimate in tensor tomography. *Journal of Physics: Conference Series*, 124(1):012007, 2008.
[33] P. Stefanov and G. Uhlmann. Stability estimates for the X-ray transform of tensor fields and boundary rigidity. *Duke Math. J.*, 123(3):445–467, 2004.
[34] P. Stefanov and G. Uhlmann. Stability estimates for the X-ray transform of tensor fields and boundary rigidity. *Duke Mathematical Journal*, **123**(3):445–467, 2004.
[35] P. Stefanov and G. Uhlmann. Boundary rigidity and stability for generic simple metrics. *J. Amer. Math. Soc.*, 18(4):975–1003, 2005.
[36] P. Stefanov and G. Uhlmann. Integral geometry of tensor fields on a class of non-simple Riemannian manifolds. *Amer. J. Math.*, 130(1):239–268, 2008.
[37] P. Stefanov and G. Uhlmann. Local lens rigidity with incomplete data for a class of non-simple Riemannian manifolds. *J. Differential Geom.*, 82(2):383–409, 2009.
[38] P. Stefanov and G. Uhlmann. The geodesic X-ray transform with fold caustics. *Anal. PDE*, 5-2:219–260, 2012.
[39] P. Stefanov, G. Uhlmann, and A. Vasy. Local and global boundary rigidity and the geodesic X-ray transform in the normal gauge. *arXiv:1702.03638*, 2017.
[40] P. Stefanov, G. Uhlmann, and A. Vasy. Inverting the local geodesic X-ray transform on tensors. *Journal d'Analyse Mathématique, to appear*, 2018.

[41] M. Taylor. *Partial differential equations I: Basic theory*, volume **115** of *Applied Mathematical Sciences*. Springer-Verlag New York, 2011.

[42] M. E. Taylor. Pseudodifferential operators. volume 34 of Princeton Mathematical Series, 1981.

[43] A. Tiero. On korn's inequality in the second case. *Journal of elasticity*, **54**(3):187–191, 1999.

[44] F. Trèves. *Introduction to pseudodifferential and Fourier integral operators*. Springer Science & Business Media, 1980.

[45] G. Uhlmann and A. Vasy. The inverse problem for the local geodesic ray transform. *Invent. Math.*, 205(1):83–120, 2016.

Department of Mathematics, Northeastern University, Boston, MA 02115, USA
E-mail address: y_assylbekov@yahoo.com

Department of Mathematics, Purdue University, West Lafayette, IN 47907, USA
E-mail address: stefanov@math.purdue.edu