Boundary rigidity of Riemannian manifolds

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Domain $\Omega \subset \mathbf{R}^n$, $\partial\Omega \in C^\infty$. Let $g = \{g_{ij}\}$ be a Riemannian metric in Ω. Distance function: $\rho_g(x,y)$.

Boundary rigidity: *Does $\rho_g(x,y)$, known for all x, y on $\partial\Omega$, determine g, up to an isometry?*

In other words, if $\rho_{g_1} = \rho_{g_2}$ on $\partial\Omega^2$, is there a diffeo $\psi : \Omega \to \Omega$, $\psi|_{\partial\Omega} = Id$, such that $\psi_* g_1 = g_2$?

No, in general, but may be yes for *simple* metrics. A metric g is simple in Ω, if the latter is strictly convex w.r.t. g, and for any $x \in \bar{\Omega}$, the exp map is a diffeo on $\exp_x^{-1}(\bar{\Omega})$.

Equivalent formulation for (simple metrics): Knowing the *scattering relation* σ, can we recover the metric g?

$$\sigma : (x, \xi) \to (y, \eta)$$

This information is contained in the (hyperbolic) Dirichlet to Neumann map; in the scattering kernel. Possible applications: in medical imaging, in geophysics, etc.

Some history:

Mukhometov; Mukhometov & Romanov, Bernstein & Gerver, Croke, Gromov, Michel, Pestov & Sharafutdinov

Results for g conformal; flat; of negative curvature.

S-Uhlmann '98: for g close to the Euclidean one.

Croke, Dairbekov and Sharafutdinov '00: locally, near metrics with small enough curvature.

Lassas, Sharafutdinov & Uhlmann '03: one metric with small curvature, one close to the Euclidean.

Pestov & Uhlmann '03: $n = 2$, simple metrics (no smallness assumptions)

Linearized problem: *Recover a tensor f_{ij} from the geodesic X-ray transform*

$$I_g f(\gamma) = \int f_{ij}(\gamma(t))\dot{\gamma}^i(t)\dot{\gamma}^j(t)\,dt$$

known for all max geodesics γ in Ω.

Every tensor admits an orthogonal decomposition into a *solenoidal* part f^s and a *potential* part $d^s v$,

$$f = f^s + d^s v, \quad v|_{\partial\Omega} = 0.$$

Here $\delta^s f^s = 0$. The divergence δ is given by: $[\delta f]_i = g^{jk}\nabla_k f_{ij}$. We have $I_g(d^s v) = 0$. More precise formulation of the linearized problem: Does $I_g f = 0$ imply $f^s = 0$? We will call this *s-injectivity* of I_g. True at least for g Euclidean.

Estimates? **V. Sharafutdinov**: if the curvature is small enough, then I_g is s-injective and

$$\begin{aligned}\|f^s\|^2_{L^2(\Omega)} \;\le\; C\Big(&\|j_\nu f|_{\partial\Omega}\|_{H^{1/2}(\partial\Omega)}\|I_g f\|_{L^2(\Gamma_-)}\\ &+\|I_g f\|^2_{H^1(\Gamma_-)}\Big).\end{aligned}$$

Here $j_\nu f = f_{ij}\nu^j$, and ν is the normal.

The small curvature condition was the largest known class of (simple) metrics with s-injective I_g.

In case of 1-tensors (differential forms or vector fields) and functions, s-injectivity/injectivity is known for all simple metrics. Non-sharp stability estimates are also known and our methods allow us to obtain sharp estimates.

A typical plan of attack is as follows:

(1) injectivity of the linear problem (LP)

(2) Stability (estimate) of the LP

(3) Local uniqueness of the non-linear problem (NLP)

(4) Stability estimate for the NLP

$$(1) + (2) \Longrightarrow (3) + (4)$$

In our case, injectivity of LP is s-injectivity; local uniqueness of NLP is mod isometry; and we show that

$$(1) \Longrightarrow (2) + (3) + (4)$$

Sketch of the main results (I):

- Study the linear problem in detail, show that $N_g := I_g^* I_g$ is a ΨDO near Ω

- Find the principal symbol of N_g, identify the kernel. Then N_g is elliptic on $(\text{Ker}\, N_g)^\perp$

- Construct a parametrix of N_g on $(\text{Ker}\, N_g)^\perp$ to recover f^s, i.e., $f^s = AN_g f + Kf$.
  Note: the projection $f \mapsto f^s$ is not a ΨDO

- Prove an estimate of the type
$$\|f^s\| \le C\|N_g f\|_* + C_s\|f\|_{H^{-s}}, \quad \forall s > 0$$

- If I_g is s-injective, show that
$$\|f^s\| \le C\|N_g f\|_*$$

- Show that s-injectivity of I_g implies local uniqueness for the non-linear boundary rigidity problem near g.

To illustrate the approach, consider a model problem: On a compact manifold M without boundary, assume that A is an elliptic pseudo-differential operator (ΨDO), i.e., locally,

$$(Af)(x) = \frac{1}{(2\pi)^n} \int e^{-ix\cdot\xi} a(x,\xi) \widehat{f}(\xi)\, d\xi.$$

If $a(x,\xi) \neq 0$ for $|\xi| \gg 0$, then a and A are called elliptic. Then one can construct a *parametrix* B such that

$$BA = Id + K,$$

where K is smoothing, i.e., it sends "everything" into smooth functions. We construct B by iterations:

$$B = B_1 + B_2 + \dots,$$

where B_1 is a ΨDO with symbol $b = 1/a$, etc.

Now, since

$$BA = Id + K,$$

the problem of invertibility of A is reduced to that of $Id+K$, where K is compact. It is known that if the latter is injective (i.e., if -1 is not an eigenvalue of K), then $(Id + K)^{-1}$ is bounded!

Therefore, injectivity of A implies existence of A^{-1} and the estimate

$$\|A^{-1}\| \leq C.$$

If, in addition, $A = A(g)$ depends continuously on a parameter g (in our case, this is the metric g), then $K = K(g)$ has the same property, and C can be chosen locally uniform for g close to g_0, under the condition that $A(g_0)$ is injective.

Sketch of the main results (II):

About the linear problem:

- Show that N_g is s-injective for real analytic simple metrics using analytic ΨDO calculus.

- Show that the constant C in

$$\|f^s\| \leq C\|N_g f\|_*$$

is locally uniform in g, provided that g is near a metric for which I_g is s-injective.

- As a result show that I_g is injective (with a stability estimate) for an open dense set $\mathcal{G}$ of metrics.

Sketch of the main results (III):

About the non-linear problem:

- *Strong* local uniqueness near $g \in \mathcal{G}$

- Hölder type of stability estimate near any $g \in \mathcal{G}$

Representation for N_g:

$$(N_g f)_{kl}(x) = \frac{1}{\sqrt{\det g}} \int \frac{f^{ij}(y)}{\rho(x,y)^{n-1}} \frac{\partial \rho}{\partial y^i} \frac{\partial \rho}{\partial y^j} \frac{\partial \rho}{\partial x^k} \frac{\partial \rho}{\partial x^l} \times \det \frac{\partial^2(\rho^2/2)}{\partial x \partial y}\, dy,$$

Here $\rho(x,y)$ is the distance in the metric.

Principal symbol of N_g:

$$\sigma_p(N_g)^{ijkl}(x,\xi) = c_n\,|\xi|^{-1}\sigma(\varepsilon^{ij}\varepsilon^{kl}),$$

where $\varepsilon^{ij} = \delta^{ij} - \xi^i\xi^j/|\xi|^2$, and σ is symmetrization, i.e., the average over all permutations of i, j, k, l.

$\sigma_p(N_g)$ is not elliptic, it vanishes on the range of $\sigma_p(d^s)$. However, $\sigma_p(N_g)$ is elliptic on the range of $\sigma_p(\delta^s)$.

Define $\Delta^s = \delta^s d^s$. Then

$$v = v_\Omega = (\Delta^s_D)^{-1}\delta^s f,$$

and

$$f^s = f^s_\Omega = f - d^s(\Delta^s_D)^{-1}\delta^s f.$$

The Euclidean Case: Let $g = e$. Define

$$v_{\mathbf{R}^n} = (\Delta^s)^{-1}\delta^s f,$$

and

$$f^s_{\mathbf{R}^n} = f - d^s(\Delta^s)^{-1}\delta^s f.$$

Assume $I_e f = 0$. Then $N_e f = 0$. $\exists$ parametrix $A = A(D)$, such that $AN_e f = f^s_{\mathbf{R}^n} \Longrightarrow f^s_{\mathbf{R}^n} = 0$.

For $x \notin \Omega$, $0 = f = f^s_{\mathbf{R}^n} + d^s v_{\mathbf{R}^n}$, therefore, $d^s v_{\mathbf{R}^n} = 0$ there. This easily implies $v_{\mathbf{R}^n} = 0$ outside Ω, so

$$v_{\mathbf{R}^n} = v_\Omega \quad (!)$$

(provided $I_e f = 0$). Then $f^s_{\mathbf{R}^n} = f^s_\Omega = 0$, and the s-injectivity of I_e is proved.

For general simple metrics:

$\exists$ parametrix $A = A(x, D)$, such that

$$f = \tilde{f}^s + d^s\tilde{v} \quad \text{with} \quad \tilde{f}^s := AN_g f,$$

and, loosely speaking, $\tilde{f}^s$ and $d^s\tilde{v}$ have all properties needed modulo smoothing operators (Ψ^∞), except that $\tilde{v}$ does not vanish on $\partial\Omega$! (Not even mod Ψ^∞.) They are analogues of $f^s_{\mathbf{R}^n}$ and $v_{\mathbf{R}^n}$. To make $\tilde{v}$ vanish on $\partial\Omega$, we subtract a corrective term $d^s w$ (replace $\tilde{v}$ by $\tilde{v} - w$) with w such that

$$\Delta^s w = 0 \quad \text{in } \Omega, \quad w|_{\partial\Omega} = \tilde{v}|_{\partial\Omega}.$$

To get $\tilde{v}|_{\partial\Omega}$, we use the fact that $d^s\tilde{v} = -\tilde{f}^s = -AN_g f$ outside Ω, so $\tilde{v}|_{\partial\Omega}$ can be expressed as certain integrals of $AN_g f$ along geodesics connecting outside points with points on $\partial\Omega$. This gives

$$f^s = A' N_g f \quad \text{in } \Omega \text{ mod } \Psi^\infty.$$

with A' of order 2 (not 1, unfortunately).

Let $\Omega_1 \supset\supset \Omega$. In boundary local coordinates, set

$$\|f\|_{\tilde{H}^2(\Omega_1)} = \|x^n \partial_n f\|_{H^1(\Omega_1)} + \sum_{j=1}^{n-1} \|\partial_j f\|_{H^1(\Omega_1)} + \|f\|_{H^1(\Omega_1)}.$$

Theorem 1 (S-Uhlmann, '03, '04) *Let g be simple, extended as a simple metric in Ω_1.*

(a) The following estimate holds for each symmetric 2-tensor f in $H^1(\Omega)$:

$$\|f^s_\Omega\|_{L^2(\Omega)} \le C\|N_g f\|_{\tilde{H}^2(\Omega_1)} + C_s\|f\|_{H^{-s}(\Omega_1)}, \quad \forall s.$$

(b) Ker $I_g \cap \mathcal{S}L^2(\Omega)$ *is finite dimensional and included in $C^\infty(\bar{\Omega})$.*

(c) Assume that I_g is s-injective in Ω, i.e., that Ker $I_g \cap \mathcal{S}L^2(\Omega) = \{0\}$. *Then for any symmetric 2-tensor f in $H^1(\Omega)$ we have*

$$\|f^s\|_{L^2(\Omega)} \le C\|N_g f\|_{\tilde{H}^2(\Omega_1)}.$$

C is locally uniform as a function of g. ('04)

The boundary rigidity problem:

Theorem 2 (S-Uhlmann, '03, '04) *Let* g_0 *be a simple metric in* Ω. *Assume that* I_{g_0} *is s-injective. Then there exists* $\varepsilon > 0$ *and* $k > 0$*, such that if for* $\tilde{g}_j$, $j = 1, 2$ *we have*

$$\|\tilde{g}_j - g_0\|_{C^k} \leq \varepsilon,$$

and

$$\rho_{g_1} = \rho_{g_2} \quad \text{on } \partial\Omega^2,$$

then there exists a diffeomorphism $\psi : \bar{\Omega} \to \bar{\Omega}$ *with* $\psi|_{\Omega} = Id$*, such that*

$$g_2 = \psi_* g_1.$$

Sketch of the proof. Choose first semi-geodesic coordinates in Ω_1, such that for g, $\tilde{g}$:

$$g_{in} = g_{ni} = \delta_{in}, \quad i = 1, \dots, n.$$

Goal: under the assumption that I_g is s-injective, if $\rho_{\tilde{g}} = \rho_g$ on $\partial\Omega^2$, and $\tilde{g}$ is close to g, show that $\tilde{g} = g$ (no additional diffeo).

Linearize:

$$\rho_{\tilde{g}}(x,y) - \rho_g(x,y) = I_g f(x,y) + R_g(f)(x,y)$$

$\forall (x,y) \in \partial\Omega^2$, where $f = \tilde{g} - g$ is of the form $f_{in} = f_{ni} = 0$. The remainder R_g is quadratic:

$$|R_g(f)(x,y)| \le C|x-y| \|f\|^2_{C^1(\bar{\Omega})}, \quad \forall (x,y) \in \partial\Omega^2.$$

If $\rho_{\tilde{g}} = \rho_g$, we get

$$|I_g f(x,y)| \le C|x-y| \|f\|^2_{C^1(\bar{\Omega})}, \quad \forall (x,y) \in \partial\Omega^2.$$

For f of the special form above,

$$\|f\| \le C\|f^s\|_{H^2}.$$

Now, this, the estimate on $\|N_g f\|$ from below in Thm 1, and interpolation inequalities imply $f = 0$ for $\|f\| \ll 1$.

Byproducts:

X-ray transform of functions.

$$I_g f(\gamma) = \int f(\gamma(t))\, dt$$

Mukhometov; Romanov; Bernstein and Gerver: I_g is injective for simple metrics. A non-sharp estimate is known. As a consequence of the injectivity:

Theorem 3 *Let g be a simple metric in Ω and assume that g is extended smoothly as a simple metric near the convex domain $\Omega_1 \supset\supset \Omega$. Then for any function $f \in L^2(\Omega)$,*

$$\|f\|/C \le \|N_g f\|_{H^1(\Omega_1)} \le C\|f\|.$$

Moreover, in Ω, $f = c_n^{-1}|D|\chi N_g f$ mod $H^1(\Omega)$.

X-ray transform of 1-forms.

$$I_g f(\gamma) = \int f_j(\gamma(t))\dot{\gamma}^j(t)\,dt$$

As before, for each $f = f_j dx^j$,

$$f = f^s + d\phi, \quad \delta f^s = 0, \quad \phi|_{\partial\Omega} = 0.$$

For simple metrics, $I_g f = 0 \Longrightarrow f^s = 0$ with a non-sharp estimate (Anikonov-Romanov).

Theorem 4 *Assume that g is simple metric in Ω and extend g as a simple metric in $\Omega_1 \supset\supset \Omega$. Then for any 1-form $f = f_i dx^i$ in $L^2(\Omega)$ we have*

$$\|f^s\|_{L^2(\Omega)}/C \le \|N_g f\|_{H^1(\Omega_1)} \le C\,\|f^s\|_{L^2(\Omega)}.$$

Moreover, $f^s = c_n^{-1}|D|\chi N_g f$ mod $H^1(\Omega)$.

Results for generic simple metrics

Results for the linear problem

There was a large class of simple metrics (all with not so small curvature) for which s-injectivity of I_g and local (and global, of course) uniqueness of the rigidity problem were not known.

We show next that this is true for an open dense set of such metrics.

This is based on the following observations:

(1) We have s-injectivity for any simple analytic metric.

(2) The metrics for which I_g is s-injective, form an open set.

(3) Analytic metric are dense (of course) in the space of simple C^k metrics.

The idea for proving (1) is to use analytic ΨDOs.

Consider a **model problem** first: Let f be a function, not 2-tensor, and assume that

$$I_g f(\gamma) = \int f(\gamma(t))\,dt = 0 \quad \forall\gamma.$$

Assume that g is simple and analytic. Analytic weight function is also allowed. Then N_g is an analytic ΨDO (roughly speaking, a ΨDO with a real-analytic amplitude). Then one can construct a parametrix A to $N_g = I_g^* I_g$, such that

$$AN_g f = f + Rf \quad \text{in } \Omega_1 \supset\supset \Omega.$$

with R *analytic-regularizing*, i.e., Rf is analytic $\forall f$.

Assume now that $I_g f = 0$. Then $N_g f = 0$, so

$$f = -Rf \quad \text{in } \Omega_1.$$

The l.h.s. is compactly supported, the r.h.s. is analytic. Therefore, $f = 0$.

Not such an impressive result, since we know that I_g is injective for functions, for all simple metrics by energy estimates (Mukhometov et al.)

In case of analytic weight however, this is the way to prove injectivity and support theorems (Quinto and Boman '87, '91, Euclidean g, analytic weight).

In our case things are more complicated because we have non-Euclidean (analytic) metric, and we work with tensors. Injectivity is replaced by s-injectivity. We need recovery to infinite order at the boundary first.

As a byproduct, one can generalize all of the integral geometry results above to analytic simple metrics and analytic weights.

Formal formulation of the results above (for the linear problem)

Theorem 5 (S-Uhlmann '04) *Let g be a simple metric in Ω, real analytic in $\bar{\Omega}$. Then I_g is s-injective.*

Theorem 6 (S-Uhlmann '04) *$\exists k_0$ such that for each $k \geq k_0$, the set $\mathcal{G}^k(\Omega)$ of simple $C^k(\Omega)$ metrics in Ω for which I_g is s-injective is open and dense in the $C^k(\Omega)$ topology. Moreover, for any $g \in \mathcal{G}^k$,*

$$\|f^s\|_{L^2(\Omega)} \leq C\|N_g f\|_{\tilde{H}^2(\Omega_1)}, \quad \forall f \in H^1(\Omega),$$

with a constant $C > 0$ that can be chosen locally uniform in $\mathcal{G}^k$ in the $C^k(\Omega)$ topology.

Generic results for the non-linear problem

The s-injectivity of N_g and the stability estimate imply local uniqueness and Hölder stability for the non-linear problem.

Generic local uniqueness:

Theorem 7 (S-Uhlmann '04) *Let k_0 and $\mathcal{G}^k(\Omega)$ be as in Theorem 6. Then $\exists k \geq k_0$, such that $\forall g_0 \in \mathcal{G}^k$, $\exists \varepsilon > 0$, such that for any two metrics g_1, g_2 with*

$$\|g_m - g_0\|_{C^k(\Omega)} \leq \varepsilon, \quad m = 1, 2,$$

we have the following:

$$\rho_{g_1} = \rho_{g_2} \quad \text{on } (\partial\Omega)^2$$

implies

$$g_2 = \psi_* g_1$$

with some diffeomorphism $\psi : \Omega \to \Omega$ fixing the boundary.

Stability estimate:

Theorem 8 (S-Uhlmann '04) *Let* k_0 *and* $\mathcal{G}^k(M)$ *be as in Theorem 6. Then for any* $\mu < 1$, *there exits* $k \geq k_0$ *such that for any* $g_0 \in \mathcal{G}^k$, *there is an* $\varepsilon_0 > 0$ *and* $C > 0$ *with the property that that for any two metrics* g_1, g_2 *with*

$$\|g_m - g_0\|_{C(\Omega)} \leq \varepsilon_0,$$

and

$$\|g_m\|_{C^k(M)} \leq A,$$

$m = 1, 2$, *with some* $A > 0$, *we have the following stability estimate*

$$\|g_2 - \psi_* g_1\|_{C(\Omega)} \leq C(A) \|\rho_{g_1} - \rho_{g_2}\|^{\mu}_{C(\partial\Omega\times\partial\Omega)}$$

with some diffeomorphism $\psi : \Omega \to \Omega$ *fixing the boundary.*