The identification problem in SPECT: Uniqueness, non-uniqueness and stability

Plamen Stefanov

Purdue University

Numerical results obtained with Sonting Luo and Jianliang Qian

The Identification Problem in SPECT

This problem arises in SPECT. We measure the radiation emitted by radioactive markers in the patient's body, modeled by a source distribution $f(x)$, attenuated by the body, with attenuation $a(x)$. We want to recover f but we know neither f, nor a. So the question is: can we recover both? — but we care about f only.

Math Model

The attenuated X-ray transform

$$X_a f(x, \theta) = \int e^{-Ba(x+t\theta,\theta)} f(x + t\theta)\, \mathrm{d}t, \quad x \in \mathbf{R}^2,\ \theta \in S^1,$$

in the plane.

We use the notation

$$Ba(x, \theta) = \int_0^\infty a(x + t\theta)\, \mathrm{d}t$$

to denote the "beam transform" of a, usually denoted by Da.

The Identification Problem in SPECT

This problem arises in SPECT. We measure the radiation emitted by radioactive markers in the patient's body, modeled by a source distribution $f(x)$, attenuated by the body, with attenuation $a(x)$. We want to recover f but we know neither f, nor a. So the question is: can we recover both? — but we care about f only.

Math Model

The attenuated X-ray transform

$$X_a f(x,\theta) = \int e^{-Ba(x+t\theta,\theta)} f(x+t\theta)\,\mathrm{d}t, \quad x \in \mathbf{R}^2,\ \theta \in S^1,$$

in the plane.

We use the notation

$$Ba(x,\theta) = \int_0^\infty a(x+t\theta)\,\mathrm{d}t$$

to denote the "beam transform" of a, usually denoted by Da.

Identification Problem

Given $X_a f$, recover both a and f.

Can we recover both?

- Short answer: Depends.
- Sometimes we can, even in a stable way; sometimes we cannot.
- There is a hidden dynamical system (in the phase space). Generally speaking, the problem is well posed, locally near some (a, f), if the perturbation δa is supported in a set which is *non-trapping* w.r.t. that flow.
- If δa is supported in a *trapping set* ; well posedness and uniqueness (even up to a finite dimensional set) *may be* lost. In the radial case, at least, **they are** lost.
- There are various degrees of instability when the non-trapping condition fails.

Identification Problem

Given $X_a f$, recover both a and f.

Can we recover both?

- Short answer: Depends.
- Sometimes we can, even in a stable way; sometimes we cannot.
- There is a hidden dynamical system (in the phase space). Generally speaking, the problem is well posed, locally near some (a, f), if the perturbation δa is supported in a set which is *non-trapping* w.r.t. that flow.
- If δa is supported in a *trapping set* ; well posedness and uniqueness (even up to a finite dimensional set) *may be* lost. In the radial case, at least, **they are** lost.
- There are various degrees of instability when the non-trapping condition fails.

Identification Problem

Given $X_a f$, recover both a and f.

Can we recover both?

- Short answer: Depends.
- Sometimes we can, even in a stable way; sometimes we cannot.
- There is a hidden dynamical system (in the phase space). Generally speaking, the problem is well posed, locally near some (a, f), if the perturbation δa is supported in a set which is *non-trapping* w.r.t. that flow.
- If δa is supported in a *trapping set* ; well posedness and uniqueness (even up to a finite dimensional set) *may be* lost. In the radial case, at least, **they are** lost.
- There are various degrees of instability when the non-trapping condition fails.

Identification Problem

Given $X_a f$, recover both a and f.

Can we recover both?

- Short answer: Depends.
- Sometimes we can, even in a stable way; sometimes we cannot.
- There is a hidden dynamical system (in the phase space). Generally speaking, the problem is well posed, locally near some (a, f), if the perturbation δa is supported in a set which is *non-trapping* w.r.t. that flow.
- If δa is supported in a *trapping set* ; well posedness and uniqueness (even up to a finite dimensional set) *may be* lost. In the radial case, at least, **they are** lost.
- There are various degrees of instability when the non-trapping condition fails.

Identification Problem

Given $X_a f$, recover both a and f.

Can we recover both?

- Short answer: Depends.
- Sometimes we can, even in a stable way; sometimes we cannot.
- There is a hidden dynamical system (in the phase space). Generally speaking, the problem is well posed, locally near some (a, f), if the perturbation δa is supported in a set which is *non-trapping* w.r.t. that flow.
- If δa is supported in a *trapping set* ; well posedness and uniqueness (even up to a finite dimensional set) *may be* lost. In the radial case, at least, **they are** lost.
- There are various degrees of instability when the non-trapping condition fails.

Identification Problem

Given $X_a f$, recover both a and f.

Can we recover both?

- Short answer: Depends.
- Sometimes we can, even in a stable way; sometimes we cannot.
- There is a hidden dynamical system (in the phase space). Generally speaking, the problem is well posed, locally near some (a, f), if the perturbation δa is supported in a set which is *non-trapping* w.r.t. that flow.
- If δa is supported in a *trapping set* ; well posedness and uniqueness (even up to a finite dimensional set) *may be* lost. In the radial case, at least, **they are** lost.
- There are various degrees of instability when the non-trapping condition fails.

If we know a, problem solved...

When a is known, it is well known that f can be reconstructed uniquely, even by means of explicit formulas: Bukhgeim, Kazantsev & Arbuzov; Novikov; Natterer.

For this reason, some of the numerical attempts to do a reconstruction are focused on recovery, or getting a good approximation of a first, instead of treating (a, f) as a pair. Then they get a better approximation for a, etc. Sometimes this is called *attenuation correction*. In clinical applications, additional X-rays are taken to reconstruct $a(x)$ first. Eliminating or reducing those additional X-rays remains an important problem.

If we know a, problem solved...

When a is known, it is well known that f can be reconstructed uniquely, even by means of explicit formulas:
Bukhgeim, Kazantsev & Arbuzov; Novikov; Natterer.

For this reason, some of the numerical attempts to do a reconstruction are focused on recovery, or getting a good approximation of a first, instead of treating (a, f) as a pair. Then they get a better approximation for a, etc. Sometimes this is called *attenuation correction*. In clinical applications, additional X-rays are taken to reconstruct $a(x)$ first. Eliminating or reducing those additional X-rays remains an important problem.

Clinical scans with the wrong and the right attenuation

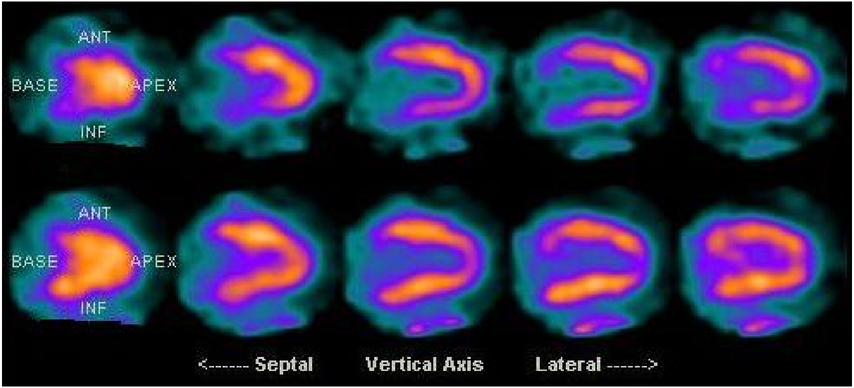


Figure : SPECT cardiac scans reconstructed assuming $a = 0$ (top), and with an approximation of the actual a (bottom)

a SPECT/CT scanner

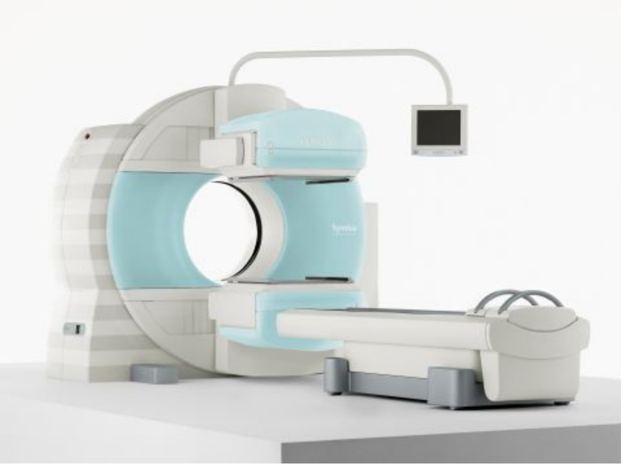


Figure : The Siemens Symbia SPECT/CT scanner

a CT (only) scanner

Figure : The Siemens Somatom Sensation Spirit CT scanner

. . . but we do not know a

No much progress in the mathematical understanding of the identification problem so far. A related but not identical problem for finding both a constant attenuation and the source in the exponential X-ray transform has been solved by SOLMON and HERTLE. The main result in SOLMON is, roughly speaking, that specific pairs of constant a and radial f cannot be distinguished but all other pairs can.

The identification problem with f a finite sum of delta sources has been studied by NATTERER, also BOMAN, but the results there do not (and cannot) imply uniqueness.

. . . but we do not know *a*

No much progress in the mathematical understanding of the identification problem so far. A related but not identical problem for finding both a constant attenuation and the source in the exponential X-ray transform has been solved by Solmon and Hertle. The main result in Solmon is, roughly speaking, that specific pairs of constant a and radial f cannot be distinguished but all other pairs can.

The identification problem with f a finite sum of delta sources has been studied by Natterer, also Boman, but the results there do not (and cannot) imply uniqueness.

NATTERER also viewed the problem as a range characterization problem: if the ranges of X_{a_1} and X_{a_2} happen to be the same, for example, then there cannot be uniqueness. If they intersect at the origin only, there is. Range conditions, in example, in a work by NOVIKOV, have been viewed as a possible tool for solving the problem, both numerically, for example by BRONNIKOV; and analytically, as in the recent work by JOLLLIVET & BAL. Numerical reconstructions have been tried, too, with variable success, by

CENSOR et al., MANGLOS et al., WELCH et al., RAMLAU et al., BRONNIKOV, ZAIDI,

for example. Some of them use clinical data.

A. L. BUKHGEIM recently outlined a recovery algorithm if a is a priori known to be a constant multiple of the characteristic function of a star-shaped domain.

NATTERER also viewed the problem as a range characterization problem: if the ranges of X_{a_1} and X_{a_2} happen to be the same, for example, then there cannot be uniqueness. If they intersect at the origin only, there is. Range conditions, in example, in a work by NOVIKOV, have been viewed as a possible tool for solving the problem, both numerically, for example by BRONNIKOV; and analytically, as in the recent work by JOLLLIVET & BAL. Numerical reconstructions have been tried, too, with variable success, by

CENSOR et al., MANGLOS et al., WELCH et al., RAMLAU et al., BRONNIKOV, ZAIDI,

for example. Some of them use clinical data.

A. L. BUKHGEIM recently outlined a recovery algorithm if a is a priori known to be a constant multiple of the characteristic function of a star-shaped domain.

Linearization

Linearize. Notation: $\delta X_{a,f}$ acting on $(\delta a, \delta f)$. Another notation:

$$I_w f(x,\theta) = \int w(x+t\theta,\theta) f(x+t\theta)\,\mathrm{d}t,$$

which is the weighted X-ray transform of f with weight $w(x,\theta)$. Note that X_a is of the same type but with a more special weight: $X_a = I_{e^{-Ba}}$.

Linearization

$$\delta X_{a,f}(\delta a, \delta f) = I_w \delta a + X_a \delta f,$$

where

$$w(x,\theta) = -\int_{-\infty}^{0} e^{-Ba(x+t\theta,\theta)} f(x+t\theta)\,\mathrm{d}t.$$

Another way to write w:

$$w = -e^{-Ba} u,$$

with u solving the transport equation $(\theta \cdot \partial_x + a) = f$, $u = 0$ for $\theta \cdot x \ll 1$:

$$u(x,\theta) = \int_{-\infty}^{0} e^{-\int_t^0 a(x+s\theta)\,\mathrm{d}s} f(x + t\theta)\,\mathrm{d}t.$$

A more general problem: inverting a sum of two weighted X-ray transforms

In other words,

$$\delta X_{a,f}(\delta a, \delta f) = I_{w_1}\delta a + I_{w_2}\delta f,$$

where

$$w_1 = -e^{-Ba}u, \quad w_2 = e^{-Ba}.$$

This brings us to the more general problem:

Inverting a sum of two weighted X-ray transforms

Given two weights $w_{1,2}(x,\theta)$, and

$$\mathcal{I}(g_1, g_2) = I_{w_1}g_1 + I_{w_2}g_2,$$

find g_1 and g_2.

A more general problem: inverting a sum of two weighted X-ray transforms

In other words,

$$\delta X_{a,f}(\delta a, \delta f) = I_{w_1}\delta a + I_{w_2}\delta f,$$

where

$$w_1 = -e^{-Ba}u, \quad w_2 = e^{-Ba}.$$

This brings us to the more general problem:

Inverting a sum of two weighted X-ray transforms

Given two weights $w_{1,2}(x,\theta)$, and

$$\mathcal{I}(g_1, g_2) = I_{w_1}g_1 + I_{w_2}g_2,$$

find g_1 and g_2.

The first impression is that this might be too much to ask for, but the second impression is that we seem to have two equations (integrals in the θ and the $-\theta$ directions) for two unknowns. So this might work, if a certain determinant is not zero. On the other hand, that determinant vanishes for any x (and some θ), as we will see in a moment.

To study Ker$\,\mathcal{I}$, take the Fourier transform of

$$\left(I_{w_1}g_1 + I_{w_2}g_2\right)(z, \pm\theta) = 0, \quad z \perp \theta,$$

w.r.t. z, to get

$$\Big(w_1(x, \pm D^\perp/|D|) + \text{l.o.t.}\Big)\, g_1 + \Big(w_2(x, \pm D^\perp/|D|) + \text{l.o.t.}\Big)\, g_2 = 0.$$

Here, "l.o.t." = "lower order terms". This is actually a 2×2 system of ΨDO equations.

The first impression is that this might be too much to ask for, but the second impression is that we seem to have two equations (integrals in the θ and the $-\theta$ directions) for two unknowns. So this might work, if a certain determinant is not zero. On the other hand, that determinant vanishes for any x (and some θ), as we will see in a moment.

To study Ker $\mathcal{I}$, take the Fourier transform of

$$\left(I_{w_1} g_1 + I_{w_2} g_2\right)(z, \pm\theta) = 0, \quad z \perp \theta,$$

w.r.t. z, to get

$$\Big(w_1(x, \pm D^\perp/|D|) + \text{l.o.t.}\Big)\, g_1 + \Big(w_2(x, \pm D^\perp/|D|) + \text{l.o.t.}\Big)\, g_2 = 0.$$

Here, "l.o.t." = "lower order terms". This is actually a 2×2 system of ΨDO equations.

The first impression is that this might be too much to ask for, but the second impression is that we seem to have two equations (integrals in the θ and the $-\theta$ directions) for two unknowns. So this might work, if a certain determinant is not zero. On the other hand, that determinant vanishes for any x (and some θ), as we will see in a moment.

To study $\mathrm{Ker}\,\mathcal{I}$, take the Fourier transform of

$$(I_{w_1} g_1 + I_{w_2} g_2)\,(z, \pm\theta) = 0, \quad z \perp \theta,$$

w.r.t. z, to get

$$\Big(w_1(x, \pm D^\perp/|D|) + \text{l.o.t.} \Big)\, g_1 + \Big(w_2(x, \pm D^\perp/|D|) + \text{l.o.t.} \Big)\, g_2 = 0.$$

Here, "l.o.t." = "lower order terms". This is actually a 2×2 system of ΨDO equations.

The determinant of the principal symbol is given by the following

Hamiltonian

$$p_0(x,\xi) = W(x, \xi^{\perp}/|\xi|),$$

where

$$W(x,\theta) = w_1(x,\theta)w_2(x,-\theta) - w_1(x,-\theta)w_2(x,\theta).$$

This function is of fundamental importance. Since W is an odd function of θ, it has zeros for any x! Therefore, p_0 cannot be elliptic in any domain. The Hamiltonian flow of p_0 then plays a fundamental role by the Hörmander's propagation of singularities theorem. We call the projections of the Hamiltonian curves on the x-space **rays**.

The determinant of the principal symbol is given by the following

Hamiltonian

$$p_0(x, \xi) = W(x, \xi^\perp/|\xi|),$$

where

$$W(x, \theta) = w_1(x, \theta)w_2(x, -\theta) - w_1(x, -\theta)w_2(x, \theta).$$

This function is of fundamental importance. Since W is an odd function of θ, it has zeros for any x! Therefore, p_0 cannot be elliptic in any domain. The Hamiltonian flow of p_0 then plays a fundamental role by the Hörmander's propagation of singularities theorem. We call the projections of the Hamiltonian curves on the x-space **rays**.

The determinant of the principal symbol is given by the following

Hamiltonian

$$p_0(x, \xi) = W(x, \xi^{\perp}/|\xi|),$$

where

$$W(x, \theta) = w_1(x, \theta) w_2(x, -\theta) - w_1(x, -\theta) w_2(x, \theta).$$

This function is of fundamental importance. Since W is an odd function of θ, it has zeros for any x! Therefore, p_0 cannot be elliptic in any domain. The Hamiltonian flow of p_0 then plays a fundamental role by the Hörmander's propagation of singularities theorem. We call the projections of the Hamiltonian curves on the x-space **rays**.

A radial example

Choose

$$w_1 = \frac{1}{2}\theta \cdot x, \quad w_2 = 1.$$

Then

$$W(x,\theta) = \theta \cdot x, \quad |\xi| p_0 = x_1\xi_2 - x_2\xi_1.$$

Therefore,

$$|D| p_0(x, D) = x_1 D_2 - x_2 D_1 = -\mathrm{i}\partial/\partial\phi,$$

where ϕ is the polar angle. Bicharacteritics:

$$x = R(\cos t, \sin t), \quad \xi = \lambda(\sin t, \cos t), \quad R \geq 0, \lambda \neq 0.$$

The rays are the circles $|x| = R \geq 0$, including the origin.

A radial example

Choose

$$w_1 = \frac{1}{2}\theta \cdot x, \quad w_2 = 1.$$

Then

$$W(x,\theta) = \theta \cdot x, \quad |\xi| p_0 = x_1\xi_2 - x_2\xi_1.$$

Therefore,

$$|D| p_0(x, D) = x_1 D_2 - x_2 D_1 = -\mathrm{i}\partial/\partial\phi,$$

where ϕ is the polar angle. Bicharacteritics:

$$x = R(\cos t, \sin t), \quad \xi = \lambda(\sin t, \cos t), \quad R \geq 0, \lambda \neq 0.$$

The rays are the circles $|x| = R \geq 0$, including the origin.

One can easily show that

$$\operatorname{Ker}\mathcal{I} = \{(g_1, 0);\ g_1 \text{ is radial}\}.$$

In other words, there is an infinite dimensional kernel, and the rays here appear as level curves of the functions in the kernel.

One can easily show that

$$\operatorname{Ker}\mathcal{I} = \{(g_1, 0);\ g_1 \text{ is radial}\}.$$

In other words, there is an infinite dimensional kernel, and the rays here appear as level curves of the functions in the kernel.

ΨDOs of real principal type

We will apply the theory of ΨDOs of real principal type. If P is such an operator, singularities (points of the wave front set $\mathrm{WF}(g)$) of the solution $Pg = 0$ occupy whole bicharacteristics. Under the a priori assumption $\operatorname{supp} g \subset K$, we can actually recover $\mathrm{WF}(g)$ if all bicharacteristics over K leave K eventually.

Definition 1

We call K non-trapping (for p_0) if there is no complete bicharacteristic over K.

ΨDOs of real principal type

We will apply the theory of ΨDOs of real principal type. If P is such an operator, singularities (points of the wave front set $\mathrm{WF}(g)$) of the solution $Pg = 0$ occupy whole bicharacteristics. Under the a priori assumption $\operatorname{supp} g \subset K$, we can actually recover $\mathrm{WF}(g)$ if all bicharacteristics over K leave K eventually.

Definition 1

We call K non-trapping (for p_0) if there is no complete bicharacteristic over K.

The non-trapping condition plays a fundamental role in the theory of solvability of $Pu = h$ in K. In our case, we get

If a priori, $\operatorname{supp} g \in K$, and K is non-trapping, then $\mathcal{I}g = 0 \Longrightarrow g \in C^\infty$.

We can make it more precise:

If $\operatorname{supp} g \in K$, and K is non-trapping, then $\mathcal{I}g \in H^s \Longrightarrow g \in H^{s-3/2}$.

The operator $\mathcal{I}$ is of order $-1/2$, so there is a loss of one derivative there.

The non-trapping condition plays a fundamental role in the theory of solvability of $Pu = h$ in K. In our case, we get

If a priori, $\operatorname{supp} g \in K$, and K is non-trapping, then $\mathcal{I}g = 0 \Longrightarrow g \in C^\infty$.

We can make it more precise:

If $\operatorname{supp} g \in K$, and K is non-trapping, then $\mathcal{I}g \in H^s \Longrightarrow g \in H^{s-3/2}$.

The operator $\mathcal{I}$ is of order $-1/2$, so there is a loss of one derivative there.

The non-trapping condition plays a fundamental role in the theory of solvability of $Pu = h$ in K. In our case, we get

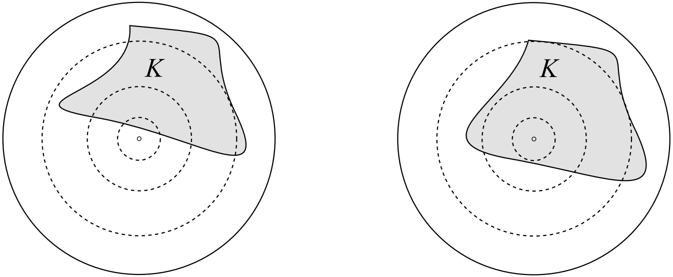
If a priori, $\operatorname{supp} g \in K$, and K is non-trapping, then $\mathcal{I}g = 0 \Longrightarrow g \in C^\infty$.

We can make it more precise:

If $\operatorname{supp} g \in K$, and K is non-trapping, then $\mathcal{I}g \in H^s \Longrightarrow g \in H^{s-3/2}$.

The operator $\mathcal{I}$ is of order $-1/2$, so there is a loss of one derivative there.

Trapping and non-trapping sets in the "radial example"

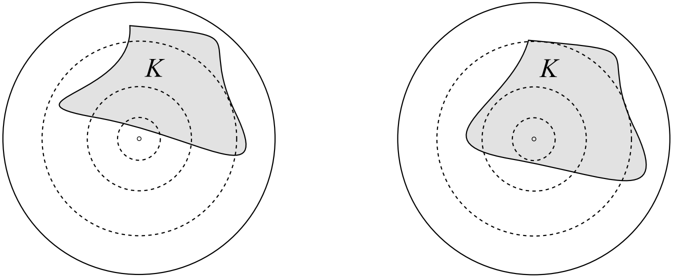


The rays are the circles $|x| = R$.

Non-trapping, left; trapping, right.

In the first case, the problem of inverting $\mathcal{I}$ is well posed (uniqueness and stability); in the second one — it is not (infinite dimensional kernel).

Trapping and non-trapping sets in the “radial example”

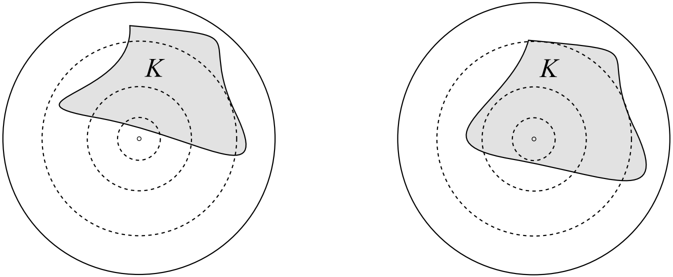


The rays are the circles $|x| = R$.

Non-trapping, left; trapping, right.

In the first case, the problem of inverting $\mathcal{I}$ is well posed (uniqueness and stability); in the second one — it is not (infinite dimensional kernel).

Trapping and non-trapping sets in the "radial example"



The rays are the circles $|x| = R$.

Non-trapping, left; trapping, right.

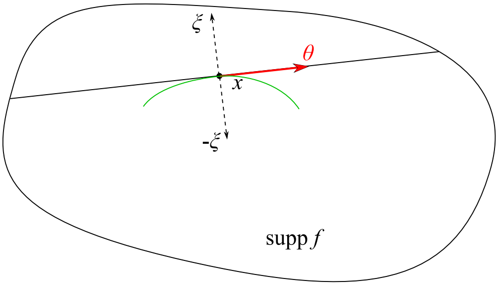
In the first case, the problem of inverting $\mathcal{I}$ is well posed (uniqueness and stability); in the second one — it is not (infinite dimensional kernel).

Back to the linearized map $\delta X_{a,f}$

What does this mean for $\delta X_{a,f}$? Look for (x, θ) so that

$$u(x, \theta) = u(-x, \theta),$$

where, as before, u is the solution of the transport eqn. (an attenuated integral of f). If $a = 0$, then this just means that x is the midpoint of the chord below. Then (x, θ) is a zero of W, and $(x, \pm\theta^{\perp})$ are characteristic. The green curve represents a ray.

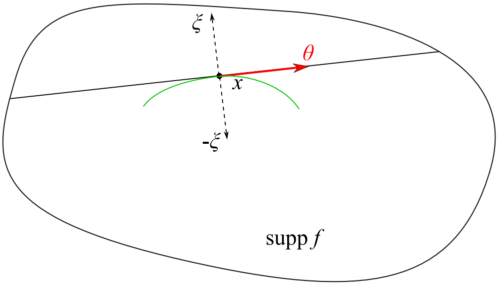


Back to the linearized map $\delta X_{a,f}$

What does this mean for $\delta X_{a,f}$? Look for (x, θ) so that

$$u(x, \theta) = u(-x, \theta),$$

where, as before, u is the solution of the transport eqn. (an attenuated integral of f). If $a = 0$, then this just means that x is the midpoint of the chord below. Then (x, θ) is a zero of W, and $(x, \pm\theta^{\perp})$ are characteristic. The green curve represents a ray.

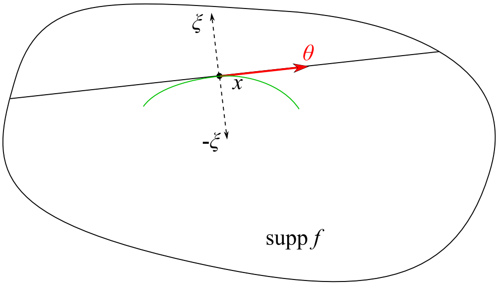


Back to the linearized map $\delta X_{a,f}$

What does this mean for $\delta X_{a,f}$? Look for (x, θ) so that

$$u(x, \theta) = u(-x, \theta),$$

where, as before, u is the solution of the transport eqn. (an attenuated integral of f). If $a = 0$, then this just means that x is the midpoint of the chord below. Then (x, θ) is a zero of W, and $(x, \pm\theta^{\perp})$ are characteristic. The green curve represents a ray.



Main results for the linearized map $\delta X_{a,f}$

Assume δa supported in a non-trapping compact set K. Then

- Knowing $\delta X_{a,f}(\delta a, \delta f)$, we can recover the singularities with a loss of one derivative; there is an estimate.
- $\operatorname{Ker} \delta X_{a,f}$ is finite dimensional and is in $C_0^\infty(K)$.
- If $\delta X_{a,f}$ is injective on K, then there is a stability estimate:

$$\|\delta a\|_{H^s(K)} + \|\delta a\|_{H^s(K)} \le C\|\delta X_{a,f}(\delta a, \delta f)\|_{H^{s+3/2}(Z)}.$$

- This estimate is preserved with a uniform C under a slightly stronger condition: $K \subset \Omega$ with Ω *pseudo-convex*: (any compact subset is non-trapping, and $\forall$ compact $K_1 \subset \Omega$, there exists a compact set $K_2 \subset \Omega$ so that every ray in Ω having endpoints over K_1, lies entirely in K_2).

Why δa must be supported in a non-trapping set only? The reason is that $\delta X_{a,f}$ is elliptic where $\delta a = 0$ because it reduces to X_a then.

Main results for the linearized map $\delta X_{a,f}$

Assume δa supported in a non-trapping compact set K. Then

- Knowing $\delta X_{a,f}(\delta a, \delta f)$, we can recover the singularities with a loss of one derivative; there is an estimate.
- $\operatorname{Ker} \delta X_{a,f}$ is finite dimensional and is in $C_0^\infty(K)$.
- If $\delta X_{a,f}$ is injective on K, then there is a stability estimate:

$$\|\delta a\|_{H^s(K)} + \|\delta a\|_{H^s(K)} \le C\|\delta X_{a,f}(\delta a, \delta f)\|_{H^{s+3/2}(Z)}.$$

- This estimate is preserved with a uniform C under a slightly stronger condition: $K \subset \Omega$ with Ω *pseudo-convex*: (any compact subset is non-trapping, and $\forall$ compact $K_1 \subset \Omega$, there exists a compact set $K_2 \subset \Omega$ so that every ray in Ω having endpoints over K_1, lies entirely in K_2).

Why δa must be supported in a non-trapping set only? The reason is that $\delta X_{a,f}$ is elliptic where $\delta a = 0$ because it reduces to X_a then.

Main results for the linearized map $\delta X_{a,f}$

Assume δa supported in a non-trapping compact set K. Then

- Knowing $\delta X_{a,f}(\delta a, \delta f)$, we can recover the singularities with a loss of one derivative; there is an estimate.
- $\operatorname{Ker} \delta X_{a,f}$ is finite dimensional and is in $C_0^\infty(K)$.
- If $\delta X_{a,f}$ is injective on K, then there is a stability estimate:

$$\|\delta a\|_{H^s(K)} + \|\delta a\|_{H^s(K)} \le C\|\delta X_{a,f}(\delta a, \delta f)\|_{H^{s+3/2}(Z)}.$$

- This estimate is preserved with a uniform C under a slightly stronger condition: $K \subset \Omega$ with Ω *pseudo-convex*: (any compact subset is non-trapping, and $\forall$ compact $K_1 \subset \Omega$, there exists a compact set $K_2 \subset \Omega$ so that every ray in Ω having endpoints over K_1, lies entirely in K_2).

Why δa must be supported in a non-trapping set only? The reason is that $\delta X_{a,f}$ is elliptic where $\delta a = 0$ because it reduces to X_a then.

Main results for the linearized map $\delta X_{a,f}$

Assume δa supported in a non-trapping compact set K. Then

- Knowing $\delta X_{a,f}(\delta a, \delta f)$, we can recover the singularities with a loss of one derivative; there is an estimate.
- $\operatorname{Ker} \delta X_{a,f}$ is finite dimensional and is in $C_0^\infty(K)$.
- If $\delta X_{a,f}$ is injective on K, then there is a stability estimate:

$$\|\delta a\|_{H^s(K)} + \|\delta a\|_{H^s(K)} \le C \|\delta X_{a,f}(\delta a, \delta f)\|_{H^{s+3/2}(Z)}.$$

- This estimate is preserved with a uniform C under a slightly stronger condition: $K \subset \Omega$ with Ω *pseudo-convex*: (any compact subset is non-trapping, and $\forall$ compact $K_1 \subset \Omega$, there exists a compact set $K_2 \subset \Omega$ so that every ray in Ω having endpoints over K_1, lies entirely in K_2).

Why δa must be supported in a non-trapping set only? The reason is that $\delta X_{a,f}$ is elliptic where $\delta a = 0$ because it reduces to X_a then.

Main results for the linearized map $\delta X_{a,f}$

Assume δa supported in a non-trapping compact set K. Then

- Knowing $\delta X_{a,f}(\delta a, \delta f)$, we can recover the singularities with a loss of one derivative; there is an estimate.
- $\operatorname{Ker} \delta X_{a,f}$ is finite dimensional and is in $C_0^\infty(K)$.
- If $\delta X_{a,f}$ is injective on K, then there is a stability estimate:

$$\|\delta a\|_{H^s(K)} + \|\delta a\|_{H^s(K)} \leq C\|\delta X_{a,f}(\delta a, \delta f)\|_{H^{s+3/2}(Z)}.$$

- This estimate is preserved with a uniform C under a slightly stronger condition: $K \subset \Omega$ with Ω *pseudo-convex*: (any compact subset is non-trapping, and $\forall$ compact $K_1 \subset \Omega$, there exists a compact set $K_2 \subset \Omega$ so that every ray in Ω having endpoints over K_1, lies entirely in K_2).

Why δa must be supported in a non-trapping set only? The reason is that $\delta X_{a,f}$ is elliptic where $\delta a = 0$ because it reduces to X_a then.

Explicit conditions for injectivity of $\delta X_{a,f}$ (and therefore, for stability):

- **Local Condition:** With $W_0 = u(x,\theta) - u(x,-\theta)$, if

$$W_0(x_0,\theta) = 0 \quad \Longrightarrow \quad \partial_{\theta^\perp} W_0(x_0,\theta) \neq 0,$$

  then $\delta X_{a,f}$ is injective on some neighborhood of x_0. This property guarantees that the rays have non-zero speed at x_0, i.e., x_0 is non-trapping.

- If Ba and u are analytic in K, then $\delta X_{a,f}$ is injective on K.

Explicit conditions for injectivity of $\delta X_{a,f}$ (and therefore, for stability):

- **Local Condition:** With $W_0 = u(x,\theta) - u(x,-\theta)$, if

$$W_0(x_0,\theta) = 0 \quad \Longrightarrow \quad \partial_{\theta^\perp} W_0(x_0,\theta) \neq 0,$$

  then $\delta X_{a,f}$ is injective on some neighborhood of x_0. This property guarantees that the rays have non-zero speed at x_0, i.e., x_0 is non-trapping.
- If Ba and u are analytic in K, then $\delta X_{a,f}$ is injective on K.

Main results for the non-linear map $(a, f) \mapsto X_a f$

Theorem 2 (local uniqueness and stability)

Fix a_0, f_0. Let

- *$a_j - a_0$, $f_j - f_0$, $j = 1, 2$ be supported in a compact set $K \subset \Omega$ — non-trapping (a bit more is needed: pseudo-convex).*
- *$\delta X_{a,f}$ be injective on K.*
- *a_j, f_j are close to a_0, f_0 in the sense*

$$\|B(a_j - a_0)\|_{C^k(\bar{\Omega}\times S^1)} + \|u_j - u_0\|_{C^k(\bar{\Omega}\times S^1)} \leq \varepsilon, \quad j = 1, 2, \quad k \gg 1.$$

Then, if $\varepsilon \ll 1$, $X_{a_1} f_1 = X_{a_2} f_2$ implies $a_1 = a_2$ and $f_1 = f_2$.
Moreover, there is Hölder stability.

The estimate above is just a smallness condition for $a_j - a_0$ and $f_j - f_0$ but a weaker one.

Conditions for injectivity were given before (small K satisfying the local injectivity condition, or analyticity).

Main results for the non-linear map $(a, f) \mapsto X_a f$

Theorem 2 (local uniqueness and stability)

Fix a_0, f_0. Let

- *$a_j - a_0$, $f_j - f_0$, $j = 1, 2$ be supported in a compact set $K \subset \Omega$ — non-trapping (a bit more is needed: pseudo-convex).*
- *$\delta X_{a,f}$ be injective on K.*
- *a_j, f_j are close to a_0, f_0 in the sense*

$$\|B(a_j - a_0)\|_{C^k(\bar{\Omega}\times S^1)} + \|u_j - u_0\|_{C^k(\bar{\Omega}\times S^1)} \le \varepsilon, \quad j = 1, 2, \quad k \gg 1.$$

Then, if $\varepsilon \ll 1$, $X_{a_1} f_1 = X_{a_2} f_2$ implies $a_1 = a_2$ and $f_1 = f_2$. Moreover, there is Hölder stability.

The estimate above is just a smallness condition for $a_j - a_0$ and $f_j - f_0$ but a weaker one.

Conditions for injectivity were given before (small K satisfying the local injectivity condition, or analyticity).

Main results for the non-linear map $(a, f) \mapsto X_a f$

Theorem 2 (local uniqueness and stability)

Fix a_0, f_0. Let

- $a_j - a_0$, $f_j - f_0$, $j = 1, 2$ *be supported in a compact set* $K \subset \Omega$ *— non-trapping (a bit more is needed: pseudo-convex).*
- $\delta X_{a,f}$ *be injective on* K.
- a_j, f_j *are close to* a_0, f_0 *in the sense*

$$\|B(a_j - a_0)\|_{C^k(\bar{\Omega}\times S^1)} + \|u_j - u_0\|_{C^k(\bar{\Omega}\times S^1)} \leq \varepsilon, \quad j = 1, 2, \quad k \gg 1.$$

Then, if $\varepsilon \ll 1$, $X_{a_1} f_1 = X_{a_2} f_2$ *implies* $a_1 = a_2$ *and* $f_1 = f_2$. *Moreover, there is Hölder stability.*

The estimate above is just a smallness condition for $a_j - a_0$ and $f_j - f_0$ but a weaker one.

Conditions for injectivity were given before (small K satisfying the local injectivity condition, or analyticity).

Main results for the non-linear map $(a, f) \mapsto X_a f$

Theorem 2 (local uniqueness and stability)

Fix a_0, f_0. Let

- $a_j - a_0$, $f_j - f_0$, $j = 1, 2$ *be supported in a compact set* $K \subset \Omega$ *— non-trapping (a bit more is needed: pseudo-convex).*
- $\delta X_{a,f}$ *be injective on* K.
- a_j, f_j *are close to* a_0, f_0 *in the sense*

$$\|B(a_j - a_0)\|_{C^k(\bar{\Omega}\times S^1)} + \|u_j - u_0\|_{C^k(\bar{\Omega}\times S^1)} \le \varepsilon, \quad j = 1, 2, \quad k \gg 1.$$

Then, if $\varepsilon \ll 1$, $X_{a_1} f_1 = X_{a_2} f_2$ implies $a_1 = a_2$ and $f_1 = f_2$. Moreover, there is Hölder stability.

The estimate above is just a smallness condition for $a_j - a_0$ and $f_j - f_0$ but a weaker one.

Conditions for injectivity were given before (small K satisfying the local injectivity condition, or analyticity).

A radial example

We study the linearization δX w.r.t. (a, f) near

$$a = 0, \quad f = \mathbf{1}_{B(0,1)},$$

for perturbations of those a and f supported in $B(0,1)$ only. Recall

$$\delta X_{a,f}(\delta a, \delta f) = I_w \delta a + I_1 \delta f$$

with

$$w(x,\theta) = -\sqrt{1-(\theta^\perp \cdot x)^2} - \theta \cdot x.$$

Then

$$W_0 = -2\theta \cdot x.$$

The Hamiltonian H, up to a constant factor, is as in the previous example. Therefore, $-2|\xi|H$ is the symbol of

$$x_1 D_2 - x_2 D_1 = -\mathrm{i}\partial/\partial\phi,$$

ϕ is the polar angle in the x space. The rays are the concentric circles $|x| = R$, $R \geq 0$, including the degenerate case $x = 0$. As before, $K \subset B(0,1)$ is non-trapping, if and only if K does not contain an entire circle of that kind.

A radial example

We study the linearization δX w.r.t. (a, f) near

$$a = 0, \quad f = \mathbf{1}_{B(0,1)},$$

for perturbations of those a and f supported in $B(0,1)$ only. Recall

$$\delta X_{a,f}(\delta a, \delta f) = I_w \delta a + I_1 \delta f$$

with

$$w(x,\theta) = -\sqrt{1-(\theta^\perp \cdot x)^2} - \theta \cdot x.$$

Then

$$W_0 = -2\theta \cdot x.$$

The Hamiltonian H, up to a constant factor, is as in the previous example. Therefore, $-2|\xi|H$ is the symbol of

$$x_1 D_2 - x_2 D_1 = -\mathrm{i}\partial/\partial\phi,$$

ϕ is the polar angle in the x space. The rays are the concentric circles $|x| = R$, $R \geq 0$, including the degenerate case $x = 0$. As before, $K \subset B(0,1)$ is non-trapping, if and only if K does not contain an entire circle of that kind.

A radial example

We study the linearization δX w.r.t. (a, f) near

$$a = 0, \quad f = \mathbf{1}_{B(0,1)},$$

for perturbations of those a and f supported in $B(0, 1)$ only. Recall

$$\delta X_{a,f}(\delta a, \delta f) = I_w \delta a + I_1 \delta f$$

with

$$w(x, \theta) = -\sqrt{1 - (\theta^\perp \cdot x)^2} - \theta \cdot x.$$

Then

$$W_0 = -2\theta \cdot x.$$

The Hamiltonian H, up to a constant factor, is as in the previous example. Therefore, $-2|\xi|H$ is the symbol of

$$x_1 D_2 - x_2 D_1 = -\mathrm{i}\partial/\partial\phi,$$

ϕ is the polar angle in the x space. The rays are the concentric circles $|x| = R$, $R \geq 0$, including the degenerate case $x = 0$. As before, $K \subset B(0, 1)$ is non-trapping, if and only if K does not contain an entire circle of that kind.

Non-trapping and trapping sets:

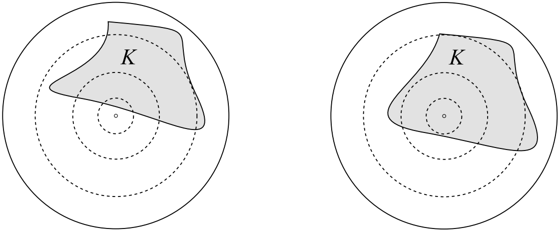


Figure : Left: A non-trapping (and pseudo-convex) set. Small enough perturbations supported in a non-trapping set are recoverable. Actually, $\{\operatorname{supp}\delta a \subset \text{non-trapping}\}$ only is enough.

In the whole ball (which is trapping) the kernel of $\delta X(\delta a, \delta f)$ consists of radial δa and δf supported there connected by (using the Radon transform notation $Rf(p,\omega)$)

$$\sqrt{1-p^2}R\delta a - R\delta f = 0,$$

which is an infinite dimensional space. Indeed, given a radial δa supported in $B(0,1)$, we can solve for δf explicitly (an Abel type of integral operator).

Non-trapping and trapping sets:

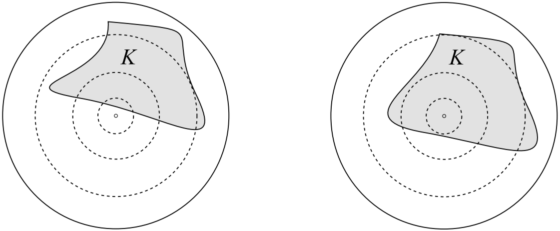


Figure : Left: A non-trapping (and pseudo-convex) set. Small enough perturbations supported in a non-trapping set are recoverable. Actually, $\{\operatorname{supp} \delta a \subset \text{non-trapping}\}$ only is enough.

In the whole ball (which is trapping) the kernel of $\delta X(\delta a, \delta f)$ consists of radial δa and δf supported there connected by (using the Radon transform notation $Rf(p, \omega)$)

$$\sqrt{1-p^2} R\delta a - R\delta f = 0,$$

which is an infinite dimensional space. Indeed, given a radial δa supported in $B(0,1)$, we can solve for δf explicitly (an Abel type of integral operator).

Non-trapping and trapping sets:

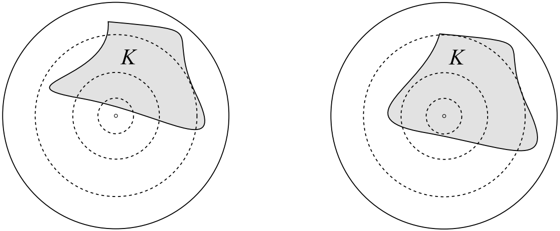


Figure : Left: A non-trapping (and pseudo-convex) set. Small enough perturbations supported in a non-trapping set are recoverable. Actually, $\{\operatorname{supp}\delta a \subset \text{non-trapping}\}$ only is enough.

In the whole ball (which is trapping) the kernel of $\delta X(\delta a, \delta f)$ consists of radial δa and δf supported there connected by (using the Radon transform notation $Rf(p,\omega)$)

$$\sqrt{1-p^2}R\delta a - R\delta f = 0,$$

which is an infinite dimensional space. Indeed, given a radial δa supported in $B(0,1)$, we can solve for δf explicitly (an Abel type of integral operator).

Non-trapping and trapping sets:

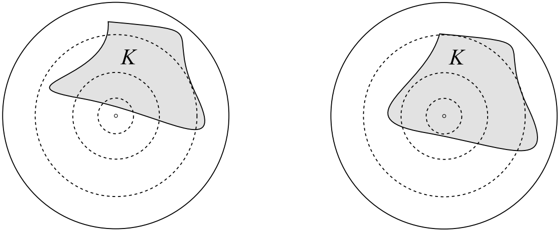


Figure : Left: A non-trapping (and pseudo-convex) set. Small enough perturbations supported in a non-trapping set are recoverable. Actually, $\{\mathrm{supp}\, \delta a \subset$ non-trapping$\}$ only is enough.

In the whole ball (which is trapping) the kernel of $\delta X(\delta a, \delta f)$ consists of radial δa and δf supported there connected by (using the Radon transform notation $Rf(p, \omega)$)

$$\sqrt{1-p^2} R\delta a - R\delta f = 0,$$

which is an infinite dimensional space. Indeed, given a radial δa supported in $B(0,1)$, we can solve for δf explicitly (an Abel type of integral operator).

Non-uniqueness for general radial a and f

Next two theorems serve as an example that for non-trapping domains, the uniqueness may fail (by more than a finite dimensional space).

Theorem 3 (non-uniqueness for the linearization)

Let $f \in C_0^\infty$ be radial. Then $\delta X_{0,f}$ has an infinite dimensional kernel.

One can write an explicit integral formula to compute δf, given δa.

Non-uniqueness for general radial a and f

Next two theorems serve as an example that for non-trapping domains, the uniqueness may fail (by more than a finite dimensional space).

Theorem 3 (non-uniqueness for the linearization)

Let $f \in C_0^\infty$ be radial. Then $\delta X_{0,f}$ has an infinite dimensional kernel.

One can write an explicit integral formula to compute δf, given δa.

Theorem 4 (non-uniqueness for the non-linear problem)

Let $a \in C_0^\infty$ and $f \in C_0^\infty$ be radial. Then there exists a radial $f_0 \in C_0^\infty$ so that

$$X_a f = X_0 f_0.$$

Again, one can write an explicit integral formula to compute f_0, given a and f. A simple but a non-constructive proof is to observe that the l.h.s. is an even C_0^∞ function of $r = |x|$, and therefore, in the range of X_0.

The method (developed with Luo and Jianliang) works!

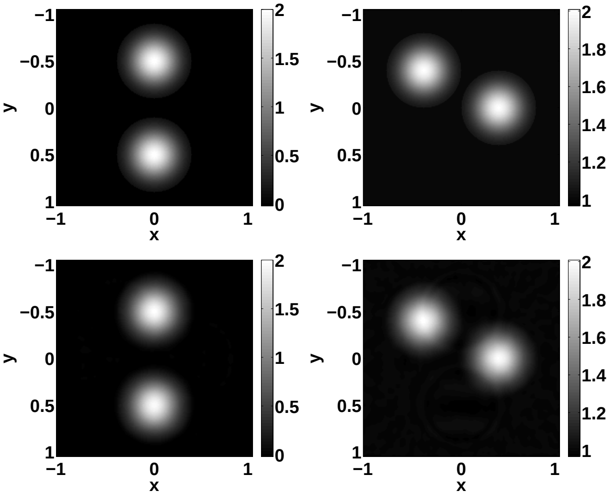


Figure : "Good guess" $a_0 = 0$, $f_0 = 1$. Top row: original a and f; bottom row: the reconstructed ones with good guesses

Non-uniqueness for radial (a, f)

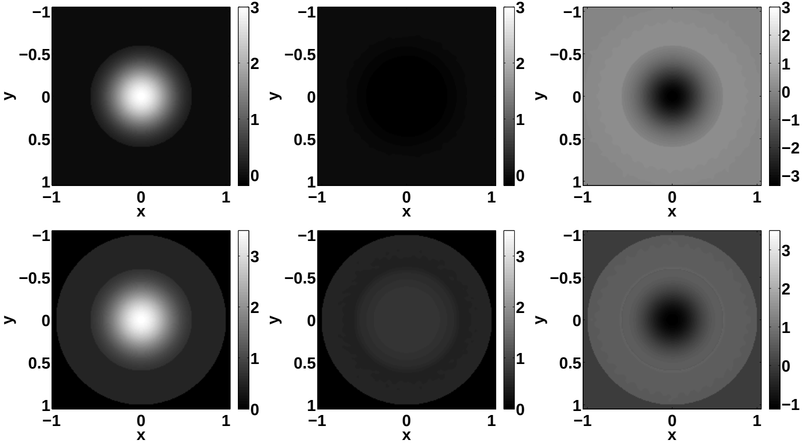


Figure : Non-uniqueness for radial (a, f). Top row: a, bottom row: f. Fist column: exact a and f; second and third columns: computed a, f with different initial guesses. The reconstructions are totally wrong.

Ill-posedness for perturbed radial (a, f)

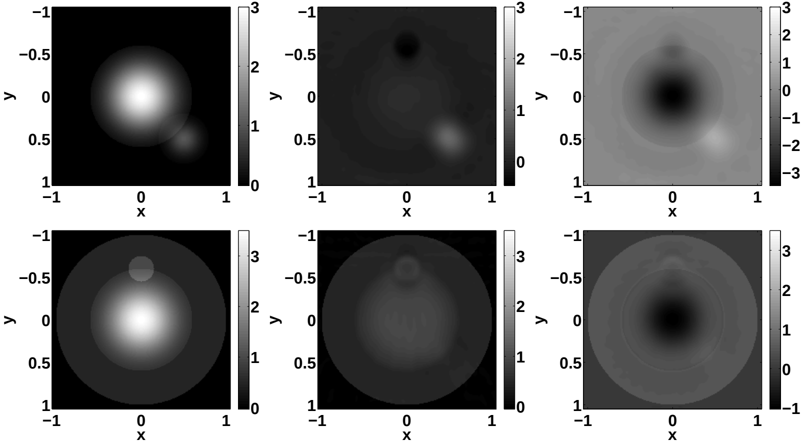


Figure : Ill-posedness for perturbed radial (a, f). Fist column: exact a and f; second and third columns: computed a, f with different choices of initial guesses. The reconstructions are very poor.

Ill posedness with circular rays

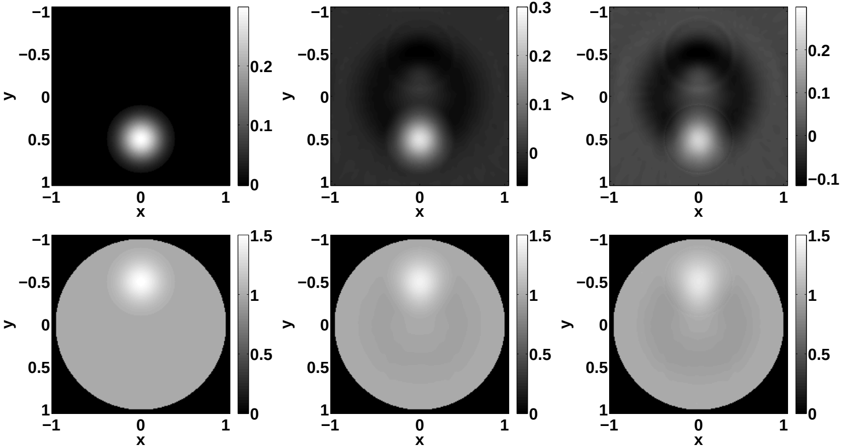


Figure : Ill posedness with circular rays. Top row: a, bottom row: f. Fist column: exact a and f; second and third columns: computed a, f with different initial guesses. The artifacts are circular, dictated by the Hamiltonian flow.

Stabilized example with circular rays

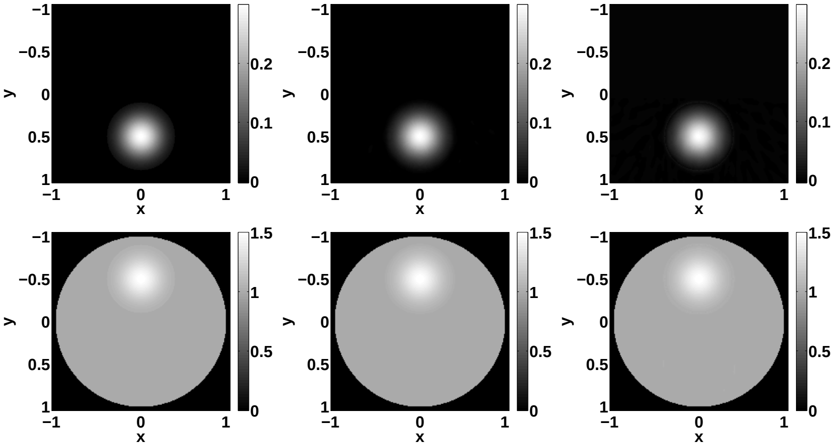


Figure : A stabilized example with circular rays. A constraint condition for $\operatorname{supp} a$ to be in $y < 0$ was used. The initial guesses were the same as in Figure 8. Top row: a, bottom row: f. Fist column: exact a and f; second and third columns: computed a, f with different initial guesses.

In the examples above, the determinant $W(x, \xi)$ has a non-degenerate characteristic variety $W = 0$. The Hamiltonian flow (projected on the x-space) consists of concentric circles.

It is possible to have an open set in the phase space, where $W = 0$. Then those zeros are stationary and we can expect high instability.

In the examples above, the determinant $W(x, \xi)$ has a non-degenerate characteristic variety $W = 0$. The Hamiltonian flow (projected on the x-space) consists of concentric circles.

It is possible to have an open set in the phase space, where $W = 0$. Then those zeros are stationary and we can expect high instability.

Large open set of zeros of W

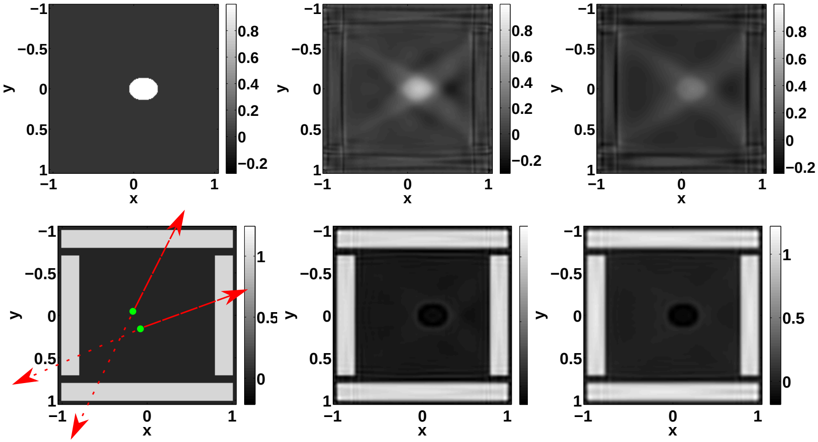


Figure : Large open set of unstable points. Top row: a, bottom row: f. Fist column: exact a and f; 2nd and 3rd column: computed with guess zero for both a and f with the LBFGS/AG solver. With the LBFGS solver, the recovered a takes values in the range $(-0.31, 0.70)$ instead of $(0, 1)$, while with the AG solver, that range is $(-0.25, 0.34)$. Note the black spot in the reconstructed f.

A smaller open set of zeros of W

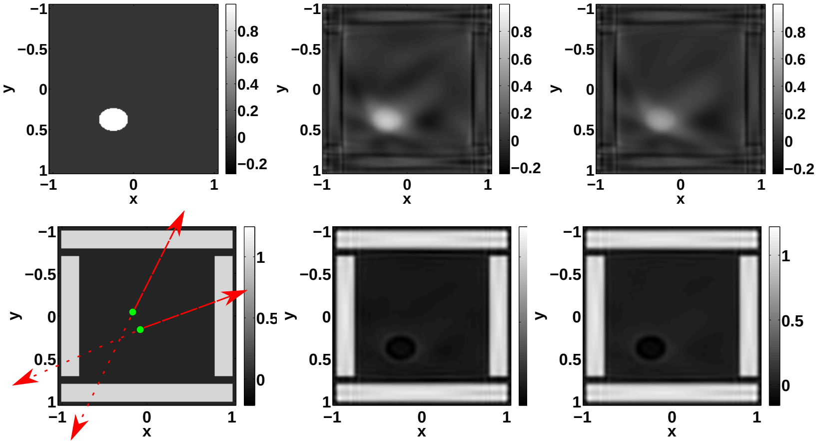


Figure : The phantom was moved closer to a corner compared to the previous example. This decreases the set of the unstable points (the zeros of W. Top row: a, bottom row: f. Fist column: exact a and f; second column: computed with guess zero for both a and f with the LBFGS solver; computed with guess zero for both a and f with the AG solver.

A less unstable case; two walls removed

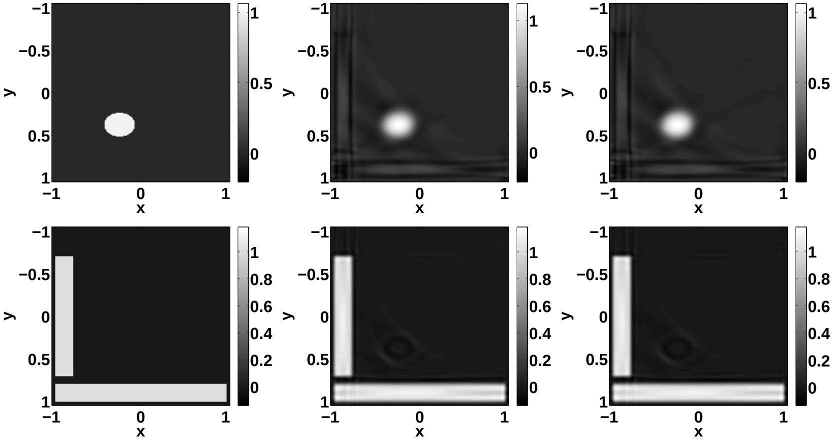


Figure : We removed two "walls". This changes the properties of W dramatically. The set of zeros is much smaller.

A less unstable case; two walls removed

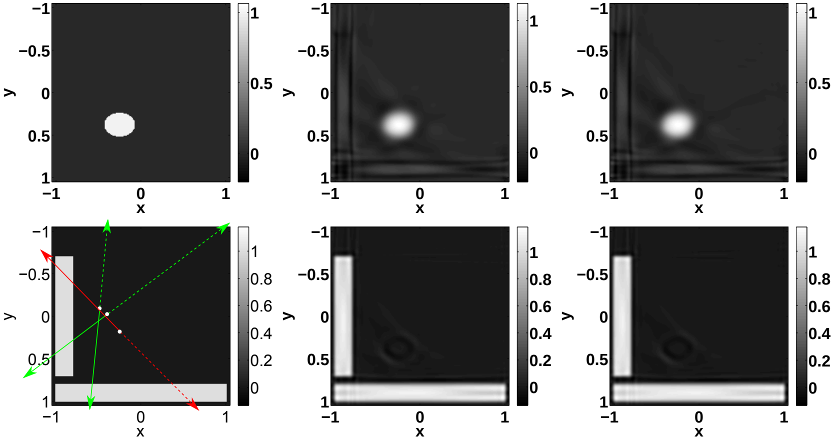


Figure : We removed two “walls”. This changes the properties of W dramatically. The set of zeros is much smaller.

Another less unstable case

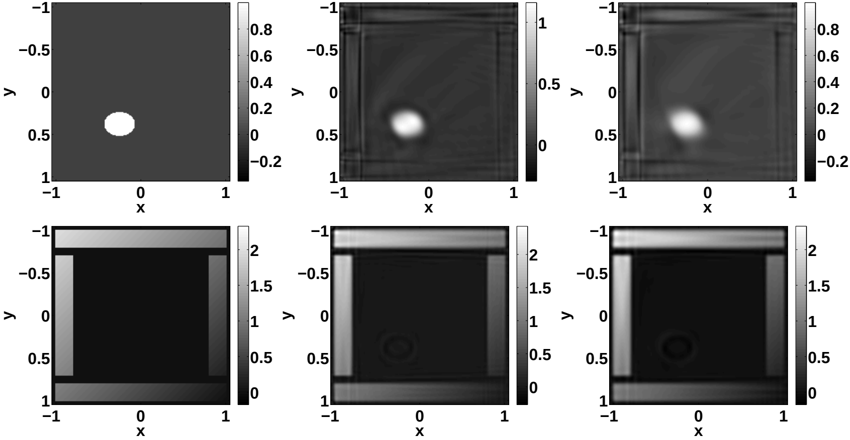


Figure : Back to four "walls" but the have linearly changing density. The zeros of W are (almost) a set of lower dimension: two opposite θ's at each x.

Another less unstable case

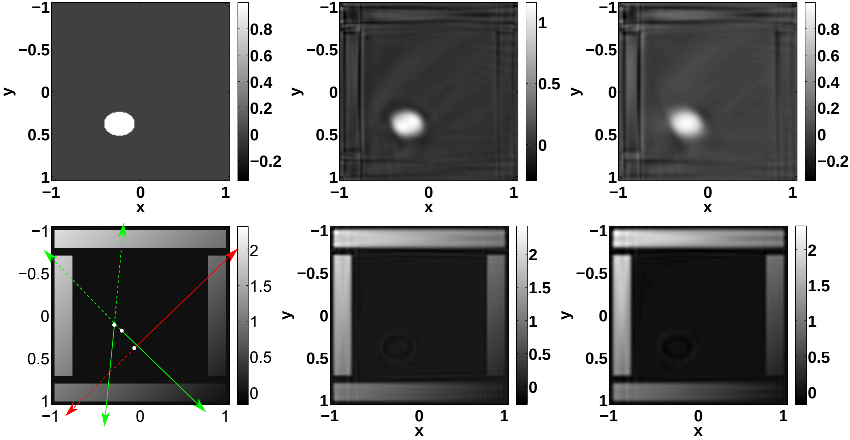


Figure : Back to four "walls" but the have linearly changing density. The zeros of W are (almost) a set of lower dimension: two opposite θ's at each x.