Traveltime Tomography with Local Data

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Joint work with Gunther Uhlmann and Andras Vasy

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The Boundary Rigidity Problem (Travel Time Tomography)

Let M be a compact domain (manifold) with boundary. Let $g = \{g_{ij}\}$ be a Riemannian metric on M. Let $d_g(x, y)$ be the distance between any two boundary points x, y (in the metric g).

Boundary Rigidity: *Does d_g, known on $\partial M \times \partial M$,* determine g uniquely?

In fact, any isometry fixing ∂M pointwise does not change d_g, so in the question above, we want to determine g up to such an isometry.

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Travel Time Seismology

This problem was first studied in the beginning of the 20th century by **Herglotz**, and **Wiechert & Zoeppritz** in an attempt to recover the inner structure of the Earth from travel times of seismic waves. They solved explicitly a partial case of this problem: when M is a ball, and g is a radially symmetric isotropic metric, i.e.,

$$ds^2 = c^{-2}(r)dx^2, \quad r := |x|.$$

They imposed the following condition:

$$\frac{d}{dr}\frac{r}{c(r)} > 0.$$

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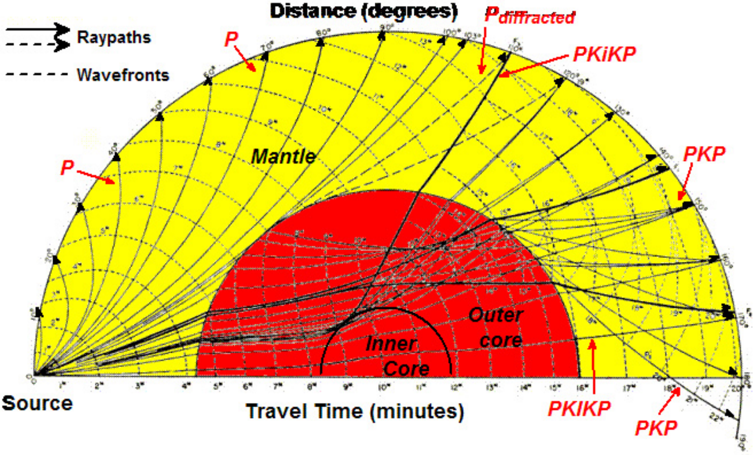

Figure : Travel times of P-waves through Earth. Picture taken from the web page of L. Braile, Purdue University.

Boundary rigidity has obvious counter-examples, like a very slow region inside, which would not affect the boundary distance function. Some conditions are needed.

One of those conditions is g to be simple :

- $\forall(x, y) \in \partial M \times \partial M$, $\exists$ unique minimizing geodesic connecting x, y, depending smoothly on (x, y) (i.e., no caustics);
- ∂M is strictly convex.

The best known results so far are

- boundary rigidity for simple metrics in 2D (Pestov & Uhlmann);
- boundary rigidity for simple metrics near a metric in a generic set, including the real analytic (simple) ones (S & Uhlmann);
- near metrics with an explicit upper bound on the curvature (lassas, Sharafutdinov & Uhlmann).

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Assume that g is isotropic, i.e., $g_{ij}(x) = c^{-2}(x)\delta_{ij}$. Physically, this corresponds to a variable wave speed that does not depend on the direction of propagation. In the class of the isotropic metrics, we do not have the freedom to apply isometries and we would expect g to be uniquely determined. This is known to be true for simple metrics (MUKHOMETOV, ROMANOV, et al.) More generally, we can fix g_0 and we have uniqueness of the recovery of the conformal factor $c(x)$ in $c^{-2}g_0$.

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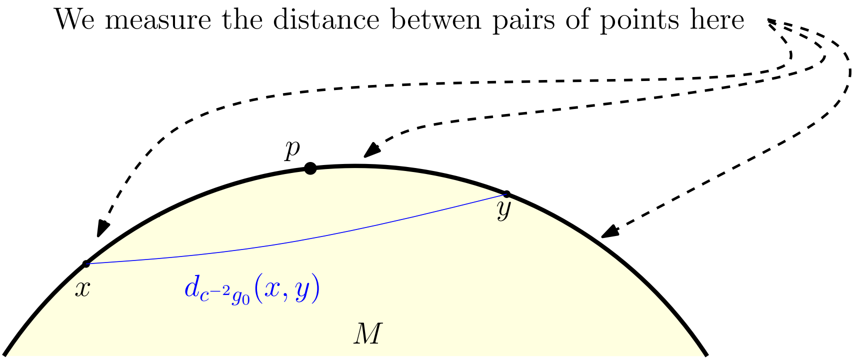
The problem we consider in this work is the partial data problem of recovery of a conformal factor:

Boundary Rigidity with partial data:

Does $d_{c^{-2}g_0}$, known on $\partial M \times \partial M$ near some p, determine $c(x)$ near p uniquely?



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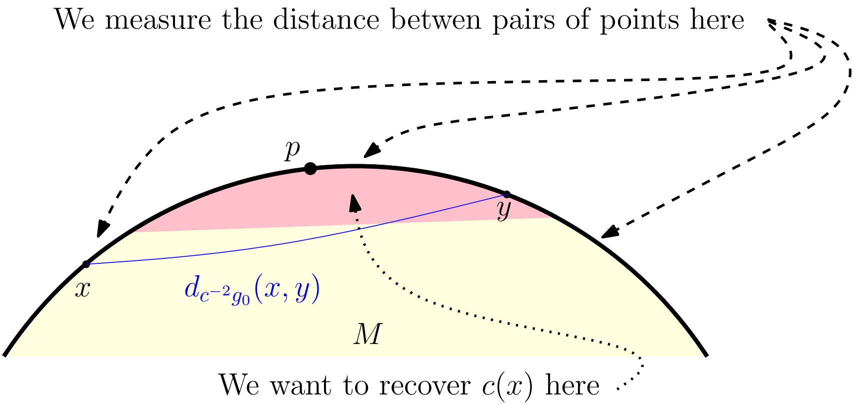

Local boundary rigidity

Theorem 1 (S-Uhlmann-Vasy)

Let $\dim M \geq 3$*. If* ∂M *is strictly convex near* p*, and* $d = \tilde{d}$ *near* (p, p)*, then* $c = \tilde{c}$ *near* p*.*

Moreover, there is conditional Hölder stability.

In particular, this theorem shows that the local geophysics problem (modeled that way) is solvable. Of course, not knowing that fact until now did not prevent many people from "solving it".

The only results so far of similar nature is for real analytic metrics (Lassas, Sharafutdinov & Uhlmann). We can recover the whole jet of the metric at ∂M and then use analytic continuation. This is the first local result without analyticity assumptions.

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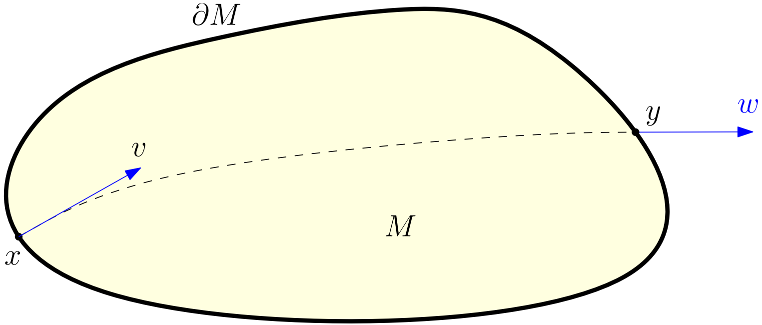
Lens Rigidity

Define the scattering relation L and the length (travel time) function ℓ:



$$L : (x, v) \to (y, w), \quad \ell(x, v) \to [0, \infty].$$

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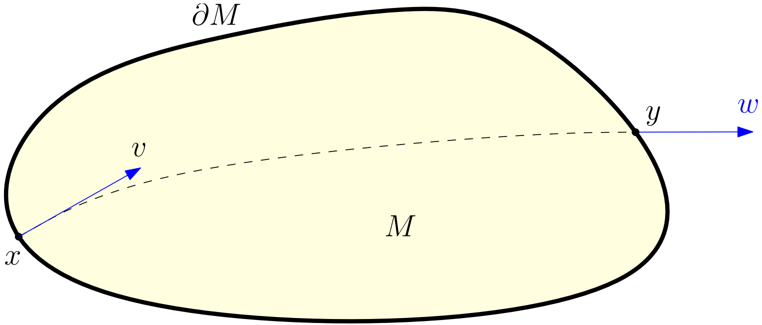

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No, in general but the counterexamples are harder to construct.

The lens rigidity problem and the boundary rigidity one are equivalent for simple metrics! Indeed, then $d_g(x, y)$, known for x, y on ∂M determines σ, ℓ uniquely, and vice-versa. This is also true locally, near a point p where ∂M is strictly convex.

For non-simple metrics (caustics and/or non-convex boundary), the Lens Rigidity is the right problem to study.

There are fewer results: local generic rigidity near a class of non-simple metrics (S & Uhlmann) , the torus is lens rigid (Croke).

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Does the lens relation L, known for points near p, and "almost tangent directions" determine $c(x)$ near p uniquely?

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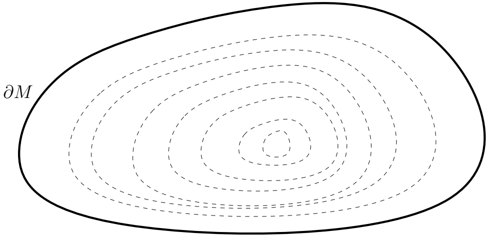
Global Lens Rigidity under the foliation condition

We could use a layer stripping argument to get deeper and deeper in M and prove that one can determine c in the whole M.

Foliation condition:

M is foliated by strictly convex hypersurfaces if, up to a nowhere dense set, $M = \cup_{t\in[0,T)}\Sigma_t$, where Σ_t is a smooth family of strictly convex hypersurfaces and $\Sigma_0 = \partial M$.



A more general condition: several families, starting form outside M. Also, enough $M \setminus$ foliation to be simple.

Global Lens Rigidity under the foliation condition

Theorem 2 (S-Uhlmann-Vasy)

Let $\dim M \geq 3$*, let* $c = \tilde{c}$ *on* ∂M*, let* ∂M *be strictly convex with respect to both* $g = c^{-2}g_0$ *and* $\tilde{g} = \tilde{c}^{-2}g_0$*. Assume that* M *can be foliated by strictly convex hypersurfaces for* g*. Then if* $L = \tilde{L}$ *on* $\partial_- SM$*, we have* $c = \tilde{c}$ *in* M*.*

Moreover, there is conditional Hölder stability.

It is enough to be able to foliate just a part of M, if what is left is a simple manifold.

A more general result compared to MUKHOMETOV'S one: conjugate points inside are allowed, or even trapped geodesics. Example: a tubular neighborhood of a periodic geodesic on a negatively curved manifold.

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Idea of the proof

The proof is based on two main ideas.

First, we use the approach in a recent paper by Uhlmann and Vasy on the linear integral geometry problem.

Second, we convert the non-linear boundary rigidity problem to a "pseudo-linear" one. Straightforward linearization, which works for the problem with full data, fails here.

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Uhlmann and Vasy's result

They consider the inversion of the geodesic ray transform

$$If(\gamma) = \int f(\gamma(s))\, ds$$

known for geodesics intersecting some neighborhood of $p \in \partial M$ (where ∂M is strictly convex) "almost tangentially". Then they prove that those integrals determine f near p uniquely. It is a Helgason support type of theorem for non-analytic curves! This was extended recently by H. Zhou for arbitrary curves (∂M must be strictly convex w.r.t. them) and non-vanishing weights.

The main trick in Uhlmann and Vasy is the following revolutionary idea: Introduce an artificial, still strictly convex boundary near p which cuts a small subdomain near p. Then use Melorse's scattering calculus to show that the I, composed with a suitable "back-projection" is elliptic in that calculus. Since the subdomain is small, it would be invertible as well.

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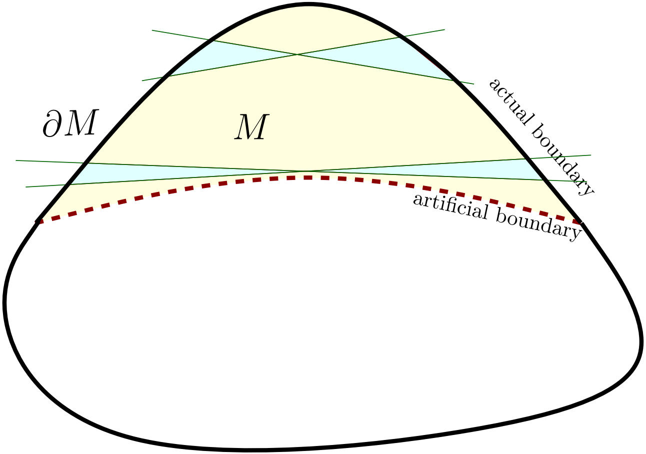
Consider

$$Pf(z) = \int_{S_z M} x^{-2} \chi(x, z, \gamma_{z,v}) If(\gamma_{z,v}) dv,$$

where χ is a smooth cutoff sketched below (angle $\sim x$), and x is the distance to the artificial boundary. In fact, $P = I^* x^{-2} \chi I$.



Main idea of Uhlmann and Vasy

Note that the support of the weight χ decreases when we approach the artificial boundary. From classical ΨDO point of view, this is an elliptic ΨDO away from the artificial boundary (here, $n \geq 3$ is important!) since the conormals of the geodesic we use cover the conormal bundle in the region of interest. The classical ΨDO calculus however is not good (without modifications, at least) to work on manifolds with boundary.

Instead, Uhlmann and Vasy proposed that we use Melrose's scattering calculus! This calculus works perfectly, and P is elliptic there. Next, if the artificial boundary is close enough to the real one (small domain), elliptic operators are actually invertible. Hence the uniqueness, and even stability!

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The scattering ΨDO calculus can be defined as a version of the classical ΨDO calculus on $\mathbf{R}^n_x$ with compactification of $\mathbf{R}^n_x \times \mathbf{R}^n_\xi$.
Consider ΨDOs with symbols $a(z,\zeta)$ satisfying symbol-like estimates both w.r.t. z and ζ (Hörmander, Parenti, Shubin)

$$|\partial_z^\alpha \partial_\zeta^\beta a(z,\zeta)| \le C_{\alpha\beta} \langle z\rangle^\ell \langle \zeta\rangle^m.$$

This defines the class $S^{\ell,m}(\mathbf{R}^n \times \mathbf{R}^n)$. Lower order in this calculus means both lower order of differentiation and smaller growth at ∞.

The scattering class of ΨDOs is obtained from this one by compactification of both $\mathbf{R}^n_x$ and $\mathbf{R}^n_\xi$.

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In polar coordinates $r\omega$, $r > 0$, $\omega \in S^{n-1}$, replace r by $x := 1/r$. Then a neighborhood of ∞ becomes a neighborhood of 0, i.e., $0 < x < C$; and $x = 0$ is the "infinite boundary".

If one parameterizes S^{n-1} locally by $y \in \mathbf{R}^{n-1}$, then we have coordinates

$$(x, y) \in \mathbf{R}_+ \times \mathbf{R}^{n-1} =: \mathbf{R}^n_+.$$

The standard basic vector fields $\partial/\partial r$, $\partial/\partial(ry^j)$ take the form:

$$x^2 \frac{\partial}{\partial x}, \quad x \frac{\partial}{\partial y^j},$$

and they are complete, tangent to $x = 0$ and unit.

We do that both for z and its dual, ζ. Then the class $\Psi^{\ell,m}(\mathbf{R}^n)$ becomes the class $\Psi^{\ell,m}_{\mathrm{sc}}(\mathbf{R}^n_+)$ with symbols in $S^{\ell,m}_{\mathrm{sc}}(\mathbf{R}^n_+ \times \mathbf{R}^n_+)$. This can be done on manifolds with boundary, as well.

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$$(x, y) \in \mathbf{R}_+ \times \mathbf{R}^{n-1} =: \mathbf{R}^n_+.$$

The standard basic vector fields $\partial/\partial r$, $\partial/\partial(ry^j)$ take the form:

$$x^2 \frac{\partial}{\partial x}, \quad x\frac{\partial}{\partial y^j},$$

and they are complete, tangent to $x = 0$ and unit.

We do that both for z and its dual, ζ. Then the class $\Psi^{\ell,m}(\mathbf{R}^n)$ becomes the class $\Psi^{\ell,m}_{\mathrm{sc}}(\mathbf{R}^n_+)$ with symbols in $S^{\ell,m}_{\mathrm{sc}}(\mathbf{R}^n_+ \times \mathbf{R}^n_+)$. This can be done on manifolds with boundary, as well.

The scattering ΨDO calculus of Melrose

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"Pseudo-linearization"

Back to the non-linear boundary (or lens) rigidity problem. We may try to solve it by linearization, which is just the operator I. This works well for the global problem (full data) but it is very problematic for the local one (partial data). The problem is that the stability for the linear problem degenerates at the artificial boundary. Next, different speed have different geodesics, so that remainder in the linearization cannot be restricted to the same domain.

Instead, we reduce the problem to the inversion of a weighted version of I acting on $c - \tilde{c}$ (two speeds with the same data), and the weight depends of those speeds as well. This operator comes from an identity in a 1998 paper by S & Uhlmann.

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The "pseudo-linear identity"

Let V and $\tilde{V}$ be two vector fields, and $X(t, X^{(0)})$ be the solution with initial condition $X^{(0)}$. Then

$$\boxed{\begin{aligned}&\tilde{X}\left(t, X^{(0)}\right) - X\left(t, X^{(0)}\right)\\ &\quad = \int_0^t \frac{\partial \tilde{X}}{\partial X^{(0)}}\left(t - s, X(s, X^{(0)})\right)\left(V - \tilde{V}\right)\left(X(s, X^{(0)})\right) ds.\end{aligned}}$$

The beauty of this identity is that it is linear in $V - \tilde{V}$ (with weight depending on V and $\tilde{V}$).

We take V, $\tilde{V}$ to be the Hamiltonian vector fields. Same lens data $\Longrightarrow$ the l.h.s. is zero. We get a weighed integral of the $V - \tilde{V}$ then. The last n components give us $\partial(c - \tilde{c})$. But we can invert this. It is weighted version of Uhlmann and Vasy's result.

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The proof is just to apply the fundamental theorem of calculus

$$F(t) - F(0) = \int_0^t F'(s)\,ds$$

to the function

$$F(s) = \tilde{X}\Big(t - s, X(s, X^{(0)})\Big)\,.$$

Example

HERGLOTZ and WIECHERT & ZOEPPRITZ showed that one can determine a radial speed $c(r)$ in the ball $B(0,1)$ satisfying

$$\boxed{\frac{d}{dr}\frac{r}{c(r)} > 0.}$$

The uniqueness is in the class of the radial speeds.

One can check directly that their condition is equivalent to the following one: the Eucledian spheres $\{|x| = t\}$, $t \le 1$ are strictly convex for $c^{-2}dx^2$ as well. Then $B(0,1)$ satisfies the foliation condition. Therefore, if $\tilde{c}(x)$ is another speed, not necessarily radial, with the same lens relation, equal to c on the boundary, then $c = \tilde{c}$. There could be conjugate points.

Therefore, speeds satisfying the HERGLOTZ and WIECHERT & ZOEPPRITZ condtion are conformally lens rigid.

Also, so is some neighborhood of those speeds in C^k, $k \gg 1$.

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