Travel Time Tomography and Tensor Tomography, II

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Solving an Inverse Problem through linearization

First, this is not the only way to solve an inverse problem. But it is one of the most used.

For more details, see Linearizing non-linear inverse problems and an application to inverse backscattering (with Gunther Uhlmann), *J. Funct. Anal.* 256(9)(2009), 2842–2866.

Consider the following "inverse problem." Let $\mathcal{A} : \mathcal{B}_1 \to \mathcal{B}_2$ (Banach spaces).

$$\text{Given } h \in \operatorname{Ran}(\mathcal{A}), \text{ find } f \text{ so that } \mathcal{A}(f) = h. \tag{1}$$

We want to prove local uniqueness near some (and hopefully all) f_0, i.e., that

$$\mathcal{A}(f_1) = \mathcal{A}(f_2) \quad \text{for } f_{1,2} \text{ close to } f_0 \quad \Longrightarrow \quad f_1 = f_2.$$

The first thing that comes to mind is to see if the derivative is "non-zero." Let A_f be the differential of $\mathcal{A}$ (the Gâteaux derivative) at f (we assume that it exists), i.e., A_f is a linear operator given by

$$A_f h = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}(\mathcal{A}(f + \varepsilon h) - \mathcal{A}(f)).$$

Then the local uniqueness near f_0 is often associated with the injectivity of A_{f_0}. That can be wrong!

A finitely dimensional "inverse problem."

Let $\mathcal{A} : \mathbf{R}^n \to \mathbf{R}^m$, $\mathcal{A} \in C^2$. Then

$$\mathcal{A}(x) = \mathcal{A}(x_0) + A_{x_0}(x - x_0) + R_{x_0}(x) \quad \text{with } |R_{x_0}(x)| \le C_{x_0}|x - x_0|^2,$$

for x near x_0. Assume now that A_{x_0} is injective (then $m \ge n$). This immediately implies

$$|h| \le C|A_{x_0}h|, \quad \forall h \in \mathbf{R}^n.$$

Therefore,

Injectivity implies stability (of the linear problem) in finite dimensions.

Also, it implies local uniqueness and stability for the original non-linear problem. Indeed, assuming $\mathcal{A}(x) = \mathcal{A}(x_0)$, we get

$$|x - x_0| \le C|A_{x_0}(x - x_0)| \le C|x - x_0|^2$$

and the local uniqueness follows easily. Similarly, if $|x - x_0| \ll 1$, one gets

$$|x - x_0| \le 2C|\mathcal{A}(x) - \mathcal{A}(x_0)|. \tag{2}$$

One can replace $\mathbf{R}^m$ here by an ∞-dim space. In particular, we get (in the original formulation, $\mathcal{A} : \mathcal{B}_1 \to \mathcal{B}_2$), that if $\mathcal{B}_1$ is finite dimensional (there is no need $\mathcal{B}_2$ to be finitely dimensional, too), we have trivially Lipschitz stability for any inverse problem with an injective differential!

Now, if $\dim \mathcal{B}_1 = \infty$, this is no longer true in general. First,

Injectivity does not imply stability,

i.e., an estimate of the type

$$\boxed{\|h\|_{\mathcal{B}_1} \leq C\|A_{f_0}h\|_{\mathcal{B}_2}.}$$

Second,

Without stability, we may not have local uniqueness (forget about any type of stability) for the non-linear problem.

So if we want to use those type of arguments, we need to prove injectivity **and stability** of A_f. Under those conditions, the arguments above work. This is known as *the local injectivity theorem*.

Most inverse problems are ill-posed however, and this approach rarely works the way described. It is typical that we can only prove stability in different (reasonable) norms:

$$\|h\|_{\mathcal{B}_1'} \le C\|A_{f_0}h\|_{\mathcal{B}_2'}, \quad \forall h \in \mathcal{B}_1,$$

where $\mathcal{B}_{1,2}$ are different Banach spaces that cannot be replaced with the original ones. Then we can still prove local uniqueness (and Hölder stability) under an additional regularity condition. Remember, the decisive argument was that the linearization holds up to a quadratic error $O(|f-f_0|^2)$, while on the left we had $|f-f_0|$. So at some point we get

$$\|f-f_0\| \le C\|f-f_0\|^2,$$

that implies $f=f_0$ if $\|f-f_0\| \ll 1$. We can afford to replace the power 2 by $1+\varepsilon$, $\varepsilon>0$, and this argument still works!

This suggests that we should use interpolation estimates of the kind

$$\|f\|_{H^s} \le C\|f\|_{H^{s_1}}^{\alpha_1}\|f\|_{H^{s_2}}^{\alpha_2},$$

where $\alpha_1 s_1 + \alpha_2 s_2 = 1$, $\alpha_1+\alpha_2=1$, $\alpha_j>0$.

Here is an example. Suppose that $\mathcal{A} : L^2 \to L^2$. On the other hand, assume that A_f is injective for any f but we can only show that

$$\|h\|_{L^2} \le C\|A_f h\|_{H^1}. \tag{3}$$

What we really want is

$$\|h\|_{L^2} \le C\|A_f h\|_{L^2}, \tag{4}$$

but that might not be true. If we can show that $\mathcal{A} : L^2 \to H^1$, then we just choose $\mathcal{B}_2 = H^1$, and we can proceed as before. But if we cannot, it is time to use interpolation estimates:

$$\|A_f h\|_{H^1} \le C\|A_f h\|_{L^2}^{\alpha}\,\|A_f h\|_{H^s}^{1-\alpha}$$

with $s(1-\alpha) = 1$. Now, if we assume that h belongs to a subspace so that

$$\|A_f h\|_{H^s} \le C_0,$$

then

$$\|A_f h\|_{H^1} \le C_0'\|A_f h\|_{L^2}^{\alpha}.$$

Then we get from (3),

$$\|h\|_{L^2} \le C\|A_f h\|_{L^2}^{\alpha}.$$

Compare this to (4). It is similar, but the power 1 is replaced by $\alpha < 1$.

We can still use

$$\|h\|_{L^2} \leq C\|A_f h\|_{L^2}^{\alpha}.$$

in our proof of uniqueness, where an important role was played by the inequality

$$\|f - f_0\| \leq C\|A_{f_0}(f - f_0)\| \leq C\|f - f_0\|^2.$$

In our case, we have instead

$$\|f - f_0\| \leq C\|A_{f_0}(f - f_0)\| \leq C\|f - f_0\|^{2\alpha}$$

and it is enough to have $2\alpha > 1$. This can be achieved if $s > 2$ (remember that $s(1 - \alpha) = 1$). So, for this to work we need

$$\|A_f h\|_{H^{2+\delta}} \leq C_0, \quad \text{with some } \delta > 0.$$

A typical situation is that A_f is e ΨDO of order -1; then the estimate above holds if

$$\|f\|_{H^{1+\delta}} \leq C_1, \tag{5}$$

for f near f_0, with some $C_1 > 0$. Let us say that we work on a compact domain (manifold) with or without boundary. Then (5) restricts f to a compact subset and appears as an additional assumption. The resulting stability estimate is called a *conditional stability estimate*.

Conditions for local uniqueness and stability

Theorem 1 (weak local uniqueness and stability)

Assume that $\mathcal{A}$ has a derivative at f_0 with quadratic estimate on the remainder. Let

$$\|h\|_{\mathcal{B}_1'} \le C\|A_{f_0}h\|_{\mathcal{B}_2'}, \quad \forall h \in \mathcal{B}_1. \tag{6}$$

Assume also that there exist Banach spaces $\mathcal{B}_2'' \subset \mathcal{B}_2'$, $\mathcal{B}_1'' \subset \mathcal{B}_1$ so that $A_{f_0} : \mathcal{B}_1'' \to \mathcal{B}_2''$ and the following interpolation estimates hold

$$\|u\|_{\mathcal{B}_2'} \le C\|u\|_{\mathcal{B}_2}^{\mu_2}\|u\|_{\mathcal{B}_2''}^{1-\mu_2}, \quad \|h\|_{\mathcal{B}_1} \le C\|h\|_{\mathcal{B}_1'}^{\mu_1}\|h\|_{\mathcal{B}_1''}^{1-\mu_1} \quad \mu_1, \mu_2 \in (0,1], \quad \mu_1\mu_2 > 1/2.$$

Then for any $K > 0$ there exists $\epsilon > 0$, so that for any f with

$$\|f - f_0\|_{\mathcal{B}_1} \le \epsilon, \quad \|f\|_{\mathcal{B}_1''} \le K, \tag{7}$$

one has the conditional stability estimate

$$\|f - f_0\|_{\mathcal{B}_1} \le C(K)\|\mathcal{A}(f) - \mathcal{A}(f_0)\|_{\mathcal{B}_2}^{\mu_1\mu_2}, \quad C(K) = CK^{2-\mu_1-\mu_2}. \tag{8}$$

In particular, there is a weak local uniqueness near f_0, i.e., if $\mathcal{A}(f) = \mathcal{A}(f_0)$, then $f = f_0$.

Theorem 2 (Strong local uniqueness and stability)

Assume in addition that there is a Banach space $\mathcal{K} \subset \mathcal{B}_1''$ so that (6) holds for f_0 replaced with f close enough to f_0 in $\mathcal{K}$, and $A_f : \mathcal{B}_1'' \to \mathcal{B}_2''$ is uniformly bounded for such f. Then there exists $\epsilon > 0$, so that for any f_1, f_2 with

$$\|f_1 - f_0\|_{\mathcal{K}} \le \epsilon, \quad \|f_2 - f_0\|_{\mathcal{K}} \le \epsilon, \tag{9}$$

one has the conditional stability estimate

$$\|f_1 - f_2\|_{\mathcal{B}_1} \le C\|\mathcal{A}(f_1) - \mathcal{A}(f_2)\|_{\mathcal{B}_2}^{\mu_1\mu_2}. \tag{10}$$

In particular, there is a strong local uniqueness near f_0, i.e., if $\mathcal{A}(f_1) = \mathcal{A}(f_2)$, then $f_1 = f_2$.

Here the compactness type of condition is hidden in the assumption that $f_{1,2} \in \mathcal{K} \subset \mathcal{B}_2''$.

A short version of what we did so far:

- Proving injectivity of the linearization is not enough.
- We need also a stability estimate.
- That estimate may not be in the original norms. As long as it is in some H^s or C^k spaces, the whole approach still works.
- If the estimate is not in the original norms, we pay a price: we have to assume higher regularity of the coefficients.

Back to Tensor Tomography

The linearized problem

If g is simple, and for the geodesic X-ray transform

$$I_g f(\gamma) = \int f_{ij}(\gamma(t))\dot\gamma^i(t)\dot\gamma^j(t)\,dt$$

we have $I_g f(\gamma) = 0$ for all max geodesics γ in M, show that $f = dv$ with $v|_{\partial M} = 0$.

First, we will project onto the space of tensors (called solenoidal) orthogonal to all such dv's (called potential). There is a natural definition of L^2 spaces of tensors of a fixed order involving the volume measure. We want to describe all f so that $(f, dv) = 0$. Integrate by parts to get

$$\delta f = 0, \tag{11}$$

where δ is the divergence operator sending 2-tensors to 1-tensors (forms); given in local coordinates by

$$[\delta f]_i = \nabla^j f_{ij}.$$

Therefore, solenoidal tensors are given by the condition (11).

Solenoidal-Potential decomposition

Based on that, we can decompose orthogonally any symmetric 2-tensor f into a solenoidal part f^s and a potential one dv with $v|_{\partial M} = 0$:

$$f = f^s + dv.$$

To find f^s and v, use the condition $\delta f^s = 0$ to get

$$\delta dv = \delta f, \quad v|_{\partial M} = 0.$$

The is an elliptic 2-nd order differential equation (system) with Dirichlet b.c. It has unique solution.

Reformulation of the Tensor Tomography Problem

For g simple,

$$I_g f = 0 \quad \Longrightarrow \quad f^s = 0?$$

It makes sense to study this for tensors of any order; in particular for Order 1: 1-forms, Order 0: functions (then $I_g f = 0 \Rightarrow f = 0$).

For 1-forms and functions, the answer is known to be affirmative.

For 2-tensors, it is still an open problem, with several partial results.

Let $\partial_- SM$ consists of points x on ∂M and unit vectors $\theta \in S_x$ pointing into M. There is a natural measure $d\mu$ on $\partial_- SM$ defined locally as the product of the surface measures (more precisely, the induced measure on the submanifold ∂SM) times the factor $\langle \nu, \theta \rangle$, where ν is a unit normal to ∂M. Then on can view I_g as the operator

$$I_g : L^2(M, d\mathrm{Vol}) \Longrightarrow L^2(\partial_- SM, d\mu).$$

Instead of studying I_g, we will study $N := I_g^* I_g$. It is much more convenient object to study and we do not lose much. S-injectivity of N is equivalent to s-injectivity of I_g; stability estimates for N can be translated into stability estimates for I_g.

The next step is to extend M slightly to another manifold M_1 with boundary (domain) and extend all tensors as zero there. Now, study N in M_1 acting on tensors supported in M. So we can think of N as the operator

$$N : L^2(M) \mapsto L^2(M_1).$$

Again, no real loss of generality. S-injectivity of N is equivalent to s-injectivity of I_g. Similarly for stability estimates.

Example: functions and straight lines

Let us pause for a moment and consider the simplest case: integrals of functions in $\mathbf{R}^n$ over straight lines. Then it is well known that

$$Nf(x) = I^* If(x) = c_n \int \frac{f(y)}{|x-y|^{n-1}}\, dy.$$

Note that the singularity is integrable. Since Nf is a convolution, N is a Fourier multiplier:

$$Nf = c_n' \mathcal{F}^{-1} |\xi|^{-1} \mathcal{F},$$

because, up to a multiplication by a non-zero constant, the Fourier transform of $|z|^{-n+1}$ is $|\xi|^{-1}$. Therefore, N is a ΨDO of order -1 with a symbol proportional to $|\xi|^{-1}$. To invert it, we just apply $c_n''|D|$, that is the ΨDO with symbol $|\xi|$, i.e.,

$$f = c_n''|D|Nf.$$

Note that this forces us to consider Nf in the whole $\mathbf{R}^n$ even if we start with f supported in a fixed M. The equivalent to that in the Riemannian case will be M_1.

The Schwartz kernel of N

We sketch the main steps in the analysis of N next. What we need is to prove injectivity under some assumptions, and stability.

Consider the more general weighed geodesic transform

$$I_g f(\gamma) = \int \alpha(\gamma(t), \dot\gamma(t)) f_{ij}(\gamma(t)) \dot\gamma^i(t) \dot\gamma^j(t)\, dt$$

Theorem 3 (S-Uhlmann, Duke Math. J. 2004)

For any symmetric 2-tensor $f \in C(M)$ we have

$$(N_g f)_{kl}(x) = \frac{1}{\sqrt{\det g}} \int A(x,y) \frac{f^{ij}(y)}{\rho(x,y)^{n-1}} \frac{\partial \rho}{\partial y^i} \frac{\partial \rho}{\partial y^j} \frac{\partial \rho}{\partial x^k} \frac{\partial \rho}{\partial x^l} \left| \det \frac{\partial^2(\rho^2/2)}{\partial x \partial y} \right| dy, \quad x \in M_1,$$

with

$$A(x,y) = \bar\alpha(x, -\nabla_x \rho(x,y)) \alpha(y, \nabla_y \rho(x,y)) + \bar\alpha(x, \nabla_x \rho(x,y)) \alpha(y, -\nabla_y \rho(x,y)). \quad (12)$$

Note that it is enough to prove the theorem for the weighted geodesic X-ray transform of functions (that are tensors, too, but of order 0) because we can think of $\alpha(\gamma(t), \dot\gamma(t)) \dot\gamma^i(t) \dot\gamma^j(t)$ as a weight multiplying f_{ij}.

Here is some very brief sketch of the proof. Fix a point $x \in M_1$. Then $I^* If(x)$ is a weighted integral of f along any geodesic through x (because of If); integrated additionally w.r.t. all unit directions of all geodesics through x (because of the presence of I^* there). The latter is also a weighted integral, when $\alpha \neq 1$. Split the integration of each geodesic in two parts: starting from x into each of the two possible directions. This is integration in geodesic polar coordinates but instead of the measure $\rho^{n-1}\, d\rho\, d\sigma_x(\theta)$, we have $d\rho\, d\sigma_x(\theta)$. Divide and multiply by the factor ρ^{n-1} to get $\rho^{1-n}(x,y)$ after change of variables $y = \exp_x(\rho\theta) = \gamma_{x,\theta}(\rho)$. The determinant is just the Jacobian of this change.

So for example when $\alpha = 1$ we get

$$(N_g f)_{kl}(x) = \frac{2}{\sqrt{\det g(x)}} \int \frac{f(y)}{\rho(x,y)^{n-1}} \left| \det \frac{\partial^2(\rho^2/2)}{\partial x \partial y} \right| dy, \quad x \in M_1,$$

that is a straightforward generalization of what happens in the Euclidean case.

Remark. Note that the integral in the theorem is not written in an invariant form. For an invariant formula, we need to replace dy by $(\det g(y))^{-1/2}\, d\mathrm{Vol}(y)$. It is easy to see then that the quantity

$$\frac{1}{\sqrt{\det g(x) \det g(y)}} \det \frac{\partial^2(\rho^2/2)}{\partial x \partial y}$$

is invariantly defined.

Pseudodifferential operators

The kernel of N has a singularity of the type

$$\frac{1}{|x-y|^{n-1}}.$$

That suggests that N might be a ΨDO. We are going to prove this now.

What is a ΨDO? Start with the observation that

$$\frac{1}{i}\partial_{x^j} f = (2\pi)^{-n}\int e^{i(x-y)\cdot\xi}\xi_j f(y)\,dy\,d\xi.$$

We can easily generalize this to differential operators $P(x,D)=\sum_{|\alpha|\le m} c_\alpha(x)D^\alpha$, where $D=\frac{1}{i}\partial$:

$$Pf = (2\pi)^{-n}\int e^{i(x-y)\cdot\xi}p(x,\xi)f(y)\,dy\,d\xi, \tag{13}$$

where $p(x,\xi)=\sum_{|\alpha|\le m} c_\alpha(x)\xi^\alpha$ is the *symbol* of P.

A **pseudodifferential operator** is given by (13) but $p(x,\xi)$ does not need to be a polynomial in ξ. Instead, we assume that p belongs to a certain class.

One such class is the class of the polyhomogeneous symbols:

$$p(x,\xi) \sim p_m(x,\xi) + p_{m-1}(x,\xi) + \cdots ,$$

where p_k is homogeneous in ξ of order k for $|\xi| \gg 1$. The term p_m is called a *principal symbol* of P.

Composition of ΨDOs: Let P, Q be ΨDOs of orders m_1, m_2 in the class above. Then $P(x,D)Q(x,D)$ is a ΨDO again of order $m = m_1 + m_2$. Its principal symbol is given by

$$p_{m_1}(x,\xi)q_{m_2}(x,\xi)$$

but the formula for its (full) symbol is more complicated.

Mapping properties in Sobolev spaces:

$$P(x,D) : H^s(M) \to H^{s-m}(M), \quad M \text{ compact.}$$

Elliptic ΨDOs: $P(x,D)$ is elliptic, if $p_m(x,\xi) \neq 0$ for $|\xi| \gg 1$. If P is matrix-valued, then we want $\det p_m(x,\xi) \neq 0$ for $|\xi| \gg 1$.

"Negligible Operators": Those are the ΨDOs of order $-\infty$, sending any H^s to C^∞ (smoothing operators).

Amplitudes and symbols. One can replace $p(x,\xi)$ in the definition of P by $a(x,y,\xi)$ having a similar polyhomogeneous expansion. One can show that modulo smoothing operators, P is equivalent to an operator of the previous type, with some symbol $p(x,\xi)$ and

$$p_m(x,\xi) = a_m(x,x,\xi).$$

"Inversion" of elliptic ΨDOs. Let $P(x,D)$ be elliptic. One can construct a "parametrix" by using the following construction. Take Q (of order $-m$ with principal symbol p_m^{-1} (or p^{-1}) smoothly cut for ξ in a compact where p_m may vanish. Then QP will have principal symbol 1 (as a 1st order ΨDO). That means that

$$QP = Id + K_{-1},$$

where K is 1 degree smoothing (i.e., of order -1). We are not claiming that K_{-1} is small! Now, one can iterate this procedure to modify the lower order part of Q and make K of order $-\infty$. For our purposes however, this will not be necessary.

Compactness. Restricted to any compact, K_{-1} is a compact operator in L^2 (or any H^s).

Fredholm equations

This has the following consequence. Let us say that we want to solve the equation

$$Pf = h, \quad \text{with } h \in L^2(M) \text{ given, and } P \text{ elliptic.}$$

What we really want is to write something like $f = P^{-1}h$. What we can do is to construct a parametrix to get

$$(Id + K_{-1})f = Qh.$$

Since K_{-1} is compact, this is Fredholm equation (actually, $Pf = h$ is Fredholm, too). This has the following nice consequences

- Uniqueness: $Id + K_{-1}$ may have a kernel, but is is always finite dimensional.
- The cokernel of $Id + K_{-1}$ is finitely dimensional, too.
- If we take away the kernel from $L^2(M)$ and consider $Id + K_{-1}$ as a map from there to the complement of the cokernel, this map is invertible (with a bounded inverse).
- Next,

$$\|f\| \le C\|Pf\|, \quad f \perp \operatorname{Ker} P.$$

- In particular, if we know somehow that P is injective, then

$$\|f\| \le C\|Pf\|, \quad \forall f.$$

- The estimate above (hence injectivity) is preserved under small perturbations of P.

Schwartz kernels of ΨDOs

Let A be a ΨDO

$$Af(x) = \frac{1}{(2\pi)^n} \int e^{i(x-y)\cdot\xi} a(x,y,\xi) f(y)\, dy\, d\xi.$$

Perform the ξ integral to get that A has Schwartz kernel

$$\check{a}(x, y, x-y), \tag{14}$$

where $\check{a}$ is the inverse Fourier transform of a w.r.t. ξ.

We are in a situation where we know the Schwartz kernel, and we want to find the amplitude a (and we hope that a would be an amplitude, indeed). So we just need to write the kernel of N in the form (14) and take the Fourier transform in the third variable.

We start with the observation

$$\rho^2(x,y) = G_{ij}(x,y)(x-y)^i(x-y)^j$$

with some smooth G so that $G(x,x) = g_{ij}(x)$. This allows us to write

$$\frac{1}{\rho^{n-1}(x,y)} = \frac{b(x,y,x-y)}{|x-y|^{n-1}},$$

where $b(x,y,\theta)$ is smooth and homogeneous of order 0 w.r.t. θ. The coefficient b is the price that we pay for having variable coefficients (non-Euclidean metric).

We can express now the whole kernel of N (remember, we consider the case when f is a function now) in the form (with a different b of the same type)

$$\frac{A(x,y)}{\rho^{n-1}(x,y)} \det \frac{\partial^2(\rho^2/2)}{\partial x \partial y} = \frac{b(x,y,x-y)}{|x-y|^{n-1}},$$

Then N is a formal ΨDO with amplitude

$$a(x,y,\xi) = \mathcal{F}_{z\to\xi}\left(b(x,y,z)/|z|^{n-1}\right).$$

The Fourier transform above is easy to evaluate in polar coordinates for z to get

$$a(x,y,\xi) = \pi \int_{|\theta|=1} b(x,y,\theta)\delta(\xi\cdot\theta)\, d\theta$$

Note that $a(x,y,\xi)$ is homogeneous of order -1. To get the principal symbol, we set $y = x$. Recall that $G_{ij}(x,x) = g_{ij}(x)$, and this simplifies considerably the formula for the principal symbol. The next step is to write the integral over the sphere S_xM (in the metric) instead of the Euclidean one. This can be done with the change $S^{n-1} \ni \theta \mapsto g^{-1/2}(x)\theta \in S_xM$.

After skipping some details, we summarize what we proved so far for the weighted ray transform of functions (we will return to tensors later).

Theorem 4

Let

$$I_g f(\gamma) = \int \alpha(\gamma(t),\dot\gamma(t)) f(\gamma(t))\, dt.$$

Then $I_g^* I_g$ is a ΨDO of order -1 in some neighborhood M_1 of M with principal symbol

$$p(x,\xi) = 2\pi \int_{S_xM} |\alpha(x,\theta)|^2 \delta(\xi\cdot\theta)\, d\sigma(\theta).$$

Here, the factor $2|\alpha(x,\theta)|^2$ came from (12).

Is I_g (respectively, $I_g^* I_g$) invertible? If so, is it stable?

If $\alpha = 1$ (and the metric is simple), yes (Mukhometov, Romanov, ...). On the other hand, even if g is Euclidean, there exists $\alpha > 0$ so that I_g has a non-trivial kernel (Boman). If g and α are real analytic in M, then - yes.

We however want more — stability. Remember, this is just a model for the tensor transform, and the later linearizes the boundary rigidity problem. So we need an estimate as well, not only uniqueness.

The central idea is to find out whether $I_g^* I_g$ is elliptic. If it is, we can apply the Fredholm theory as above.

Ellipticity Condition

$$\forall (x, \xi) \in T^* M \setminus 0,\ \exists \theta \perp \xi, \text{ so that } \alpha(x, \theta) \neq 0.$$

In particular, α that never vanishes is enough.

This also confirms a basic fact in integral geometry (under some conditions, in our case: simplicity):

Ellipticity Condition

Integrals over open sets of geodesics (or geodesic-like curves) determine conormal singularities to them.

Wave Front Sets and Integral Geometry

The whole idea of microlocal analysis is to look at functions/distributions not only near points but near directions. Smoothness is tied to a rapid decay of the Fourier transform. We first localize f near x_0 and then study the behavior of the FT for large ξ in a small cone near some ξ_0. If the decay is rapid, we say that $(x_0, \xi_0) \not\in \mathrm{WF}(f)$.

More precisely, $(x_0, \xi_0) \not\in \mathrm{WF}(f)$, if

$$\widehat{\chi f}(\xi) \le C_N |\xi|^{-N}, \quad \forall N > 0, \ \left| \frac{\xi}{|\xi|} - \frac{\xi_0}{|\xi_0|} \right| \ll 1$$

for some $\chi \in C_0^\infty$ with $\chi(x_0) \neq 0$.

Lemma 5

Let g be simple in M, and let I_g be the weighted ray transform with $\alpha \neq 0$. Let $I_g f(\gamma) = 0$ (or let it be smooth) for γ close to some γ_0. Then

$$WF(f) \cap N^*\gamma_0 = \emptyset.$$

Theorem 6

let g be simple, and let I_g be the weighted ray transform of functions. Assume the ellipticity condition (satisfied, if $\alpha \neq 0$). Then
(a) I_g has a finitely dimensional smooth kernel.
(b)

$$\|f\| \leq C\|I_g^* I_g f\|_{H^1(M_1)}, \quad \forall f \in (Ker\, I_g)^\perp.$$

(c) If I_g is injective, then

$$\|f\| \leq C\|I_g^* I_g f\|_{H^1(M_1)}, \quad \forall f.$$

(d) The estimate in (c) is preserved under a small perturbation of g and α in $C^k(M)$, $k \gg 1$.

So, injectivity implies stability (in this case, because we are inverting an elliptic operator). Moreover, injectivity is preserved under a small perturbation! Therefore,

The set of (g, α) with an injective ray transform is open in some C^k.

It is certainly non-empty (Euclidean metric, constant weight). So in particular we get a stable uniqueness for metrics close to the Euclidean one, and weights close to a constant.

Injectivity and generic uniqueness

Injectivity is know if g is Euclidean and $\alpha(x,\theta)$ is of special type:

$$\alpha(x,\theta) = e^{-\int_0^\infty a(x-s\theta)\,ds}$$

(the attenuated X-ray transform). There are even inversion formulas (Bukgheim, Novikov). In general it fails. However,

Theorem 7

Let g, $\alpha \neq 0$ be real analytic in M. Then I_g is injective (and therefore, stable).

Corollary 8

The set of (g,α) (with g simple) for which I_g is injective is an open and dense set in some $C^k(M)$. Moreover, for any (g,α) in this set,

$$\|f\| \leq C\|I_g^* I_g f\|_{H^1(M_1)},$$

with a constant $C>0$ that can be chosen locally uniform.