Travel Time Tomography and Tensor Tomography, II

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Mini Course, MSRI 2009, Lecture 2
First, this is not the only way to solve an inverse problem. But it is one of the most used.

For more details, see Linearizing non-linear inverse problems and an application to inverse backscattering (with Gunther Uhlmann), *J. Funct. Anal.* 256(9)(2009), 2842–2866.

Consider the following “inverse problem.” Let $A : B_1 \to B_2$ (Banach spaces).

Given $h \in \text{Ran}(A)$, find $f$ so that $A(f) = h$. \hfill (1)

We want to prove local uniqueness near some (and hopefully all) $f_0$, i.e., that

$A(f_1) = A(f_2)$ for $f_1, f_2$ close to $f_0 \implies f_1 = f_2$.

The first thing that comes to mind is to see if the derivative is “non-zero.” Let $A_f$ be the differential of $A$ (the Gâteaux derivative) at $f$ (we assume that it exists), i.e., $A_f$ is a linear operator given by

$$A_f h = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} (A(f + \varepsilon h) - A(f)).$$

Then the local uniqueness near $f_0$ is often associated with the injectivity of $A_{f_0}$. That can be wrong!
Let $\mathcal{A} : \mathbb{R}^n \rightarrow \mathbb{R}^m$, $\mathcal{A} \in C^2$. Then

$$\mathcal{A}(x) = \mathcal{A}(x_0) + A_{x_0}(x - x_0) + R_{x_0}(x) \quad \text{with} \quad |R_{x_0}(x)| \leq C_{x_0} |x - x_0|^2,$$

for $x$ near $x_0$. Assume now that $A_{x_0}$ is injective (then $m \geq n$). This immediately implies

$$|h| \leq C |A_{x_0} h|, \quad \forall h \in \mathbb{R}^n.$$

Therefore,

**Injectivity implies stability (of the linear problem) in finite dimensions.**

Also, it implies local uniqueness and stability for the original non-linear problem. Indeed, assuming $\mathcal{A}(x) = \mathcal{A}(x_0)$, we get

$$|x - x_0| \leq C |A_{x_0}(x - x_0)| \leq C |x - x_0|^2$$

and the local uniqueness follows easily. Similarly, if $|x - x_0| \ll 1$, one gets

$$|x - x_0| \leq 2C |A(x) - A(x_0)|.$$  \hspace{1cm} (2)

One can replace $\mathbb{R}^m$ here by an $\infty$-dim space. In particular, we get (in the original formulation, $\mathcal{A} : \mathcal{B}_1 \rightarrow \mathcal{B}_2$), that if $\mathcal{B}_1$ is finite dimensional (there is no need $\mathcal{B}_2$ to be finitely dimensional, too), we have trivially Lipschitz stability for any inverse problem with an injective differential!
Now, if \( \dim B_1 = \infty \), this is no longer true in general. First, injectivity does not imply stability, i.e., an estimate of the type

\[
\| h \|_{B_1} \leq C \| A_{f_0} h \|_{B_2}.
\]

Second, without stability, we may not have local uniqueness (forget about any type of stability) for the non-linear problem.

So if we want to use those type of arguments, we need to prove injectivity and stability of \( A_f \). Under those conditions, the arguments above work. This is known as the local injectivity theorem.
Most inverse problems are ill-posed however, and this approach rarely works the way described. It is typical that we can only prove stability in different (reasonable) norms:

\[ \| h \|_{B_1'} \leq C \| A_{f_0} h \|_{B_2'}, \quad \forall h \in B_1, \]

where \( B_{1,2} \) are different Banach spaces that cannot be replaced with the original ones. Then we can still prove local uniqueness (and Hölder stability) under an additional regularity condition. Remember, the decisive argument was that the linearization holds up to a quadratic error \( O(|f - f_0|^2) \), while on the left we had \( |f - f_0| \). So at some point we get

\[ \| f - f_0 \| \leq C \| f - f_0 \|^2, \]

that implies \( f = f_0 \) if \( \| f - f_0 \| \ll 1 \). We can afford to replace the power 2 by \( 1 + \varepsilon \), \( \varepsilon > 0 \), and this argument still works!

This suggests that we should use interpolation estimates of the kind

\[ \| f \|_{H^s} \leq C\| f \|_{H^{s_1}}^{\alpha_1} \| f \|_{H^{s_2}}^{\alpha_2}, \]

where \( \alpha_1 s_1 + \alpha_2 s_2 = 1, \alpha_1 + \alpha_2 = 1, \alpha_j > 0 \).
Here is an example. Suppose that \( A : L^2 \to L^2 \). On the other hand, assume that \( A_f \) is injective for any \( f \) but we can only show that

\[
\| h \|_{L^2} \leq C \| A_f h \|_{H^1}. \tag{3}
\]

What we really want is

\[
\| h \|_{L^2} \leq C \| A_f h \|_{L^2}, \tag{4}
\]

but that might not be true. If we can show that \( A : L^2 \to H^1 \), then we just choose \( B_2 = H^1 \), and we can proceed as before. But if we cannot, it is time to use interpolation estimates:

\[
\| A_f h \|_{H^1} \leq C \| A_f h \|_{L^2}^\alpha \| A_f h \|_{H^s}^{1-\alpha}
\]

with \( s(1 - \alpha) = 1 \). Now, if we assume that \( h \) belongs to a subspace so that

\[
\| A_f h \|_{H^s} \leq C_0,
\]

then

\[
\| A_f h \|_{H^1} \leq C'_0 \| A_f h \|_{L^2}^\alpha.
\]

Then we get from (3),

\[
\| h \|_{L^2} \leq C \| A_f h \|_{L^2}^\alpha.
\]

Compare this to (4). It is similar, but the power 1 is replaced by \( \alpha < 1 \).
We can still use
\[ \| h \|_{L^2} \leq C \| A_f h \|_{L^2}^\alpha. \]
in our proof of uniqueness, where an important role was played by the inequality
\[ \| f - f_0 \| \leq C \| A_{f_0} (f - f_0) \| \leq C \| f - f_0 \|^2. \]

In our case, we have instead
\[ \| f - f_0 \| \leq C \| A_{f_0} (f - f_0) \| \leq C \| f - f_0 \|^{2\alpha} \]
and it is enough to have \(2\alpha > 1\). This can be achieved if \(s > 2\) (remember that \(s(1 - \alpha) = 1\)). So, for this to work we need
\[ \| A_f h \|_{H^{2+\delta}} \leq C_0, \quad \text{with some } \delta > 0. \]

A typical situation is that \(A_f\) is \(e\ \Psi\)DO of order \(-1\); then the estimate above holds if
\[ \| f \|_{H^{1+\delta}} \leq C_1, \quad \text{(5)} \]
for \(f\) near \(f_0\), with some \(C_1 > 0\). Let us say that we work on a compact domain (manifold) with or without boundary. Then (5) restricts \(f\) to a compact subset and appears as an additional assumption. The resulting stability estimate is called a \textit{conditional stability estimate}. 
Theorem 1 (weak local uniqueness and stability)

Assume that $A$ has a derivative at $f_0$ with quadratic estimate on the remainder. Let

$$
\|h\|_{B_1^\prime} \leq C\|A_{f_0}h\|_{B_2^\prime}, \quad \forall h \in B_1.
$$

(6)

Assume also that there exist Banach spaces $B_2^\prime\prime \subset B_2^\prime$, $B_1^\prime\prime \subset B_1$ so that $A_{f_0} : B_1^\prime\prime \rightarrow B_2^\prime\prime$ and the following interpolation estimates hold

$$
\|u\|_{B_2^\prime} \leq C\|u\|_{B_2^\prime\prime}\|u\|_{B_2^\prime\prime}^{1-\mu_2}, \quad \|h\|_{B_1^\prime} \leq C\|h\|_{B_1^\prime\prime}\|h\|_{B_1^\prime\prime}^{1-\mu_1} \quad \mu_1, \mu_2 \in (0, 1], \quad \mu_1\mu_2 > 1/2.
$$

Then for any $K > 0$ there exists $\epsilon > 0$, so that for any $f$ with

$$
\|f - f_0\|_{B_1} \leq \epsilon, \quad \|f\|_{B_1^\prime\prime} \leq K,
$$

(7)

one has the conditional stability estimate

$$
\|f - f_0\|_{B_1} \leq C(K)\|A(f) - A(f_0)\|_{B_2^\prime}, \quad C(K) = CK^{2-\mu_1-\mu_2}.
$$

(8)

In particular, there is a weak local uniqueness near $f_0$, i.e., if $A(f) = A(f_0)$, then $f = f_0$. 

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Theorem 2 (Strong local uniqueness and stability)

Assume in addition that there is a Banach space \( \mathcal{K} \subset B_1'' \) so that (6) holds for \( f_0 \) replaced with \( f \) close enough to \( f_0 \) in \( \mathcal{K} \), and \( A_f : B_1'' \to B_2'' \) is uniformly bounded for such \( f \). Then there exists \( \epsilon > 0 \), so that for any \( f_1, f_2 \) with

\[
\| f_1 - f_0 \|_\mathcal{K} \leq \epsilon, \quad \| f_2 - f_0 \|_\mathcal{K} \leq \epsilon,
\]  

(9)

one has the conditional stability estimate

\[
\| f_1 - f_2 \|_{B_1} \leq C \| A(f_1) - A(f_2) \|_{B_2}^{\mu_1 \mu_2}.
\]  

(10)

In particular, there is a strong local uniqueness near \( f_0 \), i.e., if \( A(f_1) = A(f_2) \), then \( f_1 = f_2 \).

Here the compactness type of condition is hidden in the assumption that \( f_{1,2} \in \mathcal{K} \subset B_2'' \).
A short version of what we did so far:

- Proving injectivity of the linearization is not enough.
- We need also a stability estimate.
- That estimate may not be in the original norms. As long as it is in some $H^s$ or $C^k$ spaces, the whole approach still works.
- If the estimate is not in the original norms, we pay a price: we have to assume higher regularity of the coefficients.
The linearized problem

If $g$ is simple, and for the geodesic X-ray transform

$$I_g f(\gamma) = \int f_{ij}(\gamma(t)) \dot{\gamma}^i(t) \dot{\gamma}^j(t) \, dt$$

we have $I_g f(\gamma) = 0$ for all max geodesics $\gamma$ in $M$, show that $f = dv$ with $v|_{\partial M} = 0$.

First, we will project onto the space of tensors (called solenoidal) orthogonal to all such $dv$’s (called potential). There is a natural definition of $L^2$ spaces of tensors of a fixed order involving the volume measure. We want to describe all $f$ so that $(f, dv) = 0$. Integrate by parts to get

$$\delta f = 0,$$

where $\delta$ is the divergence operator sending 2-tensors to 1-tensors (forms); given in local coordinates by

$$[\delta f]_i = \nabla^j f_{ij}.$$ 

Therefore, solenoidal tensors are given by the condition (11).
Solenoidal-Potential decomposition

Based on that, we can decompose orthogonally any symmetric 2-tensor $f$ into a solenoidal part $f^s$ and a potential one $dv$ with $v|_{\partial M} = 0$:

$$f = f^s + dv.$$  

To find $f^s$ and $v$, use the condition $\delta f^s = 0$ to get

$$\delta dv = \delta f, \quad v|_{\partial M} = 0.$$  

The is an elliptic 2-nd order differential equation (system) with Dirichlet b.c. It has unique solution.

Reformulation of the Tensor Tomography Problem

For $g$ simple,

$$l_g f = 0 \quad \implies \quad f^s = 0?$$

It makes sense to study this for tensors of any order; in particular for Order 1: 1-forms, Order 0: functions (then $l_g f = 0 \Rightarrow f = 0$).

For 1-forms and functions, the answer is known to be affirmative.

For 2-tensors, it is still an open problem, with several partial results.
Let $\partial_- SM$ consists of points $x$ on $\partial M$ and unit vectors $\theta \in S_x$ pointing into $M$. There is a natural measure $d\mu$ on $\partial_- SM$ defined locally as the product of the surface measures (more precisely, the induced measure on the submanifold $\partial SM$) times the factor $\langle \nu, \theta \rangle$, where $\nu$ is a unit normal to $\partial M$. Then on can view $I_g$ as the operator

$$I_g : L^2(M, d\text{Vol}) \mapsto L^2(\partial_- SM, d\mu).$$

Instead of studying $I_g$, we will study $N := I_g^* I_g$. It is much more convenient object to study and we do not lose much. $S$-injectivity of $N$ is equivalent to $s$-injectivity of $I_g$; stability estimates for $N$ can be translated into stability estimates for $I_g$.

The next step is to extend $M$ slightly to another manifold $M_1$ with boundary (domain) and extend all tensors as zero there. Now, study $N$ in $M_1$ acting on tensors supported in $M$. So we can think of $N$ as the operator

$$N : L^2(M) \mapsto L^2(M_1).$$

Again, no real loss of generality. $S$-injectivity of $N$ is equivalent to $s$-injectivity of $I_g$. Similarly for stability estimates.
Let us pause for a moment and consider the simplest case: integrals of functions in $\mathbb{R}^n$ over straight lines. Then it is well known that

$$Nf(x) = I^* If(x) = c_n \int \frac{f(y)}{|x - y|^{n-1}} dy.$$ 

Note that the singularity is integrable. Since $Nf$ is a convolution, $N$ is a Fourier multiplier:

$$Nf = c'_n F^{-1} |\xi|^{-1} F,$$

because, up to a multiplication by a non-zero constant, the Fourier transform of $|z|^{-n+1}$ is $|\xi|^{-1}$. Therefore, $N$ is a $\Psi$DO of order $-1$ with a symbol proportional to $|\xi|^{-1}$. To invert it, we just apply $c''_n |D|$, that is the $\Psi$DO with symbol $|\xi|$, i.e.,

$$f = c''_n |D| Nf.$$ 

Note that this forces us to consider $Nf$ in the whole $\mathbb{R}^n$ even if we start with $f$ supported in a fixed $M$. The equivalent to that in the Riemannian case will be $M_1$. 

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The Schwartz kernel of $N$

We sketch the main steps in the analysis of $N$ next. What we need is to prove injectivity under some assumptions, and stability.

Consider the more general weighed geodesic transform

$$l_g f(\gamma) = \int \alpha(\gamma(t), \dot{\gamma}(t)) f_{ij}(\gamma(t)) \dot{\gamma}^i(t) \dot{\gamma}^j(t) \, dt$$


For any symmetric 2-tensor $f \in C(M)$ we have

$$\left( N_g f \right)_{kl}(x) = \frac{1}{\sqrt{\det g}} \int A(x, y) \frac{f^{ij}(y)}{\rho(x, y)^{n-1}} \frac{\partial \rho}{\partial y^i} \frac{\partial \rho}{\partial y^j} \frac{\partial \rho}{\partial x^k} \frac{\partial \rho}{\partial x^l} \left| \det \frac{\partial^2 (\rho^2/2)}{\partial x \partial y} \right| \, dy, \quad x \in M_1,$$

with

$$A(x, y) = \tilde{\alpha}(x, \nabla_x \rho(x, y)) \alpha(y, \nabla_y \rho(x, y)) + \bar{\alpha}(x, \nabla_x \rho(x, y)) \alpha(y, -\nabla_y \rho(x, y)). \quad (12)$$

Note that it is enough to prove the theorem for the weighted geodesic X-ray transform of functions (that are tensors, too, but of order 0) because we can think of $\alpha(\gamma(t), \dot{\gamma}(t)) \dot{\gamma}^i(t) \dot{\gamma}^j(t)$ as a weight multiplying $f_{ij}$. 
Here is some very brief sketch of the proof. Fix a point \( x \in M_1 \). Then \( I^* If(x) \) is a weighted integral of \( f \) along any geodesic through \( x \) (because of \( If \)); integrated additionally w.r.t. all unit directions of all geodesics through \( x \) (because of the presence of \( I^* \) there). The latter is also a weighted integral, when \( \alpha \neq 1 \). Split the integration of each geodesic in two parts: starting from \( x \) into each of the two possible directions. This is integration in geodesic polar coordinates but instead of the measure \( \rho^{n-1} d\rho d\sigma_x(\theta) \), we have \( d\rho d\sigma_x(\theta) \). Divide and multiply by the factor \( \rho^{n-1} \) to get \( \rho^{1-n}(x, y) \) after change of variables \( y = \exp_x(\rho \theta) = \gamma_{x,\theta}(\rho) \). The determinant is just the Jacobian of this change.

So for example when \( \alpha = 1 \) we get

\[
(N_g f)_{kl}(x) = \frac{2}{\sqrt{\det g(x)}} \int \frac{f(y)}{\rho(x, y)^{n-1}} \left| \det \frac{\partial^2(\rho^2/2)}{\partial x \partial y} \right| dy, \quad x \in M_1,
\]

that is a straightforward generalization of what happens in the Euclidean case.

**Remark.** Note that the integral in the theorem is not written in an invariant form. For an invariant formula, we need to replace \( dy \) by \( (\det g(y))^{-1/2} d\Vol(y) \). It is easy to see then that the quantity

\[
\frac{1}{\sqrt{\det g(x) \det g(y)}} \det \frac{\partial^2(\rho^2/2)}{\partial x \partial y}
\]

is invariantly defined.
Pseudodifferential operators

The kernel of $N$ has a singularity of the type

$$\frac{1}{|x - y|^{n-1}}.$$

That suggests that $N$ might be a $\Psi$DO. We are going to prove this now.

What is a $\Psi$DO? Start with the observation that

$$\frac{1}{i} \partial_{x^j} f = (2\pi)^{-n} \int e^{i(x-y) \cdot \xi} \xi_j f(y) \, dy \, d\xi.$$

We can easily generalize this to differential operators $P(x, D) = \sum_{|\alpha| \leq m} c_\alpha(x) D^\alpha$, where $D = \frac{1}{i} \partial$:

$$Pf = (2\pi)^{-n} \int e^{i(x-y) \cdot \xi} p(x, \xi) f(y) \, dy \, d\xi,$$

where $p(x, \xi) = \sum_{|\alpha| \leq m} c_\alpha(x) \xi^\alpha$ is the symbol of $P$.

A pseudodifferential operator is given by (13) but $p(x, \xi)$ does not need to be a polynomial in $\xi$. Instead, we assume that $p$ belongs to a certain class.
One such class is the class of the polyhomogeneous symbols:

\[ p(x, \xi) \sim p_m(x, \xi) + p_{m-1}(x, \xi) + \cdots , \]

where \( p_k \) is homogeneous in \( \xi \) of order \( k \) for \( |\xi| \gg 1 \). The term \( p_m \) is called a principal symbol of \( P \).

**Composition of \( \Psi DOs \):** Let \( P, Q \) be \( \Psi DOs \) of orders \( m_1, m_2 \) in the class above. Then \( P(x, D)Q(x, D) \) is a \( \Psi DO \) again of order \( m = m_1 + m_2 \). Its principal symbol is given by

\[ p_{m_1}(x, \xi)q_{m_2}(x, \xi) \]

but the formula for its (full) symbol is more complicated.

**Mapping properties in Sobolev spaces:**

\[ P(x, D) : H^s(M) \to H^{s-m}(M), \quad M \text{ compact.} \]

**Elliptic \( \Psi DOs \):** \( P(x, D) \) is elliptic, if \( p_m(x, \xi) \neq 0 \) for \( |\xi| \gg 1 \). If \( P \) is matrix-valued, then we want \( \det p_m(x, \xi) \neq 0 \) for \( |\xi| \gg 1 \).

**“Negligible Operators”:** Those are the \( \Psi DOs \) of order \( -\infty \), sending any \( H^s \) to \( C^\infty \) (smoothing operators).
**Amplitudes and symbols.** One can replace $p(x, \xi)$ in the definition of $P$ by $a(x, y, \xi)$ having a similar polyhomogeneous expansion. One can show that modulo smoothing operators, $P$ is equivalent to an operator of the previous type, with some symbol $p(x, \xi)$ and

$$p_m(x, \xi) = a_m(x, x, \xi).$$

**“Inversion” of elliptic $\Psi$DOs.** Let $P(x, D)$ be elliptic. One can construct a “parametrix” by using the following construction. Take $Q$ (of order $-m$ with principal symbol $p_m^{-1}$ (or $p^{-1}$) smoothly cut for $\xi$ in a compact where $p_m$ may vanish. Then $QP$ will have principal symbol $1$ (as a 1st order $\Psi$DO). That means that

$$QP = Id + K_{-1},$$

where $K$ is 1 degree smoothing (i.e., of order $-1$). We are not claiming that $K_{-1}$ is small! Now, one can iterate this procedure to modify the lower order part of $Q$ and make $K$ of order $-\infty$. For our purposes however, this will not be necessary.

**Compactness.** Restricted to any compact, $K_{-1}$ is a compact operator in $L^2$ (or any $H^s$).
Fredholm equations

This has the following consequence. Let us say that we want to solve the equation

\[ Pf = h, \quad \text{with } h \in L^2(M) \text{ given, and } P \text{ elliptic.} \]

What we really want is to write something like \( f = P^{-1} h \). What we can do is to construct a parametrix to get

\[ (Id + K_{-1})f = Qh. \]

Since \( K_{-1} \) is compact, this is Fredholm equation (actually, \( Pf = h \) is Fredholm, too). This has the following nice consequences

- **Uniqueness:** \( Id + K_{-1} \) may have a kernel, but is is always finite dimensional.
- The cokernel of \( Id + K_{-1} \) is finitely dimensional, too.
- If we take away the kernel from \( L^2(M) \) and consider \( Id + K_{-1} \) as a map from there to the complement of the cokernel, this map is invertible (with a bounded inverse).
- Next,

\[ \| f \| \leq C \| Pf \|, \quad f \perp \text{Ker } P. \]

- In particular, if we know somehow that \( P \) is injective, then

\[ \| f \| \leq C \| Pf \|, \quad \forall f. \]

- The estimate above (hence injectivity) is preserved under small perturbations of \( P \).
Let $A$ be a $\Psi$DO

$$Af(x) = \frac{1}{(2\pi)^n} \int e^{i(x-y) \cdot \xi} a(x, y, \xi) f(y) \, dy \, d\xi.$$ 

Perform the $\xi$ integral to get that $A$ has Schwartz kernel

$$\tilde{a}(x, y, x - y), \quad (14)$$

where $\tilde{a}$ is the inverse Fourier transform of $a$ w.r.t. $\xi$.

We are in a situation where we know the Schwartz kernel, and we want to find the amplitude $a$ (and we hope that $a$ would be an amplitude, indeed). So we just need to write the kernel of $N$ in the form (14) and take the Fourier transform in the third variable.
We start with the observation

\[ \rho^2(x, y) = G_{ij}(x, y)(x - y)^i(x - y)^j \]

with some smooth \( G \) so that \( G(x, x) = g_{ij}(x) \). This allows us to write

\[ \frac{1}{\rho^{n-1}(x, y)} = \frac{b(x, y, x - y)}{|x - y|^{n-1}}, \]

where \( b(x, y, \theta) \) is smooth and homogeneous of order 0 w.r.t. \( \theta \). The coefficient \( b \) is the price that we pay for having variable coefficients (non-Euclidean metric).

We can express now the whole kernel of \( N \) (remember, we consider the case when \( f \) is a function now) in the form (with a different \( b \) of the same type)

\[ A(x, y) \det \frac{\partial^2 (\rho^2 / 2)}{\partial x \partial y} = \frac{b(x, y, x - y)}{|x - y|^{n-1}}, \]

Then \( N \) is a formal \( \Psi \)DO with amplitude

\[ a(x, y, \xi) = \mathcal{F}_{z \rightarrow \xi}(b(x, y, z)/|z|^{n-1}). \]

The Fourier transform above is easy to evaluate in polar coordinates for \( z \) to get

\[ a(x, y, \xi) = \pi \int_{|\theta|=1} b(x, y, \theta) \delta(\xi \cdot \theta) \, d\theta \]
Note that $a(x, y, \xi)$ is homogeneous of order $-1$. To get the principal symbol, we set $y = x$. Recall that $G_{ij}(x, x) = g_{ij}(x)$, and this simplifies considerably the formula for the principal symbol. The next step is to write the integral over the sphere $S_x M$ (in the metric) instead of the Euclidean one. This can be done with the change $S^{n-1} \ni \theta \mapsto g^{-1/2}(x) \theta \in S_x M$.

After skipping some details, we summarize what we proved so far for the weighted ray transform of functions (we will return to tensors later).

**Theorem 4**

Let

$$l_g f(\gamma) = \int \alpha(\gamma(t), \dot{\gamma}(t)) f(\gamma(t)) \, dt.$$  

Then $l_g^* l_g$ is a $\Psi DO$ of order $-1$ in some neighborhood $M_1$ of $M$ with principal symbol

$$p(x, \xi) = 2\pi \int_{S_x M} |\alpha(x, \theta)|^2 \delta(\xi \cdot \theta) \, d\sigma(\theta).$$

Here, the factor $2|\alpha(x, \theta)|^2$ came from (12).
Is $l_g$ (respectively, $l_g^* l_g$) invertible? If so, is it stable?

If $\alpha = 1$ (and the metric is simple), yes (Mukhometov, Romanov, ...). On the other hand, even if $g$ is Euclidean, there exists $\alpha > 0$ so that $l_g$ has a non-trivial kernel (Boman). If $g$ and $\alpha$ are real analytic in $M$, then - yes.

We however want more — stability. Remember, this is just a model for the tensor transform, and the later linearizes the boundary rigidity problem. So we need an estimate as well, not only uniqueness.

The central idea is to find out whether $l_g^* l_g$ is elliptic. If it is, we can apply the Fredholm theory as above.

**Ellipticity Condition**

\[
\forall (x, \xi) \in T^* M \setminus 0, \ \exists \theta \perp \xi, \text{ so that } \alpha(x, \theta) \neq 0.
\]

In particular, $\alpha$ that never vanishes is enough.

This also confirms a basic fact in integral geometry (under some conditions, in our case: simplicity):

**Ellipticity Condition**

Integrals over open sets of geodesics (or geodesic-like curves) determine conormal singularities to them.
The whole idea of microlocal analysis is to look at functions/distributions not only near points but near directions. Smoothness is tied to a rapid decay of the Fourier transform. We first localize $f$ near $x_0$ and then study the behavior of the FT for large $\xi$ in a small cone near some $\xi_0$. If the decay is rapid, we say that $(x_0, \xi_0) \not\in WF(f)$.

More precisely, $(x_0, \xi_0) \not\in WF(f)$, if

$$\widehat{\chi f}(\xi) \leq C_N |\xi|^{-N}, \quad \forall N > 0, \quad \left| \frac{\xi}{|\xi|} - \frac{\xi_0}{|\xi_0|} \right| \ll 1$$

for some $\chi \in C_0^\infty$ with $\chi(x_0) \neq 0$.

**Lemma 5**

*Let $g$ be simple in $M$, and let $I_g$ be the weighted ray transform with $\alpha \neq 0$. Let $I_g f(\gamma) = 0$ (or let it be smooth) for $\gamma$ close to some $\gamma_0$. Then*

$$WF(f) \cap N^* \gamma_0 = \emptyset.$$
Theorem 6

Let $g$ be simple, and let $I_g$ be the weighted ray transform of functions. Assume the ellipticity condition (satisfied, if $\alpha \neq 0$). Then

(a) $I_g$ has a finitely dimensional smooth kernel.

(b) 
\[ \|f\| \leq C \|I_g^* I_g f\|_{H^1(M_1)}, \quad \forall f \in (\text{Ker } I_g)^\perp. \]

(c) If $I_g$ is injective, then 
\[ \|f\| \leq C \|I_g^* I_g f\|_{H^1(M_1)}, \quad \forall f. \]

(d) The estimate in (c) is preserved under a small perturbation of $g$ and $\alpha$ in $C^k(M)$, $k \gg 1$.

So, injectivity implies stability (in this case, because we are inverting an elliptic operator). Moreover, injectivity is preserved under a small perturbation! Therefore,

The set of $(g, \alpha)$ with an injective ray transform is open in some $C^k$.

It is certainly non-empty (Euclidean metric, constant weight). So in particular we get a stable uniqueness for metrics close to the Euclidean one, and weights close to a constant.
Injectivity and generic uniqueness

Injectivity is know if $g$ is Euclidean and $\alpha(x, \theta)$ is of special type:

$$\alpha(x, \theta) = e^{- \int_0^\infty a(x-s\theta) \, ds}$$

(the attenuated X-ray transform). There are even inversion formulas (Bukgheim, Novikov). In general it fails. However,

**Theorem 7**

*Let $g, \alpha \neq 0$ be real analytic in $M$. Then $I_g$ is injective (and therefore, stable).*

**Corollary 8**

*The set of $(g, \alpha)$ (with $g$ simple) for which $I_g$ is injective is an open and dense set in some $C^k(M)$. Moreover, for any $(g, \alpha)$ in this set,

$$\|f\| \leq C \|I_g^* I_g f\|_{H^1(M_1)},$$

with a constant $C > 0$ that can be chosen locally uniform.*