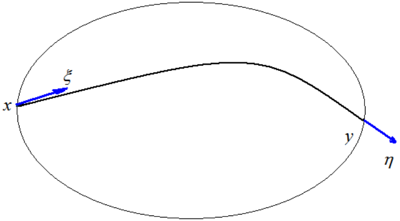
Recovery of anisotropic metrics from travel times

Plamen Stefanov

Purdue University

The Lens Rigidity and the Boundary Rigidity Problems

Let M be a bounded domain with boundary. Let g be a Riemannian metric on M. Define the scattering relation σ and the length (travel time) function ℓ:



$$\sigma : (x, \xi) \to (y, \eta), \quad \ell(x, \xi) \to [0, \infty].$$

Diffeomorphisms preserving ∂M pointwise do not change σ, ℓ! Actually, they do, if their differential on ∂M is not identity. One can either change slightly the definition of σ, ℓ, or simply assume that g is known on ∂M.

Lens rigidity: *Do σ, ℓ determine uniquely g, up to an isometry?*

In other words, if $\sigma_{\hat{g}} = \sigma_g$, $\ell_{\hat{g}} = \ell_g$, is there a diffeo $\psi : M \to M$, $\psi|_{\partial M} = Id$, such that $\psi_* \hat{g} = g$?

No, in general, so some assumptions are needed.

Boundary rigidity:

Does the boundary distance function $\rho(x, y)$, known for all x, y on ∂M, determine uniquely g, up to an isometry?

In general, the answer to the boundary rigidity problem is negative as well, with obvious counterexamples.

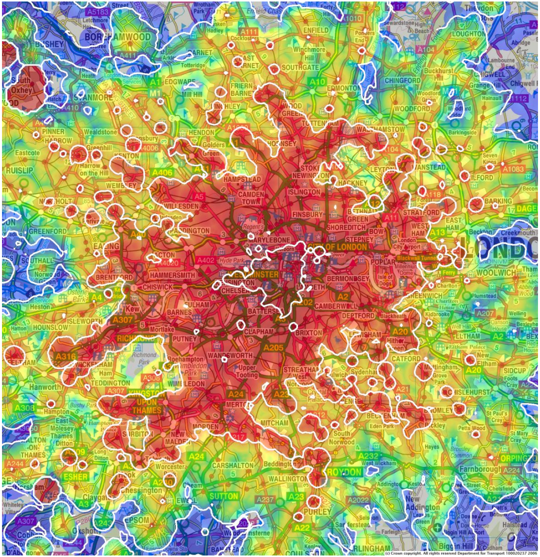


Figure: Travel times through London

A metric g is called simple in M, if the latter is strictly convex w.r.t. g, and if there are no conjugate points (no caustics). In other words, any two pints are connected by a unique geodesic.

The lens rigidity problem and the boundary rigidity one are equivalent for simple metrics! Indeed, then $\rho(x, y)$, known for x, y on ∂M determines σ, ℓ uniquely, and vice-versa.

For non-simple metrics (caustics and/or non-convex boundary), the Lens rigidity is the right problem to study.

Even for non-simple metrics, one can still recover σ, ℓ from the travel times, but we need multiple arrival times, and a non-degeneracy assumption.

We propose a reconstruction algorithm, based on the work by S. and Gunther Uhlmann on the theoretical aspects of this problem. We will concentrate on simple metrics. So the goal is: given $\rho(x,y)$, reconstruct g, up to an isometry.

Note that $\rho(x,y)$ depends on $2(n-1)$ variables, while $g(x)$ depends on n variables. So the problem is formally determined in 2D and overdetermined in 3D. In 3D, we have results about uniqueness with partial data. Perhaps first the problem should be tried numerically in 2D. There are results already (Zhao et al.) for isotropic metrics, i.e., metrics of the form

$$g(x) = c(x)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then one needs to recover a single function $c(x)$.

Our goal is to recover anisotropic metrics, i.e., metrics of the form

$$g(x) = \begin{pmatrix} \alpha(x) & \beta(x) \\ \beta(x) & \gamma(x) \end{pmatrix}.$$

- Based on linearization, i.e., given a known g_0 and ρ_g for g close to g_0, we want to recover a matrix $f(x)$ so that $g = g_0 + f + O(\|f\|^2)$.
- Because of the non-uniqueness due to isometries, there are infinitely many "right answers" f (even up to a quadratic error).
- The problem of finding f reduces to a (linear) Fredholm equation

$$(Id + K)f = h,$$

  with K a compact operator and h known.
- As a result, the problem is relatively stable. We proved a conditional Hölder stability estimate for simple metrics.

Linearized problem:

Recover a tensor field f_{ij} from the geodesic X-ray transform

$$I_g f(\gamma) = \int f_{ij}(\gamma(t))\dot\gamma^i(t)\dot\gamma^j(t)\,dt$$

known for all (or some) max geodesics γ in M.

Every tensor admits an orthogonal decomposition into a *solenoidal* part f^s and a *potential* part dv,

$$f = f^s + dv, \quad v|_{\partial M} = 0.$$

where $\delta f^s = 0$.

Here the symmetric differential dv is given by $[dv]_{ij} = (\nabla_i v_j + \nabla_j v_i)/2$, and the divergence δ is given by: $[\delta f]_i = \sum_{kj} g^{jk}\nabla_k f_{ij}$. We have $I_g(dv) = 0$.

More precise formulation of the linearized problem: Given $I_g f$ recover f^s.

Dealing with the non-uniqueness due to isometries

Recall our goal: g_0 is known (for example, the Euclidean metric). We are given ρ_g with g close to g_0, and want to find $f \approx g - g_0$.

Too many f's. Suppose we have computed numerically one of them. How do we know it is among the many right ones?

Usually, we start with a metric g (that results in some difference $f = g - g_0$) that we actually know but pretend we do not. Then we compute $\tilde{f}$ and want to see if it is close to f.

One way to deal with the non-uniqueness problem is to look for a special f with some specific property. Then we just compare the computed one and the actual one.

Using global semi-geodesic coordinates

For simple metrics g, one can always choose a metric $\psi^* g$ (isometric to g) that looks like this

$$\psi^* g(x) = \begin{pmatrix} 1 & 0 \\ 0 & \gamma(x) \end{pmatrix}.$$

This also requires a change of M because ψ does not fix ∂M. One can simply assume that g is of the kind above, keep M fixed, and not worry about diffeomorphisms ψ.

The advantage now is that there is one function γ only to recover. The reconstruction then is designed in a way that would produce a metric of the same type. This is similar to the approach by S. and G. Uhlmann in the 1996 MRL paper.

Disadvantage: The reconstruction loses stability for codirections close to $(0, 1)$. This is similar to the usual tomography problem with some small angle of measurements near $(1, 0)$ missing.

Using solenoidal tensors (matrices)

For any metric g_0, and g close enough to g_0, among the many f's that approximate $g - g_0$, we will chose the solenodial one. More precisely, there is ψ so that

$$\psi^* g = g_0 + f.$$

with $\delta_{g_0} f = 0$. Such an f is unique. This suggests the following.

- Since we have the freedom to replace g by $\psi^* g$, we therefore start with a "unknown" g so that $g - g_0$ is solenoidal.
- We then use a reconstruction algorithm that is designed to produce always solenoidal metrics.
- We compute numerically an approximation $\tilde{f}$ for f.
- Compare $\tilde{f}$ and f to see the error.

Advantage: Stable in all codirections (no preferred direction)

Disadvantage: What we reconstruct is a symmetric 2×2 matrix, i.e., three scalar functions. On the other hand, anyone of them determines the other two.

Step 1: Choose a solenoidal variation of g_0

How do we do the first step above: given g_0, choose f close to g_0 so that f is solenoidal, i.e., $\delta_{g_0} f = 0$?

Start with f small enough. Then project it to the solenoidal tensors to get f^s that is solenoidal and $g_0 + f^s$ has the same data (up to a quadratic error) as $g_0 + f$.

This is done like this: Take $f^s = f + dv$. We want $\delta f^s = 0$, therefore,

$$\delta dv = -\delta f, \text{(known)} \quad v|_{\partial M} = 0.$$

This a wel posed Dirichlet problem for a 2×2 elliptic system.
If g_0 is Euclidean, then

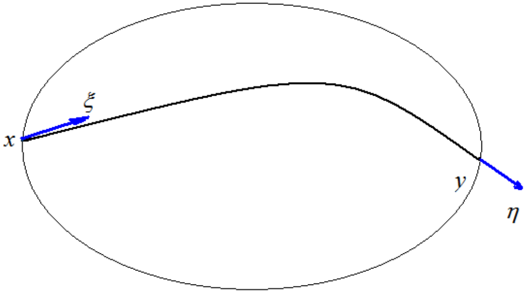
$$2(\delta dv)_i = \Delta v_i + \partial_i \sum_j \partial_j v_j, \qquad (\delta f)_i = \sum_j \partial_j v_{ij}.$$

Then the perturbed metric we want to reconstruct will be $g_0 + f^s$, i.e., we want to reconstruct f^s (that will be denoted f again below).

Step 2: Given $\rho(x, y)$, compute an approximation for $N = I^*I$

$$If(x,\xi) \approx \rho_g^2(x,y) - \rho_{g_0}^2(x,y),$$

where $I = I_{g_0}$, $f \approx g - g_0$, and we know g_0. Also, $y =_x \xi$.



We then compute $Nf = I^*If$. Here $Nf(x)$ can be interpreted as the the integrals of f along all geodesics through x, integrated w.r.t. the initial direction.

Step 3: Compute a parametrix for N

3a: First parametrix Q_1

Let g_0 be the Euclidean metric. Then compute first

$$[Q_1 f]_{ij} = \sum_{kl} \mathcal{F}^{-1}\, a_{ijkl}(\xi)\mathcal{F}(Nf)^{kl},$$

in a slightly larger domain $M_1 \Supset M$. Here $a_{ijkl}(\xi)$ is a rational function, homogeneous of order 1 singular only at $\xi = 0$ with explicit form

$$a_{ijkl} = |\xi| \left(c_1 \delta_{ik}\delta_{jl} + c_2(\delta_{ij} - |\xi|^{-2}\xi_i\xi_j)\delta_{kl} \right).$$

The coefficients c_1 and c_2 depend on n only.

Now, $Q_1 f$ is a parametrix for $f^s_{M_1}$ but not for f^s yet.

3b: Second parametrix Q_2

Apply Q_2 to $Q_1 Nf$ to get a parametrix for f^s:
To this end: compare $f^s_{M_1}$ and f^s in M. They differ by dw, and w can be computed in terms of $Q_1 Nf$ only.
This involves (1) integration of $Q_1 Nf$ from any $x \in \partial M$ in two linearly independent directions, and (2) solving the same elliptic problem as above but with with zero r.h.s. and non-zero BC.

More precisely,

$$v_0(x) \cdot \xi = \int_0^\infty (Q_1 Nf)(x + tv)\, dt, \quad \forall (x, \xi) \in \partial_+ SM,\ s > 0.$$

Then we solve

$$\delta dw = 0, \quad w|_{\partial M} = v_0$$

and set

$$Q_2 Q_1 f = Q_1 Nf + dw.$$

Now,

$$h := Q_2 Q_1 N f = (Id + K) f,$$

where K is a compact operator. So, we have to solve a Fredholm equation, with h known.

Is K known? We have $K = Q_2 Q_1 N - Id$, and Q_2, N are ΨDOs with constant coefficients (Fourier multipliers). Next, Q_2 is explicit, too. So K is known.

Now, solve for f and $g \approx g_0 + f$.

This procedure can now be iterated to get a Newton-like scheme.