Thermoacoustic tomography, variable sound speed

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Based on a joint work with Gunther Uhlmann

Thermoacoustic Tomography

In thermoacoustic tomography, a short electro-magnetic pulse is sent through a patient's body. The tissue reacts and emits an ultrasound wave form any point, that is measured away from the body. Then one tries to reconstruct the internal structure of a patient's body form those measurements.

The Mathematical Model

$$P = c^2 \frac{1}{\sqrt{\det g}} \left(\frac{1}{\mathrm{i}}\frac{\partial}{\partial x^i} + a_i\right) g^{ij} \sqrt{\det g} \left(\frac{1}{\mathrm{i}}\frac{\partial}{\partial x^j} + a_j\right) + q.$$

Let u solve the problem

$$\begin{cases} (\partial_t^2 + P)u = 0 & \text{in } (0,T)\times \mathbf{R}^n, \\ u|_{t=0} = f, \\ \partial_t u|_{t=0} = 0, \end{cases} \tag{1}$$

where $T>0$ is fixed.

Assume that f is supported in $\bar\Omega$, where $\Omega \subset \mathbf{R}^n$ is some smooth bounded domain. The measurements are modeled by the operator

$$\Lambda f := u|_{[0,T]\times\partial\Omega}.$$

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If $T = \infty$, we can just solve a Cauchy problem backwards with zero initial data.
One of the most common methods when $T < \infty$ is to do the same (time reversal). Solve

$$\begin{cases} (\partial_t^2 + P)v_0 & = & 0 \quad \text{in } (0,T)\times\Omega, \\ v_0|_{[0,T]\times\partial\Omega} & = & h, \\ v_0|_{t=T} & = & 0, \\ \partial_t v_0|_{t=T} & = & 0. \end{cases} \tag{2}$$

Then we define the following

"Approximate Inverse"

$$A_0 h := v_0(0,\cdot) \quad \text{in } \bar{\Omega}.$$

Most (but not all) works are in the case of constant coefficients, i.e., when $P = -\Delta$. If n is odd, and $T > \operatorname{diam}(\Omega)$, this is an exact method by the Hyugens' principle.

In that case, this is actually an integral geometry problem because of Kirchoff's formula — recovery of f from integrals over spheres centered at $\partial\Omega$.

When n is even, or when the coefficients are not constant, this is an "approximate solution" only. As $T \to \infty$, the error tends to zero by finite energy decay. The convergence is exponentially fast, when the geometry is non-trapping.

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Known results

KRUGER; AGRANOVSKY, AMBARTSOUMIAN, FINCH, GEORGIEVA-HRISTOVA, JIN, HALTMEIER, KUCHMENT, NGUYEN, PATCH, QUINTO, WANG, XU ...

The time reversal method is frequently used in a slightly modified way. The boundary condition h is first cut-off near $t = T$ in a smooth way. Then the compatibility conditions at $\{T\} \times \partial\Omega$ are satisfied and at least we stay in the energy space.

When T is fixed, there is no control over the error (unless n is odd and $P = -\Delta$). There are other methods, as well, for example a method based on an eigenfunctions expansion; or explicit formulas in the constant coefficient case (with $T = \infty$ in even dimensions), that just give a computable version of the time reversal method.

Results for variable coefficients exists but not so many. FINCH AND RAKESH (2009) proved uniqueness when $T > \mathrm{diam}(\Omega)$, based on Tataru's uniqueness theorem (that we use, too). Reconstructions for finite T have been tried numerically, and they "seem to work" at least for non-trapping geometries.

Another problem of a genuine applied interest is uniqueness and reconstruction with measurements on a part of the boundary. There were no results so far for the variable coefficient case, and there is a uniqueness result in the constant coefficients one by Finch, Patch and Rakesh (2004).

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The main results in a nutshell

- We study the general case of variable coefficients and fixed $T > T(\Omega)$ (the longest geodesics of $c^{-2}g$).

Measurements on the whole boundary:

- we write an explicit solution formula in the form of a converging Neumann series (hence, uniqueness and stability).

Measurements on a part of the boundary:

- We give an almost "if and only if" condition for uniqueness, stable or not.
- We give another almost "if and only if" condition for stability.
- We describe the observation operator Λ as an FIO, and under the condition above, we show that it is elliptic.
- Then we show that the problem reduces to solving a Fredholm equation with a trivial kernel.

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We assume here that (Ω, g) is non-trapping, i.e., $T(\Omega) < \infty$, and that $T > T(\Omega)$.

A new pseudo-inverse

Given h (that eventually will be replaced by Λf), solve

$$\begin{cases} (\partial_t^2 + P)v &= 0 \quad \text{in } (0,T)\times\Omega,\\ v|_{[0,T]\times\partial\Omega} &= h,\\ v|_{t=T} &= \phi,\\ \partial_t v|_{t=T} &= 0, \end{cases} \tag{3}$$

where ϕ solves the elliptic boundary value problem

$$P\phi = 0, \quad \phi|_{\partial\Omega} = h(T,\cdot).$$

Note that the initial data at $t = T$ satisfies compatibility conditions of first order (no jump at $\{T\}\times\partial\Omega$). Then we define the following pseudo-inverse

$$Ah := v(0,\cdot) \quad \text{in } \bar{\Omega}.$$

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Given $U \subset \mathbf{R}^n$, the energy in U is given by

$$E_U(t,u) = \int_U \left(|Du|^2 + c^{-2} q|u|^2 + c^{-2}|u_t|^2 \right) \mathrm{d}\,\mathrm{Vol},$$

where $D_j = -\mathrm{i}\partial/\partial x^j + a_j$, $D = (D_1, \dots, D_n)$, $|Du|^2 = g^{ij}(D_i u)(D_j u)$, and $\mathrm{d}\,\mathrm{Vol}(x) = (\det g)^{1/2} \mathrm{d}x$. In particular, we define the space $H_D(U)$ to be the completion of $C_0^\infty(U)$ under the Dirichlet norm

$$\|f\|_{H_D}^2 = \int_U \left(|Du|^2 + c^{-2} q|u|^2 \right) \mathrm{d}\,\mathrm{Vol}\,.$$

The norms in $H_D(\Omega)$ and $H^1(\Omega)$ are equivalent, so

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Main results, whole boundary

Theorem 1

Let $T > T(\Omega)$. Then $A\Lambda = Id - K$, where K is compact in $H_D(\Omega)$, and $\|K\|_{H_D(\Omega)} < 1$. In particular, $Id - K$ is invertible on $H_D(\Omega)$, and the inverse thermoacoustic problem has an explicit solution of the form

$$f = \sum_{m=0}^{\infty} K^m A h, \quad h := \Lambda f.$$

Some numerical experiments (with Peijun Li, see next slide) show that even the first term Ah only works quite well. In the case, we have the following error estimate:

Corollary 2

$$\|f - A\Lambda f\|_{H_D(\Omega)} \le \left(\frac{E_\Omega(u,T)}{E_\Omega(u,0)} \right)^{\frac{1}{2}} \|f\|_{H_D(\Omega)}, \quad \forall f \in H_{D(\Omega)},\ f \ne 0,$$

where u is the solution with Cauchy data $(f, 0)$.

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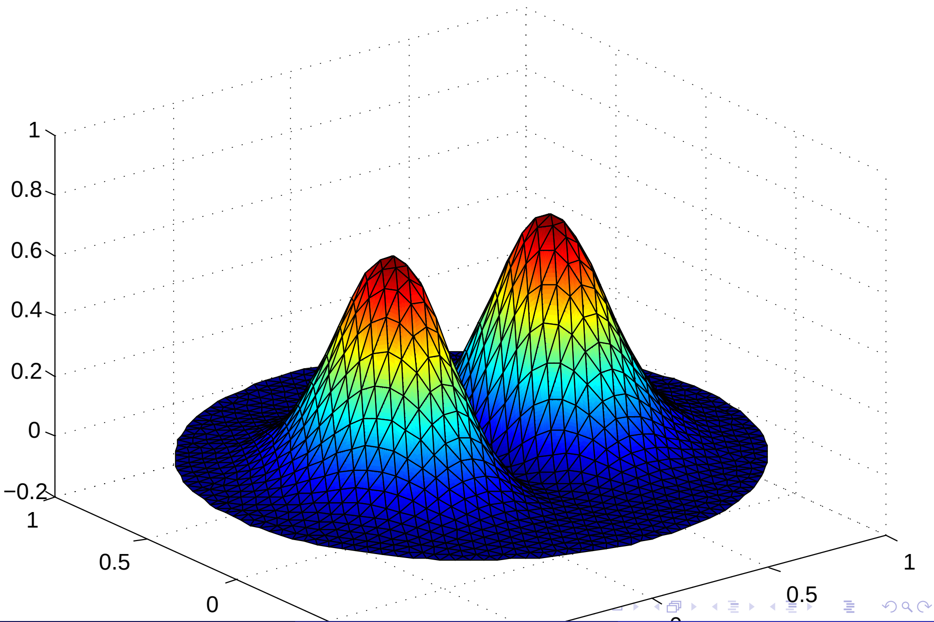
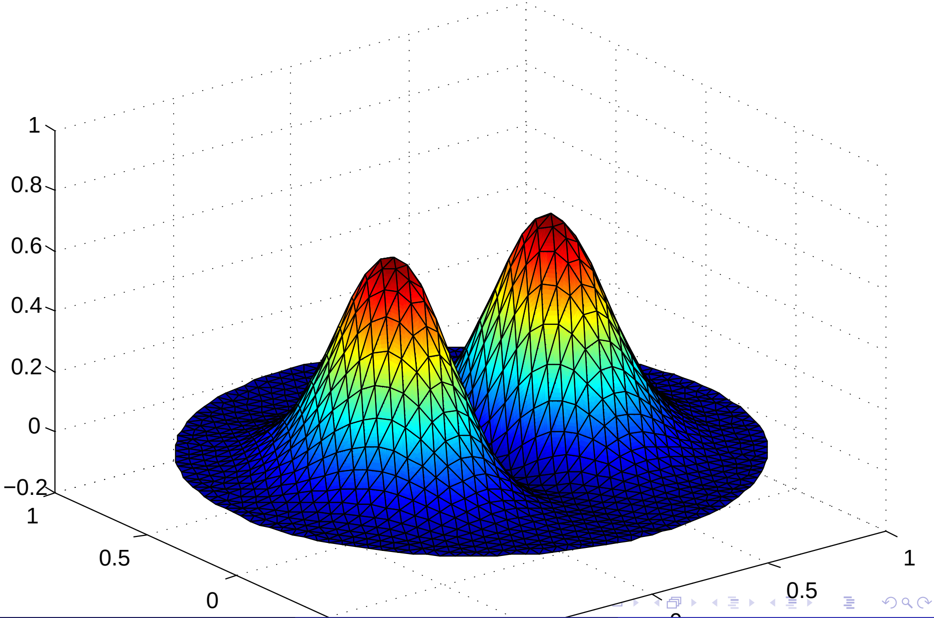
where u is the solution with Cauchy data $(f, 0)$.

Here, $\Omega = B(0,1)$, $T = 2$. Based on the 1st term only. **Original:**



Here, $\Omega = B(0,1)$, $T = 2$. Based on the 1st term only. **Reconstruction:**



Measurements on a part of the boundary

Assume that $P = -\Delta$ outside Ω. Let $\Gamma \subset \partial\Omega$ be a relatively open subset of $\partial\Omega$. Set

$$\mathcal{G} := \{(t,x);\ x \in \Gamma,\ 0 < t < s(x)\},$$

where s is a fixed continuous function on Γ. This corresponds to measurements taken at each $x \in \Gamma$ for the time interval $0 < t < s(x)$. The special case studied so far is $s(x) \equiv T$, for some $T > 0$; then $\mathcal{G} = [0, T] \times \Gamma$.

We assume now that the observations are made on $\mathcal{G}$ only, i.e., we assume we are given

$$\Lambda f|_{\mathcal{G}}.$$

We consider f's with

$$\operatorname{supp} f \subset \mathcal{K},$$

where $\mathcal{K} \subset \Omega$ is a fixed compact.

Uniqueness?

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Heuristic arguments for uniqueness: To recover f from Λf on $\mathcal{G}$, we must at least be able to get a signal from any point, i.e., we want for any $x \in \mathcal{K}$, at least one signal from x to reach some $z \in \Gamma$ for $t < s(z)$. In other words, we should at least require that

Condition A

$$\forall x \in \mathcal{K}, \exists z \in \Gamma \text{ so that } \operatorname{dist}(x, z) < s(z).$$

Theorem 3

Let $P = -\Delta$ outside Ω, and let $\partial\Omega$ be strictly convex. Then under Condition A, if $\Lambda f = 0$ on $\mathcal{G}$ for $f \in H_D(\Omega)$ with $\operatorname{supp} f \subset \mathcal{K}$, then $f = 0$.

Proof based on Tataru's uniqueness continuation results. Generalizes a similar result for flat geometry by Finch et al.

It is worth mentioning that without Condition A, one can recover f on the reachable part of $\mathcal{K}$. Of course, one cannot recover anything outside it, by finite speed of propagation. Thus, up to replacing $<$ with $\leq$,

Condition A is an "if and only if" condition for uniqueness.

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Stability

Heuristic arguments for stability: To be able to recover f from Λf on $\mathcal{G}$ *in a stable way*, we should be able to recover all singularities. In other words, we should require that

Condition B

$$\forall (x,\xi)\in S^*\mathcal{K},\ (\tau_\sigma(x,\xi),\gamma_{x,\xi}(\tau_\sigma(x,\xi))\in\mathcal{G} \text{ for either } \sigma=+ \text{ or } \sigma=- \text{ (or both)}.$$

We show next that this is an "if and only if" condition (up to replacing an open set by a closed one, as before) for stability. Actually, we show a bit more.

Proposition 1

Assume formally $T=\infty$. Then $\Lambda=\Lambda_+ + \Lambda_-$, where $\Lambda_\pm$ are elliptic Fourier Integral Operators of zeroth order with canonical relations given by the graphs of the maps

$$(y,\xi)\mapsto\left(\tau_\pm(y,\xi),\gamma_{y,\pm\xi}(\tau_\pm(y,\xi)),|\xi|,\dot\gamma'_{y,\pm\xi}(\tau_\pm(y,\xi))\right),$$

where $|\xi|$ is the norm in the metric $c^{-2}g$, and the prime in $\dot\gamma'$ stands for the tangential projection of $\dot\gamma$ on $T\partial\Omega$.

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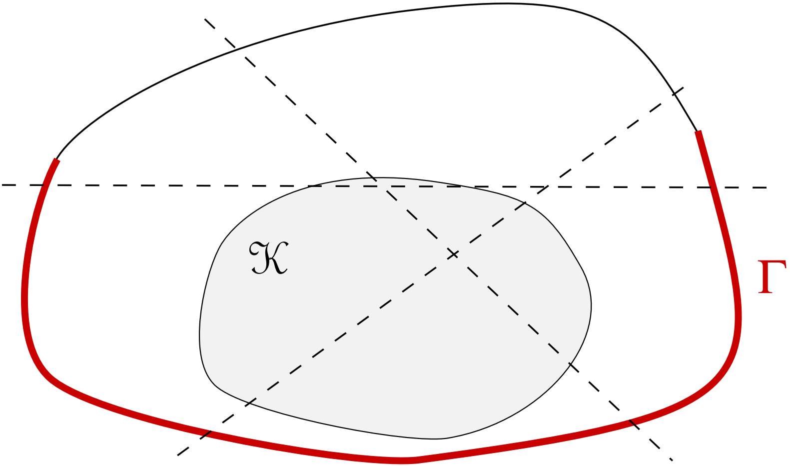
Let us say that $c = 1$, and we take measurements on $[0, T] \times \Gamma$, $T > \mathrm{diam}(\Omega)$. Then Condition B is equivalent to the following:

Every line through $\mathcal{K}$ intersects Γ.



Choose and fix $T > \sup_\Gamma s$. Let A be the "time reversal" operator as before (ϕ will be 0 because of χ below). Let $\chi(t) \in C^\infty$ be a cutoff equal to 1 near $[0, T(\Omega)]$, and equal to 0 close to $t = T$.

Theorem 4

$A\chi\Lambda$ is a zero order classical ΨDO in some neighborhood of $\mathcal{K}$ with principal symbol

$$\frac{1}{2}\left(\chi(\gamma_{x,\xi}(\tau_+(x,\xi))) + \chi(\gamma_{x,\xi}(\tau_-(x,\xi)))\right).$$

If $\mathcal{G}$ satisfies Condition B, then
(a) $A\chi\Lambda$ is elliptic,
(b) $A\chi\Lambda$ is a Fredholm operator on $H_D(\mathcal{K})$, and
(c) there exists a constant $C > 0$ so that

$$\|f\|_{H_D(\mathcal{K})} \le C\|\Lambda f\|_{H^1(\mathcal{G})}.$$

(b) follows by building a parametrix, and (c) follows from (b) and from the uniqueness result.

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Reconstruction

One can constructively write the problem in the form

Reducing the problem to a Fredholm one

$$(Id - K)f = BA\chi\Lambda f \quad \text{with the r.h.s. given,}$$

i.e., B is an explicit operator (a parametrix), where K is compact with 1 not an eigenvalue.

Reconstructing the acoustic speed c

Let f be known first. Linearize Λ near some background c. Then $\delta\Lambda[f, \delta c]$ is a bilinear form. Then

$$\Delta f \neq 0 \quad \text{on } \operatorname{supp} \delta c$$

is a sufficient condition for $\delta\Lambda[f, \cdot]$ to be Fredholm. On the other hand, if $\Delta f = 0$ in an open set inside $\operatorname{supp} \delta c$, then that map, even if it happens to be injective, will be unstable in any pair of Sobolev spaces.

We still do not know if $\delta\Lambda[f, \cdot]$ is injective. If so, one would have local uniqueness and Hölder stability.

The recovery of both f and c is not so clear. Preliminary calculations show that the linearization $\delta\Lambda$ may have a huge kernel. One could try to use more than one measurements but how realistic is that?

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An alternative way to recover c

Recovery of sound speed and more generally, a metric, from travel times is well developed and there are numerical results. Why not reduce the problem to this one?

Place a small object with thermoacoustic properties "f_0" different from the surrounding media. That means: replace f by $f + f_0$ with $\operatorname{supp} f_0$ non intersecting $\operatorname{supp} f$. Take your measurements $\Lambda(f + f_0)$. Subtract Λf from that. Then we get

$$\Lambda f_0$$

without the need to alter the patient. Now, from Λf_0, we can get the travel times from $\partial \operatorname{supp} f$ through Ω. If $\operatorname{supp} f_0$ is small enough, then just measure the first arrival time at each point on the boundary.

Then repeat this with f_0 supported elsewhere, etc. Then recover c from the travel times. Moreover, we do not need to know f for that. Once we know c, we can recover f.

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