Microlocal Methods in X-ray Tomography

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Lecture I: Euclidean X-ray tomography

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References

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X-ray transform

X-ray transform of a function f in $\mathbf{R}^n$:

$$Xf(\ell) = \int_\ell f \, \mathrm{d}s \tag{1}$$

along any given (undirected) line ℓ in $\mathbf{R}^n$. Here $\mathrm{d}s$ is the unit length measure on ℓ. Lines in $\mathbf{R}^n$ can be parameterized by initial points $x \in \mathbf{R}^n$ and directions $\theta \in S^{n-1}$, thus we can write, without changing the notation,

$$Xf(x,\theta) = \int_{\mathbf{R}} f(x+s\theta)\, \mathrm{d}s, \quad (x,\theta) \in \mathbf{R}^n \times S^{n-1}. \tag{2}$$

That parameterization is not unique because for any x, θ, t,

$$Xf(x,\theta) = Xf(x+t\theta,\theta), \quad Xf(x,\theta) = Xf(x,-\theta). \tag{3}$$

The latter identity reflects the fact that we consider the lines as undirected ones.

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Motivation

Motivating example: X-ray medical imaging. An X-ray point source is placed at different points around patient's body, and the intensity I of the rays is measured after the rays go through the body. The intensity depends on the position x and the direction θ of the rays. It solves the transport equation

$$(\theta \cdot \nabla_x + \sigma(x))\, I(x,\theta) = 0, \tag{4}$$

where σ is the absorption of the body. Equation (4) simply says that the directional derivative of I in the direction θ equals $-\sigma I$. A natural initial/boundary condition is to require that

$$\lim_{s\to -\infty} I(x+s\theta,\theta) = I_0,$$

where I_0 is the source intensity, that may depend on the line. Since f is of compact support in this case, the limit above is trivial. Then (4) has the explicit solution

$$I(x,\theta) = e^{-\int_{-\infty}^{0} \sigma(x+s\theta,\theta)\,\mathrm{d}s} I_0.$$

The measurement outside patent's body is modeled by

$$\lim_{s\to\infty} I(x+s\theta,\theta) =: I_1,$$

and this limit is trivial as well. Since both I_1 and I_0 are known, we may form the quantity

$$-\log(I_1/I_0) = \int_{-\infty}^{\infty} \sigma(x+s\theta,\theta)\,\mathrm{d}s$$

That is exactly $X\sigma(x,\theta)$. The problem then reduces to recovery of σ given $X\sigma$.

Count the number of variables that to parameterize Xf. For any θ, it is enough to restrict x to a hyperplane perpendicular to θ, that takes away one dimension. One such hyperplane is

$$\theta^{\perp} := \{x|\; x\cdot\theta = 0\}. \tag{5}$$

Then $Xf(x,\theta)$ is an even (w.r.t. θ) function of $2n-2$ variables, while f depends on n variables. Therefore, if $n=2$, Xf and f depends on the same number of variables, 2. We say that the problem of inverting X is then a *formally determined* problem. If $n\geq 3$, then Xf depends on more variables, making the problem *formally overdetermined*. On the other hand, in dimensions $n\geq 3$, if we know $Xf(\ell)$ for all lines, we also know $Xf(\ell)$ for the n-dimensional family of lines that consists of all ℓ parallel to a fixed 2-dimensional plane, say the one spanned by $(1,0,\dots,0)$ and $(0,1,0,\dots,0)$. It is then enough to solve the 2-dimensional problem of inverting R on each such plane. This is one way one can reduce the problem of inverting X to a formally determined one (that we can solve, as we will see later) using partial data. For this reason, very often the X-ray transform in analyzed in two dimensions only.

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The Radon Transform

The Radon transform Rf of a function f: integrals of f over all hyperplanes π in $\mathbf{R}^n$:

$$Rf(\pi) = \int_\pi f \, \mathrm{d}S. \tag{6}$$

Here $\mathrm{d}S$ is the Euclidean surface measure on each such hyperplane. Each such hyperplane can be written in exactly two different ways in the form

$$\pi = \{x | \ x \cdot \omega = p\} = \{x | \ x \cdot (-\omega) = -p\}$$

with $p \in \mathbf{R}$, $\omega \in S^{n-1}$. We then write

$$Rf(p, \omega) = \int_{x \cdot \omega = p} f \, \mathrm{d}S_x. \tag{7}$$

Then Rf is an even function on $\mathbf{R} \times S^{n-1}$.

The problem of finding f given Rf is always a formally determined one since both f and Rf are functions of n variables.
In $\mathbf{R}^2$, those two transforms are the same but parameterized differently.

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The transpose X'

Since (x,θ) and $(x+s\theta,\theta)$ define the same line, a natural parameterization is

$$\Sigma = \left\{(z,\theta)|\ \theta\in S^{n-1},\ z\in\theta^\perp\right\}.$$

Define a measure $\mathrm{d}\sigma$ on Σ:

$$\mathrm{d}\sigma(z,\theta) = \mathrm{d}S_z\,\mathrm{d}\theta,$$

where, $\mathrm{d}\theta$ is the standard measure on S^{n-1}, and $\mathrm{d}S_z$ is the Euclidean measure on the hyperplane $\theta^\perp$. In this parameterization, each directed line has unique coordinates but each undirected one has two pairs of coordinates.

Another parameterization: Assume that we will apply X only to functions supported in some bounded domain Ω with a strictly convex smooth boundary. The strict convexity assumption is not restrictive since we can always enlarge the domain to a strictly convex one, for example a ball, that contains the domain of interest.

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Set

$$\partial_- S\Omega = \left\{(x,\theta) \in \partial\Omega \times S^{n-1} | \; \nu(x)\cdot\theta < 0\right\}, \tag{8}$$

where ν is the exterior unit normal to $\partial\Omega$. On $\partial_- S\Omega$, define the measure

$$\mathrm{d}\mu(x,\theta) = |\nu(x)\cdot\theta| \mathrm{d}S_x \, \mathrm{d}\theta, \tag{9}$$

where $\mathrm{d}S_x$ is the surface measure on $\partial\Omega$. There is a natural map

$$\partial_- S\Omega \ni (x,\theta) \longmapsto (z,\theta) \in \Sigma, \tag{10}$$

where z is the intersection of the ray $\{x + s\theta | \; s \in \mathbf{R}\}$ with $\theta^\perp$. The map (10) is invertible on its range. Given (z,θ), x is the intersection of the ray $\{z + s\theta | \; s \in \mathbf{R}\}$ with $\partial\Omega$ having the property that at x, the vector θ points into Ω.

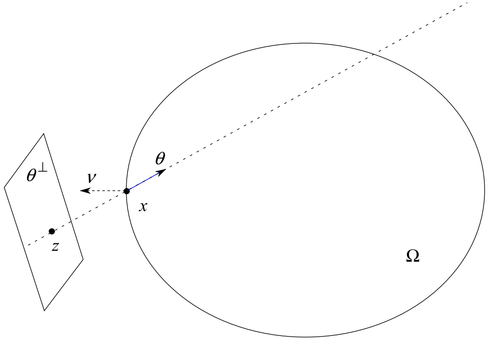
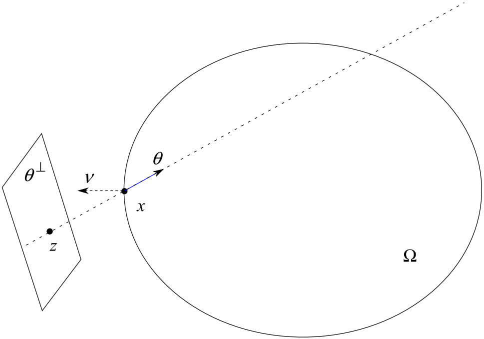


Figure: Two ways to parameterize a line

Proposition 1

The map (10) and its inverse are isometries.

The proof is immediate. Fix θ, and project locally $\partial\Omega$ on $\theta^\perp$, in the direction of θ, near some point x so that $(x,\theta) \in \partial_- S\Omega$. The Jacobian of that projection is $1/|\nu(x)\cdot\theta|$.



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We can compute now the transpose X' w.r.t. either parameterization:

$$X'\psi(x) = \int_{S^{n-1}} \psi(x - (x\cdot\theta)\theta, \theta)\,\mathrm{d}\theta. \tag{11}$$

We can interpret this formula in the following way. The function ψ is a function on the manifold of lines. Given $x \in \mathbf{R}^n$, for any $\theta \in S^{n-1}$ we evaluate ψ on the line through x in the direction of θ, and then integrate over θ. In other words, $X'\psi(x)$ is an integral of $\psi = \psi(\ell)$ over all lines ℓ through x

$$X'\psi(x) = \int_{\ell\ni x} \psi(\ell)\,\mathrm{d}\ell_x,$$

where $\mathrm{d}\ell_x$ is the unique measure on $\{\ell \ni x\}$ invariant under orthogonal transformations, with total measure $|S^{n-1}|$, i.e., $\mathrm{d}\ell_x = \mathrm{d}\theta$ in the parameterization that we use. Compare this to (1) which can also be written in the form

$$Xf(\ell) = \int_{x\in\ell} f(x)\,\mathrm{d}s \tag{12}$$

The transform X' is often called a backprojection — it takes a function defined on lines to a function defined on the "x-space" $\mathbf{R}^n$.

X can be extended to compactly supported distributions

Note that X' does not preserve the compactness of the support, i.e., for $\psi \in C_0^\infty$, $X'\psi$ may not be of compact support!

By duality, we define X on the space $\mathcal{E}'(\mathbf{R}^n)$ of compactly supported distributions but we cannot do this on $\mathcal{D}'(\mathbf{R}^n)$, as it could be expected (even for smooth functions we need a certain decay at infinity).

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The transpose of R

There is a natural measure on $\mathbf{R} \times S^{n-1}$, where Rf lives. The transpose R' w..r.t. it is well defined. A simple calculation yields, for $\psi \in C_0^\infty(\mathbf{R} \times S^{n-1})$,

$$R'\psi(x) = \int_{S^{n-1}} \psi(x \cdot \omega, \omega)\, \mathrm{d}\omega.$$

Similarly to what we had before, ψ is a function on the set of oriented hyperplanes (and on the set of hyperplanes when ψ is even). Then we can think of $\mathbf{R}'\psi$ as an integral of $\psi = \psi(\pi)$ over the set of all hyperplanes π through x. Similarly to X', R' is also called sometimes a backprojection.

We extend R to compactly supported distributions as before.

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The Fourier Slice Theorem

Theorem 1 (The Fourier Slice Theorem)

For any $f \in L^1(\mathbf{R}^n)$*,*

$$\hat{f}(\zeta) = \int_{\theta^\perp} e^{-\mathrm{i}z\cdot\zeta} Xf(z,\theta)\,\mathrm{d}S_z, \quad \forall\theta \perp \zeta,\ \theta \in S^{n-1}.$$

Denote by $\mathcal{F}_{\theta^\perp}$ the Fourier transform in the z variable on $\theta^\perp$. With this notation, the Fourier Slice Theorem reads: for any θ, $\hat{f}|_{\theta^\perp} = \mathcal{F}_{\theta^\perp} Xf$.

Proof.

The integral on the right equals $\int_{\theta^\perp}\int_{\mathbf{R}} e^{-\mathrm{i}z\cdot\zeta} f(z+s\theta)\,\mathrm{d}s\,\mathrm{d}S_z$. Set $x = z + s\theta$ and note that $x\cdot\zeta = z\cdot\zeta$ when $\zeta\perp\theta$. Then we see that the integral above equals $\hat{f}(\zeta)$. □

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Theorem 1 immediately implies injectivity of X on $L^1(\mathbf{R}^n)$ (also, on $\mathcal{E}'(\mathbf{R}^n)$). T

Also, it says that knowing Xf for a fixed θ recovers $\hat{f}$ for $\xi \perp \theta$. This has a micolocal equivalent, as we will see later.

In fact, for compactly supported functions, the theorem implies a bit more. The decisive argument in the proof is the analyticity of the Fourier transform of compactly supported functions.

Corollary 2

Let $f \in L^1(\mathbf{R}^n)$ have compact support and let $Xf(\cdot, \theta) = 0$ for θ in an infinite set of (distinct) unit vectors, then $f = 0$.

Finitely many "roentgenograms" however are not enough to recover f.

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Intertwining properties

Intertwining properties for R

$$R\Delta = d_p^2 R, \quad R' d_p^2 = \Delta R',$$

on $C_0^\infty(\mathbf{R}^n)$ and on $C_0^\infty(\mathbf{R}\times S^{n-1})$, respectively.

Proof: straightforward, either by direct computations or by using the Fourier Slice Theorem.

Let Δ_z denote the Laplacian in the z variable on each $\theta^\perp$. Set $|D_z| = (-\Delta_z)^{1/2}$. Similarly,

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The Kernel of $X'X$

Proposition 2

$$X'Xf(x) = 2\int_{\mathbf{R}^n} \frac{f(y)}{|x-y|^{n-1}}\mathrm{d}y, \quad \forall f \in \mathcal{S}(\mathbf{R}^n)$$

This is a convolution and therefore a Fourier multiplier! We need the Fourier transform of $|x|^{-(n-1)}$ to find it, and the answer is $c|\xi|^{-1}$. Therefore, $X'X = c|D|^{-1}$, and to invert it, we need to apply $c'|D|$.

Theorem 3

For any $f \in \mathcal{S}(\mathbf{R}^n)$,

$$f = c_n|D|X'Xf, \quad c_n = \left(2\pi|S^{n-2}|\right)^{-1}.$$

For $n=2$, $|S^{n-2}| = 2$, so $c_2 = (4\pi)^{-1}$.

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The Fourier multiplier $|D|$ is a non-local operator. Therefore, if we want to recover f only in a neighborhood of some x_0 by means of formula above, it is not enough to know Xf for all lines ℓ that intersect that neighborhood.

In particular, if f is compactly supported, we need to compute $X'Xf$ for all $x \in \mathbf{R}^n$ to be able to apply $|D|$. Numerically, one would just truncate the computational region but $X'Xf$ does not decay very fast (only like $|x|^{-n+1}$), and the error will be not so small. But such a truncated recovery would still be a parametrix (more — later).

This calls for another inversion formula.

Theorem 4 (A filtered back-projection)

For any $f \in \mathcal{S}(\mathbf{R}^n)$,

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The non-local operator $|D_z|$ now appears between X' and X. Is that a progress (for computational purposes)? Yes — if f is compactly supported, we need to evaluate the result in a compact set, i.e., we need to know $\chi X'|D_z|Xf$ for some χ of compact support. This means that we need to evaluate $|D_z|Xf$ on a compact set as well. But Xf is compactly supported. So we need to apply $|D_z|$ from a comact set to a compact set.

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The Schwartz kernel of $R'R$

Proposition 3

For any $f \in \mathcal{S}(\mathbf{R}^n)$,

$$R'Rf(x) = |S^{n-2}| \int_{\mathbf{R}^n} \frac{f(y)}{|x-y|} \mathrm{d}y.$$

We have a convolution again, with the Fourier transform of $|x|^{-1}$. The latter is $c|\xi|^{-n+1}$. Therefore,

Theorem 5

For any $f \in \mathcal{S}(\mathbf{R}^n)$,

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Let H be the Hilbert transform

$$Hg(p) = \frac{1}{\pi} \,\mathrm{pv}\int_{\mathbf{R}} \frac{g(s)}{p-s}\,\mathrm{d}s, \qquad (13)$$

where "pv$\int$" stands for an integral in a principal value sense.

Theorem 6 (filtered backprojection)

For any $f \in \mathcal{S}(\mathbf{R}^n)$,

$$f = \begin{cases} C_n' R' d_p^{n-1} Rf, & n \text{ odd}, \\ C_n' R' H d_p^{n-1} Rf, & n \text{ even}, \end{cases} \qquad (14)$$

where d_p stands for the derivative of $Rf(p,\omega)$ w.r.t. p, H is the Hilbert transform w.r.t. p and

$$C_n' = \begin{cases} (-1)^{(n-1)/2} C_n, & n \text{ odd}, \\ (-1)^{(n-2)/2} C_n, & n \text{ even}, \end{cases}$$

with $C_n = \frac{1}{2}(2\pi)^{1-n}$ (as in Theorem 5).

The appearance of the Hilbert transform H for n even, and the different constants for n odd/even may look strange at first glance, especially when compared to the inversion formula in Theorem 5, that looks the same for all $n \geq 2$. For n even, note first that $H = -\mathrm{i}\,\mathrm{sgn}(D_p)$, $D_p = -\mathrm{i}d_p$, therefore,

$$(-1)^{(n-2)/2} H d_p^{n-1} = |D_p|^{n-1}, \quad n \text{ even.}$$

On the other hand,

$$(-1)^{(n-1)/2} d_p^{n-1} = |D_p|^{n-1}, \quad n \text{ odd.}$$

Therefore, in both cases, (14) can be written as

$$f = C_n R' |D_p|^{n-1} R f \tag{15}$$

Stability estimates

Set

$$\|g\|_{\bar{H}^s(\Sigma)} = \left\|(1-\Delta_z)^{s/2} g\right\|_{L^2(\Sigma)},$$
$$\|g\|_{\bar{H}^s(\mathbf{R}\times S^{n-1})} = \left\|(1-\mathrm{d}_p^2)^{s/2} g\right\|_{L^2(\mathbf{R}\times S^{n-1})}, \quad (16)$$

Theorem 7 (Stability estimates)

For any bounded domain $\Omega \subset \mathbf{R}^n$ with smooth boundary, and any s, we have

$$\|f\|_{H^s(\mathbf{R}^n)}/C \le \|Xf\|_{\bar{H}^{s+1/2}(\Sigma)} \le C\|f\|_{H^s(\mathbf{R}^n)}, \quad (17)$$
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Theorem 7 shows that we "gain 1/2 derivative" with the operator X, and $(n-1)/2$ derivatives with the operator R. Each one of those two operators involves an integration that has a smoothing effect. The gain is a half of the dimension of the linear submanifolds over which we integrate.

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Stability estimates in terms of $X'X$ and $R'R$

Why $X'X$ (the normal operator)? It is the first step of the inversion; it lives in the same space; $X'X$ is injective if and only if X is.

Theorem 8

Let $\Omega \subset \mathbf{R}^n$ be open and bounded, and let $\Omega_1 \supset \bar{\Omega}$ be another such set. Then for any integer $s = 0, 1, \dots,$ there is a constant $C > 0$ so that for any $f \in H^s(\mathbf{R}^n)$ supported in $\bar{\Omega}$, we have

$$\|f\|_{H^s(\mathbf{R}^n)}/C \le \|X'Xf\|_{H^{s+1}(\Omega_1)} \le C\|f\|_{H^s(\mathbf{R}^n)}, \tag{19}$$

$$\|f\|_{H^s(\mathbf{R}^n)}/C \le \|R'Rf\|_{H^{s+n-1}(\Omega_1)} \le C\|f\|_{H^s(\mathbf{R}^n)} \tag{20}$$

The proof (of the inequalities on the left) seems to be straightforward as well — we have a formula for f in terms of $X'Xf$ and $R'Rf$. Just apply $c|D|$ to $X'Xf$, and we get f. Problem: we need $X'Xf$ on the whole $\mathbf{R}^n$ for that! The theorem requires to know this on Ω_1 only. ???

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To prove the first estimate, use the inversion formula $f = c_n|D|X'Xf$ and write

$$\|f\|^2_{H^s(\mathbf{R}^n)} \le c_n\|X'Xf\|^2_{H^{s+1}(\mathbf{R}^n)} = c_n\|X'Xf\|^2_{H^{s+1}(\Omega_1)} + c_n\|X'Xf\|^2_{H^{s+1}(\mathbf{R}^n\setminus\Omega_1)}.$$

We want to get rid of the last term. The following lemma solves the problem:

Lemma 9

Let X, Y, Z be Banach spaces, let $A : X \to Y$ be a bounded linear operator, and $K : X \to Z$ be a compact linear operator. Let

$$\|f\|_X \le C\left(\|Af\|_Y + \|Kf\|_Z\right), \quad \forall f \in X. \tag{21}$$

Assume that A is injective. Then there exists $C' > 0$ so that

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We do know that $X'X : H^s_0(\Omega) \to H^{s+1}(\Omega_1)$ is injective. So we can apply the lemma and get rid of that term (no control over C though!)

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Support theorems

Theorem 10 (Support theorem)

Let $f \in C(\mathbf{R}^n)$ be such that

(i) $|x|^k f(x)$ *is bounded for any integer* k,

(ii) *there exists a constant* $A > 0$ *so that* $Rf(p, \omega) = 0$ *for* $|p| > A$.

Then $f(x) = 0$ *for* $|x| > A$.

Corollary 11

Let $K \subset \mathbf{R}^n$ be a convex compact set. Let $f \in C(\mathbf{R}^n)$ satisfy the assumption (i) above. Assume also that $Rf(\pi) = 0$ for any hyperplane π not intersecting K. Then $f = 0$ outside K.

Support theorems for X can be derived directly from those for R by working in various 2D planes, where R and X are the same transforms. On the other hand, one can formulate stronger results for X since the lines in $\mathbf{R}^n$ are "thinner" and can fit into smaller "holes."

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The proof (due to Helgason) is not short — first we prove it for radially symmetric functions, and then we manage to reduce the general case to the radial one. Later we will present a microlocal explanation which can be generalized to more general curves. In some sense, there is kind of analytic continuation from infinity, where f is assumed to be small, to the exterior of the ball.

The rapid decay condition is essential. For any $N > 0$, there is a function with $|f| \leq C(1 + |x|)^{-N}$ of infinite support with Radon transform compactly supported.

An important part of the theory which we will skip is Range Conditions.

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