Microlocal Methods in X-ray Tomography

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Lecture II: The weighted X-ray transform

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Local and Lambda tomography

Let us say that we are interested in a certain "region of interest" only (the heart for example). Regardless of what formula we use, for recovery of f we need to make global computations. We would like to measure Xf for all lines through the region only. This problem has no unique solution (there are counterexamples). In medical imaging, we would like to do it anyway.

We can use for reconstruction

$$f \sim X'Xf$$

(instead of $c_n|D|X'Xf$). Up to a constant, this is $|D|^{-1}f$ — a smoothened version of f. It has the same singularities (same WF(f)) but of lower order. Another option is to use

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Neither formula is an approximation, and it distorts the constant (smooth) parts of f. The Lambda tomography Faridani et al. tries to combine both:

$$\begin{aligned} f &\sim -\Delta X'Xf + \mu X'Xf \\ &= |D|f + \mu|D|^{-1}f. \end{aligned}$$

Here $\mu > 0$ is a constant depending somehow on the domain where $\operatorname{supp} f$ is, and is chosen experimentally.

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Definition

The weighted X-ray transform:

$$X_w f(x,\theta) = \int_{\mathbf{R}} w(x+t\theta,\theta) f(x+t\theta)\,\mathrm{d}t, \quad (x,\theta)\in \mathbf{R}^n \times S^{n-1}.$$

Here $w = w(x,\theta)$ is a smooth weight depending in general not only on the point x but also on the direction θ. We can parametrize X_w by $(z,\theta)\in\Sigma$ as before. In general, $X_w f$ is not an even function of θ any more.

What is the motivation?

- The usual X-ray transform with incomplete data can be reduced to a weighted one, with a weight equal to zero where we have no data.
- There are natural examples, like the attenuated X-ray transform.

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The attenuated X-ray transform

Arises in SPECT: We measure a source outside patient's body but the signal attenuates before being measured.

The model is the transport equation:

$$(\theta \cdot \nabla + a(x))u = f(x), \quad u|_{\Gamma_-} = 0.$$

Here, $\Gamma_- = \{(x,\theta) \in \partial\Omega \times S^{n-1};\ \theta \text{ points towards the interior}\}$. What we measure is $u|_{\Gamma_+}$ (for $x \in \partial\Omega$, θ outgoing). Then $u|_{\Gamma_+}$ is given by the attenuated X-ray transform with weight

$$w(x,\theta) = \exp\left(-\int_0^\infty a(x+t\theta)\,\mathrm{d}t\right).$$

The derivation is simple: the transport equation is a 1st order linear ODE along the lines parallel to θ, find the integrating factor, etc.

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The transpose X'_w

We use either of the two parameterizations which we used for X.

Proposition 1

$$X'_w\psi(x) = \int_{S^{n-1}} w(x,\theta)\psi(x-(x\cdot\theta)\theta,\theta)\,\mathrm{d}\theta, \quad \forall\psi\in C^\infty(\Sigma).$$

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Proof.

Let $\phi \in C_0^\infty(\mathbf{R}^n)$, $\psi \in C^\infty(\Sigma)$. We have

$$\int_\Sigma (X_w\phi)\psi \,\mathrm{d}\sigma = \int_\Sigma \int_{\mathbf{R}} w(z+s\theta,\theta)\phi(z+s\theta)\,\psi(z,\theta)\,\mathrm{d}s\,\mathrm{d}S_z\,\mathrm{d}\theta. \tag{1}$$

Set $x = z + s\theta$, where $z \in \theta^\perp$. For a fixed $\theta \in S^{n-1}$, $(z,s) \mapsto x$ is an isomorphism with a Jacobian equal to 1. The inverse is given by

$$z = x - (x\cdot\theta)\theta, \quad s = x\cdot\theta.$$

We therefore have

$$\int_\Sigma (X_w\phi)\psi \,\mathrm{d}\sigma = \int_{S^{n-1}} \int_{\mathbf{R}^n} w(x,\theta)\phi(x)\,\psi(x-(x\cdot\theta)\theta,\theta)\,\mathrm{d}x\,\mathrm{d}\theta.$$

This completes the proof. □

Proposition 2

For any two L^∞ weights a, b,

$$X_b' X_a f(x) = \int \frac{W\left(x, y, \frac{x-y}{|x-y|}\right)}{|x-y|^{n-1}} f(y)\,\mathrm{d}y,$$

where

$$W(x,y,\theta) = b(x,\theta)a(y,\theta) + b(x,-\theta)a(y,-\theta).$$

Compare this to the $w = 1$ case; then $W = 2$.

The proof is now a two slide proof.

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Proof

By the proposition,

$$X_b' X_a f(x) = \int_{S^{n-1}} b(x,\theta) \int a(x-(x\cdot\theta)\theta + t\theta, \theta) f(x-(x\cdot\theta)\theta + t\theta)\,\mathrm{d}t\,\mathrm{d}\theta$$
$$= \int_{S^{n-1}} b(x,\theta) \int a(x+t\theta,\theta) f(x+t\theta)\,\mathrm{d}t\,\mathrm{d}\theta.$$

Split the t-integral in two parts: for $t>0$ and for $t<0$, and replace t by $-t$ in the second one to get

$$X_b' X_a f(x) = \int_{S^{n-1}} b(x,\theta) \int a(x+t\theta,\theta) f(x+t\theta)\,\mathrm{d}t\,\mathrm{d}\theta$$
$$= \int_{S^{n-1}} b(x,\theta) \int_0^\infty a(x+t\theta,\theta) f(x+t\theta)\,\mathrm{d}t\,\mathrm{d}\theta$$
$$+ \int_{S^{n-1}} b(x,\theta) \int_0^\infty a(x-t\theta,\theta) f(x-t\theta)\,\mathrm{d}t\,\mathrm{d}\theta.$$

Proof, continued

Replace $-\theta$ by θ in the second integral to get

$$X_b' X_a f(x) = \int_{S^{n-1}} \int_0^\infty \Big[b(x,\theta) a(x+t\theta,\theta) \\ + b(x,-\theta) a(x+t\theta,-\theta) \Big] f(x+t\theta)\, \mathrm{d}t\, \mathrm{d}\theta.$$

Pass to polar coordinates $y = x + t\theta$, centered at x to finish the proof.

making it a ΨDO

To write $X_b' X_a$ as a ΨDO, recall that if the Schwartz kernel of a linear operator is given by $K(x,y,x-y)$, then it is a formal ΨDO with an amplitude given by the Fourier transform of K w.r.t. the third variable. Therefore, $X_a X_b$ is a formal ΨDO with amplitude

$$\int_{\mathbf{R}^n} e^{-\mathrm{i}z\cdot\xi} \frac{W(x,y,z/|z|)}{|z|^{n-1}}\,\mathrm{d}z = \int_{\mathbf{R}_+\times S^{n-1}} e^{-\mathrm{i}r\theta\cdot\xi} W(x,y,\theta)\,\mathrm{d}r\,\mathrm{d}\theta$$
$$= \pi \int_{S^{n-1}} W(x,y,\theta)\delta(\theta\cdot\xi)\,\mathrm{d}\theta.$$

We used here the fact that W is an even function of θ and that the inverse Fourier transform of 1 is δ.

If $n = 2$, we get

$$\int_{\mathbf{R}^n} e^{\mathrm{i}z\cdot\xi} \frac{W(x,y,z/|z|)}{|z|^{n-1}}\,\mathrm{d}z = \frac{\pi}{|\xi|}\Big(W(x,y,\xi^\perp/|\xi|) + W(x,y,-\xi^\perp/|\xi|)\Big) = \frac{2\pi}{|\xi|} W(x,y,\xi^\perp/|\xi|), \tag{2}$$

where $\xi^\perp := (-\xi_2, \xi_1)$. Since this is a homogeneous function of ξ, with an integrable singularity that can be cut-off resulting in a smoothing operator, this completes the proof.

Theorem 1

Let a, b be smooth. Then $X_b' X_a$ is a classical ΨDO of order -1 with amplitude given by (2) and a principal symbol

$$\sigma_p(X_b' X_a) = \pi \int_{S^{n-1}} W(x,y,\theta)\delta(\theta\cdot\xi)\,\mathrm{d}\theta.$$

If $n = 2$, the integral is understood in the sense (2), i.e.,

$$\begin{aligned}\frac{2\pi}{|\xi|}W(x,y,\xi^\perp/|\xi|) = &\frac{2\pi}{|\xi|}b(x,\xi^\perp/|\xi|)a(y,\xi^\perp/|\xi|)\\ &+\frac{2\pi}{|\xi|}b(x,-\xi^\perp/|\xi|)a(y,-\xi^\perp/|\xi|)\end{aligned}$$

Note also that we can write

$$\sigma_p(X_b'X_a) = 2\pi\int_{S^{n-1}} b(x,\theta)a(x,\theta)\delta(\theta\cdot\xi)\,\mathrm{d}\theta.$$

Our goal now is to recover the wave front set $\mathrm{WF}(f)$ of f given $X_w f$. A good choice of "back-projection" is X_w^* (which equals X_w'). If $X_w^* X_w$ is elliptic, we can use a parametrix for that.

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Theorem 1 implies a necessary and sufficient condition for ellipticity: $X_b' X_a$ is an elliptic ΨDO of order -1 at (x, ξ) if and only if the average of $(ab)(x, \theta)$ over the $(n-2)$-dimensional sphere $|\theta| = 1$, $\theta \perp \xi$ is not zero. If $n = 2$, there are only two such $\theta's$, namely $\pm\xi^\perp/|\xi|$.

We return the the analysis of the operator $X_w^* X_w$. Then $a = \bar{b} = w$, so $ab = |w|^2$.

Corollary 2

Let $w \in C^\infty(\mathbf{R}^n \times S^{n-1})$. Then $X_w^* X_w$ is an elliptic ΨDO of order -1 at (x, ξ) if and only if there exists a unit $\theta \perp \xi$ so that $w(x, \theta) \neq 0$. In particular, let $\Omega \subset \mathbf{R}^n$ be open and bounded. Then $X_w^* X_w$ is an elliptic ΨDO of order -1 in a neighborhood of $\bar{\Omega}$ if and only if

$$\forall (x, \xi) \in \bar{\Omega} \times \mathbf{R}^n \setminus 0, \exists \theta \perp \xi \text{ so that } w(x, \theta) \neq 0. \tag{3}$$

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Limited angle X-ray transform

Let $w = 1$. Assume that we know $Xf(z, \theta)$ for θ restricted to an open set $U \subset S^1$ given by $0 \le \alpha < \arg\theta < \beta \le 2\pi$, and all corresponding $z \in \theta^\perp$, knowing a priori that f is continuous and of compact support.

Is that enough to recover f? By the Fourier Slice Theorem, we can uniquely determine $\hat{f}(\xi)$ for all ξ so that $\xi \cdot \theta = 0$ for some θ as above. In particular, of $\beta - \alpha > \pi$ (we have more than "half" of the angles), we can recover $\hat{f}(\xi)$ for all ξ. Of course, then we have all the lines as well, and we have stability.

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What if we know $Xf(z,\theta)$ for $\alpha < \arg\theta < \beta$ with $\beta - \alpha < \pi$? Then we can still recover easily $\hat f(\xi)$ for all $\xi \in U^\perp$ but the latter does not cover the whole $\mathbf{R}^n$.

If $\alpha = 0$, $\beta = \pi/2$, for example, then we only get $\hat f(\xi)$ for $\arg\xi$ in $[\pi/2,\pi] \cup [3\pi/2, 2\pi]$. On the other hand, $\hat f(\xi)$ is real analytic, and then by analytic continuation, we can recover $\hat f(\xi)$ even for ξ in the missing sector. Therefore, the so restricted Xf recovers f uniquely.

We have an even stronger uniqueness statement (the infinite many θ's theorem), and we could have used the support theorem to get the same conclusion.

The use of analytic continuation is a strong suggestion (but not a proof!) of possible instability. As we will see below, in the second case, $\beta - \alpha < \pi$, stability is lost, indeed.

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Theorem 3

Let Ω and w be as in Corollary 2, and $\Omega_1 \Supset \Omega$. Assume that w satisfies the ellipticity condition (3). Then
(a) $\operatorname{Ker} X_w \cap L^2(\Omega)$ *is a finite dimensional subspace of* $C_0^\infty(\mathbf{R}^n)$.
(b) For any $s \geq 0$, there exists constants $C > 0$ (independent of s) and C_s so that

$$\|f\|_{L^2(\Omega)} \leq C\|X_w^* X_w f\|_{H^1(\Omega_1)} + C_s\|f\|_{H^{-s}(\mathbf{R}^n)}, \quad \forall f \in L^2(\Omega).$$

(c) If X_w is injective on $L^2(\Omega)$, then the estimate above holds without the last term (and possibly a different C), i.e.,

$$\|f\|_{L^2(\Omega)} \leq C\|X_w^* X_w f\|_{H^1(\Omega_1)}, \quad \forall f \in L^2(\Omega).$$

(d) If the ellipticity condition (3) fails in an open set, then the estimate

$$\|f\|_{H^{s_1}(\Omega)} \leq C\|X_w^* X_w f\|_{H^{s_2}(\Omega_1)}, \quad \forall f \in L^2(\Omega).$$

does not hold regardless of the choice of s_1 and s_2.

What does this theorem tell us?

Assume $w \neq 0$ for simplicity and assume that we have an open set of lines. Choose a weight w constant along each line, supported in that set, and positive in "a smaller one". Then apply the theorem to get

- In 2D, we need a bit more of "half' of the lines for stability. Notice that for general weights, lines are directed.
- In 2D, if you are missing an open set of (undirected) lines, there is no stability, and there is nothing you can do about it.
- In higher dimensions ($n \geq 3$) a much smaller set of lines is enough for stability.

What do we do if we have limited angle data, and there is no stability? The common approach is to minimize a certain functional with a regularizing term. This produces a picture with some blurred "features" (singularities). This brings us to the notion of the "visible singularities".

Visible Singularities

Let U be an open set of lines on which Xf is known. The visible singularities (x, ξ) are the ones (co)-normal to U.

In the more general case (a non-trivial weight), we should in addition require that $w(x, \theta) \neq 0$ for at least one $\theta \perp \xi$.

The analysis so far shows that

What we can recover

Given $X_w f$, we can recover "in a stable way" the visible singularities only.

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Given $X_w f$, we can recover "in a stable way" the visible singularities only.

What does “in a stable way” mean? It means we can do it by a parametrix construction (not ba analytic continuation, for example). Remember, if the ΨDO P is elliptic, we can construct a right and a left inverse

$$QP = I + R_1, \quad PQ = I + R_2, \quad R_{1,2} \text{ smoothing.}$$

There is a notion of microlocal ellipticity. Fix a conic set $V \subset T^*\mathbf{R}^n$ ($(x,\xi) \in V \Rightarrow (x, s\xi) \in V$, $\forall s > 0$). Let the principal symbol p_m of P satisfy

$$|p_m(x,\xi)| \geq C(1+|\xi|)^m, \quad (x,\xi) \in V.$$

Then we say that P is elliptic on V. Then one can construct a ΨDO Q so that

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In other words, $\mathrm{WF}(R_{1,2}f) \cap V = \emptyset$ for any f.

Back to $X_w f$ known for lines in U: Multiply w by another smooth weight χ cutting in U. Then $X^*_{\chi w} X_{\chi w}$ is elliptic on $U^\perp$ (in a slightly smaller cone, actually, depending on χ). Then we can construct a parametrix on $U^\perp$.

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Sketch of the proof of the theorem

(a): $X_w^* X_w$ is elliptic and has a parametrix, then $QX_w^* X_w = I + K$, where K is smoothing and hence compact. If $X_w f = 0$,

$$(I + K)f = 0.$$

This is a Fredholm equation, and has a finitely dimensional kernel. It is smooth, because $f = -Kf$.

(b) The estimate follows from $f = QX_w^* X_w f - Kf$.

(c) follows from the trick in Lecture 1.

(d) requires a bit more work. If that estimate were true, $X_w^* X_w f \in H^{s_2}$ would imply $f \in H^{s_1}$. But we can take any distribution f with $\mathrm{WF}(f)$ in the (open) invisible set, of any negative order, and then $X_w^* X_w f \in C^\infty$ because the whole symbol $X_w^* X_w$ vanishes in the invisible set.

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We had to assume above that X_w was injective. When is that true?

Boman's example

There is a smooth positive weight $w \in C^\infty(\bar{B}(0,1) \times S^1)$ so that X_w is not injective on $\bar{B}(0,1)$.

An injectivity example is the attenuated X-ray transform.

Actually, this is a generic property of non-vanishing weights, as we will see below.

To get there, we will first show that perturbing w of an injective X_w preserves injectivity.

Starting point:

$$\|f\|_{L^2(\Omega)} \le C\|X_w^* X_w f\|_{H^1(\Omega_1)}, \quad \forall f \in L^2(\Omega).$$

If $C^k \ni w \mapsto X_w^* X_w \in \mathcal{L}(L^2(\Omega); H^1(\Omega_1))$ is continuous, for some k, we can just perturb that estimate. It is essential here that the estimate is sharp, and a perturbation of the kind $\epsilon\|f\|_{L^2(\Omega)}$ can be absorbed by the l.h.s.

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There are two ways (at least) to show that $X_w^* X_w$ depends continuously on w. The first one follows more or less directly from the calculus. $X_w^* X_w$ is a ΨDO with an amplitude $a(x, y, \xi)$ expressed directly in terms of w. One can show that $X_w^* X_w$ has a norm bounded by a constant depending on a finitely many derivatives of w.

Downside: requires too many derivatives, $k = 2n + 1$.

Another approach: start with

$$X_w^* X_w f(x) = \int \frac{W\left(x, y, \frac{x-y}{|x-y|}\right)}{|x-y|^{n-1}} f(y)\, \mathrm{d}y,$$

where

$$W(x, y, \theta) = \bar{w}(x, \theta) w(y, \theta) + \bar{w}(x, -\theta) w(y, -\theta).$$

Choose w_1 and w_2 and estimate $(X_{w_1}^* X_{w_1} - X_{w_2}^* X_{w_2}) f$ directly.

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The latter can be done with tools from the Calderón-Zygmund theory.

Proposition 3

Let A be the operator

$$[Af](x) = \int \frac{\alpha\Big(x, y, |x-y|, \frac{x-y}{|x-y|}\Big)}{|x-y|^{n-1}} f(y)\,\mathrm{d}y$$

with $\alpha(x, y, r, \theta)$ *compactly supported in* x, y. *Then*
(a) If $\alpha \in C^2$, *then* $A : L^2 \to H^1$ *is continuous with a norm not exceeding* $C\|\alpha\|_{C^2}$.
(b) Let $\alpha(x, y, r, \theta) = \alpha'(x, y, r, \theta)\phi(\theta)$. *Then* $\|A\|_{L^2 \to H^1} \le C\|\alpha'\|_{C^2}\|\phi\|_{H^1(S^{n-1})}$.

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This implies the following:

Theorem 4

If w_0 satisfies the ellipticity condition, and X_{w_0} is injective on $L^2(\Omega)$, and therefore,

$$\|f\|_{L^2(\Omega)} \le C\|X_{w_0}^* X_{w_0} f\|_{H^1(\Omega_1)}, \quad \forall f \in L^2(\Omega),$$

then X_w is injective on $L^2(\Omega)$ for $\|w - w_0\|_{C^2} \ll 1$ as well, and the constant in the stability estimate can be chosen locally uniform in w.

This shows that the set of weights satisfying the ellipticity condition for which X_w is injective, is open in C^2. For example, if $w(x) \neq 0$, $\forall x$, the ellipticity condition is automatically satisfied.

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Why is this set dense? This is much harder to prove and requires tools from the Analytic Microlocal Analysis. The bottom line is:

Generic injectivity and stability of X_w

Assume the ellipticity condition (or just assume $w(x) \neq 0, \forall x$). Then X_w is injective and stable on Ω for an open dense set of w in C^2 including the real analytic ones.

Analytic ΨDOs

The theory of analytic ΨDOs is much more delicate than the classical one. It can be used, for example, to prove the elliptic analytic regularity property of (elliptic) differential operators: Pf analytic implies f analytic.

Analytic diff. operators in Ω are the ones with analytic coefficients. Analytic ΨDOs are the ones with (pseudo) analytic amplitudes. Then the symbol estimates look like this:

$$\left|D_\xi^\alpha a(x,y,\xi)\right| \le C^{|\alpha|+1}\alpha!|\xi|^{m-|\alpha|}, \quad |\xi| \ge R_0 \sup(|\alpha|,1)$$

for (x,y) in a complex neighborhood of $\bar\Omega \times \bar\Omega$, $\xi \in \mathbf{R}^n$.

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"Negligible" operators are the ones that send distributions to analytic functions.

An elliptic analytic ΨDO has a parametrix that inverts it up to an analytic regularizing operator. Now, suppose that $P : L^2(\Omega) \to L^2(\Omega_1)$ is an elliptic ΨDO. Then there is a parametrix Q so that $QN = I + K$ when acting of functions of compact support in Ω, and K is analytic regularizing in Ω. If $Pf = 0$, then $f = -Kf$, where, as always, we extend f as zero outside Ω. Therefore, f is real analytic in Ω_1, and vanishes in $\Omega_1 \setminus \Omega$. Therefore, $f = 0$ by analytic continuation. So, N has a trivial kernel.

The same idea is applied to $X_w^* X_w f$, $\operatorname{supp} f \subset \bar{\Omega}$ under the ellipticity condition. Then $X_w^* X_w$ is elliptic near $\bar{\Omega}$, therefore, f is analytic there. But $f = 0$ away from $\bar{\Omega}$, hence $f = 0$. This argument goes back to Boman and Quinto.

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Analytic ΨDOs and the support theorem

Analytic ΨDO calculus gives an alternative proof of the support theorem. Assume that f is compactly supported for simplicity and all integrals over all lines outside $B(0,1)$ vanish. We cannot apply directly the arguments above because the ellipticity condition is not satisfied. On the other hand, we can prove *microlocal analyticity* at the visible singularities. This is related to the notion of the analytic wave front set.

So we are in a situation where, outside $B(0,1)$, we cannot show that f is analytic but we can show that it is microlocally analytic at (co)directions (x,ξ) so that the line through x normal to ξ does not hit $B(0,1)$. We need an analytic continuation theorem based on microlocal (only) analyticity. Such theorem is the Sato-Kawai-Kashiwara theorem. It says that if S is a hypersurface and $f=0$ on one side of it, and f is analytic at (x,ξ) with $x\in S$, $\xi\perp S$, then $f=0$ near x.

By the way, the support theorem is "unstable". We know that because there is an open set of invisible singularities.

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The geodesic X-ray transform

One generalization: The geodesic X-ray transform. Let g be a Riemannian metric on $\bar{\Omega}$. Set

$$Xf(\gamma) = \int f(\gamma(s))\,\mathrm{d}s,$$

where γ is any unit speed maximal geodesic. Can we invert X?

Yes, and this has been done by Mukhometov and other Russian mathematicians in he 80s. They used the energy method which gives un-sharp estimates as well. We are interested in partial data problems, weighted transforms and transforms of tensor fields. For some of them, energy estimates work well but there are results that we can get with microlocal methods only.

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Compute fist the Schwartz kernel of $X'X$. To be a ΨDO, we need all singularities to be on the diagonal. It is fairly easy to show that

$$X'\psi(x) = \int_{\gamma \ni x} \psi(\gamma)$$

w.r.t. a "natural" measure. In the Euclidean case, we used polar coordinates centered at x. Can we do this now?

Only if there are no conjugate points! The essential condition is that the exponential map $\exp_x \xi$ is a diffeomorphism on Ω. This is an analog of the Bolker condition. Then we have an analog of the results above. We can apply the analytic ΨDO calculus, if the metric g and the weight w are analytic.

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The X-ray transform over a general family of curves

Assume that we have a general family of "geodesic-like" curves: for any point x and direction θ, there is a unique curve of that family through x in the direction θ. One way to define it is to define a generator (a vector field) in the phase space. Then we set

$$X_w f(\gamma) = \int w(\gamma, \dot{\gamma}) f(\gamma(s))\, \mathrm{d}s.$$

An essential requirement for the whole machinery to works is lack of conjugate points. This means that "polar coordinates" centered at any point are global coordinates in $\bar{\Omega}$. Then we have similar results.

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The X-ray transform over a general family of curves: results

- The visible singularities are (x, ξ) with the property that $\exists\theta \perp \xi$ so that $w(x,\theta) \neq 0$. Assume next that all singularities are visible (the ellipticity condition).
- The kernel is smooth and finitely dimensional.
- If X_w is injective, there is stability.
- If w and the family of the curves are analytic, X_w is injective, and therefore stable.
- A small perturbation of w and of the family preserves that property.
- X_w is injective, and therefore stable for a dense open set of weights and families of curves.
- If w and the family of the curves are analytic, there are support theorems.

A result of Bela Frigyik, S, and Gunther Uhlmann; also Venky Krishnan.