Microlocal Analysis of Thermoacoustic (or Multiwave) Tomography, I

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Purdue University

The model, the direct problem, Uniqueness

Mini Course, Fields Institute, 2012

References

Those lectures are based on several joint works with Gunther Uhlmann; one of them joint with Qian, Uhlmann, and Zhao. They are available at my web page.

- Thermoacoustic tomography with variable sound speed (with Gunther Uhlmann), *Inverse Prob.* 25(2009), 075011.
- Thermoacoustic tomography arising in brain imaging (with Gunther Uhlmann), *Inverse Prob.* 27(2011), 045004.
- An efficient Neumann-series based algorithm for Thermoacoustic and Photoacoustic Tomography with variable sound speed (with Jianliang Qian, Gunther Uhlmann and Hongkai Zhao), *SIAM J. Imaging Sciences* 4(2011), 850-883. (many pictures)
- Multi-wave methods via ultrasound (with Gunther Uhlmann), to appear in *Inside Out, proceedings of a 2010 MSRI workshop.* (a survey)

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More references

A useful reading is also the following:

- Linearizing non-linear inverse problems and an application to inverse backscattering (with Gunther Uhlmann), *J. Funct. Anal.* 256(9)(2009), 2842–2866.

Thermo- and photo- acoustic Tomography

In thermo/photo-acoustic tomography (PAT/TAT), a short electro-magnetic pulse/laser beam is sent through a patient's body. The tissue reacts and emits an ultrasound wave form any point, that is measured away from the body. Then one tries to reconstruct the internal structure of a patient's body form those measurements.

This imaging method is a “hybrid/multiwave” one. We send one wave (electromagnetic or laser) and measure another one, acoustic. Other types of waves might be used to create the ultrasound response, like elastic one, variable magnetic field. etc.

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Why? We could just use MRI or CT scan, they have wonderful resolution, what else could we want? CT scans have great resolution but cancer cells have low contrast w.r.t. X-rays. They do not look so much different than anything else. Ultrasound waves have very good resolution, too but the same problem with contrast.

PET/TAT sends waves that heat up the cancer cells and they absorb photo/EM waves much more that the rest (big contrast). Then those cells emit ultra-sound waves with good resolution. So we combine the contrast of the waves that we send with the resolution of the waves that we get back.

We are interested now in the recovery of the ultrasound source. The next phase, which we will not discuss is knowing the source, to recover the tissue properties. This is an inverse problem with internal measurements taking into account the interaction between the two waves.

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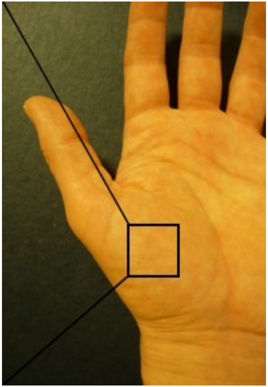
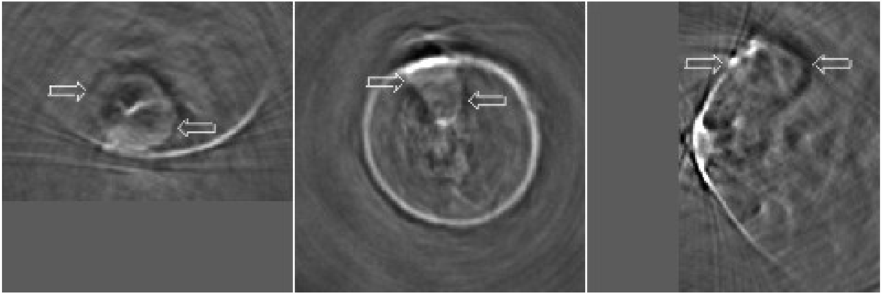
PET: Real life image



Taken from the UCL webpage: Volume rendered in vivo photoacoustic image of the vascular anatomy in the palm of the hand. Image volume: 20mm x 20mm x 6mm.



Wikipedia: First 3D thermoacoustic image of breast cancer. From left to right: images depict axial, coronal and sagittal views of the cancer (arrows).

The Mathematical Model

Let $c(x) > 0$ be the acoustic speed. Let u solve the problem

$$\begin{cases} (\partial_t^2 - c^2\Delta)u = 0 & \text{in } (0,T)\times \mathbf{R}^n, \\ u|_{t=0} = f, \\ \partial_t u|_{t=0} = 0, \end{cases} \tag{1}$$

where $T > 0$ is fixed.
Assume that f is supported in $\bar{\Omega}$, where $\Omega \subset \mathbf{R}^n$ is some smooth bounded domain. The measurements are modeled by the operator

$$\Lambda f := u|_{[0,T]\times\partial\Omega}.$$

The problem is to reconstruct the unknown f.

Note that the wave equation is solved in the whole space, and $\partial\Omega$ is "invisible" to the solution.

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How good is that model?

- Well, it produces nice pictures in real world applications
- But what you see often are only the good pictures . . .
- The variable nature of the speed is a significant enough factor to be included
- Attenuation is neglected
- In brain imaging, the speed is discontinuous, jumps by a factor of two
- We may have measurements of a part of the boundary only (does not change the model, just the data)
- We may not know the speed either
- All the usual practical annoyances: discrete measurements, noise, etc.

What can microlocal analysis do?

- It works equally well for variable coefficients and non-Euclidean geometry.
- It recovers the singularities (called "features" in applications, usually discontinuities, boundary of layers) in a more or less direct way; there is a calculus.
- Recovery of singularities is an important problem by itself. In geophysics, this is all they do.
- Roughly speaking, it deals with the infinite dimensionality of function spaces. Elliptic ΨDOs or FIOs, for examples, are invertible modulo finitely dim. spaces. Ellipticity is much easier to check than injectivity.
- In particular, stability of inverse problems is answered by whether one can recover all singularities.
- Analytic microlocal analysis (for analytic coefficients) can prove injectivity, too.

If $T=\infty$, we can just solve a Cauchy problem backwards with zero initial data.

One of the most common methods when $T<\infty$ is to do the same (time reversal). Solve

$$\begin{cases} (\partial_t^2 - c^2\Delta)v_0 &= 0 \quad \text{in } (0,T)\times\Omega,\\ v_0|_{[0,T]\times\partial\Omega} &= \chi h,\\ v_0|_{t=T} &= 0,\\ \partial_t v_0|_{t=T} &= 0, \end{cases} \tag{2}$$

where h will be taken to be $h=\Lambda f$. Here χ cuts off smoothly near $t=T$ so that the 1st order compatibility condition is satisfied.

Then we define the following

Time Reversal

$$f \approx A_0 h := v_0(0,\cdot) \quad \text{in } \bar\Omega, \text{ where } h=\Lambda f.$$

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Most (but not all) works are in the case of constant coefficients, i.e., when $c = 1$. If n is odd, and $T > \operatorname{diam}(\Omega)$, this is an exact method by the Hyugens' principle.

In that case, this is actually an integral geometry problem because of Kirchoff's formula — recovery of f from integrals over spheres centered at $\partial\Omega$.

When n is even, or when the coefficients are not constant, this is an "approximate solution" only. As $T \to \infty$, the error tends to zero by finite energy decay. When the geometry is non-trapping, the convergence is uniform and exponentially fast for n odd and $O(t^{1-n})$ for n even ([Hristova], based on classical local energy decay estimates).

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Prior results

Kruger; Agranovsky, Ambartsoumian, Finch, Georgieva-Hristova, Jin, Haltmeier, Kuchment, Nguyen, Patch, Quinto, Wang, Xu ...

The time reversal (but not only) is often used for reconstruction. It is exact only when $T=\infty$ but above some critical time T_1, it is a parametrix.

When T is fixed, there is no good control over the error (unless n is odd and $c=\mathrm{const}$). There are other methods, as well, for example a method based on an eigenfunctions expansion; or explicit formulas if $c=\mathrm{const}$ and Ω is a ball (with $T=\infty$ in even dimensions).

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Results for variable coefficients existed but not so many. Finch and Rakesh (2009) proved uniqueness when $T > \operatorname{diam}(\Omega)$, based on Tataru's uniqueness theorem (that we use, too). Reconstructions for finite T have been tried numerically, and they "seem to work" at least for non-trapping geometries.

Another problem of a genuine applied interest is uniqueness and reconstruction with measurements on a part of the boundary. There were no results so far for the variable coefficient case, and there is a uniqueness result in the constant coefficients one by Finch, Patch and Rakesh (2004).

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Ω=ball, constant speed

The simplest case is when $c = 1$ and Ω is the unit ball. Let also $n = 3$. Then there are explicit reconstruction formulas (FINCH, HALTMEIER, KUNYANSKY, NGUEN, PATCH, RAKESH, XU, WANG).

Let $g(x,t) = \Lambda f$, $x \in S^{n-1}$, be the data. Then, in 3D,

$$f(x) = -\frac{1}{8\pi^2}\Delta_x \int_{|y|=1} \frac{g(y,|x-y|)}{|x-y|}\mathrm{d}S_y.$$

Also,

$$f(x) = -\frac{1}{8\pi^2}\int_{|y|=1} \left(\frac{1}{t}\frac{d^2}{dt^2}g(y,t)\right)\Big|_{t=|y-x|} \mathrm{d}S_y.$$

Yet another one, a partial case of an explicit formula in any dimension (KUNYANSKY):

$$f(x) = \frac{1}{8\pi^2}\nabla_x \cdot \int_{|y|=1} \left(\nu(y)\frac{1}{t}\frac{d}{dt}\frac{g(y,t)}{t}\right)\Big|_{t=|y-x|} \mathrm{d}S_y.$$

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When $c =$ const., an n is odd, this is also an integral geometry problem. By the Kirchhoff's formula, up to time derivatives, **in odd dimensions**, what we measure are the spherical means of f centered at point on $\partial\Omega$:

$$\Lambda f \sim \int_{|\omega|=1} f(x + t\omega)\,\mathrm{d}\omega, \quad t \in [0, T],\ x \in \partial\Omega.$$

Now, we have to invert it. This transform can be (and has been) studied with microlocal methods that in particular answer some questions about stability and recovery of singularities, including cases with partial data (but c still constant). One can also use analytic microlocal analysis for uniqueness.

Our initial interest in this problem was motivated by extending this approach to non Euclidean transforms over geodesic spheres.

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Analysis of the acoustic equation

Before we can do an intelligent analysis of the inverse problem, we need a deep understanding of the direct one.

Is the direct problem solvable in the first place? In what spaces, etc.?

Rewrite the acoustic equation as a system:

$$\frac{\partial}{\partial t}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 & I \\ c^2\Delta & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2, \end{pmatrix} \qquad \Longleftrightarrow \qquad \begin{array}{l} u_1 = u, \quad u_2 = u_t \\ (\partial_t^2 - c^2\Delta)u = 0. \end{array}$$

Cauchy data for the acoustic equation become initial data for the system above. Set

$$\mathbf{P} = \begin{pmatrix} 0 & I \\ c^2\Delta & 0 \end{pmatrix}, \qquad \text{then} \qquad \mathbf{u}_t = \mathbf{P}\mathbf{u}.$$

The energy of a solution u in the domain U is given by

$$E_U(t,u) = \int_U \left(|Du|^2 + c^{-2}|u_t|^2\right)\mathrm{d}x.$$

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$$\mathbf{P} = \begin{pmatrix} 0 & I \\ c^2\Delta & 0 \end{pmatrix}, \qquad \text{then} \qquad \mathbf{u}_t = \mathbf{P}\mathbf{u}.$$

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$$E_U(t,u) = \int_U \left(|Du|^2 + c^{-2}|u_t|^2\right) \mathrm{d}x.$$

Analysis of the acoustic equation

Before we can do an intelligent analysis of the inverse problem, we need a deep understanding of the direct one.

Is the direct problem solvable in the first place? In what spaces, etc.?

Rewrite the acoustic equation as a system:

$$\frac{\partial}{\partial t}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 & I \\ c^2\Delta & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2, \end{pmatrix} \qquad \Longleftrightarrow \qquad \begin{array}{l} u_1 = u, \quad u_2 = u_t \\ (\partial_t^2 - c^2\Delta)u = 0. \end{array}$$

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A fundamental property of the energy in the whole space (say, for initial data in $C_0^\infty \times C_0^\infty$) is that it is preserved by the solution (just differentiate w.r.t. t and integrate by parts).

Energy norm for the Cauchy data (f, h), that we denote by $\|\cdot\|_{\mathcal{H}}$:

$$\|(f,h)\|_{\mathcal{H}}^2 = \int_U \left(|Df|^2 + c^{-2}|h|^2\right) \mathrm{d}x,$$

and this defines a Hilbert space $\mathcal{H}(U)$ as the completion of $C_0^\infty(U) \times C_0^\infty(U)$ under that norm.

In particular, when $g = 0$, we get the space $H_D(U)$ with the norm

$$\|f\|_{H_D(U)}^2 = \int_U |Df|^2 \,\mathrm{d}x.$$

Next, $H_D(U) \subset H^1(U)$, if U is bounded with smooth boundary, therefore, $H_D(U)$ is topologically equivalent to $H_0^1(U)$. Note that

$$\|f\|_{H_D(U)}^2 = (-Pf, f)_{H_D(U, c^{-2}\mathrm{d}x)},$$

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So the natural setup is to solve

$$\mathbf{u}_t = \mathbf{P}\mathbf{u}, \quad \mathbf{u}(0) = \mathbf{f} \in \mathcal{H}.$$

$\mathbf{P}$ extends to a skew-selfadjoint operator in $\mathcal{H}$. Existence of a solution

$$\mathbf{u}(t) = e^{t\mathbf{P}}\mathbf{f}$$

follows by Stone's theorem (usual notation $u = e^{\mathrm{i}tA}f$, $A^* = A$). Next, $e^{t\mathbf{P}}$ is a unitary group (energy conservation).

In the TAT problem, the Cauchy data is $(f, 0)$, and the energy norm of that is $\|f\|_{H_D(\Omega)}$.

By a result of Lasiecka, Lions and Triggiani, $\Lambda : H_D(\Omega) \to H^1_{(0)}([0,T] \times \partial\Omega)$ is bounded, where the subscript (0) indicates the subspace of functions vanishing for $t = 0$. So the data Λf is well defined for $f \in H_D(\Omega)$.

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The wave equation has the finite speed of propagation property: "signals" propagate with speed no greater that 1, in the metric $c^{-2}dx^2$ (or with speed c, in the metric $\mathrm{d}x^2$); i.e., if u has Cauchy data (f,h) for $t=0$, then

$$u(t,x)=0 \quad \text{for} \quad t>\text{dist}(x,\mathrm{supp}\,(f,h)), \tag{3}$$

where "dist" is the distance in the metric $c^{-2}\mathrm{d}x^2$. Another way to say this is that any solution at (t_0,x_0) has a domain of dependence given by the characteristic cone (possibly, non-smooth!)

$$\{(t,x);\ \text{dist}(x,x_0)\le|t-t_0|\}\,. \tag{4}$$

The forward part of this cone is given by $t>t_0$, and the backward one by $t<t_0$.

Proof: easy — integration by parts in the characteristic cone (for small time interval first); then the Cauchy inequality (Evans or Taylor books).

One can generalize this to the wave equation $(\partial_t^2 - c^2\Delta_g + \text{l.o.t.})u=0$; then we replace $c^{-2}\mathrm{d}x^2$ by $c^{-2}g$.

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Recall that given two subsets A and B of a metric space, the distance $\mathrm{dist}(A, B)$ is defined by

$$\mathrm{dist}(A, B) = \sup(\mathrm{dist}(a, B);\ a \in A). \tag{5}$$

This function is not symmetric in general, and the Hausdorff distance is defined as

$$\mathrm{dist}_{\mathrm{H}}(A, B) = \max\left(\mathrm{dist}(A, B), \mathrm{dist}(B, A)\right).$$

The finite speed of propagation property can then be formulated in the following form:

Finite speed of propagation

If u has Cauchy data (f, h) at $t = 0$ supported in the set U, then $u(t, x) = 0$ when $\mathrm{dist}(x, U) > |t|$.

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What if c is constant?

When c is constant, and $n \geq 3$ is odd, we have:

Huygens' principle

If u has Cauchy data (f, h) at $t = 0$ supported in the set U, then

$$\operatorname{supp} u(t, \cdot) \subset \{x;\ \exists y \in U, |x - y| = c|t|\}.$$

The physical interpretation is that signals propagate with speed c (vs. less or equal than c).

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Unique continuation

Due mainly to Tataru. Proof: hard. Local version: unique continuation across every surface non-characteristic for $\partial_t^2 - c^2\Delta$; generalizes to $\partial_t^2 - c^2\Delta_g +$ l.o..t.. One of its global versions, presented below, follows from its local version by Holmgren's type of arguments.

Theorem 1

Assume that $u \in H^1_{\mathrm{loc}}$ satisfies

$$(\partial_t^2 - c^2\Delta_g + l.o.t.)u = 0,$$

near the set in (6) and vanishes in a neighborhood of $[-T, T] \times \{x_0\}$, with some $T > 0$, $x_0 \in \mathbf{R}^n$. Then

$$u(t,x) = 0 \quad \textit{for} \quad |t| + \mathrm{dist}_{c^{-2}g}(x_0, x) < T. \tag{6}$$

For the partial data analysis we need a version restricted to a bounded (connected) domain Ω. The inconvenience of the theorem above is that it requires u to solve the wave equation in a cone that may not fit in $\mathbf{R} \times \Omega$. Next theorem shows unique continuation of Cauchy data on $\mathbf{R} \times \partial\Omega$ to their domain of influence.

Proposition 1

$\Omega \subset \mathbf{R}^n$: domain; let $u \in H^1$ solve the homogeneous wave equation in $[-T, T] \times \Omega$. Let u have Cauchy data zero on $[-T, T] \times \Gamma$, where $\Gamma \subset \partial\Omega$ is open. Then $u = 0$ in the domain of influence $\{(t, x) \in [-T, T] \times \Omega;\ \mathrm{dist}(x, \Gamma) < T - |t|\}$.

A possible proof, using unique continuation: extend u as zero in a one sided neighborhood of Γ, in the exterior of Ω (by extending g and c there first), and this extension will still be a solution. Then we apply unique continuation along a curve connecting that exterior neighborhood with an arbitrary point x so that $\mathrm{dist}(x, \Gamma) < T$. To make sure that we always stay in some neighborhood of that curve in the x space, we need to apply the unique continuation Theorem 1 in small increments.

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Back to TAT: Uniqueness

We have all ingredients in place to prove sharp uniqueness results. The underlying metric is $c^{-2}\mathrm{d}x^2$. Set

$$T_0 := \mathrm{dist}(\Omega, \partial\Omega) = \max_{x\in\bar{\Omega}} \mathrm{dist}(x, \partial\Omega).$$

Theorem 2

(i) $T \geq T_0 \implies$ *uniqueness.*
(ii) $T < T_0 \implies$ *no uniqueness. We can recover* $f(x)$ *for* $\mathrm{dist}(x, \partial\Omega) \leq T$ *and nothing else.*

The proof of (i) is based on the unique continuation property. The proof of (ii) (the second statement) is just finite speed of propagation.

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Sketch of the proof

Observation: knowing Λf on $[0,T]\times\partial\Omega$, we know the Dirichlet data only. For unique continuation, we need the Neumann data as well. Can we recover it? Yes! Note that u is not just any solution in Ω — it actually extends to a solution in the whole space! Solve the wave equation outside Ω with Dirichlet data Λf and zero initial data. That will give us u outside Ω. Take the normal derivative — and we get the Neumann data as well.

In other words, solve

$$\begin{cases} (\partial_t^2-\Delta)w &= 0 \quad \text{in } (0,T)\times\mathbf{R}^n\setminus\bar\Omega,\\ w|_{[0,T]\times\partial\Omega} &= h,\\ w|_{t=0} &= 0,\\ \partial_t w|_{t=0} &= 0. \end{cases} \tag{7}$$

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The uniqueness proof is immediate now. The solution u extends in an even way w.r.t. t in $[-T, T]$ as a solution again. On $[-T, T] \times \partial\Omega$, u has zero Cauchy data. By unique continuation, for any point x inside, that can be reached by time T from $\partial\Omega$, we get $u = 0$ near that point. By finite speed of propagation, we cannot say anything about the other points.

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Data on a part of $\partial\Omega$

Let $\Gamma \subset \partial\Omega$ be a relatively open subset of $\partial\Omega$. Measurements on

$$\mathcal{G} := \{(t,x);\ x \in \Gamma,\, 0 < t < s(x)\}\,, \tag{8}$$

where s is a fixed continuous function on Γ. This corresponds to measurements taken at each $x \in \Gamma$ for the time interval $0 < t < s(x)$. The special case studied so far is $s(x) \equiv T$, for some $T > 0$; then $\mathcal{G} = [0, T] \times \Gamma$, and this is where our main interest is.

So the partial data problem is: given

$$\Lambda f|_{\mathcal{G}},$$

recover f.

Data on a part of $\partial\Omega$

Let $\Gamma \subset \partial\Omega$ be a relatively open subset of $\partial\Omega$. Measurements on

$$\mathcal{G} := \{(t,x);\ x \in \Gamma,\, 0 < t < s(x)\}\,, \tag{8}$$

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$$\forall x \in \mathcal{K}, \exists z \in \Gamma \text{ so that } \operatorname{dist}(x, z) < s(z). \tag{9}$$

In Theorem 1 below, we show that this is a necessary condition, up to replacing the $<$ sign by the $\leq$ one, is sufficient, as well.

Another way to formulate this condition is to say that $f = 0$ in the domain of influence

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Sharp uniqueness result, that in particular generalizes the result by Finch, Patch and Rakesh to the variable coefficients case.

Theorem 3

Let $P = -\Delta$ outside Ω, and let $\partial\Omega$ be strictly convex. Then under the assumption (9), if $\Lambda f = 0$ on $\mathcal{G}$ for $f \in H_D(\Omega)$ with $\operatorname{supp} f \subset \mathcal{K}$, then $f = 0$.

As above, we can make this more precise.

Proposition 2

Let $P = -\Delta$ outside Ω, and let $\partial\Omega$ be strictly convex. Assume that $\Lambda f = 0$ on $\mathcal{G}$ for some $f \in H_D(\Omega)$ with $\operatorname{supp} f \subset \Omega$ that may not satisfy (9). Then $f = 0$ in $\Omega_{\mathcal{G}}$. Moreover, no information about f in $\Omega \setminus \bar{\Omega}_{\mathcal{G}}$ is contained in $\Lambda f|_{\mathcal{G}}$.

Proof: delicate! We can only recover the Neumann derivative on $\mathcal{G}$ for small t. Then we show that $f = 0$ near Γ only. That however gives us more data, we repeat the argument, etc.

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