Microlocal Analysis of Thermoacoustic (or Multiwave) Tomography, III

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TAT of brain imaging (discontinuous wave speed)

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Discontinuous speeds, modeling Brain Imaging

The following model appears in brain imaging and was proposed by Lihong Wang in May 2010, during a Banff meeting. Let c be piecewise smooth with a non-zero jump across a smooth closed surface Γ. How much of all that is preserved? The direct problem is a transmission problem

$$\begin{cases} (\partial_t^2 - c^2\Delta)u &= 0 \qquad \text{in } (0,T)\times \mathbf{R}^n, \\ u\big|_{\Gamma_{\text{int}}} &= u\big|_{\Gamma_{\text{ext}}}, \\ \frac{\partial u}{\partial \nu}\Big|_{\Gamma_{\text{int}}} &= \frac{\partial u}{\partial \nu}\Big|_{\Gamma_{\text{ext}}}, \\ u\big|_{t=0} &= f, \\ \partial_t u\big|_{t=0} &= 0, \end{cases} \tag{1}$$

where $T>0$ is fixed, $u|_{\Gamma_{\text{int,ext}}}$ is the limit value (the trace) of u on Γ when taking the limit from the "exterior" and from the "interior" of Γ, respectively, and f is the source that we want to recover. We similarly define the interior/exterior normal derivatives, and ν is the exterior unit (in the Euclidean metric) normal to Γ.

Assume that f is supported in $\bar{\Omega}$, where $\Omega \subset \mathbf{R}^n$ is some smooth bounded domain. The measurements are modeled by the operator Λf. The problem is to reconstruct the unknown f.

In brain imaging, the interface is the skull. The sound speed jumps by about a factor of 2 there. Experiments show that the ray that arrives first carries about 20% of the energy.

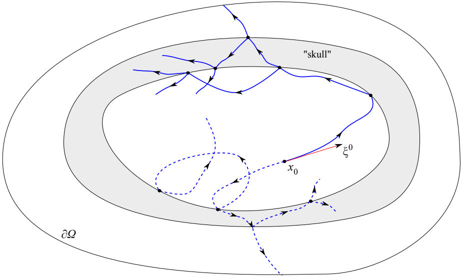

Figure: Propagation of singularities in the "skull" geometry

Propagation of singularities (an example is shown on the previous slide) is well understood away from tangent rays. When a ray approaches Γ from the side with the higher speed, there are always a reflected and a refracted rays. When the ray is coming from a slower to a faster region, we may or may not have a refracted one, but we always have a reflected one. If there is only a reflected one, this is known as full internal reflection. The energy (at high frequencies) naturally splits into fractions of the total one. So a single singularity may exit at several different places with different amplitudes.

There might be trapping singularities, as well, that remain invisible. But even the visible ones, are visible with a fraction of their amplitudes only! In a way, all singularities inside Γ are partly invisible, some — totally invisible.

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Transmission and reflection

Let us see how we can construct a parametrix of the solution with boundary data on $\mathbf{R}\times\Gamma$.
The microlocal regions of $T^*(\mathbf{R}\times\Gamma)\ni(t,x,\tau,\xi')$ (here, ξ' stands for a projection) w.r.t. the sound speed c_{int}, i.e., in $\bar\Omega_{\text{int}}$, are defined as follows:

hyperbolic region: $c_{\text{int}}(x)|\xi'|<\tau$,

glancing manifold: $c_{\text{int}}(x)|\xi'|=\tau$,

elliptic region: $c_{\text{int}}(x)|\xi'|>\tau$.

One has a similar classification of $T^*\Gamma$ with respect to the sound speed c_{ext}. A ray that hits Γ transversely, coming from Ω_{int}, has a tangential projection on $T^*(\mathbf{R}\times\Gamma)$ in the hyperbolic region relative to c_{int}. If $c_{\text{int}}<c_{\text{ext}}$, that projection may belong to any of the three microlocal regions w.r.t. the speed c_{int}. If $c_{\text{int}}>c_{\text{ext}}$, then that projection is always in the hyperbolic region for c_{ext}. When we have a ray that hits Γ from Ω_{ext}, then those two cases are reversed.

Hyperbolic region on both sides

Let's stay near a point belonging to the hyperbolic region w.r.t. both c_{int} and c_{ext}, i.e., we have a ray hitting from the "interior" and

$$c_{\text{int}}^{-2}\tau^2 - |\xi'|^2 > 0, \quad c_{\text{ext}}^{-2}\tau^2 - |\xi'|^2 > 0. \tag{2}$$

This is always the case with a ray coming from a fast and going to a slow region.

The analysis also applies to the opposite case, under the assumption above. We will confirm below in this setting the well known fact that under that condition, such a ray splits into a reflected ray with the same tangential component of the velocity that returns to the interior Ω_{int}, and a transmitted one, again with the same tangential component of the velocity, that propagates in Ω_{ext}. We will also compute the amplitudes and the energy at high frequencies of the corresponding asymptotic solutions.

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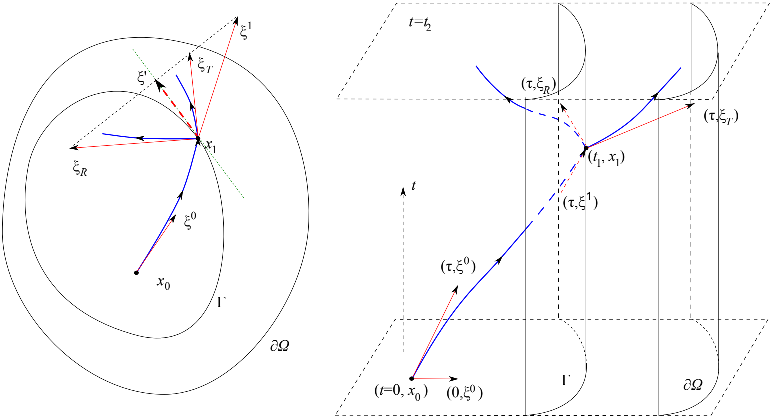

Figure: The reflected and the transmitted rays. Left: in the x space. Right: in the (t, x) space.

Choose boundary normal coordinates (x', x^n). We will express the incoming solution u_+ in $\mathbf{R} \times \bar{\Omega}_{\text{int}}$ the reflected one u_R in the same set; and the transmitted one u_T in $\mathbf{R} \times \bar{\Omega}_{\text{ext}}$, up to smoothing terms in the form

$$u_\sigma = (2\pi)^{-n} \int e^{\mathrm{i}\varphi_\sigma(t,x,\tau,\xi')} b_\sigma(t,x,\tau,\xi') \hat{h}(\tau,\xi') \,\mathrm{d}\tau \,\mathrm{d}\xi', \quad \sigma = +, R, T, \tag{3}$$

where $\hat{h} := \int_{\mathbf{R}\times\mathbf{R}^{n-1}} e^{-\mathrm{i}(-t\tau + x'\cdot\xi')} h(t,x')\mathrm{d}t\,\mathrm{d}x'$. We chose to alter the sign of τ so that if $c = 1$, then the phase function in (3) would equal φ_+, i.e., then $\varphi_+ = -t\tau + x\cdot\xi$. Next, φ_+, φ_R, φ_T solve the eikonal equation

$$\partial_t \varphi_\sigma + c(x)|\nabla_x \varphi_\sigma| = 0, \quad \varphi_\sigma|_{x^n=0} = -t\tau + x'\cdot\xi'. \tag{4}$$

The boundary condition comes from the first transmission condition. The right choice of the sign in front of $\partial_t\varphi_+$ is the positive one because $\partial_t\varphi_+ = -\tau < 0$ for $x^n = 0$, and that derivative must remain negative near the boundary as well. We see below that $\varphi_{R,T}$ have the same boundary values on $x^n = 0$, therefore they satisfy the same eikonal equation, with the same choice of the sign.

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The first transmission condition implies

$$b_T^{(0)} - b_R^{(0)} = 1 \quad \text{for } x^n = 0.$$

The second transmission condition implies

$$\frac{\partial \varphi_T}{\partial x^n} b_T^{(0)} - \frac{\partial \varphi_R}{\partial x^n} b_R^{(0)} = \frac{\partial \varphi_+}{\partial x^n} \quad \text{for } x^n = 0.$$

This is a linear system for $b_R^{(0)}|_{x^n=0}$, $b_T^{(0)}|_{x^n=0}$ with determinant

$$-\left(\frac{\partial \varphi_T}{\partial x^n} - \frac{\partial \varphi_R}{\partial x^n} \right) \Big|_{x^n=0}.$$

We can compute the normal derivatives of the phase functions for $x^n = 0$:

$$\frac{\partial \varphi_R}{\partial x^n} = -\frac{\partial \varphi_+}{\partial x^n} = -\sqrt{c_{\text{int}}^{-2}\tau^2 - |\xi'|^2}, \quad \frac{\partial \varphi_T}{\partial x^n} = \quad \sqrt{c_{\text{ext}}^{-2}\tau^2 - |\xi'|^2}.$$

In particular, the determinant above is positive, so there is a solution.

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One can compute the amplitudes of each waves, and the high-frequency energy that they carry. They turn out to be all positive. When the transmitted ray tends to a tangential one, the energy tends to 0.

Snell's Law. Let α, respectively, β, be the angle that $\xi^1 = \partial\varphi_+/\partial x^n$, respectively, $\xi_T := \partial\varphi_T/\partial x^n$, makes with the (co)-normal. Then

$$|\xi'| = |\xi^1|\sin\alpha = c_{\text{int}}^{-1}\tau\sin\alpha, \quad |\xi'| = |\xi_T|\sin\beta = c_{\text{ext}}^{-1}\tau\sin\beta \tag{5}$$

By (5), we recover Snell's law

$$\frac{\sin\alpha}{\sin\beta} = \frac{c_{\text{int}}}{c_{\text{ext}}}, \tag{6}$$

Assume now that $c_{\text{int}} < c_{\text{ext}}$, which is the case where there might be no transmitted ray. Denote by

$$\alpha_0(x) = \arcsin(c_{\text{int}}(x)/c_{\text{ext}}(x)) \tag{7}$$

the critical angle at any $x \in \Gamma$ that places $(\xi^1)'$ in the glancing manifold w.r.t. c_{ext}. No transmitted wave when $\alpha > \alpha_0$; more precisely, no real phase function φ_T in that case. It exists, when $\alpha < \alpha_0$. In the critical case $\alpha = \alpha_0$: an outgoing ray tangent to Γ that we are not going to analyze.

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Full internal reflection case.

Assume now that $(\xi^1)'$ is in the elliptic region w.r.t. c_{ext}, then there is no transmitted singularity, but one can still construct a parametrix for the "evanescent" wave in Ω_{ext}; and there is a reflected ray. This is known as a full internal reflection.

We proceed as above with one essential difference. There is no real valued solution φ_T to the eikonal equation in the exterior. Formally,

$$\frac{\partial \varphi_T}{\partial \nu} = \quad \mathrm{i}\sqrt{|\xi'|^2 - c_{\text{ext}}^{-2}\tau^2} \quad \text{for } x^n = 0. \tag{8}$$

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In general, the eikonal equation may not be solvable but one can still construct solutions modulo $O((x^n)^\infty)$. The same applies to the transport equations. One can show that the $O((x^n)^\infty)$ error does not change the properties of u_T to be a parametrix. In particular, for the transmitted energy, on gets

$$E_{\Omega_{\text{ext}}}(\mathbf{u}_T(t_2)) \cong 0. \tag{9}$$

Moreover, u_T is smooth in $\bar{\Omega}_{\text{ext}}$. Therefore, no energy, as far as the principal part only is concerned, is transmitted to Ω_{ext}. That does not mean that the solution vanishes there, of course.

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Glancing, gliding rays and other cases

We do not analyze the cases where $(\xi^1)'$ is in the glancing manifold w.r.t. to one of the speeds. We can do that because the analysis of those cases is not needed because of our assumptions guaranteeing no tangent rays. The analysis there is more delicate. We do not analyze either the case where $(\xi^1)'$ is in the elliptic region with respect to either speed.

Finally, we need to justify the parametrix. We satisfied the equation, including the transmission conditions up to smooth errors. By a non-trivial result of Mark Williams, this introduces a smooth error.

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Back to the inverse problem

(Completely) trapped singularities are a problem, as before. Let $\mathcal{K} \subset \Omega$ be a compact set such that all rays originating from it are never tangent to Γ and not completely trapped up to time T. For f satisfying

$$\operatorname{supp} f \subset \mathcal{K}$$

the Neumann series above still converges (uniformly to f).

We need a small modification to keep the support in $\mathcal{K}$ all the time. We use the projection $\Pi_{\mathcal{K}} : H_D(\Omega) \to H_D(\mathcal{K})$ for that purpose.

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Reconstruction

Theorem 1

Let any ray from $\mathcal{K}$ have a path never tangent to Γ that reaches $\partial\Omega$ at time $|t| < T$. Then

$$\Pi_{\mathcal{K}} A \Lambda = I - K \text{ in } H_D(\mathcal{K}), \text{ with } \|K\|_{H_D(\mathcal{K})} < 1.$$

In particular, $I - K$ is invertible on $H_D(\mathcal{K})$, and Λ restricted to $H_D(\mathcal{K})$ has an explicit left inverse of the form

$$f = \sum_{m=0}^{\infty} K^m \Pi_{\mathcal{K}} A h, \quad h = \Lambda f. \tag{10}$$

The assumption $\operatorname{supp} f \subset \mathcal{K}$ means that we need to know f outside $\mathcal{K}$; then we can subtract the known part.

In the numerical experiments below, we do not restrict the support of f, and still get good reconstruction images but the invisible singularities remain invisible.

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Brain imaging of square headed people

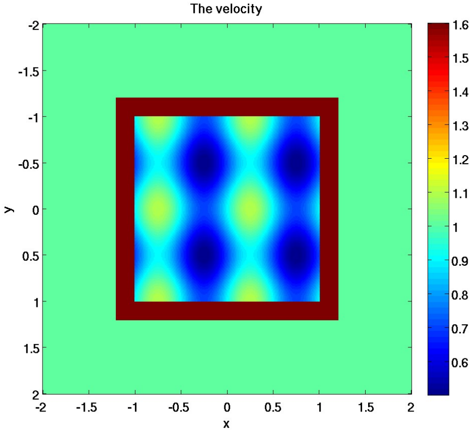
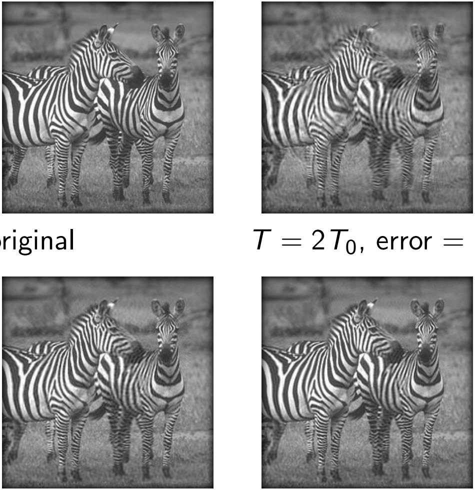


Figure: The speed jumps by a factor of 2 in average at the interior boundary of the "skull".

A “skull” speed, Neumann series



original

$T = 2T_0$, error $= 15\%$

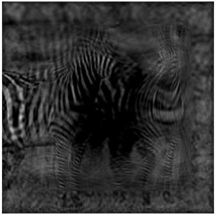
$T = 4T_0$, error $= 9.75\%$

$T = 8T_0$, error $= 7.55\%$

Figure: Neumann Series, 15 steps

A "skull" speed, Time Reversal

original



$T = 2T_0$, error $= 68\%$

$T = 4T_0$, error $= 23.7\%$

$T = 8T_0$, error $= 78.5\%$

Figure: Time Reversal. "White clipping" in the last image, many values in $[1, 1.6]$

A “skull” speed, Time Reversal

original

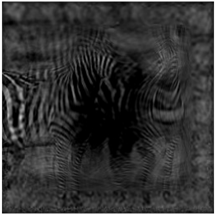


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Figure: Time Reversal. Values in last image compressed from $[0, 1]$ to $[-0.05, 1.6]$

Original vs. Neumann Series vs. Time Reversal

original | NS, error $= 7.55\%$ | TR, error $= 78.5\%$

Figure: $T = 8T_0$. Original vs. Neumann Series vs. Time Reversal (the latter compressed from $[0, 1]$ to $[-0.05, 1.6]$)

Note that the TR image will always have fake singularities (artifacts), but they become weaker and weaker as $T \to \infty$.

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