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Sharp upper bounds on the number of resonances near the real axis for trapping systems

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Abstract

We study resonances near the real axis ($|\mathrm{Im}\, z| = O(h^N)$, $N \gg 1$) and the corresponding resonant states for semiclassical long range operators $P(h)$. Without a priori assumptions on the distribution or on the multiplicities of the resonances, we show that the truncated resonant states form a family of quasimode states for $P(h)$, stable under small perturbations. As a consequence, they form also a family of quasimode states for any suitably defined (self-adjoint) reference operator $P^\#(h)$, therefore, those resonances are perturbed eigenvalues of $P^\#(h)$. Next we show that the semiclassical wave front set of the resonant states is contained in the set of trapped directions $\mathcal{T}$. We construct a suitable reference operator from $P(h)$ by imposing a microlocal barrier outside $\mathcal{T}$ to show that the counting function for those resonances admits an upper bound of Weyl's type connected with the measure of $\mathcal{T}$. We give an example of system for which this bound is optimal and also prove similar bound in case of classical scattering by obstacle.

1 Introduction

This paper is devoted to a detailed study of the behavior of the resonances and resonant states near the real axis. We work mainly in the semi-classical setting but most results can be easily translated into the classical one. By resonances near the real axis we mean resonances in a "box" $\Omega(h) = [a_0, b_0] + i[-S(h), 0]$, where $0 < S(h) = O(h^K)$, $K \gg 1$. Such resonances may exist only for trapping geometries. We accept the convention here that resonances lie in the lower half-plane.

For simplicity of the exposition, we consider compact perturbation $P(h)$ of the long range Schrödinger operator $-h^2\Delta + V(x)$. Our results however hold for general long range perturbations of the Laplacian (see section 8), i.e., when the long range perturbation is in the second and first order part as well. The basic properties are established in the abstract "black box scattering" setting introduced by Sjöstrand and Zworski [SjZ] (see next section). It is well known that if $z(h)$ is a resonance, then there exists a $z(h)$-outgoing resonant state $u(h)$ satisfying the equation $(P(h) - z(h))u(h) = 0$. By [B1], [St3], if $-\mathrm{Im}\, z = O(h^\infty)$, and if $P(h) = -h^2\Delta$ for large x, then

$$u(h) = O(h^\infty) \quad \text{for } R_1 \le |x| \le R_2, \tag{1.1}$$

where $R_0 < R_1 < R_2$ are such that the scatterer is included in the ball $B(0, R_0)$. For simplicity, in this introduction we will assume $S(h) = O(h^\infty)$. This does not immediately imply that the same is true for the generalized "eigenfunctions", i.e., for the solutions of $(P(h) - z(h))^k u(h) = 0$ with some $k > 1$. We call those generalized "eigenfunctions" (with infinite energy) resonant states as well. Since $-\mathrm{Im}\, z = O(h^\infty)$, one can expect that

$$(P(h) - z(h))u(h) = O(h^\infty) \quad \text{for any resonant state.} \tag{1.2}$$

The estimate above has to be considered in the following sense — u is normalized in $B(0, R)$, $R \gg 1$ and then the r.h.s. has to be $O(h^\infty)$ in the same ball. If one tries to carry out some recursive procedure for proving those two

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estimates, it is quite likely to get exponentially big (with respect to $1/h$) terms , because for the number of steps k we have $k = O(h^{-n^\#})$, $n^\# \geq n$, where $n \geq 2$ is the dimension. However obtaining such estimates even under the assumption that $m(h)$ is uniformly bounded does not follow easily from known results. The third question we are trying to answer is about the degree of linear independence of resonant states corresponding to resonances too close to each other, for example two resonances $z_1(h)$ and $z_2(h)$ in $\Omega(h)$ with $z_z(h) - z_2(h) = O(h^\infty)$. In other words, can we control how small is the angle between two (or more) resonant states $u_1(h)$, $u_2(h)$ corresponding to such resonances? If we could, then we would know that a linear combination of $u_1(h)$, $u_2(h)$ satisfying (1.1) would also satisfy (1.1) for any $z(h)$ that is $O(h^\infty)$ close to $z_1(h)$ and $z_2(h)$. This is also related to the stability of the property of linear independence of such resonant states and is crucial for providing link between resonances and eigenvalues of a reference operator below.

Instead of working with single resonances, we work with clusters of resonances. We exploit the following argument: since the number of resonances in $\Omega(h)$ is $O(h^{-n^\#})$, we can always group the resonances in clusters of diameter $d(h) = O(h^\infty)$ with distance between two clusters at least $ch^{n^\#}d(h)$ with some $c > 0$. Then we prove (see Proposition 3.3 and the remark at the end of section 4) estimates (1.1) and (1.2) for any linear combination of resonant states corresponding to resonances in such cluster. For technical reasons we work with the complex scaled operator $P_\theta(h)$. In Proposition 3.4 we give an affirmative answer to the third question above for resonant states associated with different clusters. Within a single cluster, we still cannot control the angle between resonant states corresponding to two different resonances in this cluster, but we can simply choose an orthonormal basis and our results show that we still get states satisfying (1.1) and (1.2). Note that in particular, even though we work with clusters, we get those estimates for a resonant state corresponding to any single resonance $z(h)$.

In Theorem 3.1 and the proof of Theorem 3.2 we show that existence of resonances in $\Omega(h)$ near the real axis implies existence of at least the same number of real quasimodes on an interval slightly wider than the projection of $\Omega(h)$ on the real axis. The corresponding quasimode states are linearly independent in a stable way under small perturbation and this allows us to prove in Theorem 3.2 that they generate at least the same number of eigenvalues of a suitably chosen reference operator $P^\#(h)$. This generalizes the result in [St4], where the number of quasimodes is not estimated. In some sense, this is a result converse to that in [St1], that says that existence of asymptotically orthogonal real quasimodes implies existence of at least the same number of resonances nearby. It is implicit in [St1] that the asymptotic orthogonality can be replaced by the condition of linear independence stable under certain small perturbations. By quasimodes states we mean approximate solutions with error $O(h^N)$, $N \gg 1$ or $N = \infty$ supported in a fixed compact.

In sections 4–6 we study a differential elliptic second order operator $P(h)$ that is a compact perturbation of the long range Scrödinger operator $-h^2\Delta + V(x)$ as before and satisfies the black box assumptions. Using (1.1) and propagation of singularities arguments, we show that the wave front set of any linear combination of resonant states as above is contained in the set $\mathcal{T}$ of trapped bicharacteristics of $P(h)$. In particular, this is true for any resonant state related to a single resonance $z(h)$, as $h \to 0$. The upper bound for $N(\Omega(h))$ established in Theorem 3.2 in terms of upper spectral bound for a self-adjoint reference operator $P^\#(h)$ says that the number of eigenvalues of $P^\#(h)$ in a small neighborhood of $[a_0, b_0]$ is at least the same as the number of resonances $N(\Omega(h))$ in $\Omega(h)$. This provides us with effective ways to get sharp upper bounds of the counting function $N(\Omega(h))$ by using suitably chosen reference operators. One obvious choice for $P^\#(h)$ is the Dirichlet realization of $P(h)$ in the ball $B(0, R)$, $R > R_0$ but it does not provide sharp bounds. We choose $P^\#(h)$ to be $P(h)$ plus additional h-ΨDO with principal symbol that vanishes near the trapped rays and increases quickly outside some small neighborhood of them. This allows us to use known spectral asymptotics for $P^\#(h)$ to get a Weyl type upper bound

$$N(\Omega(h)) \leq (2\pi h)^{-n} \left(\text{meas}\left(\mathcal{T} \cap p_0^{-1}[a_0, b_0]\right) + o(1)\right), \tag{1.3}$$

where p_0 is the principal symbol of $P(h)$, and $S(h) = h^K$, $K \gg 1$ (see Theorem 5.1). In section 6 we give an example of a semi-classical system that admits a *lower bound* of the type (1.3). In one special case we can actually prove an asymptotic formula for $N(\Omega(h))$. Let $P(h)$ be a second order elliptic semi-classical differential operator with principal symbol p_0 and assume that for some energy $b_0 > 0$, the set $\{p_0 \leq b_0\} \subset T^*\mathbf{R}^n$ has a bounded component. Then this component consists of trapped points only. Let a_0, b_0 be non-critical values of p_0. The we have a lower bound of $N(\Omega(h))$ in terms of the volume of this component restricted to the energy levels above a_0.

If we assume also that the unbounded one is non-trapping, then the inequality above turns into asymptotics, in this case actually, $\mathcal{T} \cap p_0^{-1}[a_0, b_0]$ is the volume of the union of the compact components of $p_0^{-1}[a_0, b_0]$. To prove the asymptotic formula, we use the bound (1.3) and the fact that the eigenfunctions of a suitable reference operator serve as quasimodes for $P(h)$, therefore we get a sharp lower bound for $N(\Omega(h))$ as well using the results in [St1]. We also prove that under the assumption that the unbounded component is non-trapping, there is a resonance free strip $O(h^\infty) = S(h) \leq -\operatorname{Im} z \leq Mh$, $\forall M > 0$, $0 < h \leq h(M)$, $a_0 \leq \operatorname{Re} z \leq b_0$. This is done in Theorem 6.1.

In section 7, we prove an upper bound on the number of resonances in classical scattering in a neighborhood of the real line of the kind $0 < -\operatorname{Im}\lambda \leq S(\lambda) = O(\lambda^{-\infty})$, $\operatorname{Re}\lambda \gg 1$ in terms of the measure of the trapped set similar to Theorem 5.1. We consider a compactly supported metric perturbation of the Laplacian in the exterior of a bounded obstacle with Dirichlet boundary conditions.

We notice that the idea that the counting function of the resonances (not only near the real line) is essentially bounded by the spectral counting function of certain reference operator, which is $P(h)$, modified for $|x| > R_0$, has been used implicitly or explicitly [Z1], [SjZ], [V], [Sj2] in the proof of the polynomial bound of this function (see (2.2), (2.3) in next section). Sjöstrand [Sj1], under certain assumptions involving analyticity and hyperbolicity of the bicharacteristic flow, showed that the resonances in a box of height δ, $C_0 h \leq \delta \leq 1/C_0$ is $O(\delta^{d-\varepsilon} h^{-n})$, $\varepsilon > 0$, where d is the Minkowski codimension of the set of the trapped rays. Numerical study of this and other phenomena can be found in [Li], [LiZ]. M. Zerzeri [Ze] obtained in the classical case an upper bound in a sector in $\mathbf{C}$ related to the measure of the trapped rays but the notion of trapped rays that he uses is weaker than the common one, in particular, $-\Delta + V(x)$ with $0 \neq V \in C_0^\infty$ can be trapping, also a non-trapping kidney-shaped domain is trapping according to that definition. Lower bounds of the type ch^{-n} near the real axis can be obtained any time we have asymptotically orthogonal quasimodes with the same density [St1]. Such bound in terms of the measure of the periodic trajectories is proven in [PeZ] with different methods. In obstacle classical scattering under the assumption of existence of elliptic degenerate periodic ray, one has lower bound cr^n with c equal to the measure of invariant tori up to a constant factor. One can interpret this measure as the measure of a subset of the trapped rays near the elliptic one and this well corresponds to the classical version of (1.3).

Resonances connected to potential well for the Schrödinger operator $-h^2\Delta + V(x)$, which is included in the situation considered in section 6 have been studied extensively, see e.g. [HSj]. There is a full asymptotic expansion of the resonances near the energy level equal to a non-degenerate local minimum of $V(x)$. If the Hamiltonian is real analytic, quasimodes with exponentially small error have been constructed in [Po2], which makes possible to get expansions with exponentially small error. In section 6 we consider more general second order semiclassical differential operators and energy levels not necessarily close to the bottom of the well.

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2 Assumptions, Black-Box Scattering and preliminaries

We work in the general framework of *black-box scattering* proposed by Sjöstrand and Zworski [SjZ] (see also [Sj2], [TZ1]). For simplicity of the exposition, we consider only compactly supported perturbations of the long range Schrödinger operator $-h^2\Delta + V(x)$, i.e., only the zero order term is allowed to be long range. The general long range case is discussed in section 8. Let $\mathcal{H}$ be a complex Hilbert space of the form

$$\mathcal{H} = \mathcal{H}_{R_0} \oplus L^2(\mathbf{R}^n \setminus B(0, R_0)),$$

where $R_0 > 0$ is fixed and $B(0, R_0)$ is the ball centered at the origin with radius R_0. We consider a family of self-adjoint unbounded operators $P(h)$ in $\mathcal{H}$ with common domain $\mathcal{D}$, whose projection onto $L^2(\mathbf{R}^n \setminus B(0, R_0))$ is $H^2(\mathbf{R}^n \setminus B(0, R_0))$. Denote by $\mathbf{1}_{B(0,R_0)}$ the orthogonal projector onto $\mathcal{H}_{R_0}$ and similarly, we define $\mathbf{1}_{\mathbf{R}^n \setminus B(0,R_0)}$. Then we assume that

$$\mathbf{1}_{B(0,R_0)}\,(P(h) + i)^{-1} : \mathcal{H} \to \mathcal{H}$$

is compact. Outside $\mathcal{H}_{R_0}$, $P(h)$ is assumed to coincide with the semiclassical Schrödinger operator, i.e.,

$$\mathbf{1}_{\mathbf{R}^n\setminus B(0,R_0)}P(h)u = (-h^2\Delta + V(x,h))\left(u|_{\mathbf{R}^n\setminus B(0,R_0)}\right),$$

where (for $|x| > R_0$),

$$|V(x,h)| \le C|x|^{-\beta}, \quad \beta > 0, \tag{2.1}$$

uniformly in $h \in (0, h_0]$, $h_0 > 0$, and $V(x)$ extends analytically in x in the domain $\{r\omega \in \mathbf{C}\times\mathbf{C}^n,\ |r| > R_0,\ \mathrm{dist}(\omega, S^{n-1}) < d_0,\ r \in \mathbf{C},\ \arg(r) \in (-\tilde\theta_0, \tilde\theta_0)\}$ with some $\tilde\theta_0 > 0$, $d_0 > 0$. Finally, we assume that $P(h) > -C_0$, $C_0 > 0$. Under those assumptions, one can define (the semi-classical) resonances $\mathrm{Res}\,P(h)$ of $P(h)$ in a conic neighborhood of the real axis by the method of complex scaling (see [SjZ], [Sj2]). An outline of the complex scaling technique is given below. Resonances are also poles of the meromorphic continuation of the resolvent $(P(h)-z)^{-1} : \mathcal{H}_{\mathrm{comp}} \to \mathcal{H}_{\mathrm{loc}}$ from $\mathrm{Im}\, z > 0$ into a conic neighborhood of the real line. We will denote the so continued resolvent by $R(z,h)$. In this paper we adopt the convention that resonances lie in the lower half-plane $\mathrm{Im}\, z < 0$. In classical scattering, we consider P as above independent of h by formally assuming that $h = 1$. Then P has classical resonances $\mathrm{Res}\,P$ defined as the poles of the meromorphic continuation of the resolvent $(P-\lambda^2)^{-1} : \mathcal{H}_{\mathrm{comp}} \to \mathcal{H}_{\mathrm{loc}}$ from $\mathrm{Im}\,\lambda > 0$ to a neighborhood of the real line. For such P we then set $P(h) = h^2P$ and define resonances $z(h)$ as above. Then the semi-classical resonances and the classical ones are related by $\lambda^2 = h^{-2}z$.

As in [SjZ], [Sj2], we construct a reference selfadjoint operator $P^\#(h)$ from $P(h)$ on $\mathcal{H}^\# = \mathcal{H}_{R_0} \oplus L^2(M \setminus B(0,R_0))$, where $M = (\mathbf{R}/R\mathbf{Z})^n$ for some $R \gg R_0$. Then for the number of eigenvalues of $P^\#$ in a given interval $[-\lambda, \lambda]$, we assume

$$\#\{z \in \mathrm{Spec}\,P^\#(h);\ -\lambda \le z \le \lambda;\ \} \le C(\lambda/h^2)^{n^\#/2}, \quad \lambda \ge 1,$$

with some $n^\# \ge n$. This implies (see [SjZ] and [Sj2]) that

$$\#\{z \in \mathrm{Res}\,P(h);\ 0 < a_0 \le \mathrm{Re}\, z \le b_0;\ 0 \le -\mathrm{Im}\, z \le c_0\} \le C(a_0,b_0,c_0)h^{-n^\#}, \tag{2.2}$$

$$\#\{\lambda \in \mathrm{Res}\,P;\ 1 \le |\lambda| \le r;\ 0 \le -\mathrm{Im}\,\lambda \le 1\} \le Cr^{n^\#}, \quad r > 1. \tag{2.3}$$

Polynomial estimates of this type have been proved also in [Me], [Z1], [SjZ], [V], [Sj2].

In this paper we will often omit the dependence on h, i.e., we will write P instead of $P(h)$, z_0 instead of $z_0(h)$ where it is clear from the context that we work with h-dependent objects. We denote by C various positive constants, that may change from line to line.

For any resonance $z(h)$ there is an outgoing solution $u(h)$ to $(P(h)-z(h))u(h) = 0$ and possibly "generalized eigenvectors" $v(h)$ satisfying $(P(h)-z(h))^{k(h)}v(h) = 0$. We will call u and v *resonant states*. Given $\Omega(h) \subset \mathbf{C}$, $N(\Omega(h))$ will denote the number of resonances in $\Omega(h)$ counted with their multiplicities defined as the rank of the residue of the cut-off resolvent at any resonance. Given a self-adjoint *reference operator* $P^\#(h)$ (this notion is defined later) with point spectrum, $N^\#([a,b])$ denotes the number of eigenvalues, counting multiplicities, of $P^\#(h)$ in the interval $[a,b]$. With some abuse of notation and terminology, given $\chi \in C_0^\infty(\mathbf{R}^n)$, we will denote the operator $\mathbf{1}_{B(0,R_0)} \oplus \chi|_{\mathbf{R}^n\setminus B(0,R)}$ by χ. If $\mathrm{supp}\left(\mathbf{1}_{\mathbf{R}^n\setminus B(0,R_0)}f\right) \subset K$, where $K \supset B(0,R_0)$, we will say that $\mathrm{supp}\, f \subset K$. By $H^s(\mathbf{R}^n)$, $s = 0,1,\dots$, we denote the Sobolev space with semiclassical norm $\|f\|^2_{H^s} = \sum_{|\alpha|\le s}\|(hD)^\alpha f\|^2$

We will work with pseudodifferential operators (h-ΨDOs) with small parameter h. The class that we use is equivalent to the ΨDOs with large parameter λ (see e.g. [G]) by setting $h = 1/\lambda$. Given two open sets X, Y in $\mathbf{R}^n$, for $m,k \in \mathbf{R}$, we consider the class $S^{m,k}(X\times Y)$ to be the set of all $a(x,y,\xi,h) \in C^\infty(X\times Y\times\mathbf{R}^n)$, such that for any compact $K \subset\subset X\times Y$, all $\alpha,\beta,\gamma \in \mathbf{Z}^n$, $h \in (0,h_0]$, $h_0 > 0$ fixed, we have

$$|\partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma a| \le C_{\alpha,\beta,\gamma,K} h^{-k}(1+|\xi|)^{m-|\gamma|}. \tag{2.4}$$

If $X = Y$, we set $S^{m,k}(X) = S^{m,k}(X\times X)$. Given $a \in S^{m,k}(X\times Y)$, denote by $\mathrm{Op}(a)$ the operator

$$(\mathrm{Op}(a)u)(x,h) = \frac{1}{(2\pi h)^n}\iint e^{i(x-y)\cdot\xi/h} a(x,y,\xi,h)u(y,h)\,dy\,d\xi. \tag{2.5}$$

The class of operators corresponding to $S^{m,k}$ will be denoted by $L^{m,k}$. The "negligible operators" are those in $L^{-\infty,-\infty}$. We will work mostly with operators with symbols supported in a compact in $T^*\mathbf{R}^n$. In this case, the class considered here coincides with $S^k(m)$ (see e.g. [DSj]), where the "order function" m can be chosen to be $m = 1$. In this case also the order m with respect to ξ does not matter and we will denote the corresponding classes by S^k and L^k, respectively, and we will call k order of the corresponding class.

We refer to [G], [SjV], [DSj], [I] for more details. Below we would only like to recall the formula for the symbol of the composition of two h-ΨDOs. Given $a(x, y, \xi, h) \in S^{m,k}(X)$, one can find a symbol $\sigma(A)(x, \xi, h) \in S^{m,k}(X)$, where $A = \text{Op}(a)$, depending only on x, ξ, h such that A and $\text{Op}(\sigma(A))$ differ by a negligible operator. In this case we write $A = \sigma(A)(x, hD, h)$. If $A_j \in L^{m_j,k_j}(X)$, $j = 1, 2$, then $A_1A_2 \in L^{m_1+m_2,k_1+k_2}(X)$ and

$$\sigma(A_1A_2) \sim \sigma(A_1) \circ \sigma(A_2) := \sum_{|\alpha|\geq 0} \frac{1}{\alpha!} h^{|\alpha|} \partial_\xi^\alpha \sigma(A_1) D_x^\alpha \sigma(A_2).$$

3 Basic estimates

In this section we obtain estimates on the resonant states in the semiclassical case corresponding to resonances $z(h)$ in a box $\Omega(h) = [a(h), b(h)] + i[-c(h), 0]$, where $0 < b(h) - a(h) = O(h^N)$, $0 < c(h) = O(h^N)$, $N \gg 1$. Our goal is to prove that the resonant states are essentially supported near the scatterer and solve $(P - z_0)f = O(h^{N_1})$ with $z_0 \in [a(h), b(h)]$, where $N_1 \leq N$ depends on N. As explained in the Introduction, instead of studying single resonances, we study clusters of them close to each other and the resonances in domains of the type $\Omega(h)$ will be considered later as such clusters. We prove those estimates for each linear combination of resonant states corresponding to resonances in $\Omega(h)$. In particular, this allows us to choose orthogonal system of such linear combinations that form quasimode states for $P(h)$ with quasimode $z_0(h)$ of multiplicity equal to the total multiplicity of $\text{Res}\, P(h) \cap \Omega(h)$. Next, we study wider domains where $a(h) = a_0$, $b(h) = b_0$ are independent of h by grouping them in clusters contained in "small" domains $\Omega_k(h)$ as above. We show that resonant states, cut-off for large x, are still asymptotic solutions to the equation $(P - z_0)u = 0$. We also prove an estimate in Proposition 3.4 that allows us to control the angle between two resonant states corresponding to different clusters. This allows us in Theorem 3.2 to estimate the number $N(\Omega(h))$ of resonances in $\Omega(h)$ by the number of eigenvalues of some reference operators in an interval a bit larger that $[a(h), b(h)]$.

3.1 Brief review of Complex Scaling

We follow here [SjZ] and [Sj2]. Fix $A < B$, such that

$$R_0 + 1 < A < B - 1.$$

Note that in Proposition 3.1 we impose the condition that $B \geq B_0$ for some B_0 depending on V and throughout this paper we assume that this is fulfilled. We will perform complex scaling for $r := |x| > B$. Choose a real-valued increasing C^∞- function $\kappa(r)$, $r \geq 0$, with the properties:

(i) $\kappa(r) = 0$ for $0 \leq r \leq B$,
(ii) $\kappa(r) = 1$ for $r \geq B + 1/2$,
(iii) $0 \leq \kappa(r) \leq 1$,
(iv) $\kappa(r) = e^{-1/(r-B)^2}$ for $B \leq r \leq B + \epsilon_0$ with some $\epsilon_0 \ll 1$.

We would like to note that another choice of $\kappa(r)$ in (iv) near $r = B$ might influence the exponential term in Proposition 3.1 below. Set $\theta = \theta(r) := \theta_0\kappa(r)$, where $0 < \theta_0 < \tilde{\theta}_0$ will be chosen later. Define $f_\theta(r) := re^{i\theta(r)}$. We chose $\theta_0 \ll 1$ so that $df_\theta/dr \neq 0$. As in [SjZ], [Sj2], we perform the analytic dilation by considering the map

$$\mathbf{R}^n \ni x = r\omega \longmapsto f_\theta(r)\omega \in \mathbf{C}^n, \quad \omega \in S^{n-1}. \tag{3.1}$$

Under the action of the map (3.1), the operator P is transformed into an operator P_θ on $\mathcal{H}_{R_0} \oplus L^2(\Gamma_\theta \setminus B(0, R_0))$, where Γ_θ is the image of (3.1). We refer to the above mentioned papers for details. We will identify Γ_θ with $\mathbf{R}^n$ (in other words, we parameterize Γ_θ by r and ω). Then inside the ball $B(0, B) := \{x;\ |x| < B\}$ the operator P_θ coincides with P, while outside that ball in polar coordinates we have

$$P_\theta = P(f_\theta(r), (f'_\theta(r))^{-1}D_r, D_\omega), \tag{3.2}$$

where $P(r, D_r, D_\omega)$ is the semiclassical symbol of P in polar coordinates. Since $P = -h^2\Delta + V(x)$ for $|x| > R_0$, equation (3.2) implies that

$$P_\theta|_{r>R_0} = \left(\frac{1}{f_\theta'} hD_r\right)^2 - h\frac{n-1}{f_\theta f_\theta'} ihD_r + \frac{1}{f_\theta^2}(hD_\omega)^2 + V(re^{i\theta}). \tag{3.3}$$

The operator P_θ is elliptic, closed in $\mathcal{H}_{R_0} \oplus L^2(\Gamma_\theta \setminus B(0, R_0))$ which we identify with $\mathcal{H}$ with domain $\mathcal{D}_\theta$ that is actually the same as the domain of P after that identification. It is known that for a fixed $h > 0$ and $z \neq 0$ with $\arg(z) \neq -2\theta_0$, the operator $P_\theta - z$ is Fredholm with index 0. Moreover, for $z \neq 0$ with $-\arg(z) < 2\theta_0$, we have that z is a resonance of P if and only if z is an eigenvalue of P_θ and the multiplicities coincide. Since we are going to work with resonances with $\operatorname{Im} z = o(1)$, for h small enough those resonances will be always eigenvalues of P_θ.

3.2 An absorption estimate, after N. Burq [B2]

Next proposition is a refinement of [B2, Prop. 6.1]. Let $\rho(r)$ be a smooth function equal to 1 for $r < A - 1/2$ and equal to $r^{(n-1)/2}$ for $r > A$. Set $\tilde P := \rho P \rho^{-1}$. Then $\tilde P$ is self-adjoint for the measure $d\mu := \rho^{-2} r^{n-1} dr d\omega$ and we denote by $\tilde{\mathcal{H}} = \rho\mathcal{H}$ the corresponding Hilbert space. Furthermore,

$$\tilde P|_{r>A} = h^2\biggl(-\partial_r^2 - \frac{\Delta_\omega}{r^2} + \frac{(n-1)(n-3)}{4r^2} + V(re^{i\theta}\omega, h)\biggr), \quad d\mu|_{r>A} = dr d\omega.$$

Let $\tilde P_\theta$ be the operator obtained from $\tilde P$ by analytic dilation for $r \geq B$ and denote by $\tilde{\mathcal{D}}$ its domain. In fact, $\tilde P_\theta = \rho(f_\theta) P_\theta \rho^{-1}(f_\theta)$. Here, for n even, the branch of $f_\theta^{1/2}$ is chosen in an obvious way. Fix $a_0 > 0$. Note that in the proposition above we require that $B \geq B_0$, where B_0 has to be large enough. It is not difficult to see that the B_0 that we choose guarantees that the Hamiltonian $\xi^2 + V(x)$ is non-trapping for $|x| > B_0$ for energy levels above a_0.

Proposition 3.1 *There exists $B_0 > 0$, such that if $B \geq B_0$, for $h > 0$ and $\theta_0 > 0$ small enough,* $\operatorname{Re} z \geq a_0$, $\operatorname{Im} z \leq 0$, *and for any $u \in \tilde D_\theta$ we have*

$$\begin{aligned} C\int \bigl((\theta + r\theta')|h\partial_r u|^2 + \theta(|hr^{-1}\nabla_\omega u|^2 + |u|^2)\bigr) dr d\omega \\ \leq \quad -\operatorname{Im}\left(e^{i\theta}(\tilde P_\theta - z)u, u\right)_{\tilde{\mathcal{H}}} + \left(-\operatorname{Im} z + e^{-h^{-1/3}}\right)\|u\|^2_{\tilde{\mathcal{H}}}, \end{aligned} \tag{3.4}$$

where $C = \min(a_0, 1)/2$ and the inner product and the norm are taken in $\tilde{\mathcal{H}}$.

Remark. It follows from the proof that $e^{-h^{-1/3}}\|u\|^2_{\tilde{\mathcal{H}}}$ can be replaced by $e^{-h^{-1/3}}\int_{B\leq r\leq B+ch^{1/6}} |u|^2 dr d\omega$ and that $e^{-h^{-1/3}}$ can be replaced by $e^{-h^{-2/3+\epsilon}}$, $\epsilon > 0$, by choosing $\kappa(r)$ near $r = B$ in a different way.

Proof. As mentioned earlier, we follow the proof of [B2, Prop. 6.1]. Write

$$\begin{aligned} -\operatorname{Im}\left(e^{i\theta}(\tilde P_\theta - z)u, u\right)_{\tilde{\mathcal{H}}} &= -\operatorname{Im}\left(e^{i\theta}(\tilde P_\theta - \operatorname{Re} z)u, u\right)_{\tilde{\mathcal{H}}} - (-\operatorname{Im} z)(\cos\theta u, u)_{\tilde{\mathcal{H}}} \\ &\geq -\operatorname{Im}\left(e^{i\theta}(\tilde P_\theta - \operatorname{Re} z)u, u\right)_{\tilde{\mathcal{H}}} - (-\operatorname{Im} z)\|u\|^2_{\tilde{\mathcal{H}}}. \end{aligned} \tag{3.5}$$

Therefore, it is enough to prove the proposition for z real. So, let $z \geq a_0$. Note next that one can assume that u is supported in $r > A$, where in particular, $d\mu = dr d\omega$. Indeed, choose a smooth cut-off function $0 \leq \chi \leq 1$ such that $\chi(x) = 0$ for $|x| < A$ and $\chi(x) = 1$ for $|x| > A + 1/2$. Since $e^{i\theta}(\tilde P_\theta - z)$ is symmetric on $\operatorname{supp}(1-\chi)$, writing $u = \chi u + (1-\chi)u$ we can see as in [B2] that $(1-\chi)u$ contributes nothing to $\operatorname{Im}(e^{i\theta}(\tilde P_\theta - z)u, u)_{\tilde{\mathcal{H}}}$, so we may replace u by χu there.

For $e^{i\theta}\tilde P_\theta$ we have

$$e^{i\theta}\tilde P_\theta = -\frac{1}{1+ir\theta'}h\partial_r\frac{e^{-i\theta}}{1+ir\theta'}h\partial_r - e^{-i\theta}\frac{h^2\Delta_\omega}{r^2} + e^{i\theta}h^2\frac{(n-1)(n-3)}{4r^2} + e^{i\theta}V(r^{i\theta}r\omega,h). \tag{3.6}$$

Integrating by parts we get

$$\begin{aligned} -\mathrm{Im}\left(e^{i\theta}(\tilde P_\theta - z)u,u\right)_{\tilde{\mathcal H}} &= \int\left(\mathrm{Im}\left(-\frac{e^{-i\theta}}{(1+ir\theta')^2}\right)|h\partial_r u|^2 + \sin\theta|hr^{-1}\nabla_\omega u|^2\right)drd\omega \qquad (3.7)\\ &\quad + \int \mathrm{Im}\left(e^{i\theta}z - e^{-i\theta}h^2\frac{(n-1)(n-3)}{4r^2} - e^{i\theta}V(r^{i\theta}r\omega,h)\right)|u|^2drd\omega\\ &\quad - h\mathrm{Im}\,(g(r)h\partial_r u,u) \qquad (3.8)\\ &= I_1+I_2+I_3\end{aligned}$$

with

$$g(r) = \frac{d}{dr}\left(\frac{1}{1+ir\theta'}\right)\frac{e^{-i\theta}}{1+ir\theta'} = \frac{-i(r\theta''+\theta')e^{-i\theta}}{(1+ir\theta')^3}. \tag{3.9}$$

It is easy to see that if $\theta_0>0$ is small enough we have

$$I_1 \ge \frac34\int\left((\theta+2r\theta')|h\partial_r u|^2 + \theta|hr^{-1}\nabla_\omega u|^2\right)drd\omega. \tag{3.10}$$

To estimate I_2, we use the fact that (see [B2])

$$V(re^{i\theta}) = V(r) + r\left(e^{i\theta}-1\right)\partial_r V(z), \quad z\in(r,re^{i\theta}), \tag{3.11}$$

as a consequence of the fact that the derivatives of V admit a symbol-like estimates because of the analyticity assumption, and that $|r\partial_r V(z)| = O(r^{-\beta})$, as $r\to\infty$. Therefore, for $r\ge B$, $B\gg 1$,

$$|\mathrm{Im}\left(e^{i\theta}V(r^{i\theta}r\omega,h)\right)| \le \theta|V(r)| + C\theta r^{-\beta} \le C\theta r^{-\beta} \le \frac{a_0}{8}\theta(r).$$

Therefore, for I_2 we get for h small enough,

$$I_2 \ge \int\left(z\sin\theta - \frac{a_0}{8}\theta\right)|u|^2drd\omega - Ch^2\int\theta|u|^2drd\omega \ge \frac34 a_0\int\theta|u|^2drd\omega, \tag{3.12}$$

provided that $\theta_0\ll 1$. For I_3 we obtain

$$I_3 = -\frac{h}{i}((\mathrm{Re}\,g)h\partial_r u,u)_{\tilde{\mathcal H}} - \frac{h^2}{2i}(\mathrm{Re}\,g'u,u)_{\tilde{\mathcal H}} + \frac{h^2}{2}(\mathrm{Im}\,g'u,u)_{\tilde{\mathcal H}}. \tag{3.13}$$

Since I_3 is real, we have

$$I_3 = -\mathrm{Im}\,h((\mathrm{Re}\,g)h\partial_r u,u)_{\tilde{\mathcal H}} + \frac{h^2}{2}(\mathrm{Im}\,g'u,u)_{\tilde{\mathcal H}} = I_3^{(1)} + I_3^{(2)}. \tag{3.14}$$

The function g admits the following estimates

$$\begin{aligned} |\mathrm{Re}\,g| &\le C(|\theta'|+|\theta''|)(|\theta|+|\theta'|) \le C|\theta|, \qquad (3.15)\\ |g'| &\le C(|\theta'|+|\theta''|+|\theta'''|), \qquad (3.16)\end{aligned}$$

The second estimate (3.16) and the first part of (3.15) follow directly from (3.9). The second part of (3.15) holds trivially for $A<r<B$, where $\theta=0$ and for $B+\epsilon_0<r$, where $|\kappa|>1/C$, and is also true for $B<r<B+\varepsilon_0$, where $\theta=\theta_0e^{-1/(r-B)}$. Now (3.15) implies that $I_3^{(1)}$ in (3.13) can be estimated by

$$|I_3^{(1)}| \le Ch\int\theta(|h\partial_r u|^2 + |u|^2)drd\omega \tag{3.17}$$

and for $h \ll 1$ this can be absorbed by the r.h.s. of (3.10) and (3.12). Next, to estimate $I_3^{(2)}$, we will show that for $h \ll 1$

$$|g'| \le e^{-h^{-1/3}} + h^{-3/2}\theta, \quad \text{for } B \le r. \tag{3.18}$$

Set $t = r - B \ge 0$. Then, since $\theta = \theta_0 e^{-1/t^2}$ for $0 \le t \ll 1$, (3.16) implies that for $0 \le t^2 \le h^{1/3}/2$ we have $|g'| \le Ct^{-9}e^{-1/t^2} \le e^{-1/(2t^2)} \le e^{-h^{-1/3}}$ for $h \ll 1$. On the other hand, for $h^{1/3}/2 \le t^2 \le \epsilon_0^2$ we have $|g'| \le Ct^{-9}e^{-1/t^2} \le Ch^{-3/2}\theta$. Since (3.18) is trivially true for $r > B + \varepsilon_0$, this proves it for all $r \ge B$. Therefore,

$$|I_3^{(2)}| \le e^{-h^{-1/3}} \|u\|^2_{\tilde{\mathcal{H}}} + h^{1/2} \int \theta |u|^2 dr d\omega \tag{3.19}$$

and the integral above can be absorbed by the r.h.s. of (3.12). Combining (3.7) with (3.10), (3.12), (3.13), (3.17) and (3.19), we complete the proof. □

Remark. Estimate (3.4) remains true for $\operatorname{Im} z > 0$ if we replace $-\operatorname{Im} z$ there by $-\cos\theta_0 \operatorname{Im} z$ (see (3.5)). In particular, this implies that for $\theta_0 \ll 1$ and any $u \in \tilde{\mathcal{D}}$

$$\frac{1}{2}\left(\operatorname{Im} z - e^{-h^{-1/3}}\right)\|u\|^2_{\tilde{\mathcal{H}}} \le -\operatorname{Im}\left(e^{i\theta}(\tilde{P}_\theta - z)u, u\right)_{\tilde{\mathcal{H}}} \le \|(\tilde{P}_\theta - z)u\|_{\tilde{\mathcal{H}}}\|u\|_{\tilde{\mathcal{H}}}$$

and after replacing u by ρu, we get

$$\|(P_\theta - z)^{-1}\| \le \frac{2}{\operatorname{Im} z - e^{-h^{-1/3}}}, \quad \operatorname{Im} z > e^{-h^{-1/3}}. \tag{3.20}$$

3.3 Estimates on the resonant states

Fix $0 < a_0 < b_0$. Choose some $a(h)$, $b(h)$ and $c(h)$ such that

$$0 < a_0 \le a(h) \le b(h) \le b_0, \quad b(h) - a(h) = o(1), \quad 2e^{-h^{-1/3}} \le c(h) \le o(1)h^{(5n^\#+1)/2}. \tag{3.21}$$

and let

$$\Omega(h) := [a(h), b(h)] + i[-c(h), 0]. \tag{3.22}$$

Let $z_1(h), \dots, z_p(h)$ be all distinct resonances in $\Omega(h)$ with multiplicities $m_1(h), \dots, m_p(h)$. Set $m(h) := m_1 + \ldots + m_p = N(\Omega(h))$. Assume that there are no resonances on $\partial\Omega$. Consider the spectral projector associated with the eigenvalues of P_θ in Ω

$$\Pi_\Omega := \frac{1}{2\pi i}\oint_{\partial\Omega}(z - P_\theta)^{-1} dz,$$

where $\partial\Omega$ is assumed to be positively oriented. Denote $\mathcal{H}_\Omega := \operatorname{Ran}\Pi_\Omega$. Then it is well known (see e.g. [K]) that P_θ acts invariantly on $\mathcal{H}_\Omega$ that is the span of all eigenvectors and generalized eigenvectors of P_θ with eigenvalues in Ω. Generalized eigenvectors corresponding to distinct eigenvalues $z_1, z_2, \dots z_p$ in Ω are linearly independent because for the corresponding spectral projectors we have $\Pi_{z_i}\Pi_{z_j} = \delta_{ij}\Pi_{z_i}$. The dimension $m(h)$ of $\mathcal{H}_\Omega$ is finite, bounded by $Ch^{-n^\#}$, and equal to the sum of the multiplicities of $z_j \in \Omega$. Set $P_\Omega := P_\theta|_{\mathcal{H}_\Omega}$. Then P_Ω is a finite rank operator (matrix) and we denote by $\|\cdot\|_{\mathcal{H}_\Omega}$ the operator norm in $\mathcal{H}_\Omega$. The spectrum of P_Ω consists of $\{z_1, \dots, z_p\}$ with the same multiplicities. The following estimate is due to Zworski [Z2] (in this generality, see the proof of Lemma 1 in [TZ1])

$$\|(z - P_\theta)^{-1}\| \le Ce^{Ch^{-n^\#}\log(1/g)} \quad \text{for } z \in \Omega_0,\ \operatorname{dist}(z, \operatorname{Res} P(h)) \ge g(h),\ g(h) \ll 1,$$

where Ω_0 is any simply connected precompact subset of $-\pi < -\arg z < 2\theta_0$ (independent of h). In our analysis we always work in domains included in the box $\Omega_0 := [a_0/2, 2b_0] + i[-c_0, c_0]$ with fixed $0 < a_0 < b_0$ and fixed $0 < c_0 \ll 1$, therefore the constant C above will be uniform. As a consequence, the resolvent of P_Ω satisfies the following estimate

$$\|(z - P_\Omega)^{-1}\|_{\mathcal{H}_\Omega} \le Ce^{Ch^{-n^\#}\log(1/g)} \quad \text{for } z \in \Omega_0,\ \operatorname{dist}(z, \operatorname{Res} P(h)) \ge g(h),\ g(h) \ll 1. \tag{3.23}$$

This allows us to apply the "semiclassical maximum principle" ([TZ1], [TZ2]) as in [St3, Lemma 2] to get the following.

Proposition 3.2 *Assume that $c(h) \le S(h) \le h^{(5n^\#+1)/2}w(h)$, $w(h) = o(1)$, as $h \to 0$, where $a(h)$, $b(h)$ and $c(h)$ are as in (3.21). Then*

$$\|(z-P_\Omega)^{-1}\|_{\mathcal{H}_\Omega} \le \frac{C}{S(h)} \quad \text{on } \partial\tilde\Omega,$$

where $\tilde\Omega := [a(h)-w(h), b(h)+w(h)] + [-h^{-n^\#}S(h), S(h)]$

Proof: We follow closely the proof of [St3, Lemma 2]. Let

$$\tilde z_j(h) := \bar z_j(h) + 2iS(h), \quad j = 1\dots p,$$

where the bar denotes complex conjugate. Then z_j and $\tilde z_j$ are symmetric about the line $\operatorname{Im} z = S(h)$ and on that line we have $\|(z-P_\Omega)^{-1}\|_{\mathcal{H}_\Omega} \le 4/S(h)$ by (3.21) and (3.20) . Set

$$G(z,h) := \frac{(z-z_1)^{m_1}\dots(z-z_p)^{m_p}}{(z-\tilde z_1)^{m_1}\dots(z-\tilde z_p)^{m_p}}.$$

We observe first that

$$|G(z,h)| \le 1 \quad \text{for } \operatorname{Im} z \le S(h). \tag{3.24}$$

The function $F := G(z-P_\Omega)^{-1}$ is holomorphic below the line $\operatorname{Im} z = S(h)$, and in particular in $\Omega(h)$. Our goal is to apply the "semiclassical maximum principle" [TZ1] in the form presented in [St3, Lemma 1] to the function F in the domain $\Omega_1 := [a(h)-5w(h), b(h)+5w(h)]+i[-S(h)h^{-2n^\#-1}, S(h)]$. To this end, we need to modify $w(h)$ and $S(h)$ to be sure that F satisfies an exponential estimate in this region with a constant independent of the region. The only obstacle to that would be existence of resonances too close to the boundary. To this end we extend Ω_1 by shifting the sides, staying in the fixed h-independent neighborhood Ω_0 of $\Omega(h)$, such that the closest resonance stays at distance at least $g(h) = h^{n^\#+1}$. This is possible in view of (2.2). Then we apply (3.23) with $\log(1/g) = (n^\#+1)\log(1/h)$ and using (3.24), we see that $\|F\| = O(\exp(Ch^{-n}\log h^{-1}))$ on the boundary of the extended domain. By the maximum principle, this is true inside it, and in particular in Ω_1. Now we are in position to apply [St3, Lemma 1]. Since by (3.20), $\|F\| \le 4/S(h)$ on the upper part of Ω_1, we deduce that for h small enough

$$\|G(z)(z-P_\Omega)^{-1}\|_{\mathcal{H}_\Omega} \le 2e^3/S(h), \quad \forall z \in \tilde\Omega(h). \tag{3.25}$$

Next step is to show that

$$1/C \le |G(z,h)| \quad \text{on } \partial\tilde\Omega(h). \tag{3.26}$$

It is enough to estimate $(z-\tilde z_j)/(z-z_j)$ on $\partial\tilde\Omega(h)$. Observe first that $|z_j - \tilde z_j| \le 4S(h), \forall j$. Next, the distance from each z_j from the three sides $\operatorname{Im} z = -S(h)h^{-n^\#}$, $\operatorname{Re} z = a-w$, $\operatorname{Re} z = b+w$ of $\tilde\Omega$ is bounded below by $S(h)h^{-n^\#}/2$ for $h \ll 1$. Therefore,

$$\left|\frac{z-\tilde z_j}{z-z_j} - 1\right| = \left|\frac{z_j-\tilde z_j}{z-z_j}\right| \le \frac{4S(h)}{S(h)h^{-n^\#}/2} = 8h^{n^\#}, \quad \forall z \in \partial\tilde\Omega(h)\setminus\{\operatorname{Im} z = S(h)\}.$$

This yields

$$\left|\frac{z-\tilde z_j}{z-z_j}\right|^{m_j} \le (1+8h^{n^\#})^{m_j}, \quad \forall z \in \partial\tilde\Omega(h)\setminus\{\operatorname{Im} z = S(h)\}. \tag{3.27}$$

On the fourth side $\operatorname{Im} z = S(h)$ of $\partial\tilde\Omega$ we have $|(z-\tilde z_j)/(z-z_j)| = 1$, thus (3.27) is trivially true there. Since $(1+x)^{1/x} < e$, $0 < x < \infty$, we get

$$|1/G(z,h)| \le (1+8h^{n^\#})^{m_1+\dots+m_p} = (1+8h^{n^\#})^m \le (1+8h^{n^\#})^{Ch^{-n^\#}} \le e^{8C}.$$

This proves (3.26). Estimates (3.25) and (3.26) together imply the proof of the proposition. □

This proposition allows us to estimate $\|P_\Omega - z_0\|_{\mathcal{H}_\Omega}$ for $z_0 \in [a(h), b(h)]$. We have

$$z_0 - P_\Omega = \frac{1}{2\pi i}\oint_{\partial\tilde{\Omega}}(z_0 - P_\Omega)(z - P_\Omega)^{-1}dz = \frac{1}{2\pi i}\oint_{\partial\tilde{\Omega}}(z_0 - z)(z - P_\Omega)^{-1}dz,$$

therefore,

$$\|z_0 - P_\Omega\|_{\mathcal{H}_\Omega} \le \frac{|\partial\tilde{\Omega}|}{2\pi}|z_0 - z|\|(z - P_\Omega)^{-1}\|_{\mathcal{H}_\Omega} \le C\frac{b - a + w}{2\pi}(b - a + w)\frac{1}{S} = C\frac{(b - a + w)^2}{S}. \tag{3.28}$$

Choosing $w(h) = h^{-(5n^\# + 1)/2}S(h)$, estimate (3.28) implies the following

$$\|(P_\theta - z_0)f\| \le C\left(\frac{(b(h) - a(h))^2}{S(h)} + h^{-5n^\# - 1}S(h)\right)\|f\|, \quad \forall f \in \mathrm{Ran}\Pi_\Omega. \tag{3.29}$$

If $c(h)h^{-(5n^\#+1)/2} \le b(h) - a(h) = o(1)$, then we choose $S(h) = h^{(5n^\#+1)/2}(b(h) - a(h))$ (then $S(h) \ge c(h)$ as required). If $b(h) - a(h) \le c(h)h^{-(5n^\#+1)/2}$, then we set $S(h) = c(h)$. This choice of $S(h)$ implies the following.

Proposition 3.3 *Let Ω and Π_Ω be as above. Then for $z_0 \in [a(h), b(h)]$ we have*

$$\|(P_\theta - z_0)f\| \le Ch^{-(5n^\#+1)/2}\max\left\{b(h) - a(h), h^{-(5n^\#+1)/2}c(h)\right\}\|f\|, \quad \forall f \in \mathrm{Ran}\Pi_\Omega. \tag{3.30}$$

In particular, we get the following.

Corollary 3.1 *Let $z_0(h)$ be a resonance with $0 < a_0 \le \mathrm{Re}\, z_0(h) \le b_0 < \infty$, $-\mathrm{Im}\, z_0(h) = o(1)h^{(5n^\#+1)/2}$, and let $f = f(h)$ be any generalized eigenfunction of $P_\theta(h)$ corresponding to the eigenvalue $z_0(h)$ (a function such that $(P_\theta - z_0)^k f = 0$ for some $k \ge 0$). Then*

$$\|(P_\theta(h) - z_0(h))f\| \le Ch^{-5n^\# - 1}\max\{-\mathrm{Im}\, z_0(h), e^{-h^{-1/3}}\}\|f\|.$$

Note that the r.h.s. in (3.30), measuring the "error", is "small" only if the width of $\Omega(h)$ does not exceed h^N, $N \gg (5n^\# + 1)/2$. This does not allow us to control the linear independence of the generalized eigenfunctions under small perturbation by integration by parts as in Proposition 3.5 below in wider domains, for example, if $a(h)$ and $b(h)$ are independent of h. Next proposition plays a crucial role in proving that resonances in "wide" boxes generate at least as many eigenvalues of the reference operator nearby. It states that the spectral projectors Π_Ω related to suitably chosen clusters of resonances contained in those boxes are polynomially bounded when restricted to the generalized eigenfunctions corresponding to eigenvalues in the "wide" box, see (3.34) below. Under the additional assumption of "well separated" resonances [TZ2] or, more generally, if we assume the existence of a resonance-free strip $c(h) \le -\mathrm{Im}\, z \le h^{-2n^\#-1}c(h)$ below $\Omega(h)$ [St3], then the spectral projectors Π_Ω are polynomially bounded on the whole space $\mathcal{H}$. In general, however, we do not know whether this is true, but fortunately, the proposition below is all we need for our purposes later.

3.4 Decomposition into clusters

Let

$$a_0 \le a(h) < b(h) \le b_0, \quad 2e^{-h^{-1/3}} \le c(h) \le o(1)h^{(7n^\#+1)/2} \tag{3.31}$$

(without the requirement that $b(h) - a(h) = O(h^N)$, $N \gg 1$). Let $\Omega(h)$ be as in (3.22). Assume that there are no resonances on $\partial\Omega(h)$. A direct consequence of (2.2) is that one can group the resonances in $\Omega(h)$ into clusters contained in the interiors of the boxes

$$\Omega_k(h) = [a_k(h), b_k(h)] + i[-c(h), 0], \quad k = 1, \dots, K(h), \ K = O(h^{-n^\#}), \tag{3.32}$$

where $\Omega_k(h)$ do not intersect, moreover, for $k \neq m$,

$$\mathrm{dist}\{\Omega_k, \Omega_m\} \geq 4w(h), \quad \mathrm{width}(\Omega_k) = b_k - a_k \leq Ch^{-n^\#} w(h), \tag{3.33}$$

where $0 < w(h) = o(1)h^{n^\#}$ is fixed in advance. There are no resonances in Ω outside Ω_k's. Denote as before by Π_{Ω_k} the spectral projectors related to the eigenvalues of P_θ in Ω_k and let P_Ω, $\mathcal{H}_\Omega$ be as before.

We know that the subspaces $\mathrm{Ran}\Omega_k(h)$ are linearly independent. The following proposition basically gives us control over the lower bound of the angles between them.

Proposition 3.4 *Under the assumptions above, if $w(h) = h^{-(5n^\#+1)/2}c(h)$, then there exists a constant $A = A(a_0, b_0)$, such that*

$$\|\Pi_{\Omega_k}\|_{\mathcal{H}_\Omega} \leq Ah^{-(7n^\#+1)/2}, \quad k = 1, \dots, K. \tag{3.34}$$

For any $f_k \in \mathrm{Ran}\Omega_k(h)$, $k = 1, \dots, K$, and for any k_0 we have

$$\|f_{k_0}\| \leq Ah^{-(7n^\#+1)/2}\|f_1 + \dots + f_K\|,$$

Proof. Following the proof of Proposition 3.2, we get that

$$\|(z - P_\Omega)^{-1}\|_{\mathcal{H}_\Omega} \leq \frac{C}{c(h)} \quad \text{on } \partial\tilde{\Omega}_k(h), \forall k, \tag{3.35}$$

where $\tilde{\Omega}_k(h) := [a_k(h) - w(h), b_k(h) + w(h)] + i[-h^{-n^\#}c(h), c(h)]$. Note that $\tilde{\Omega}_k(h)$ have the same properties (3.33) as $\Omega_k(h)$ concerning the distance between two such domains and their widths, with $w(h)$ replaced by $w(h)/2$. To justify (3.35), it is enough to note that in the proof of Proposition 3.2 we used the fact that there are no poles of $(z - P_\Omega)^{-1}$ below $-\mathrm{Im}\, z = c(h)$ only, and the fact that there might be poles to the left or right of Ω_k does not play any role as far as those poles are separated by distance $Cw(h)$ (see also [St3]). Notice also that the constant C in (3.35) is independent of k. Since there are no eigenvalues of P_Ω in $\tilde{\Omega}_k \setminus \Omega_k$, one can define $\Pi_{\Omega_k}|_{\mathcal{H}_\Omega}$ as integrals of $(z - P_\Omega)^{-1}$ over $\partial\tilde{\Omega}_k$. A direct estimation of that integral, using (3.35), yields the proof of the first part.

To prove the second part, write

$$f_{k_0} = \Pi_{\Omega_{k_0}}(f_1 + \dots + f_K)$$

and use the estimate on $\Pi_{\Omega_{k_0}}$. □

3.5 From resonances to quasimodes

We are ready now to formulate and prove some consequences of the estimates proven so far. The first one, roughly speaking, says that if P has m resonances in Ω, then P has m real quasimodes q_j in $[a(h), b(h)]$ with compactly supported asymptotically orthogonal quasimode states. Note that this theorem is in some sense converse to [St1, Theorem 1] that states that locally existence of quasimodes implies existence of resonances nearby.

Define the smooth cut-off function $0 \leq \chi_B(x) \leq 1$ as follows:

$$\chi_B(x) = 1 \text{ for } |x| \leq B + 3/4, \quad \chi_B(x) = 0 \text{ for } |x| > B + 1. \tag{3.36}$$

Theorem 3.1 *Let*

$$0 < a_0 \leq a(h) < b(h) \leq b_0, \quad b(h) - a(h) = o(1)h^{(5n^\#+1)/2}, \quad 2e^{-h^{-1/3}} \leq c(h) = o(1)h^{(5n^\#+1)/2}.$$

and set

$$\Omega(h) = [a(h), b(h)] + i[-c(h), 0].$$

Suppose that $P(h)$ has $m(h) = N(\Omega(h))$ resonances (counting multiplicities) in $\Omega(h)$. Fix $z_0(h) \in [a(h), b(h)]$. Then z_0 is a quasimode of multiplicity $m(h)$ for $P(h)$, in the following sense: The space $\chi_B\mathrm{Ran}\Pi_\Omega$ has dimension $m(h)$ and for any $\psi \in \chi_B\mathrm{Ran}\Pi_\Omega$ with $\|\psi\| = 1$ we have *(a)* supp $\psi \subset B(0, B+1)$,

(b) $\|(P(h) - z_0(h))\psi\| \le C\epsilon(h)$,

where $\epsilon(h) = h^{-(5n^\#+1)/4} \max\left\{(b(h)-a(h))^{1/2}, h^{-(5n^\#+1)/4} c^{1/2}(h)\right\}$.

Moreover, if $\psi = \chi_B f$ *with* $f \in \chi_B \mathrm{Ran}\Pi_\Omega$, *and* $\|\psi\| = 1$, *then*

$$\|\psi - f\|_{H^1} \le C\epsilon(h). \tag{3.37}$$

Remark. It follows from the propagation of singularities arguments in section 4 and from the theorem above, that one can cut off f for $B_0 < |x| < B$, *before* the complex scaling is performed, if $\epsilon(h) = O(h^N)$, $N > (5n^\# + 1)/2$ or $N = \infty$, see the remark at the end of section 4. In other words, one can replace f above by *non-scaled* resonant states. Our approach however allows us to do this only after we prove the theorem above for the scaled resonant states, not directly.

Proof of Theorem 3.1. Without loss of generality we can assume that there are no resonances on $\partial\Omega$. Given $f \in \mathrm{Ran}\Pi_\Omega$, set

$$\psi = \chi_B f. \tag{3.38}$$

Then ψ is supported in $B(0, B+1)$. Let $\epsilon(h)$ be as above (compare with (3.30)). First, observe that by Proposition 3.1 and Proposition 3.3,

$$\int \theta\left(|h\nabla f|^2 + |f|^2\right) dx \le C\left(\epsilon^2(h) + e^{-h^{-1/3}}\right)\|f\|^2 \le C\epsilon^2(h)\|f\|^2, \quad \forall f \in \mathrm{Ran}\Pi_\Omega \tag{3.39}$$

with $C > 0$ independent of f (and h). Since for $|x| > B + 1/2$ we have $\theta = \theta_0$, we get

$$\|\psi - f\|_{H^1} \le C\epsilon(h)\|f\|. \tag{3.40}$$

Normalize ψ so that $\|\psi\| = 1$. Our assumptions guarantee that $\epsilon(h) \to 0$, as $h \to 0$, so $\|f\| = 1 + o(1)$. In particular, this proves (3.37).

Next, $(P_\theta - z_0)\psi = [P_\theta, \chi_B]f + \chi_B(P_\theta - z_0)f$ and by (3.39),

$$\|[P_\theta, \chi_B]f\| \le C\left(\int_{B+3/4\le|x|\le B+1}\left(|h\nabla f|^2 + |f|^2\right)dx\right)^{1/2} \le C(\epsilon^2(h)/\theta_0)^{1/2}.$$

Since $\|(P_\theta - z_0)f\| \le \epsilon^2(h)$, we therefore have

$$\|(P_\theta - z_0)\psi\| \le C\epsilon(h). \tag{3.41}$$

To pass from P_θ to P, it is enough to estimate $\|(P_\theta - P)\psi\|$. As in Proposition 3.1, we will work with $\tilde\psi = \rho\psi$ and the corresponding operators $\tilde P_\theta = \rho P_\theta \rho^{-1}$, $\tilde P = \rho P \rho^{-1}$. Note that on supp θ we have $\rho = r^{(n-1)/2}$. Observe that the coefficients of $P_\theta - P$ are bounded by $C(\theta + \theta' + |\theta''|)$. More precisely, by (3.6) and (3.11),

$$\begin{aligned}\|(\tilde P_\theta - \tilde P)\tilde\psi\|_{\tilde{\mathcal H}} &\le C\left(\|(\theta+\theta')h^2\partial_r^2\tilde\psi\|_{\tilde{\mathcal H}} + \|(\theta+\theta'+|\theta''|)h\partial_r\tilde\psi\|_{\tilde{\mathcal H}} + \|\theta\frac{h^2\Delta_\omega}{r^2}\tilde\psi\|_{\tilde{\mathcal H}} + \|\theta\tilde\psi\|_{\tilde{\mathcal H}}\right)\\ &= I_1 + I_2 + I_3 + I_4.\end{aligned} \tag{3.42}$$

Here we used the fact that $0 \le \theta \le C\theta'$. We want to estimate each I_j in terms of ϵ using Proposition 3.1 and (3.39). Since $\theta \le \theta_0^{1/2}\theta^{1/2}$, we get immediately by (3.39),

$$I_4 \le C\epsilon. \tag{3.43}$$

In the same way we can treat $\|(\theta+\theta')h\partial_r\tilde\psi\|_{\tilde{\mathcal H}}$. To estimate I_2, we need to bound $\|\theta'' h\partial_r\tilde\psi\|_{\tilde{\mathcal H}}$. Using the explicit form of $\theta(r)$ near $r = B$, we get easily that $|\theta''| \le C\theta^{1/2}$. Therefore,

$$I_2 \le C\epsilon + C\|\theta'' h\partial_r\tilde\psi\|_{\tilde{\mathcal H}} \le C\epsilon + C\|\theta^{1/2}\partial_r\tilde\psi\|_{\tilde{\mathcal H}} \le C\epsilon. \tag{3.44}$$

To estimate I_1, introduce the smooth function $\eta(r)$ as follows. Let $\eta(r) = 0$ for $r < B$ and $\eta''' = \theta^{1/2}$. Then $\eta + \eta' + \eta'' \le C\theta^{1/2}(r)$ for $B \le r \le B+1$, and the integrands in I_j are supported there. Also, $\theta + \theta' + |\theta''| \le C\eta$. Therefore,

$$\begin{aligned} I_1 \le C\|(\theta+\theta')h^2\partial_r^2\tilde\psi\|_{\tilde{\mathcal{H}}} &\le C\|\eta h^2\partial_r^2\tilde\psi\|_{\tilde{\mathcal{H}}} \le C\left(\|h^2\partial_r^2(\eta\tilde\psi)\|_{\tilde{\mathcal{H}}} + h\|\eta' h\partial_r\tilde\psi\|_{\tilde{\mathcal{H}}} + h^2\|\eta''\tilde\psi\|_{\tilde{\mathcal{H}}}\right) \\ &\le C\left(\|h^2\partial_r^2(\eta\tilde\psi)\|_{\tilde{\mathcal{H}}} + h\|\theta^{1/2}h\partial_r\tilde\psi\|_{\tilde{\mathcal{H}}} + h^2\|\theta^{1/2}\tilde\psi\|_{\tilde{\mathcal{H}}}\right) \\ &\le C\left(\|h^2\partial_r^2(\eta\tilde\psi)\|_{\tilde{\mathcal{H}}} + \epsilon\right) \end{aligned} \tag{3.45}$$

To estimate $h\|h\partial_r(\eta\tilde\psi)\|_{\tilde{\mathcal{H}}}$, we use elliptic estimates (note that actually $h^2\partial_r^2(\eta\tilde\psi)$ is compactly supported) to get

$$\begin{aligned} \|h^2\partial_r^2(\eta\tilde\psi)\|_{\tilde{\mathcal{H}}} &\le C\left(\|(\tilde P_\theta - z_0)(\eta\tilde\psi)\|_{\tilde{\mathcal{H}}} + \|\eta\tilde\psi\|_{\tilde{\mathcal{H}}}\right) \le C\left(\|[\tilde P_\theta,\eta]\tilde\psi\|_{\tilde{\mathcal{H}}} + \epsilon\right) \\ &\le C\left(h^2\|\eta''\tilde\psi\|_{\tilde{\mathcal{H}}} + h\|\eta' h\partial_r\tilde\psi\|_{\tilde{\mathcal{H}}} + \epsilon\right) \\ &\le C\left(h^2\|\theta^{1/2}\tilde\psi\|_{\tilde{\mathcal{H}}} + h\|\theta^{1/2}h\partial_r\tilde\psi\|_{\tilde{\mathcal{H}}} + \epsilon\right) \\ &\le C\epsilon. \end{aligned} \tag{3.46}$$

In the same way we treat I_3. Combining this with (3.42), (3.43), (3.44), (3.45) and (3.46), we get

$$\|(\tilde P_\theta - \tilde P)\tilde\psi\|_{\tilde{\mathcal{H}}} \le C\epsilon.$$

This, together with (3.41), implies that

$$\|(P_\theta - z_0)\psi\| \le C\epsilon.$$

It remains to prove that $\chi_B\mathrm{Ran}\Pi_\Omega$ has the same dimension $m(h)$ as $\mathrm{Ran}\Pi_\Omega$. To show that, it is enough to prove that for any $0 \ne f \in \mathrm{Ran}\Pi_\Omega$, $\psi = \chi_B f \ne 0$. Assume that $\chi_B f = 0$. Then f is real analytic for $|x| > A$ and $f = 0$ for $|x| \le B + 1/2$. Therefore, $f = 0$. □

Theorem 3.1 implies immediately the following fact. Let $P^\#(h)$ be equal to $P(h)$ in $\mathcal{H}_{R_0} \oplus L^2(B(0,R)\setminus B(0,R_0))$ with Dirichlet boundary conditions on $\partial B(O,R)$, $R > B$. Then $P^\#(h)$ has at least $m(h) = N(\Omega(h))$ eigenvalues (counting multiplicities) in the interval $[a(h)-\delta(h), a(h)+\delta(h)]$, with $\delta(h) \ge Ch^{-n^\#}\epsilon(h)$, $C \gg 1$, if $\epsilon(h)h^{-n^\#} \ll 1$ (see [La, Proposition 32.4]). This is useful, only if the width $b(h)-a(h)$ of $\Omega(h)$ is $O(h^N)$, $N \gg 1$. We will generalize this in two directions. First, we will consider more general *reference operators* than the Dirichlet realization of $P(h)$ in a large ball, and secondly, we will prove this property for larger domains of width that can be independent of h, for example.

Let $\Omega(h)$ and $\Omega_k(h)$ be as in Proposition 3.4. Apply Theorem 3.1 to each Ω_k. In view of the upper bound (3.33) of $b_k(h) - a_k(h)$ that we have, we see that we can replace $\epsilon(h)$ with the function

$$\epsilon(h) = Ch^{-(3n^\#+1/2)}c^{1/2}(h), \tag{3.47}$$

where C depends only on the constant $C(a_0, b_0, c_0)$ in (2.2). This gives us a family of linear spaces $\chi_B\mathrm{Ran}\Pi_{\Omega_k}$, $k = 1, \dots, K(h)$, such that for each k, (a), (b), (c) of Theorem 3.1 are satisfied with $z_0(h)$ there replaced by $z_k(h) \in [a_k(h), b_k(h)]$ and $\epsilon(h)$ as above.

Definition 3.1 *Let $\mathcal{H}^\#$ be a Hilbert space that can be expressed as $\mathcal{H}^\# = \mathcal{H}_{R_0} \oplus L^2(B(0,R)\setminus B(0,R_0)) \oplus \mathcal{H}_{\mathrm{ext}}$, $R > B+1$, with $\mathcal{H}_{\mathrm{ext}}$ another Hilbert space, and assume that $P^\#(h)$ is a selfadjoint operator in $\mathcal{H}^\#$ with discrete spectrum in a h-independent neighborhood of the interval $[a(h), b(h)]$, where $a_0 \le a(h) < b(h) \le b_0$. We call $P^\#(h)$ a reference operator for $P(h)$ in $\Omega(h)$ with discrepancy $\delta(h)$, if for some decomposition of $\Omega(h)$ as above such that* $\mathrm{Res}P \cap \Omega = \mathrm{Res}P \cap (\cup_k\Omega_k)$ *one has $\|(P^\#(h) - z_k(h))\psi_k(h)\| \le \delta(h) \to 0$, as $h \to 0$ for any $\psi_k \in \chi_B\mathrm{Ran}\Pi_{\Omega_k}$, $\|\psi_k\| = 1$, where $z_k(h) \in \Omega_k(h) \cap \mathbf{R}$.*

The notion reference operator depends on the choice of the discrepancy function $\delta(h)$. The Dirichlet realization of $P(h)$ in a large ball, considered above, is an example of a reference operator with discrepancy function $\delta(h) = \epsilon(h)$. Clearly, we have a lot of freedom to choose the reference operator $P^\#$, for example one can impose other type of selfadjoint boundary conditions on $\partial B(0,R)$, or to extend $P(h)$ on a perturbed torus as in [SjZ]. The more complicated definition of reference operator that we give is justified by our desire later to obtain $P^\#$ from P not only by modifying it for large x but also by modifying it outside the wave front set of the resonant states.

Next theorem is a "global" version of Theorem 3.1, i.e., it applies to resonances in wider domains $\Omega(h)$.

Theorem 3.2 *Let*

$$0 < a_0 \le a(h) < b(h) \le b_0, \quad 2e^{-h^{-1/3}} \le c(h) \le Ch^{15n^\#+3}.$$

and set

$$\Omega(h) = [a(h), b(h)] + i[-c(h), 0].$$

Let $P^\#(h)$ be a reference operator in $\Omega(h)$ with discrepancy $\delta(h) \le h^{9n^\#/2+1}$. Then

(a)

$$N(\Omega(h)) \le N^\#\{[a(h) - \delta_1(h), b(h) + \delta_1(h)]\} \quad \text{for } h \ll 1,$$

where $\delta_1(h) = h^{-9n^\#/2-1}\delta(h)$.

(b) If $\delta_1(h) \ge h^{-9n^\#/2-1}\delta(h)$, then each $f \in \operatorname{Ran}\Pi_\Omega$ with $\|f\| = 1$ is a linear combination of eigenfunctions of $P^\#$ with eigenvalues in $[a(h) - \delta_1(h), b(h) + \delta_1(h)]$ up to an error that in any compact does not exceed

$$Ch^{-(9n^\#+1)/2}\left(h^{-(3n^\#+1/2)}c^{1/2}(h) + \delta(h)/\delta_1(h)\right)$$

for $0 < h \le h_0$, with C and h_0 uniform with respect to the choice of f.

Proof. The basic argument in the proof is that the property that the resonant states corresponding to resonances in different clusters in $\Omega(h)$ are linearly independent is stable under small perturbations as guaranteed by Proposition 3.4.

Let ψ_k be as in Definition 3.1. Then $\psi_k = \chi_B f$, $f_k \in \operatorname{Ran}\Pi_{\Omega_k}$ as in (3.38). By (3.37), $\|f_k - \psi_k\| \le \epsilon(h) = Ch^{-(3n^\#+1/2)}h^{(15n^\#+3)/2} \le h^{9n^\#/2+1}$ for $h \ll 1$ (see (3.47)). Let $\Pi^\#(h)\{[a,b]\}$ be the spectral projector of $P^\#(h)$ corresponding to the interval $[a,b]$. Set

$$v_k(h) = \Pi^\#(h)\big\{[a_k(h) - \delta_1(h), b_k(h) + \delta_1(h)]\big\}\psi_k(h), \quad k = 1, \dots, K(h). \tag{3.48}$$

Note that $\psi_k = 0$ outside $B(0, B+1)$, therefore they can be considered as functions in $\mathcal{H}^\#$ as well and the projection above is well-defined. We claim that $v_k(h)$ are linearly independent which would imply part (a) of the theorem because each ψ_k can be chosen freely in the space $\chi_B \operatorname{Ran}\Pi_{\Omega_k}$ and therefore, we would get that $\operatorname{span}\{v_k;\ \psi_k \in \chi_B\operatorname{Ran}\Pi_{\Omega_k}\}$ has dimension $\sum_k \operatorname{Rank}\Pi_{\Omega_k} = \operatorname{Rank}\Pi_\Omega$. Assume the opposite. Then

$$\alpha_1 v_1 + \ldots + \alpha_K v_K = 0 \tag{3.49}$$

with at least one coefficient non-zero. Recall that ψ_k are normalized and $\|(P^\#(h) - a_k(h))\psi_k\| \le \delta(h)$. We have as above $\|f_k - \psi_k\| \le \epsilon(h)$. Since $\|(P^\#(h) - a_k(h))\psi_k\| \le \delta(h)$, using the spectral theorem, we get that $\|\psi_k - v_k\|_{\mathcal{H}^\#} \le \delta(h)/\delta_1(h) = h^{9n^\#/2+1}$ as in [St1, sec. 3]. Let $0 \le \chi \le 1$ be a smooth cut-off function equal to 1 in $B(0, B+3/4)$ and vanishing outside $B(0,R)$. Set $v_k' = \chi v_k$. Then we have also $\|\psi_k - v_k'\| \le h^{9n^\#/2+1}$ Therefore, $\|f_k - v_k'\| \le 2h^{9n^\#/2+1}$.

Denote by v_k'' the orthogonal projections of v_k' onto the space $\operatorname{Ran}\Pi_\Omega$. Multiplying (3.49) by χ and projecting (3.49) onto the latter space, we get

$$\alpha_1 v_1'' + \ldots + \alpha_K v_K'' = 0 \tag{3.50}$$

and as before, $\|f_k - v_k''\| \le 2h^{9n^\#/2+1}$ since f_k already belongs to $\operatorname{Ran}\Pi_\Omega$. In particular, $\|v_k''\| = 1 + o(h)$. We may assume that the largest coefficient in (3.50) by absolute value is α_1. Dividing by it, we get

$$v_1'' = \beta_2 v_2'' + \ldots + \beta_K v_K'', \quad |\beta_j| \le 1. \tag{3.51}$$

Let us apply the projector Π_{Ω_1} to both sides of (3.51). Applying Proposition 3.4, we get

$$\begin{aligned}\|f_1\| - Ch^{9n^\#/2+1}h^{-(7n^\#+1)/2} \le \|\Pi_{\Omega_1} v_1''\| &\le Ch^{-n^\#}\max_{k\ge 2}\|\Pi_{\Omega_1} v_k''\| \\ &\le Ch^{-n^\#}\max_{k\ge 2}\|\Pi_{\Omega_1}(v_k''-f_k)\| \\ &\le Ch^{-n^\#}h^{-(7n^\#+1)/2}2h^{9n^\#/2+1} = Ch^{1/2}.\end{aligned}$$

This contradicts the fact that $\|f_k\| = 1+o(1)$, as $h\to 0$.

(b) Choose $f \in \mathrm{Ran}\Pi_\Omega$, with $\|f\| = 1$. Then $f = \sum f_k$, where $f_k \in \mathrm{Ran}\Pi_{\Omega_k}$ and $\|f_k\| \le Ch^{-(7n^\#+1)/2}$ by Proposition 3.4. The proof of (a) implies that $\|f_k - v_k'\| \le \epsilon(h) + \delta(h)/\delta_1(h)\|\psi_k\|$ with $\epsilon(h)$ as in (3.47). Here $v_k' = \chi v_k$ are cut-off linear combinations of eigenfunctions of the reference operator with eigenvalues in the desired interval. Define $v' = \sum v_k'$. Then

$$\|f - v'\| \le \sum \|f_k - v_k'\| \le Ch^{-n^\#}\left(\epsilon(h) + \delta(h)/\delta_1(h)\right)\max\|v_k'\| \le Ch^{-(9n^\#+1)/2}\left(\epsilon(h)+\delta(h)/\delta_1(h)\right).$$

This completes the proof of (b). □

3.6 Asymptotic orthogonality of resonant/quasimode states

Theorem 3.1 shows that one can choose $m(h)$ orthogonal quasimodes, $m(h) = N(\Omega(h))$ being the total multiplicity of resonances in Ω, provided that the size of Ω is "small". However, those quasimode states are not necessarily cut-off single resonant states, in fact they are cut-off linear combinations of such resonant states. This result is non-trivial, as explained in the Introduction, since we do not know how to control the angles between resonant states corresponding to resonances too close to each other ($|z_1 - z_2| \ll \min\{-\mathrm{Im}\, z_1, -\mathrm{Im}\, z_2\}$).

As mentioned in the Introduction, we do not know whether one can construct "almost orthogonal" quasimodes corresponding to $\Omega(h)$ with larger size (for example $b(h) - a(h) = O(1)$) by keeping their number the same as the total multiplicity of resonances in Ω. A simple argument based on integration by parts however, shows that resonant states or quasimode states, respectively, corresponding to $\Omega_1(h)$ and $\Omega_2(h)$ with $\mathrm{dist}\{\Omega_1,\Omega_2\} \gg \mathrm{diam}\Omega_{1,2}$ are "almost orthogonal". We are not going to use the proposition below in our analysis, its purpose is actually to stress on the fact that for domains $\Omega_1(h)$, $\Omega_2(h)$ too close to each other, integration by parts argument does not provide asymptotic orthogonality. Note that the control (3.33) that we have on the lower bound of $\mathrm{dist}\{\Omega_k,\Omega_m\}$ for the domains (3.32) is not enough to guarantee asymptotic orthogonality.

Proposition 3.5 *Let $\Omega_1(h)$ and $\Omega_2(h)$ be two domains as in Theorem 3.1 and $\epsilon_1(h)$, $\epsilon_2(h)$ be related to Ω_1, Ω_2 as in Theorem 3.1. Let $\psi_k \in \chi_B \mathrm{Ran}\Pi_{\Omega_k}$, $k = 1,2$ be two quasimode states as in Theorem 3.1 corresponding to $\Omega_k(h)$, $k = 1,2$. Then for $h \ll 1$,*

$$|(\psi_1,\psi_2)| \le 2\frac{\epsilon_1(h)+\epsilon_2(h)}{\mathrm{dist}\{\Omega_1(h),\Omega_2(h)\}}$$

Proof. We have $(P - z_{0k})\psi_k = g_k$ with $\|g_k\| \le \epsilon_k(h)$, $k = 1,2$. Therefore,

$$z_{01}(\psi_1,\psi_2) = (P\psi_1 - g_1,\psi_2) = (\psi_1, P\psi_2) - (g_1,\psi_2) = z_{02}(\psi_1,\psi_2) + (\psi_1,g_2) - (g_1,\psi_2).$$

□

4 Trapped geodesics and wave front set of the resonant states

In this section we will show that in some important situations the wave front set of the resonant states is contained in the union of the trapped rays (see also [CdV] for quasimodes related to eigenfunctions on compact Riemannian manifolds).

We consider self-adjoint differential operators of the form

$$P(h) = \sum_{i,j=1}^{n} hD_{x_i} a_{ij}(x) hD_{x_j} + \sum_{j=1}^{n} b_j(x) hD_{x_j} + V(x) + P_1(h) \tag{4.1}$$

with smooth real-valued coefficients a_{ij}, b_j, V, such that $\{a_{ij}(x)\}$ is a symmetric positively definite matrix for any $x \in \mathbf{R}^n$ and $a_{ij} - \delta_{ij} = b_j = 0$ for $|x| > R_0$ with some $R_0 > 0$ while $V(x)$ is a long range potential satisfying (2.1) and the analyticity condition after it for $|x| > R_0$. Here $P_1(h) = \sum \tilde{b}_j(x,h) hD_{x_j} + \tilde{V}(x,h)$ is assumed to be a differential operator of first order with coefficients supported in $B(0, R_0)$, such that $P_1(h) \in L^{1,-1}$ considered as an h-ΨDO (note that $P(h) \in L^{2,0}$). The operator $P(h)$ is self-adjoint in $L^2(\mathbf{R}^n)$ and satisfies the black-box assumptions. Resonances of $P(h)$ are the poles of the meromorphic extension of $(P(h) - z)^{-1} : L^2_{\text{comp}} \to L^2_{\text{loc}}$ form $\operatorname{Im} z > 0$ to $\mathbf{C}$, if n is odd, and to the logarithmic plane, if n is even. We are interested in the resonances near the real axis only.

We will use propagation of singularities results to get microlocal estimates of $\Pi_\Omega f / \|\Pi_\Omega f\|$ away from the trapping trajectories.

One can define the semi-classical wave front $\mathrm{WF}^s(u)$ and $\mathrm{WF}(u)$ of a tempered u as in [G] (see also [SjV], [I]). The wave front set lives in the space $T^*\mathbf{R}^n \cup S^*\mathbf{R}^n$, where $S^*\mathbf{R}^n$ is associated with the "infinite points". We note that here we will work with finite points of the semiclassical wave front set only, because we study operators with characteristic variety bounded in the ξ variable. Consider the bicharacteristics of $P(h)$ related to its semi-classical principal symbol $p_0(x,\xi) = \sum a_{ij}(x)\xi_i\xi_j + \sum_j b_j\xi_j + V(x)$. They are the integral curves of the Hamiltonian vector field $H_{p_0} = (\partial_\xi p_0)\partial_x - (\partial_x p_0)\partial_\xi$. We call a bicharacteristic $t \mapsto \gamma(t)$ *non-trapped*, if for any $R > 0$, there exists τ (positive or negative), such that $\gamma(\tau)$ lies outside $T^*B(0,R)$. We call all other bicharacteristics *trapped*. Denote by $\mathcal{T}$ the trapped subset of $T^*\mathbf{R}^n$, i.e, $(x,\xi) \in \mathcal{T}$ if and only if the bicharacteristic passing through (x,ξ) is trapped.

Theorem 4.1 *Let*

$$0 < a_0 \le a(h) < b(h) \le b_0, \quad b(h) - a(h) \le h^M, \quad M > (5n+1)/2,$$

and set

$$\Omega(h) = [a(h), b(h)] + i[-h^N, 0], \quad N \ge M + (5n+1)/2.$$

Let $P(h)$ be the operator defined above and let $f = f(h) \in \operatorname{Ran}\Pi_\Omega$, and $\|f\| = 1$. Then for $s = M/2 - (5n+1)/4$, $\mathrm{WF}^s(f)$ is supported in the set of trapped bicharacteristics of $P(h)$ on energy levels in $p_0^{-1}[a_0,b_0]$ uniformly with respect to the choice of f. More precisely, for any zeroth order symbol $q(x,\xi)$ with support disjoint from $\mathcal{T} \cap p_0^{-1}[a_0,b_0]$ there exists $C > 0$ such that $\|q(x,hD)f\| \le Ch^s$ for any $f(h)$ as above.

If $\Omega(h)$ is as above with $0 < a_0 \le a(h) < b(h) \le b_0$, $N > (7n+1)/2$ but without smallness assumptions on $b(h) - a(h)$, then the statement of the theorem is true with $s = N/2 - (15n+2)/2$.

The proof of Theorem 4.1 is based on a propagation of singularities argument. The following lemma follows directly from [I], if the energy level $z_0(h)$ is independent of h.

Lemma 4.1 (propagation of singularities) *Let $(P(h) - z_0(h))u(h) = g(h)$ in $B(0,R)$ with $0 < a_0 \le z_0(h) \le b_0 < \infty$, $R_0 < R$, and $\|u(h)\|_{L^2(B(0,R))} \le C$. Let $(x_0,\xi_0) \in T^*B(0,R)$. Assume that $(x_1,\xi_1) \in T^*B(0,R)$ can be connected with (x_0,ξ_0) by a bicharacteristic (of finite length) lying in $T^*B(0,R) \setminus \mathrm{WF}^{s+1}(g)$. Then, if $(x_0,\xi_0) \notin \mathrm{WF}^s(u)$, we also have $(x_1,\xi_1) \notin \mathrm{WF}^s(u)$ (and therefore, if $(x_0,\xi_0) \in \mathrm{WF}^s(u)$, then $(x_1,\xi_1) \in \mathrm{WF}^s(u)$). The estimates that those inclusions imply are uniform with respect to the choice of $u(h)$.*

Proof. Let $[0,T] \ni t \mapsto \gamma(t) \subset T^*B(0,R) \setminus \mathrm{WF}^{s+1}(g)$ be the bicharacteristic such that $\gamma(0) = (x_0,\xi_0)$ and $\gamma(T) = (x_1,\xi_1)$. Denote by Φ^t the bicharacteristic flow. Let $q_0(x,hD)$ be such that $q_0(x,hD)u(h) = O(h^s)$, $q_0(x,\xi) = 1$ near (x_0,ξ_0), and $\operatorname{supp} q_0 \subset T^*B(0,R)$. We will construct a symbol $q_t(x,\xi)$ such that $q_t = q_0$ for $t = 0$, for each $t \in [0,T]$, the principal symbol of $q_t(x,\xi)$ is equal to 1 near $\Phi^t(x_0,\xi_0)$ and q_t has support contained in a small neighborhood of $\Phi^t(x,\xi)$, such that $\operatorname{supp} q_t \subset T^*B(0,R) \setminus \mathrm{WF}^{s+1}(g)$.

Set

$$\alpha(t) = \|q_t(x,hD)u\|.$$

We know that $\alpha(0) = O(h^s)$. Our goal is to construct q_t such that $\alpha(t) = O(h^s)$, $\forall t \in [0,T]$. We have

$$\frac{d}{dt}\frac{\alpha^2(t)}{2} = \mathrm{Re}\,\left(\frac{d}{dt}q_t(x,hD)u, q_t(x,hD)u\right).$$

Set $Q_t = q_t(x,hD)$. We will choose q_t so that

$$ih\frac{d}{dt}Q_t = [P,Q_t] + R_t, \quad Q_t|_{t=0} = Q_0, \tag{4.2}$$

with R_t is of order $-s-1$. Suppose that we have Q_t with those properties. Then

$$\begin{aligned}\frac{d}{dt}\frac{\alpha^2(t)}{2} &= h^{-1}\mathrm{Im}\,(([P-z_0,Q_t]+R_t)u, Q_tu)\\ &= -h^{-1}\mathrm{Im}\,(Q_t(P-z_0)u, Q_tu) + h^{-1}\mathrm{Im}\,(R_tu, Q_tu)\\ &= -h^{-1}\mathrm{Im}\,(Q_tg, Q_tu) + h^{-1}\mathrm{Im}\,(R_tu, Q_tu).\end{aligned}$$

Since $\mathrm{supp}\, q_t$ is disjoint from $\mathrm{WF}^{s+1}(g)$, we get

$$\alpha(t)\frac{d}{dt}\alpha(t) \le Ch^s\alpha(t) + Ch^{-1}\|R_tu\|\alpha(t) \quad \Longrightarrow \quad \frac{d}{dt}\alpha(t) \le Ch^s + Ch^{-1}\|R_tu\|.$$

Since $\|R_t\| = O(h^{s+1})$,

$$\frac{d}{dt}\alpha(t) \le Ch^s \quad \Longrightarrow \quad \alpha(t) \le Ch^s \quad \text{for } 0 \le t \le T.$$

It remains to solve (4.2). Notice that Q_t is a finite expansion of the exact solution $e^{-itP/h}Q_0e^{itP/h}$ of (4.2) with $R_t = 0$, see e.g., [DSj, Ch. 11]. We look for $q_t(x,\xi)$ of the form $q_t = q_t^{(0)} + hq_t^{(1)} + \ldots + h^{s-1}q_t^{(s-1)}$. The principal symbol $q_t^{(0)}$ of Q_t must solve the equation

$$\left(\partial_t + H_{p_0}\right)q_t^{(0)} = 0, \quad q_t^{(0)}|_{t=0} = q_0.$$

Therefore, we define $q_t^{(0)}(x,\xi) = q_0(\Phi^{-t}(x,\xi))$, which is also confirmed by Egorov's theorem. This implies that

$$ih\frac{d}{dt}Q_t^{(0)} = [P,Q_t^{(0)}] + R_t^{(0)}, \quad Q_t^{(0)}|_{t=0} = Q_0,$$

where $R_t^{(0)}$ is of order -2. Note that the symbol of $R_t^{(0)}$ is included in the set $\Gamma = \cup_{0\le t\le T}\Phi^t(\mathrm{supp}\, q_0)$. We next solve

$$ih\frac{d}{dt}Q_t^{(1)} = [P,Q_t^{(1)}] - h^{-1}R_t^{(0)}, \quad Q_t^{(0)}|_{t=0} = 0 \tag{4.3}$$

on principal symbol level, which gives us the equation

$$\left(\partial_t + H_{p_0}\right)q_t^{(1)} = ir_t^{(0)}, \quad q_t^{(1)}|_{t=0} = 0,$$

where $h^2r_t^{(0)}$ is the principal symbol of $R_t^{(0)}$. This is an ODE along the bicharacteristics of p_0 and the solution is again supported in Γ. Then $Q_t^{(1)} = hq_t^{(1)}(x,hD)$ solves (4.3) up to a remainder $R_t^{(1)}$ of order -3. We complete the construction of the solution to (4.2) by induction. $\square$

Proof of Theorem 4.1. Let ψ be related to f as in (3.38). By Theorem 3.1, ψ solves $(P(h)-a(h))\psi(h) = O(h^s)$ with $s = M/2-(5n+1)/4$, and $\psi = 0$ for $|x| > B+1$. We have $s > 0$ because $M > 5n/2+3$. We get immediately that $\mathrm{WF}^s(\psi)$ is contained in $p_0^{-1}[a_0, b_0]$ because $P(h)-a(h)$ is elliptic outside any neighborhood of this set.

Fix $(x,\xi) \in p_0^{-1}[a_0,b_0]\setminus\mathcal{T}$. Then $\xi \neq 0$ and let $\gamma(t)$ be the bicharacteristic passing through (x,ξ) for $t = 0$. Then for some t (positive or negative), the x-projection of $\gamma(t)$ lies outside $B(0, B+1)$, where $\psi = 0$. By Lemma 4.1, $(x,\xi) \notin \mathrm{WF}^s(\psi)$. This shows that $\mathrm{WF}^s(\psi) \subset \mathcal{T}$. By (3.37) (and the ellipticity of $P(h)$), those statements remain true if we replace ψ by f. This proves the first part of the theorem.

To prove the second part, let us group all resonances in Ω in subdomains $\Omega_k(h)$ of width $O(h^M)$, where $M = N-(7n+1)/2$ as in (3.32). Then we have the conclusion of the theorem for each normalized $f_k \in \mathrm{Ran}\Pi_{\Omega_k}$. Let $f \in \mathrm{Ran}\Pi_\Omega$. Then $f = f_1 + \ldots + f_K$, $K = O(h^{-n})$. Therefore, for $q(x,hD)$ as in the theorem, we have by Proposition 3.4,

$$\|q(x,hD)f\| \le C\sum\|q(x,hD)f_j\| \le Ch^sh^{-n}\max\|f_j\| \le Ch^{s-(9n+1)/2}\|f\|$$

with $s = M/2-(5n+1)/4 = N/2-3n-1/2$. □

Remark. Lemma 4.1 allows us to estimate the resonant states $f \in \mathrm{Ran}\Pi_{\Omega_k}$ in the abstract black box setting considered in section 3 *before* the complex scaling but outside the ball where the Hamiltonian might be trapping. More precisely, let f, ψ and $\epsilon(h)$ be as in Theorem 3.1 (note that there Ω is a "small" domain, so we apply this theorem to Ω_k actually). Let B_0 be as in Proposition 3.1 and let $B > B_0$. Estimate (3.37) implies that $f = O(\epsilon(h))$ for $|x| > B+1$, where $\theta(r) = \theta_0$. We also have $(P-z_0)\psi = O(\epsilon(h))$ and $\psi = 0$ for $|x| > B+1$. Pick a point $(x,\xi) \in p_0^{-1}[a_0,b_0]$ with $B_0 < |x| < B$. Then (see the remark right before Proposition 3.1), (x,ξ) can be connected with some (x_0,ξ_0) with $|x_0| > B+1$ with a bicharacteristic of this Hamiltonian which x–projection does not intersect the black box. Then Lemma 4.1, localized near this geodesics, implies that $\psi = O(\epsilon(h))$ microlocally near (x_0,ξ_0), if $c(h) = O(h^N)$, $N > (5n^\# + 1)/2$. By a compactness argument, we have this estimate in any H^s norm in the annulus $B_0+\epsilon \le |x| \le B$, $0 < \epsilon \ll 1$. Therefore, we can cut f off in the annulus above *before* the complex scaling is performed if $c(h) = O(h^N)$. This shows that the *non-scaled* resonant states are "small" for large $|x|$ even in the black box setting, and that in Theorem 3.1, one can work with cut off non-scaled resonant states instead with scaled ones. This argument works if $c(h) = O(h^N)$, $N > (5n^\#+1)/2$, or $N = \infty$ but is not sensible enough to express the decay of the non-scaled resonant states if $\epsilon(h) = O(1)e^{-Ch^{-\rho}}$, for example, then it just gives $O(h^\infty)$. Another argument based on application of Green's formula for black boxes then can treat the latter case but we will not go into details.

5 Upper bounds on the number of resonances close to the real axis

Let $P(h)$ be the operator (4.1). In this section we are going to establish an upper bound of the resonances of $P(h)$ in a box of width independent of h and height h^N, $N \gg 1$ in terms of the measure of the trapped set $\mathcal{T}$, where the measure is considered in $T^*\mathbf{R}^n$. To this end we choose a suitable reference operator $P^\#(h)$ that imposes a barrier outside a small neighborhood of the trapped set $\mathcal{T}$ by modifying $P(h)$ there. Since the resonant states are "small" there, the resonant states will be quasimodes for the new operator. An application of Theorem 3.2 then will imply an upper bound and the well-known asymptotics for the eigenvalues of selfadjoint h-ΨDOs will relate this bound with $\mathrm{meas}(\mathcal{T})$.

Fix $0 < a_0 < b_0$, $N \ge 15n+3$ and let

$$\Omega(h) = [a_0,b_0] + i[-h^N, 0].$$

The resonances in $\Omega(h)$ are contained in $\cup_{k=1}^{K(h)}\Omega_k(h)$, where $\Omega_k(h)$ are as in (3.32) with $\omega(h)$ as in Proposition 3.4. Then by (3.33), $b_k - a_k \le h^{N-(7n+1)/2}$, $\mathrm{dist}\{\Omega_{k_1},\Omega_{k_2}\} \ge 4h^{N-(5n+1)/2}$. Each Ω_k satisfies the assumptions of Theorem 3.1. The corresponding discrepancy function (see (3.47)) is given by

$$\epsilon(h) = Ch^{N/2-(3n+1/2)}.$$

Denote

$$\mathcal{T}^\nu = \mathcal{T} \cap p_0^{-1}[a_0 - \nu, b_0 + \nu], \quad \mathcal{T}^\nu_\mu = \{\zeta \in T^*\mathbf{R}^n;\ \mathrm{dist}\{\zeta, \mathcal{T}^\nu\} < \mu\}, \tag{5.1}$$

where $\nu > 0$, $\mu > 0$ are small parameters. Fix $\nu > 0$. Clearly, $\mathcal{T}^\nu$ is a closed set. Assume that it is non-empty (but it may have zero measure). On the other hand, $\mathcal{T}^\nu_\mu$ is an open set of positive measure. We claim that $\mathrm{Vol}(\mathcal{T}^\nu_\mu) \to \mathrm{meas}(\mathcal{T}^\nu)$, as $\mu \to 0$. Indeed, $\forall \varepsilon > 0$, there exists an open set $U_\varepsilon \supset \mathcal{T}^\nu$, such that $\mathrm{Vol}(U_\varepsilon) \le \mathrm{meas}(\mathcal{T}^\nu) + \varepsilon$. Then $T^*\mathbf{R}^n \setminus U_\varepsilon$ is at positive distance from $\mathcal{T}^\nu$ and therefore for $\mu = \mu(\varepsilon) \ll 1$ we have $\mathcal{T}^\nu \subset \mathcal{T}^\nu_\mu \subset U_\varepsilon$ which implies that $\mathrm{meas}(\mathcal{T}^\nu) \le \mathrm{Vol}(\mathcal{T}^\nu_\mu) \le \mathrm{meas}(\mathcal{T}^\nu) + \varepsilon$. This proves our claim.

Denote $-C_0 = \min p_0(x,\xi)$ and let $q_\mu(x,\xi)$ be a smooth function such that $q_\mu = 0$ on $\mathcal{T}^\nu_\mu$, $q_\mu = 2b_0 + C_0$ outside $\mathcal{T}^\nu_{2\mu}$, and $0 \le q_\mu \le 2b_0 + C_0$. Set

$$P^\#_\mu(h) = P(h) + q^w_\mu(x, hD).$$

The principal symbol of $P^\#_\mu(h)$ is $p_\mu(x,\xi) = p_0(x,\xi) + q_\mu(x,\xi)$. The self-adjoint operator $P^\#_\mu(h)$ has discrete spectrum in $(-\infty, 2b_0)$, because $\{p_\mu(x,\xi) \le M\}$ is compact for any $M < 2b_0$. Moreover, we have the following estimate for the number $N^\#_\mu[a_0,b_0]$ of eigenvalues of $P^\#_\mu(h)$ (see [DSj])

$$\frac{1}{(2\pi h)^n}\left(V^\#_-([a_0,b_0]) + o(1)\right) \le N^\#_\mu[a_0,b_0] \le \frac{1}{(2\pi h)^n}\left(V^\#_+([a_0,b_0]) + o(1)\right), \tag{5.2}$$

where

$$V^\#_\pm([a_0,b_0]) = \lim_{\pm\epsilon \searrow 0} \int_{p_\mu(x,\xi)\in[a_0-\epsilon, b_0+\epsilon]} dx\,d\xi. \tag{5.3}$$

If a_0 and b_0 are not critical values for $p_\mu(x,\xi)$, then $V^\#_-([a_0,b_0]) = V^\#_+([a_0,b_0])$ and the remainder is actually $O(h)$. In particular,

$$N^\#_\mu([a_0 - \nu/2, b_0 + \nu/2]) \le \frac{1}{(2\pi h)^n}\left(\mathrm{Vol}\left(p_\mu^{-1}[a_0 - \nu, b_0 + \nu]\right) + o(1)\right), \quad \text{as } h \to 0. \tag{5.4}$$

Since $\mathcal{T}^\nu \subset p_\mu^{-1}[a_0 - \nu, b_0 + \nu] \subset \mathcal{T}^\nu_{2\mu}$, we obtain that $\mathrm{Vol}(p_\mu^{-1}[a_0 - \nu, b_0 + \nu]) \to \mathrm{meas}(\mathcal{T}^\nu)$, as $\mu \to 0$.

We claim that $P^\#_\mu(h)$ is a reference operator in $\Omega(h)$ with discrepancy $\epsilon(h)$. Indeed, fix k and a normalized $f_k \in \mathrm{Ran}\Omega_k(h)$. Let $\psi_k(h)$ be the corresponding cut-off resonant state given by (3.38). Then by Theorem 3.1, $(P(h) - a_k(h))\psi_k = O(\epsilon(h))$ and by Theorem 4.1, $\mathrm{WF}^s(\psi_k) \subset \mathcal{T}^0$, where $s = N/2 - 3n - 1/2$. The latter implies that replacing $P(h)$ by $P^\#_\mu(h)$ would keep the estimate $\|(P^\#_\mu(h) - a_k(h))\psi_k\| \le C\epsilon(h) = Ch^s$ with different constant C. This constant depends on μ and ν but is independent of k and on the choice of ψ_k. Therefore, $P^\#_\mu(h)$ is a reference operator with discrepancy $C_{\mu,\nu}\epsilon(h)$. We can therefore apply Theorem 3.2, if $s \ge 9n/2 + 1$, which is fulfilled in our case, to conclude that the number of eigenvalues of $P^\#_\mu(h)$ in the interval $[a_0 - \delta_1(h), b_0 + \delta_1(h)]$, where $\delta_1(h) = h^{N/2 - 15n/2 - 3/2}$, is at least equal to the number of resonances in Ω. In particular,

$$N(\Omega(h)) \le N^\#_\mu([a_0 - \nu/2, b_0 + \nu/2]) \quad \text{for } 0 < h \le h_0(\mu,\nu). \tag{5.5}$$

Relations (5.4) and (5.5) imply

$$N(\Omega(h)) \le \frac{1}{(2\pi h)^n}\left(\mathrm{Vol}\left(p_\mu^{-1}[a_0 - \nu, b_0 + \nu]\right) + o(1)\right) \quad \text{for } 0 < h \le h_0(\mu,\nu). \tag{5.6}$$

Therefore, $\limsup_{h\to 0}(2\pi h)^n N(\Omega(h)) \le \mathrm{Vol}\left(p_\mu^{-1}[a_0 - \nu, b_0 + \nu]\right)$ and taking the limit $\mu \to 0$, $\nu > 0$ fixed, we get by the remark after (5.4) that $\limsup_{h\to 0}(2\pi h)^n N(\Omega(h)) \le \mathrm{meas}(\mathcal{T}^\nu)$. Taking the limit $\nu \to 0$, we get that the latter converges to $\mathrm{meas}\left\{\mathcal{T} \cap p_0^{-1}[a_0 - 0, b_0 + 0]\right\}$, where $p_0^{-1}[a_0 - 0, b_0 + 0] := \cap_{\nu > 0} p_0^{-1}[a_0 - \nu, b_0 + \nu]$. Since p_0 is a quadratic form of ξ for each x, the limit is actually $\mathrm{meas}\left\{\mathcal{T} \cap p_0^{-1}[a_0, b_0]\right\}$. We have therefore proved the following.

Theorem 5.1 *Let $0 < E_1 < E_2$ be fixed and $N \ge 15n + 3$. Let $P(h)$ be as in (4.1) and set*

$$\Omega(h) = [E_1, E_2] + i[-h^N, 0].$$

Then

$$N(\Omega(h)) \leq \frac{1}{(2\pi h)^n}\left(\text{meas}\left\{\mathcal{T} \cap p_0^{-1}[E_1, E_2]\right\} + o(1)\right), \quad \text{as } h \to 0,$$

where $\mathcal{T}$ *is the trapped set related to* $P(h)$.

In the formulation of this theorem we passed to the commonly used notation E for the energy levels.

6 Example of sharp lower bounds, generalized potential well

In this section we study again the resonances of the operator (4.1) under the assumption that for some non-critical energy level E_2, the set $p_0^{-1}[-\infty, E_2]$ has at least one compact connected component. Then we get lower bound in terms of the volume of the compact component. If in addition we assume that the unbounded component is non-trapping, we also get an asymptotic formula for the resonances near the real line and a resonance free zone. This situation can be considered as a generalized potential well. Similar situation was studied by Shu Nakamura in [N], where he obtains asymptotic for the spectral shift function. His results, combined with existence of a resonance free zone (see the theorem below) and the techniques developed by V. Petkov and M. Zworski [PeZ] provide different approach to proving the asymptotic in the theorem below.

Fix two energy levels $0 < E_1 < E_2 < \max\{p_0(x,\xi)\}$. Assume that E_1 and E_2 are non-critical values of p_0. Assume also that $p_0^{-1}(-\infty, E_2]$ is not connected, i.e., it has a non-empty compact component (this component then must have non-empty interior because E_2 is non-critical value of p_0). Then

$$p_0^{-1}[E_1, E_2] = W_{\text{int}} \cup W_{\text{ext}},$$

is a unbounded closed set with smooth boundary, where we denote by W_{ext} the unbounded connected component and the union of the bounded ones, that is non-empty according to our assumptions, is denoted by W_{int}. Then W_{int} is a compact with smooth boundary and consists of trapped points only. The set W_{ext} contains non-trapped points and may contain trapped ones as well.

Theorem 6.1

(a) For some function $0 \leq S(h) = O(h^\infty)$ *we have as* $h \to 0$

$$\begin{aligned}
\frac{1}{(2\pi h)^n}(\text{Vol}(W_{\text{int}}) - O(h)) &\leq N([E_1, E_2] + i[-S(h), 0]) \\
&\leq N([E_1, E_2] + i[-h^{15n+3}, 0]) \\
&\leq \frac{1}{(2\pi h)^n}(\text{Vol}\,(W_{\text{int}}) + \text{meas}\{\mathcal{T} \cap W_{\text{ext}}\} + o(1))\,.
\end{aligned}$$

(b) If W_{ext} *is non-trapping, i.e., if* $W_{\text{ext}} \cap \mathcal{T} = \emptyset$*, then there exists a function* $0 < S_0(h) = O(h^\infty)$ *such that for any* $S(h)$ *such that* $S_0(h) \leq S(h) = O(h^\infty)$*,*

$$N([E_1, E_2] + i[-S(h), 0]) = \frac{1}{(2\pi h)^n}\left(\text{Vol}(W_{\text{int}}) + O(h)\right), \quad \text{as } h \to 0.$$

Moreover, if $P(h) = -h^2\Delta$ *for* $|x| > R_0$ *with some* $R_0 > 0$*, then* $\forall M > 0$ *the function* $S(h)$ *above can be chosen so that for some* $h_0 = h_0(M) > 0$*, there are no resonances in*

$$[E_1, E_2] + i[-Mh, -S(h)] \quad \text{for } 0 < h < h_0.$$

Proof. We will show first that the x-projections of W_{int} and W_{ext} do not intersect. This will allow us to use cut-off functions depending on x only.

Since $p_0(x,\xi)$ is a quadratic form with respect to ξ, we get

$$p_0(x,\xi) = \left| A(x)^{1/2}\xi + \frac{1}{2}A^{-1/2}(x)b(x)\right|^2 + \tilde V(x), \quad \tilde V(x) := V(x) - \frac{1}{4}A(x)^{-1}b(x)\cdot b(x) \tag{6.1}$$

where $A(x) := \{a_{ij}(x)\}$, $b(x) = \{b_j(x)\}$. We claim that our assumption that $p_0^{-1}(-\infty, E_2] \subset T^*\mathbf{R}^n$ is not connected implies the same for $\{\tilde V(x) \le E_2\} \subset \mathbf{R}^n$. Assume the opposite. Then for any x_0, x_1 from this set, there exists a continuous path $x = x(t)$, $0 \le t \le 1$ such that $x(0) = x_0$, $x(1) = x_1$ and $\tilde V(x(t)) \le E_2$, $0 \le t \le 1$. Fix (x_0,ξ_0) and (x_1,ξ_1) in $p_0^{-1}(-\infty, E_2]$. We will show that we can connect those two points by a path lying on or under the energy level E_2. Let $\hat\xi_0$ be the value of ξ that minimizes $p(x_0,\xi)$, i.e., $\hat\xi_0 = -A^{-1}(x_0)b(x_0)/2$, and let $\eta_0(t) = \xi_0 + t(\hat\xi_0 - \xi_0)$ be the line segment that connects ξ_0 and $\hat\xi_0$. We thus connect (x_0,ξ_0) and $(x_0,\hat\xi_0)$ by $[0,1] \ni t \mapsto (x_0,\eta(t))$. It is easy to see that $p_0(x_0,\eta_0(t))$ decreases as t increases from $t = 0$ to $t = 1$ and therefore, it stays on or under the energy level E_2. We next connect $(x_0,\hat\xi_0)$ and $(x_1,\hat\xi_1)$, where $\hat\xi_1 := -A^{-1}(x_1)b(x_1)/2$ by the path $[0,1] \ni t \mapsto (x(t), -A^{-1}(x(t))b(x(t))/2$. On this path, the quadratic part in (6.1) vanishes, therefore $p_0 = \tilde V(x(t)) \le E_2$ there. And finally, we connect $(x_1,\hat\xi_1)$ with (x_1,ξ_1) as in the first step such that p_0 increases on this path with maximum value at (x_1,ξ_1) still not exceeding E_2 by assumption. This shows that $p_0^{-1}(-\infty, E_2]$ is connected, contrary to our assumption.

Denote by X_{ext} the (unique) unbounded component of $\{\tilde V(x) \le E_2\}$ and let X_{int} be the union of the connected ones. The distance between X_{int} and X_{ext} is positive. Let $\chi_{\text{int}} + \chi_{\text{ext}} = 1$ be a partition of unity associated with those two closed sets, i.e., $\chi_{\text{int}} = 1$ in a neighborhood of X_{int}, and $\chi_{\text{ext}} = 1$ in a neighborhood of X_{ext}. Define

$$P_{\text{int}}(h) = P(h) + V_{\text{int}}(x), \qquad V_{\text{int}}(x) := \alpha\chi_{\text{ext}}(x),$$
$$P_{\text{ext}}(h) = P(h) + V_{\text{ext}}(x), \qquad V_{\text{ext}}(x) := \alpha\chi_{\text{int}}(x),$$

where $\alpha > E_2 - \inf \tilde V$. Then E_1 and E_2 are not critical values for neither symbol $p_i = p_0(x,\xi) + V_i(x)$, $i = int, ext$, and $p_{\text{int}}^{-1}[E_1,E_2] = W_{\text{int}}$, $p_{\text{ext}}^{-1}[E_1,E_2] = W_{\text{ext}}$. Moreover, $P_{\text{int}}(h)$ and $P_{\text{ext}}(h)$ are selfadjoint, $P_{\text{int}}(h)$ has discrete spectrum in $[E_1,E_2]$, while $P_{\text{ext}}(h)$ is non-trapping for energy levels in $[E_1,E_2]$.

To prove (a), note that the upper bound there follows from Theorem 5.1. It remains to prove the lower bound. To this end, we will use the eigenfunctions of $P_{\text{int}}(h)$ as quasimodes of $P(h)$. Let $v_j(h)$, $j = 1,\dots,m(h)$ be a full system of orthonormal eigenfunctions of $P_{\text{int}}(h)$ corresponding to all eigenvalues $e_j(h)$ of $P_{\text{int}}(h)$ in $[E_1+\delta(h), E_2-\delta(h)]$, where $0 < \delta = O(h^K)$, $K \gg 1$, will be chosen later. For small h, we have an asymptotics for $m(h)$ as in (b) because in intervals of length $\delta(h)$ near a non-critical energy, there are only $O(h^{1-n})$ resonances. Next, P_{int} is elliptic outside W_{int} and in particular, for x outside X_{int}. Therefore, $v_j = O(h^\infty)$ outside a neighborhood of X_{int} in each H^s norm uniformly in j. Therefore, if χ_{int} is as above, then $w_j := \chi_{\text{int}}v_j$ form quasimodes for $P(h)$, i.e., $(P(h)-e_j(h))w_j = O(h^\infty)$, $(w_i,w_j) = \delta_{ij} + O(h^\infty)$, and $\operatorname{supp} w_j \subset B(0,R)$ for any j with some $R > 0$. By [St1, Theorem 1], for $0 < h \ll 1$, $P(h)$ has at least $m(h)$ resonances in $\Omega_\delta(h) := [E_1 - \delta(h) + h^K, E_2 + \delta(h) + h^K] + i[-S(h), 0]$ with some positive $S(h) = O(h^\infty)$ and $K \ge 1$. Choose now $\delta(h)$ so that $\delta(h) = h^K$, then we have

$$N(\Omega(h)) \ge \frac{1}{(2\pi h)^n}\left(\text{Vol}(W_{\text{int}}) + O(h)\right), \quad \Omega(h) := [E_1,E_2] + i[-S(h),0]. \tag{6.2}$$

This proves (a).

To prove the first part of (b), fix $\Omega(h)$ as in (6.2) with some $0 < S(h) = O(h^\infty)$. We will first prove an upper bound for $N(\Omega)$ with remainder $O(h)$. Observe that $P_{\text{int}}(h)$ is a reference operator for $P(h)$ with discrepancy $O(h^\infty)$. Indeed, let $\Omega_k(h)$ be as in (3.32). Since $c(h) = O(h^\infty)$ in our case, we get that the error in Theorem 3.1, applied to each Ω_k, is $\epsilon(h) = O(h^\infty)$. Given $f_k(h) \in \text{Ran}\Pi_{\Omega_k}$ with $\|f_k\| = 1$, we have $\text{WF}(f_k) \subset W_{\text{int}}$ (uniformly in k) by Theorem 4.1, thus $(P_{\text{int}}(h) - z_k(h))f_k = O(h^\infty)$, where $z_k(h) \in \Omega_k(h) \cap \mathbf{R}$. According to Definition 3.1, this means that $P_{\text{int}}(h)$ is a reference operator in $\Omega(h)$ with $\delta(h) = O(h^\infty)$. By Theorem 3.2, for $h \ll 1$, $N(\Omega(h)) \le N^\#([E_1-\delta_1(h), E_2+\delta_1(h)]) \le N^\#([E_1,E_2]) + O(h^{1-n})$, with $\delta_1(h) = \delta^{1/2}(h) = O(h^\infty)$, where $N^\#$ is the counting function of the eigenvalues of $P_{\text{int}}(h)$. Here we used the fact that the number of eigenvalues of $P_{\text{int}}(h)$ in an interval of length $O(h)$ is $O(h^{1-n})$ [DSj]. Using (5.2) and the remark after it, we get

$$N(\Omega(h)) \le \frac{1}{(2\pi h)^n}\left(\text{Vol}(W_{\text{int}}) + O(h)\right), \tag{6.3}$$

where the function $S(h)$ that defines $\Omega(h)$ is any function with the property $S(h) = O(h^\infty)$. If we denote by $S_0(h)$ the function $S(h)$ for which (6.2) holds, then both (6.2) and (6.3) are true for any $S(h)$ with $0 < S_0(h) \le S(h) = O(h^\infty)$.

To prove the second part of (b), fix $M > 0$ and assume that $z(h)$ is a resonance in the domain $[E_1, E_2]+i[-Mh, 0]$. Then there exists an outgoing $u(h)$ belonging locally to the domain of $P(h)$ such that $(P(h) - z(h))u(h) = 0$. Let us normalize $u(h)$ by requiring that $\|u\|_{L^2(B(0,R))} = 1$ with a fixed $R > R_0$. Notice that $P(h) - z(h)$ is elliptic for $x \notin X_{\rm int} \cup X_{\rm ext}$. This yields $\mathrm{WF}(u|_{B(0,R_0)}) \subset T^*(X_{\rm int} \cup X_{\rm ext})$. To prove the latter, choose a cut-off function $0 \le \chi(x) \le 1$ equal to 1 in a neighborhood of $B(0, R_0)$ and having support in $B(0, R)$. Then $(P(h) - z(h))\chi u(h) = w(h)$, where $w(h) = [P(h), \chi]u(h)$ is supported in $B(0, R) \setminus B(0, R_0)$ and $\|\chi u(h)\| \le 1$. Now, $(P(h) - z(h))\chi u(h) = 0$ in a neighborhood of $\mathbf{R}^n \setminus (X_{\rm int} \cap X_{\rm ext}) \subset B(0, R_0)$, moreover $P(h) - z(h)$ is elliptic in $\mathbf{R}^n \setminus (X_{\rm int} \cap X_{\rm ext})$, and therefore for the wave front set of the compactly supported $\chi u(h)$ we get $\mathrm{WF}(\chi u) \subset T^*(X_{\rm int} \cup X_{\rm ext})$. This implies the same for $\mathrm{WF}(u|_{B(0,R_0)})$.

Choose the smooth cut-off function $\chi'_{\rm ext}$ so that $\chi_{\rm ext} = 1$ on supp $\chi'_{\rm ext}$ and $\chi'_{\rm ext} = 1$ in a neighborhood of $X_{\rm ext}$. Then $(P(h) - z(h))\chi'_{\rm ext}u(h) = v(h)$, where $v = [P(h), \chi'_{\rm ext}]u(h) = O(h^\infty)$ is supported in $B(0, R)$ and $\mathrm{WF}(\chi'_{\rm ext}u) \subset T^*X_{\rm ext}$. Then $(P(h)-z(h))\chi'_{\rm ext}u(h) = (P_{\rm ext}(h)-z(h))\chi'_{\rm ext}u(h) = v(h)$. Since $\chi'_{\rm ext}u(h) = u(h)$ for large $|x|$, we get that $\chi'_{\rm ext}u(h)$ is $z(h)$-outgoing. Therefore, $\chi'_{\rm ext}u(h) = R_{\rm ext}(z(h), h)v(h)$, where $R_{\rm ext}(z, h)$ is the outgoing resolvent of $P_{\rm ext}(h)$. Since $P_{\rm ext}$ is non-trapping for energy levels between E_1 and E_2, by [B3, Theorem 2], $\|\chi'_{\rm ext}u(h)\|_{L^2(B(0,R))} \le (C/h)\|v(h)\| = O(h^\infty)$. By the ellipticity of $P(h)$ we have similar estimate for the H^2 norm of $u(h)$ near $\partial B(0, R)$ and by the trace theorem, the H^1 norm of $u(h)$ on $\partial B(0, R)$ is $O(h^\infty)$ as well. An application of the Green's formula in the ball $B(0, R)$ then yields $-\mathrm{Im}\, z = O(h^\infty)$. This proves part (b) of the theorem. □

An example of operator satisfying the assumptions above is the Schrödinger operator $P(h) = -h^2\Delta+V(x)$, where $V(x)$ has strong local minimum, or more generally, if $\{x;\ V(x) \le E_2\}$ is not connected. In this case, the construction of the quasimodes above yields exponentially small error of the kind $e^{-d/h}$, for any d less than the Agmon distance between $X_{\rm int}$ and $X_{\rm ext}$. Therefore, there are resonances with exponentially small imaginary part with asymptotic number as in (a). Our proof does not exclude the existence of other resonances in the strip $[E_1, E_2]+i[-S(h), -e^{-d/h}]$ with some $S(h) = O(h^\infty)$ but their number does not exceed $O(h^{1-n})$. This case has been studied in much more detail in [HSj] under some analyticity assumptions on V where other precise results are obtained. In particular, it is shown in [HSj] that the resonances exponentially close to the real line are exponentially small perturbations of the eigenvalues of certain self-adjoint operator.

We would like to note also that the condition that $P(h)$ is a compact perturbation of the Laplacian is imposed in order to ensure the estimate $\|\chi R(z, h)\chi\| \le C_M/h$ for any such non-trapping $P(h)$ for energy levels between E_1 and E_2, and for $z \in [E_1, E_2] + i[-Mh, 0]$, $\forall M > 0$. As M. Zworski pointed out, such estimate for the long-range Schrödinger operator $P(h) = -h^2\Delta+V(x)$ is implicit in a recent work [Ma] by A. Martinez and then this assumption can be removed in this case.

The existence of the resonance free zone in (b) together with the results in [St3] makes it possible to get polynomial estimates on the spectral projectors Π_{Ω_k} as in Proposition 3.4 acting on the whole space $\mathcal{H}$ rather than on a space spanned by resonant states. This implies a resonance expansion of the solution of the corresponding wave equation as in [TZ2] and [St3]. Moreover, the wave front set of the spectral projectors are included in the trapped set and that gives us good control over the terms in that expansion.

7 Sharp upper bounds in the classical case

In this section we prove a result similar to Theorem 6.1 in the classical case. Let $X \subset \mathbf{R}^n$ be a domain with smooth boundary and compact complement $\mathcal{O}$. Let

$$P = \sum_{i,j=1}^n D_{x_i} a_{ij}(x) D_{x_j} + \sum_{j=1}^n b_j(x) D_{x_j} + V(x) \tag{7.1}$$

be a formally symmetric elliptic differential operator with $C^\infty(\bar{X})$ coefficients having the same properties as those of $P(h)$ in (4.1). For simplicity, assume that $P = -\Delta$ for $|x| > R_0$. We study the resonances λ of P near the real line.

Denote by P again the selfadjoint realization of P in $L^2(X)$ with Dirichlet boundary conditions on ∂X. We study the resonances λ of P near the real axis.

Define the *generalized bicharacteristic flow* of P as in [MeSj1], [MeSj2] (see also [H]). Recall that in the interior T^*X the generalized bicharacteristics are the integral curves of the Hamiltonian $p_0(x,\xi) = \sum_{ij} a_{ij}\xi_i\xi_j$. We assume that the bicharacteristics of P cannot be tangent to the boundary of infinite order. Under this assumption, any generalized bicharacteristics is uniquely determined by any of its points. Define the trapped subset $\mathcal{T}$ of T^*X as the complement of the set of all $\zeta \in T^*X$, for which any generalized bicharacteristic passing through ζ leaves $B(0,R_0)\times\mathbf{R}^n$ for either $t>0$ or $t<0$. Fix a decreasing function $0<S(r)=O(r^{-\infty})$, as $r\to\infty$. Set

$$\Omega(r) := \{\lambda\in\mathbf{C};\ 1\le \operatorname{Re}\lambda\le r,\ 0<-\operatorname{Im}\lambda< S(\operatorname{Re}\lambda)\}. \tag{7.2}$$

The main result in this section is the following.

Theorem 7.1 *Let P be the operator (7.1) and $\Omega(r)$ be as in (7.2). Then*

$$N(\Omega(r)) \le \frac{r^n}{(2\pi)^n}(\operatorname{meas}(\mathcal{T}\cap B^*X)+o(1)), \quad \text{as } r\to\infty,$$

where $B^*X = \{(x,\xi)\in T^*X;\ p_0(x,\xi)\le 1\}$.

Before proceeding with the proof of Theorem 7.1, we would like to give an example of a system with trapped set of positive measure. Let $P=-\Delta$ in the exterior of a bounded obstacle with smooth boundary and assume that there exists an elliptic periodic ray satisfying some mild degeneracy conditions (see [Po1]). Then it is known that for some $S(r)=O(r^{-\infty})$, $N(\Omega(r))$ admits a lower bound of the kind $cr^n(1+o(1))$. The constant c there is positive and is proportional to the measure of the invariant tori around the elliptic ray, which existence is guaranteed by the KAM theory. This constant can also be chosen to be $c=(2\pi)^{-n}\operatorname{meas}(\mathcal{T}_0\cap B^*X)$, where $\mathcal{T}_0\subset\mathcal{T}$ is a Cantor set of trapped rays near the periodic elliptic ray. There is no hope that $\operatorname{meas}(\mathcal{T}_0\cap B^*X)=\operatorname{meas}(\mathcal{T}\cap B^*X)$ because $\mathcal{T}_0$ is (a part of) the trapped rays that are close enough to a single periodic ray, while $\mathcal{T}$ is the set of all trapped rays. Nevertheless, this gives us a two-side estimate with different constants in the principal terms that have the same nature.

We start with a propagation of singularities result in the spirit of that in [MeSj1], [MeSj2]. We apply arguments similar to those in [Le] in order to derive the semiclassical version from the classical one. Since we are interested in operators having trapped set of positive measure, the behavior in a small neighborhood of the boundary is not important for our analysis. In the next proposition the wave front set of a tempered f is considered in the open set X, and we do not need to work with the more general $\mathrm{WF_b}(f)$. Here it is more convenient to work with ΨDOs with large parameter λ. Those operators are the same as the semiclassical ΨDOs with $h=1/\lambda$.

Let

$$\Sigma := \{(x,\xi)\in T^*X;\ p_0(x,\xi)=1\}$$

be the characteristic variety of $P-\lambda^2$. Given $f=f(x,\lambda_j)$, where $0<\lambda_j\to\infty$ we will denote $f=f(x,\lambda):=f_j$ for $\lambda\in\Lambda=\{\lambda_j\}$.

Proposition 7.1 *Let $f_j\in\mathcal{D}(P)$, $j=1,2,\dots$ be supported in $B(0,R)$, $R>R_0$, and let $0<\lambda_j\to\infty$, as $j\to\infty$. Let $(P-\lambda_j^2)f_j=O(\lambda_j^{-\infty})$ as $j\to\infty$, and $\|f_j\|=1$. Then* $\mathrm{WF}(f)\subset\mathcal{T}\cap\Sigma$.

Proof. Fix $(x_0,\xi_0)\in T^*X\setminus(\mathcal{T}\cap\Sigma)$. Let $\chi_1\in C_0^\infty(\mathbf{R}^n)$ be such that $\chi_1(x)=1$ near x_0 and $\operatorname{supp}\chi_1\subset X$ and let $\chi_2\in C_0^\infty(\mathbf{R}^n)$ be such that $\chi_2(\xi)=0$ for $|\xi|\le\delta/4$, $\chi_2(\xi)=1$ for $|\xi|\ge\delta/2$, where $0<\delta=\min\{|\xi|;\ (x,\xi)\in\Sigma\}$. Choose $q_0(x,\xi)$ homogeneous of order 0 with respect to ξ such that $q=1$ in a small conic neighborhood of (x_0,ξ_0) and $\operatorname{supp} q_0\cap\mathcal{T}=\emptyset$. Set

$$q(x,y,\xi)=\chi_1(x)q_0(x,\xi)\chi_2(\xi)\chi_1(y)$$

and let $Q=q(x,y,D)$ be the classical ΨDO with amplitude $q(x,y,\xi)$. We would like to express Q as a λ-ΨDO. In order to avoid the problem with the singularity of q_0 at $\xi=0$, for $\lambda>2$ write

$$q(x,y,\xi)=\chi_1(x)q_0(x,\xi)\chi_2(\xi)\big(1-\chi_2(\xi/\lambda)\big)\chi_1(y)+\chi_1(x)q_0(x,\xi)\chi_2(\xi/\lambda)\chi_1(y) := q_1+q_2.$$

The classical ΨDO $Q_1 := q_1(x, y, D)$, depending on λ, can be expressd as a composition of the bounded operator $Q_{11} = \chi_1(x)q_0(x, D)\chi_2(D)$, independent of λ and the λ-ΨDO Q_{12} with amplitude (as an λ-ΨDO) equal to $(1 - \chi_2(\xi))\chi_1(x)$. We have $Q_{12}f = O(\lambda^{-\infty})$, because $\mathrm{WF}(f) \subset \Sigma$, and therefore, $Q_1 f = O(\lambda^{-\infty})$. Next, the classical ΨDO Q_2 with amplitude q_2 can be expressed as a λ-ΨDO with amplitude $\chi_1 q_0(x, \xi)\chi_2(\xi)\chi_1(y)$. Our goal is to prove that $Q_2 f = O(\lambda^{-\infty})$.

Assume that the estimate above is not true. Then there exist $N > 0$ and a subsequence Λ' such that

$$\|\lambda^N Qf\| \to \infty, \quad \text{as } \Lambda' \ni \lambda \to \infty. \tag{7.3}$$

Without loss of generality we may assume that $2\lambda_j < \lambda_{j+1}$ for any 2 consecutive numbers in Λ'. Set

$$u(t, x) = \sum_{\lambda\in\Lambda'} e^{-i\lambda t}\lambda^{-1} f(x, \lambda).$$

Note that the series above is absolutely convergent. We have $(\partial_t^2 + P)u \in C^\infty(\mathbf{R} \times \bar{X})$. Since $u = 0$ for large $|x|$, by the classical propagation of singularities results for boundary value problems [MeSj1], [MeSj2], we get that the classical wave front set of u in (the interior of) $T^*(\mathbf{R} \times X)$ is contained in $\{(t, x, \tau, \xi);\ \tau^2 = p_0(x, \xi),\ (x, \xi) \in \mathcal{T}\}$. Choose $\phi \in C_0^\infty(\mathbf{R})$. Then $\phi(t)Qu \in C_0^\infty(\mathbf{R} \times \mathbf{R}^n)$ and

$$\phi(t)(Qu)(t, x) = \sum_{\lambda\in\Lambda'} e^{i\lambda t}\phi(t)\lambda^{-1} Qf(x, \lambda) = \mathcal{F}^*_{\mu\to t} \sum_{\lambda\in\Lambda'} \hat{\phi}(\mu - \lambda)\lambda^{-1} Qf(x, \lambda)$$

We get therefore that $\sum_{\lambda\in\Lambda'} \hat{\phi}(\mu - \lambda)\lambda^{-1} Qf(x, \lambda)$ is in the Schwartz class with respect to (μ, x) and in particular

$$\|\sum_{\lambda\in\Lambda'} \hat{\phi}(\mu - \lambda)\lambda^{-1} Qf(x, \lambda)\| = O(\mu^{-\infty}), \quad \text{as } \mu \to \infty.$$

We will split the sum above into two sums: one for $|\mu - \lambda| \le \mu/2$ and another for $|\mu - \lambda| > \mu/2$. In the second case, $|\hat{\phi}(\mu - \lambda)| = O(\mu^{-\infty})$ there. Thus,

$$\|\sum_{\lambda\in\Lambda', |\mu-\lambda|>\mu/2} \hat{\phi}(\mu - \lambda)\lambda^{-1} Qf(x, \lambda)\| = O(\mu^{-\infty}), \quad \text{as } \mu \to \infty.$$

When summing up for $|\mu - \lambda| \le \mu/2$, we therefore get

$$\|\sum_{\lambda\in\Lambda', |\mu-\lambda|\le\mu/2} \hat{\phi}(\mu - \lambda)\lambda^{-1} Qf(x, \lambda)\| = O(\mu^{-\infty}), \quad \text{as } \mu \to \infty.$$

For $j = 1, 2, \dots$, choose $\mu_j = \lambda_j$. The condition $2\lambda_j < \lambda_{j+1}$, $j = 1, 2, \dots$ implies that in the interval $|\lambda_j - \lambda| \le \lambda_j/2$ there is only one number in Λ' and that is λ_j. Therefore, $\|\hat{\phi}(0)\lambda_j^{-1} Qf(x, \lambda_j)\| = O(\lambda_j^{-\infty})$. One can always assume that $\hat{\phi}(0) = 1$, thus $Qf = O(\lambda^{-\infty})$, $\lambda \in \Lambda'$, contrary to (7.3). This implies that $Qf = O(\lambda^{-\infty})$, $\lambda \in \Lambda$, and therefore, $Q_2 f = O(\lambda^{-\infty})$ for $\lambda \in \Lambda$, and this completes the proof of the proposition. □

Given $r \gg 1$, set $h = 1/r$ and define $P(h) = h^2 P$. With some abuse of notation, in this section P will denote the h-independent operator (7.1), while $P(h)$ will be the operator we just defined. Any semi-classical resonance $z(h)$ is related to a classical one λ with $\mathrm{Im}\,\lambda < 0$ by the formula

$$\lambda^2 = h^{-2} z(h) = r^2 z(h). \tag{7.4}$$

Fix a small paramer $a \in (0, 1]$. First, we are going to estimate the number of resonances λ in

$$\Omega_a(r) = \{\lambda \in \mathbf{C};\ ar \le \mathrm{Re}\,\lambda \le r,\ 0 < -\mathrm{Im}\,\lambda < S(\mathrm{Re}\,\lambda)\}.$$

The image of $\Omega_a(r)$ under the map (7.4) is a curved "box" with vertices a^2, 1, $a^2 - h^2aS^2(ah^{-1}) - 2ihaS(ah^{-1})$, and $1 - h^2S^2(h^{-1}) - 2ihS(h^{-1})$. It is included in

$$\Omega(h) = [a^2/2, 1] + i[-\tilde S(h), 0], \quad \tilde S(h) := 2hS(ah^{-1}) = O(h^\infty).$$

We are going to prove first that

$$N(\Omega(h)) \le \frac{1}{(2\pi h)^n}\left(\text{meas}\{p_0^{-1}(-\infty,1] \cap \mathcal{T}\} + o(1)\right), \quad \text{as } h \to 0. \tag{7.5}$$

Here $p_0^{-1}(-\infty,1] = B^*X$. Note that without loss of generality we may assume that $\tilde S(h) \ge 2e^{-h^{-1/3}}$, so that (3.31) is satisfied.

Given $\epsilon > 0$, denote

$$X_\epsilon^+ = \{x \in \mathbf{R}^n;\ \text{dist}(x, X) < \epsilon\}, \quad X_\epsilon^- = \{x \in X;\ \text{dist}(x, \partial X) > \epsilon\}.$$

Then $X_\epsilon^- \subset X \subset X_\epsilon^+$. For $0 < \epsilon \ll 1$, $\partial X_\epsilon^\pm$ is smooth. Let us extend the coefficients of P in a smooth way outside X by keeping P self-adjoint and elliptic such that for the extension $\tilde P(h)$ we have $\tilde P(h)|_{\mathbf{R}^n\setminus X_{\epsilon_0}^+} = -h^2\Delta$. Here $\epsilon_0 > 0$ is fixed and in what follows we assume that $0 < \epsilon \le \epsilon_0$. This extension can be constructed as follows. First we extend the coefficients of P in a smooth way near X and we choose $\epsilon_0 > 0$ so small that in $X_{\epsilon_0}^+$ our operator is still elliptic. Next, we choose a smooth partition of unity $\chi_1^2 + \chi_2^2 = 1$ such that $\chi_1(x) = 1$ in $X_{\epsilon_0/2}^+$ and $\chi_1(x) = 0$ outside $X_{\epsilon_0}^+$, and set $\tilde P(h) = \chi_1 P(h)\chi_1 + \chi_2(-h^2\Delta)\chi_2$. Then the so extended $\tilde P(h)$ is elliptic self-adjoint operator in $\mathbf{R}^n$ with principal symbol $\tilde p_0 = \sum \tilde a_{ij}\xi_i\xi_j$, where $\tilde a_{ij}$ are the a_{ij} extended outside X. Clearly, every energy level $E > 0$ is non-critical for $\tilde p_0$. Our goal next is to construct a reference operator $P^\#(h)$ in the whole space that would give us a sharp bound. Similarly to (5.1), define

$$\mathcal{T}^\epsilon = \overline{T^*X_\epsilon^-} \cap \mathcal{T} \cap \tilde p_0^{-1}[a^2/2, 1], \quad \mathcal{T}_\mu^\epsilon = \{\zeta \in T^*\mathbf{R}^n;\ \text{dist}\{\zeta, \mathcal{T}^\epsilon\} < \mu\}.$$

We will define $q_\mu(x,\xi)$ in a way similar to that in section 5 by taking extra care of the behavior near the boundary. For $0 < \mu \ll 1$, choose $q_\mu \in C^\infty(T^*\mathbf{R}^n)$ so that $0 \le q_\mu \le 2$, and

$$q_\mu(x,\xi) = \begin{cases} 0 & \text{for } (x,\xi) \in \mathcal{T}_\mu^\epsilon, \\ 2 & \text{for } (x,\xi) \notin \mathcal{T}_{2\mu}^\epsilon. \end{cases}$$

We may also assume that q_μ is homogeneous in ξ of order 2 for $a^2/2 - \mu < |\xi| < 1 + \mu$, which would guarantee that any $E \in [a^2, 1]$ is a non-critical value for the principal symbol p_μ defined below. Let $0 \le V_0 \in C_0^\infty(\mathbf{R}^n)$ be such that $V_0 = 0$ in $B(0,R)$ and $V_0(x) = |x|^2$ for $|x|$ large enough. Set

$$P_\mu^\#(h) = \tilde P(h) + q_\mu^w(x, hD) + V_0(x).$$

The operator $P_\mu^\#$ is self-adjoint in $L^2(\mathbf{R}^n)$ and its principal symbol p_μ is $p_\mu = \tilde p_0 + q_\mu + V$. The spectrum of $P_\mu^\#(h)$ is discrete. Here V is added for convenience in the considerations below and is not really necessary, without it the spectrum of $P_\mu^\#(h)$ would be discrete in $(-\infty, 2)$ and that would be enough. Notice that $P_\mu^\#$ is not a reference operator in the sense of Definition 3.1 because applying $P_\mu^\#(h)$ to the resonant states of $P(h)$ would produce delta functions on the boundary ∂X. We can go around this problem if we consider the quadratic form $(P_\mu^\#(h)\psi, \psi)$, where $\psi = \psi(h)$ is related to $f = f(h) \in \text{Ran}\Pi_{\Omega(h)}$ as in (3.38) and ψ is extended as 0 outside X. Then we consider $(P_\mu^\#(h)\psi,\psi)_{L^2(\mathbf{R}^n)}$ as a quadratic form with domain larger than the domain of $P_\mu^\#(h)$. In particular, the functions ψ do not belong to the domain of $P_\mu^\#(h)$ because the normal derivative of ψ jumps at ∂X but they belong to the domain of the quadratic.

According to Proposition 7.1, $(q_\mu^w(x,hD) + V_0(x))\psi(h) = O(h^\infty)$ for any normalized $\psi \in \chi_B\text{Ran}\Pi_{\Omega_k}$, where the subdomains $\Omega_k(h)$ are as in (3.32). As in the proof of Theorem 4.1, this implies the same for any $\psi \in \chi_B\text{Ran}\Pi_{\Omega(h)}$ with $\|\psi\| = 1$.

We will show that

$$(P_\mu^\#(h)\psi,\psi)_{L^2(\mathbf{R}^n)} = (P(h)\psi,\psi)_{L^2(X)} + O(h^\infty), \quad \forall\psi\in\chi_B \mathrm{Ran}\Pi_{\Omega(h)},\ \|\psi\|=1, \tag{7.6}$$

where the l.h.s. is understood in the sense of quadratic forms. To prove (7.6), write

$$\begin{aligned}(P(h)\psi,\psi)_{L^2(X)} &= \int_X \Big(\sum_{ij} a_{ij}(hD_{x_i}\psi)(hD_{x_j}\bar\psi) + \sum_j b_j(hD_{x_j}\psi)\bar\psi + V(x)|\psi|^2\Big)dx\\ &= \int_{\mathbf{R}^n} \Big(\sum_{ij} \tilde a_{ij}(hD_{x_i}\psi)(hD_{x_j}\bar\psi) + \sum_j \tilde b_j(hD_{x_j}\psi)\bar\psi + \tilde V(x)|\psi|^2\Big)dx\\ &= (P_\mu^\#(h)\psi,\psi)_{L^2(\mathbf{R}^n)} - (q_\mu^w(x,hD)\psi,\psi)_{L^2(\mathbf{R}^n)} - (V_0\psi,\psi)_{L^2(\mathbf{R}^n)}\\ &= (P_\mu^\#(h)\psi,\psi)_{L^2(\mathbf{R}^n)} + O(h^\infty),\end{aligned}$$

where ψ, as explained above, is extended as 0 outside X.

We claim that

$$(P(h)\psi,\psi)_{L^2(X)} \le 1 + O(h^\infty), \quad \forall\psi\in\chi_B\mathrm{Ran}\Pi_{\Omega(h)},\ \|\psi\|=1. \tag{7.7}$$

To prove (7.7), we use Theorem 3.2(b). By choosing $\delta_1(h)=O(h^\infty)$ in a suitable way, we get that for any f as above, the corresponding ψ can be written as $\psi=\sum_{jk} v_{jk}+\psi_\infty$, where v_{jk} are eigenfunctions of the self-adjoint reference operator $P_0^\#(h) := P(h)+V_0(x)$ in $L^2(X)$ with eigenvalues $z_{jk}\in[a^2/2-O(h^\infty), 1+O(h^\infty)]$. For ψ_∞ we have $\|\psi_\infty\| = O(h^\infty)$ and repeating the argument in the proof of Theorem 3.2(b) based on the spectral theorem, wee see also that $\|P_0^\#(h)\psi_\infty\| = O(h^\infty)$. If among v_{jk}'s there are eigenfunctions corresponding to the same eigenvalue, we combine them as a single eigenfunction, so we may assume that v_{jk} are orthogonal to each other. Therefore,

$$\begin{aligned}(P(h)\psi,\psi)_{L^2(X)} &= (P_0^\#(h)\psi,\psi) = \Big(P_0^\#(h)\sum v_{jk}, \sum v_{jk}\Big) + O(h^\infty)\\ &= \sum z_{jk}\|v_{jk}\|^2 + O(h^\infty) \le (1+O(h^\infty))\|\psi\|^2 + O(h^\infty) = 1+O(h^\infty).\end{aligned}$$

This proves (7.7).

By (7.6) and (7.7) we get that

$$(P_\mu^\#(h)\psi,\psi)_{L^2(X)} \le 1+\alpha(h), \quad \alpha(h) = O(h^\infty) \tag{7.8}$$

in the sense of quadratic forms, for any $\psi\in\chi_B\mathrm{Ran}\Pi_\Omega$. The latter subspace is of dimension equal to $N(\Omega(h))$ by Theorem 3.1. Thus we get that (7.8) holds for any normalized ψ belonging to a subspace W of dimension $N(\Omega(h))$ included in the domain of the quadratic form in (7.8). We will show that this implies that

$$N(\Omega(h)) \le N^\#((-\infty, 1+\alpha(h)]) + O(h^{1-n}). \tag{7.9}$$

To prove (7.9), consider all eigenvalues of $P_\mu^\#$ not exceeding $1+\alpha(h)$, counted according to their multiplicities. If $1+\alpha(h)$ happens to be an eigenvalue itself, we include it according to its multiplicity, which is $O(h^{1-n})$. So the number $m(h)$ of all those eigenvalues admits an estimate like the r.h.s. of (7.9). Denote by W_0 the subspace of W that is orthogonal to the space spanned by the eigenfunctions corresponding to the $m(h)$ eigenvalues above. Assume that $\dim W = N(\Omega(h)) > m(h)$. Then W_0 is non-trivial and therefore, there exists $f\in W_0$ with $\|f\|=1$. By expanding f in terms of the normalized eigenfunctions v_j of the reference operator, we get $f=\sum_{z_j>m(h)} f_j v_j$, where $\sum z_j|f_j|^2<\infty$ (only finite number of z_j's can be negative), because f belongs to the domain of the quadratic form related to $P_\mu^\#$. We therefore get that $(P_\mu^\# f,f)_{L^2(X)} \ge z_{m(h)+1}\sum|f_j|^2 > 1+\alpha(h)$ and this contradicts (7.8). This shows that $\dim W = N(\Omega(h)) \le m(h)$, and this proves (7.9).

Next, using the fact that $E=1$ is non-critical for p_μ, as in section 5, we get that

$$N^\#((-\infty,1+\alpha(h)]) \le \frac{1}{(2\pi h)^n}\left(\mathrm{Vol}(p_\mu^{-1}(-\infty,1]) + o(h)\right), \quad \text{as } h\to 0.$$

Again as in section 5, after taking the limit $\mu \to 0$, we deduct that from this that

$$\limsup_{h\searrow 0}(2\pi h)^n N^\#((-\infty, 1+\alpha(h)]) \leq \left(\mathrm{meas}(\mathcal{T}^\epsilon \cap p_0^{-1}(-\infty, 1]) + o(1)\right), \quad \text{as } h \to 0.$$

Now we can take the limit $\epsilon \searrow 0$, and combining this with (7.9), we complete the proof of (7.5).

Using (7.5), we get that

$$N(\Omega_a(r)) \leq \frac{r^n}{(2\pi)^n}\left(\mathrm{meas}(\mathcal{T} \cap B^*X) + o(1)\right), \quad \text{as } r \to \infty. \tag{7.10}$$

With some abuse of notation, we denote by $N(\Omega)$ both the number of semiclassical and classical resonances for $\Omega = \Omega(h)$ and $\Omega = \Omega(r)$, respectively. It is easy to see that

$$N(\Omega'_a(r)) \leq C(1+ar)^n, \quad \text{where } \Omega'_a(r) := \{\lambda \in \mathbf{C};\ 1 \leq \mathrm{Re}\,\lambda < ar,\ 0 < -\mathrm{Im}\,\lambda < S(\mathrm{Re}\,\lambda)\} \tag{7.11}$$

with $C > 0$ independent of $a > 0$. This follows for example from the estimate of the number of resonances in a ball of radius r of the type (2.3) by observing that $N(\Omega'_a(r))$ above depends on r through ar only for $r \gg 1$. Then $\Omega(r) = \Omega'_a(r) \cup \Omega_a(r)$ (see (7.2)). Combining (7.10) and (7.11) together, we get that

$$N(\Omega(r)) \leq \frac{r^n}{(2\pi)^n}(A + o(1)), \quad \text{as } r \to \infty$$

for any $A > \mathrm{meas}(\mathcal{T} \cap B^*X)$. By studying $\limsup r^{-n} N(\Omega(r))$, we complete the proof of the theorem. □

8 Generalizations for general long range operators

As explained in the Introduction, we allow only the zeroth order term $V(x)$ of $P(h)$ to be long range only to simplify the exposition. One can study general long range $P(h)$ in the black box setting by requiring the coefficients of $P(h)$ outside the black box to satisfy analyticity assumptions and estimates of the type (2.1), see e.g. [B2]. Then resonances are well defined in a sector near the real axis and in particular in and near $\Omega(h)$ as shown in [Sj2]. The necessary modifications in the proofs are as follows. To prove Proposition 3.1, one needs to pass to global geodesics coordinates as done in [B2]. The absorption estimate in Proposition 3.1 then holds and the proof is the same as in Proposition 3.1, where the new terms that appear are estimated as in [B2, Proposition 7.1]. The choice of the constant B_0 then depends on the rate of decay of all coefficients outside the black box, not only on V. In estimate (3.42) those extra terms do not create additional difficulties. All results in sections 3, 4, 5 remain the same. Except for the second part of Theorem 6.1 (see also the remark at the end of section 6), $P(h)$ can be general long range operator in section 6 as well. The results in section 7 hold for long range operators as well.

References

[B1] N. BURQ, *Décroissance de lénergie locale de l'équation des ondes pour le problème extérieur et absence de résonance au voisinage du réel*, Acta Math. **180**(1998), 1–29.

[B2] N. BURQ, *Lower bounds for shape resonances widths of long range Schrödinger operators*, preprint.

[B3] N. BURQ, *Semi-classical estimates for the resolvent in non-trapping geometries*, Int. Math. Res. Not. **5**(2002), 221–241.

[BZ] N. BURQ AND M. ZWORSKI, *Resonance expansions in semi-classical propagation*, Comm. Math. Phys., to appear.

[CdV] Y. COLIN DE VERDIÈRE, *Quasimodes sur les variétés Riemanniennes*, Invent. Math. **43**(1977), 15–52.

[G] C. GÉRARD, *Asymptotique des poles de la matrice de scattering pour deux obstacles strictement convex*, Bull. Soc. Math. France, Mémoire n. 31, **116**, 1988.

[DSj] M. DIMASSI AND J. SJÖSTRAND, *Spectral Asymptotics in the Semi-Classical Limit*, London Math. Society Lecture Notes Series, No. 268, Cambridge Univ. Press, 1999.

[HSj] B. HELFFER, J. SJÖSTRAND, *Résonances en limite semi-classique*, Mém. Soc. Math. France (N.S.), **24–25**, 1986.

[H] L. HÖRMANDER, *The Analysis of Linear Partial Differential Operators*, Springer-Verlag, Berlin, 1985.

[I] V. IVRII, *Microlocal Analysis and Precise Spectral Asymptotics*, Springer-Verlag, Berlin, 1998.

[K] T. KATO, *Perturbation Theory for Linear Operators.* Springer-Verlag, New York, 1966.

[LP] P.D. LAX AND R. PHILLIPS, *Scattering Theory*, New York, Academic Press, 1967.

[La] V. LAZUTKIN, *KAM Theory and Semiclassical Approximations to Eigenfunctions*, Springer-Verlag, Berlin, 1993.

[Le] G. LEBEAU, *Contrôle de l'équation de Schrödinger*, J. Math. Pures Appl. **9**(1992), no. 3, 267–291.

[Li] K. LIN, *Numerical study of quantum resonances in chaotic scattering*, J. Comp. Phys. **176**(2002), no. 2.

[LiZ] K. LIN AND M. ZWORSKI, *Quantum resonances in chaotic scattering*, to appear in Chem. Phys. Lett.

[Ma] A. MARTINEZ, *Resonance free domains for non-analytic potentials*, preprint.

[Me] R. MELROSE, *Polynomial bounds on the distribution of poles in scattering by an obstacle*, Journeés "Equations aux Dérivees Partielles", Saint-Jean-de-Monts, 1984.

[MeSj1] R. Melrose and J. Sjöstrand, *Singularities of boundary value problems. I.* Comm. Pure Appl. Math. **31**(1978), no. 5, 593–617.

[MeSj2] R. Melrose and J. Sjöstrand, *Singularities of boundary value problems. II.* Comm. Pure Appl. Math. **35**(1982), no. 2, 129–168.

[N] S. NAKAMURA, *Spectral shift function for trapping energies in the semiclassical limit*, Commun. Math. Phys. **208**(1999), 173–193.

[PeZ] V. PETKOV AND M. ZWORSKI, *Breit-Wigner approximation and the distribution of resonances*, Comm. Math. Phys. **204**(2)(1999), 329–351.

[Po1] G. POPOV, *Quasi-modes for the Laplace operator and Glancing hypersurfaces*, in: Proceedings of Conference on Microlocal analysis and Nonlinear Waves, Minnesota 1989, M. Beals, R. Melrose and J. Rauch eds., Springer Verlag, Berlin-Heidelberg-New York, 1991.

[Po2] G. POPOV, *Invariant tori, effective stability and quasimodes with exponentially small error terms I, II*, Ann. Inst. H.Poincaré, **1**(2000), 223–248 and 249–279.

[PoV] G. POPOV AND G. VODEV, *Resonances near the real axis for transparent obstacles*, Comm. Math. Phys. **207**(1999), 411–438.

[Sj1] J. SJÖSTRAND, *Geometric bounds on the density of resonances for semiclassical problems*, Duke Math. J. **60**(1)(1990), 1–57.

[Sj2] J. SJÖSTRAND, *A trace formula and review of some estimates for resonances*, in Microlocal analysis and Spectral Theory (Lucca, 1996), 377–437, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 490, Kluwer Acad. Publ., Dordrecht, 1997.

[SjV] J. SJÖSTRAND AND G. VODEV, *Asymptotics of the number of Rayleigh resonances*, Math. Ann. **309**(1997), 287–306.

[SjZ] J. SJÖSTRAND AND M. ZWORSKI, *Complex scaling and the distribution of scattering poles*, Journal of AMS **4**(4)(1991), 729–769.

[St1] P. STEFANOV, *Quasimodes and resonances: sharp lower bounds*, Duke Math. J. **99**(1999), 75–92.

[St2] P. STEFANOV, *Lower bound of the number of the Rayleigh resonances for arbitrary body*, Indiana Univ. Math. J. **49**(2)(2000), 405–426.

[St3] P. STEFANOV, *Resonance expansions and Rayleigh waves*, Math. Res. Lett., **8**(1–2)(2001), 105–124.

[St4] P. STEFANOV, *Resonances near the real axis imply existence of quasimodes*, C. R. Acad. Sci. Paris, Série I, **330**(2)(2000), 105-108.

[StV1] P. STEFANOV AND G. VODEV, *Distribution of resonances for the Neumann problem in linear elasticity outside a strictly convex body*, Duke Math. J. **78**(1995), 677–714.

[StV2] P. STEFANOV AND G. VODEV, *Neumann resonances in linear elasticity for an arbitrary body*, Comm. Math. Phys. **176**(1996), 645–659.

[TZ1] S.-H. TANG AND M. ZWORSKI, *From quasimodes to resonances*, Math. Res. Lett., **5**(1998), 261–272.

[TZ2] S.-H. TANG AND M. ZWORSKI, *Resonance expansions of scattered waves*, Comm. Pure Appl. Math. **53**(10)(2000), 1305–1334.

[T] M. TAYLOR, *Rayleigh waves in linear elasticity as a propagation of singularities phenomenon*, in Proc. Conf. on P.D.E. and Geometry, Marcel Dekker, New York, 1979, 273–291.

[V] G. VODEV, *Sharp polynomial bounds on the number of scattering poles for perturbations of the Laplacian , Comm. Math. Phys.* **146**(1992), 39–49.

[Ze] M. ZERZERI, *Majoration du nombre des résonances près de l'axe reél pour une perturbation abstraite à support compact, du laplacien*, Comm. PDE **26**(2001), no. 11-12, 2121–2188.

[Z1] M. ZWORSKI, *Sharp polynomial bounds on the number of scattering poles*, Duke Math. J. **59**(1989), 311–323.

[Z2] M. ZWORSKI, private communication, 1992.

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