

Convergence of trapezoid rule to rough integrals

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Abstract. Rough paths techniques give the ability to define solutions of stochastic differential equations driven by signals X which are not semimartingales and whose p -variation is finite only for large values of p . In this context, rough integrals are usually Riemann–Stieltjes integrals with correction terms that are sometimes seen as unnatural. As opposed to those somewhat artificial correction terms, our endeavor in this note is to produce a trapezoid rule for rough integrals driven by general d -dimensional Gaussian processes. Namely we shall approximate a generic rough integral $\int y dX$ by Riemann sums avoiding the usual higher order correction terms, making the expression easier to work with and more natural. Our approximations apply to all controlled processes y and to a wide range of Gaussian processes X including fractional Brownian motion with a Hurst parameter $H > 1/4$. As a corollary of the trapezoid rule, we also consider the convergence of a midpoint rule for integrals of the form $\int f(X) dX$.

Résumé. La théorie des trajectoires rugueuses ouvre la possibilité de résoudre des équations différentielles stochastiques dirigées par un signal général X . Cette théorie va au-delà du cas d'une semi-martingale, et concerne un signal X dont la p -variation est finie pour p arbitrairement grand. Dans ce contexte les intégrales rugueuses sont généralement définies comme des intégrales de Riemann–Stieltjes, corrigées par des termes qui peuvent paraître non naturels. Dans la présente note nous proposons de remplacer ces termes quelque peu artificiels par une approximation trapézoïdale, dans le cas où X est un processus Gaussien d -dimensionnel général. Plus précisément, nous approchons une intégrale rugueuse de la forme $\int y dX$ par une somme de Riemann tout en évitant les termes correctifs d'ordre supérieur, ce qui rend l'expression plus facile à manipuler et plus naturelle. Nos approximations fonctionnent pour tous les processus contrôlés y , ainsi que pour une classe importante de processus Gaussiens X comprenant le mouvement brownien fractionnaire avec paramètre de Hurst $H > 1/4$. Nous énonçons aussi un résultat de convergence concernant la règle du point milieu pour des intégrales de la forme $\int f(X) dX$. Ce dernier résultat est un corollaire de notre règle trapézoïdale.

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1. Introduction

Inspired by the seminal series of papers [2,27], rough paths were first introduced in [20] in 1998 to study differential equations of the form:

$$(1.1) \quad y_t = y_0 + \sum_{j=1}^d \int_0^t V_j(y_s) dX_s^j,$$

where V_j are smooth bounded vector fields, $X : [0, T] \rightarrow \mathbb{R}^d$ is a given function with finite p -variation, usually a stochastic process, and $y : [0, T] \rightarrow \mathbb{R}^d$ is what is being solved for. Even though the final goal of the rough paths theory is to solve differential systems of the form (1.1) driven by arbitrary noisy inputs, the main step in the approach can be reduced to a proper definition of stochastic integrals like $\int_0^t V_j(y_s) dX_s^j$ above. In order to discuss this kind of integral, we consider a

generic partition $\mathcal{P} = \{0 = t_0 < t_1 < \dots < t_{n+1} = t\}$ of $[0, t]$ and the following Riemann sum:

$$(1.2) \quad \mathcal{R}_0^t(V_j(y), X^j) := \sum_{k=0}^n V_j(y_{t_k}) \delta X_{t_k t_{k+1}}^j,$$

where we define $\delta X_{st} := X_t - X_s$. In a classical setting, one obviously expects $\mathcal{R}_0^t(V_j(y), X^j)$ to converge to $\int_0^t V_j(y_s) dX_s^j$ as the mesh of \mathcal{P} goes to 0. Let us recall what this Riemann sum convergence becomes in a rougher and/or stochastic context.

- (i) When X is a semimartingale on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mu)$, we can use techniques from Itô calculus to take limits in (1.2). For example, assuming that X is a L^2 continuous martingale, then we define the stochastic integral

$$(1.3) \quad \int_0^t V_j(y_s) dX_s^j \stackrel{L^2(\Omega)}{=} \lim_{|\mathcal{P}| \rightarrow 0} \mathcal{R}_0^t(V_j(y), X^j),$$

where the limit of the Riemann sums is understood in the $L^2(\Omega)$ sense. The stochastic integral (1.3) can be extended to all processes which are adapted to the filtration \mathcal{F}_t and square integrable with respect to the bracket of X .

- (ii) When X is not a semimartingale, but is assumed to have finite p -variation for $p < 2$, one can rely on the classical Young–Stieltjes integration to take limits in (1.2). Then one invokes a result in [27] which can be summarized as follows.

Proposition 1.1. *Let $f \in C^{p\text{-var}}([0, T], \mathbb{R}^d)$ and $g \in C^{q\text{-var}}([0, T], \mathbb{R}^d)$ with $1/p + 1/q > 1$. Then the Riemann–Stieltjes integral exists:*

$$(1.4) \quad \int_0^t f_s dg_s = \lim_{|\mathcal{P}| \rightarrow 0} \mathcal{R}_0^t(f, g).$$

Proposition 1.1 can be applied directly in order to analyze (1.2). Specifically, if we assume that y should have the same p -variation as X , then we can make sense of (1.1) in Young–Stieltjes sense whenever X is a stochastic process with finite p -variation almost surely for $p < 2$. In this case we define:

$$(1.5) \quad \int_0^t V_j(y_s) dX_s^j \stackrel{\text{a.s.}}{=} \lim_{|\mathcal{P}| \rightarrow 0} \mathcal{R}_0^t(V_j(y), X^j),$$

that is $\int_0^t V_j(y_s) dX_s^j$ is defined as an almost sure limit of Riemann sums.

However, if we want to solve (1.1) beyond the semimartingale or the finite p -variation case with $p < 2$, rough paths theory is the main available path-wise type method. We give a brief introduction to this method in Section 2.1, and refer to [5,6,20] for a more comprehensive guide. At this point we just mention that rough path theory let us solve (1.1) for $X \in C^{p\text{-var}}$ and for arbitrary p , provided we can define the following stack of iterated integrals of order $n = 1, 2, \dots, \lfloor p \rfloor$:

$$(1.6) \quad \begin{aligned} X_{st}^{1,i} &= \int_s^t dX_r^i, & X_{st}^{2,ij} &= \int_s^t \int_s^r dX_u^i dX_r^j, \\ X_{st}^{3,ijk} &= \int_s^t \int_s^r \int_s^u dX_v^i dX_u^j dX_r^k, & \dots \end{aligned}$$

Notice that in this paper the analysis is restricted to the case $p < 4$, which corresponds to needing to define the first three integrals in (1.6). This is due to issues in defining these integrals for worse p -variation. As mentioned above, once the iterated integrals in (1.6) are properly defined, we can solve (1.1) thanks to the rough paths machinery whenever $X \in C^{p\text{-var}}$. In particular if y is the solution to (1.1), one can define the stochastic integral $\int y dX$ as the following limit of modified Riemann sums:

$$(1.7) \quad \int_0^t y_s dX_s = \lim_{n \rightarrow \infty} \sum_{k=0}^n y_{t_k} X_{t_k t_{k+1}}^1 + V(y_{t_k}) X_{t_k t_{k+1}}^2 + V'V(y_{t_k}) X_{t_k t_{k+1}}^3,$$

where we have considered a 1-dimensional situation (namely $d = 1$) in order to avoid cumbersome indices. Standard and relevant applications of rough paths techniques are the ability to define and solve stochastic differential equations

driven by fractional Brownian motion or other processes with low regularity that are not semimartingales. Even for equations driven by usual Brownian motions, the continuity results related to rough paths techniques bring simplifications in classical stochastic analysis results such as large deviations principles, see [17]. Other relevant applications include data analysis, see [11] and filtering theory, see [3].

Nevertheless, in spite of the rough paths theory's numerous achievements, the definition (1.7) for an integral of y with respect to X is sometimes seen as somehow not natural due to the higher order "correction" terms. In addition, the presence of the high order iterated integrals in the right hand side of (1.7) makes the rough integral approximation difficult to implement numerically. Ideally, one would thus like to take limits on simple Riemann sums like (1.2).

The natural endeavor of approximating rough (or other generalized stochastic) integrals by suitable Riemann sums has been mostly carried out in case of a 1-dimensional fractional Brownian motion B and for integrals of the form $\int f(B) dB$. The contributions in this direction includes [1,10,12,16,21–23,25]. We should also mention that the approximation of rough integrals like (1.7) by trapezoid rules, for a 1-d fractional Brownian motion and a general controlled process y , has been considered in [19]. Namely, the following approximation of a rough integral $\int y dB$ is proposed in [19]:

$$(1.8) \quad \text{tr-}\mathcal{J}_0^t(y, B) := \sum_{k=0}^n \frac{y_{t_k} + y_{t_{k+1}}}{2} \delta B_{t_k t_{k+1}},$$

where B is a one-dimensional fractional Brownian motion with Hurst parameter $H \geq 1/6$ and where y is a process whose increments are controlled by B (see Definition 2.8 below for the notion of controlled process). For $H > 1/6$ it is proven that this trapezoid rule converges to the rough integral (1.7), while in the case $H = 1/6$ an additional Brownian term pops out in the limit and the convergence holds in the weak sense only. Notice that this phenomenon had already been observed in [12,22,23] for integrands of the form $y_t = f(B_t)$ for a sufficiently smooth function f .

With those preliminary remarks in mind, the main aim of the current contribution is to extend the scope of trapezoid type approximations to rough integrals. Our generalizations will go in two directions, that is (i) we shall prove the convergence of trapezoid rules for d -dimensional Gaussian processes and (ii) we handle the case of a general class of Gaussian processes beyond the fractional Brownian case. Our prototype of convergence theorem is stated below in an informal way. The reader is referred to Theorem 3.12 for a more precise statement.

Theorem 1.2. *Let X be a centered Gaussian process on $[0, T]$ admitting a sufficiently regular covariance function R in the 2-d ρ -variation sense. Denote the rough path lift of X by $\mathbf{X} = (X^1, X^2, X^3)$. Let $\mathbf{y} = (y, y^1, y^2)$ be a process controlled by \mathbf{X} (examples of controlled processes include $y_t = f(X_t)$ and solutions of SDEs driven by \mathbf{X}). For a given partition of $[0, T]$, $\mathcal{P} = \{0 = t_0 < t_1 < \dots < t_{n+1} < T\}$, we define the trapezoidal rule:*

$$(1.9) \quad \text{tr-}\mathcal{J}_0^T(y, X) = \sum_{k=0}^n \frac{y_{t_k} + y_{t_{k+1}}}{2} X_{t_k t_{k+1}}^1.$$

Then as the mesh size $|\mathcal{P}| \rightarrow 0$ we have

$$\text{tr-}\mathcal{J}_0^T(y, X) \xrightarrow{\text{a.s.}} \int_0^T y_s d\mathbf{X}_s,$$

where the right hand side above designates the rough integral of y against X .

As mentioned above, our main Theorem 1.2 shows that one can approximate rough integrals by very natural Riemann sums, for a wide class of integrands y and Gaussian driving noises X . As a corollary of our trapezoid rule, we will also prove a midpoint rule for rough integrals of the form $\int f(X) dX$; see Corollary 3.15.

Remark 1.3. For conciseness we have assumed in the statement of our main result (see Theorem 3.12) that y and X are both paths in the space \mathbb{R}^d . In fact, the trapezoid rule holds for a general form $\int y \otimes dX$ of rough integrals as long as y is a controlled process of X . We leave the details of this simple generalization to the patient reader.

Remark 1.4. It is interesting to observe that on top of the higher order iterated integrals for the Gaussian rough paths X , the derivatives of the controlled process y are also avoided in the computation of the integral $\int y d\mathbf{X}$ in Theorem 1.2. This is part of the novelty in our main result. We will say more about this fact in Remark 3.13, once our main objects of interest are rigorously introduced.

Let us mention a few words about the techniques employed for our proofs. Indeed, most of the aforementioned 1-d contributions concerning convergences of trapezoid rules rely heavily on integration by parts techniques from Malliavin calculus, together with central limit theorems for random variables in a fixed chaos. The generalization described in Theorem 1.2 requires a new set of methods. Specifically, we shall use a combination of rough paths techniques in discrete time in order to single out the main terms in (1.8). Then we can simplify the main part of the computations by performing our integration by parts on the building blocks of our rough path \mathbf{X} only. Eventually we invoke some limit theorems for weighted sums in order to get our limit results.

Here is a brief outline of our paper. In Section 2 we set the ground for our computations by recalling some basic facts about rough paths analysis, Gaussian processes and Malliavin calculus. Section 3 is then devoted to the trapezoid rule. Namely Section 3.1 gives some preliminary results about Young integrals and convergence of random sequences. Then some random sums in a finite chaos are analyzed in Section 3.2. The corresponding weighted sums are handled in Section 3.3. With all those results in hand, our main theorem is proved in Section 3.4. Eventually we give a brief list of processes to which our general result applies in Section 3.5.

In this article, C denotes a constant which may change from line to line. In the same way, G will denote a generic integrable random variable whose value may change from line to line.

2. Preliminary material

This section contains some basic tools from Malliavin calculus and rough paths theory, as well as some analytical results which are crucial for the definition and integration of controlled processes.

2.1. Elements of rough paths

In this section we shall recall the notion of a rough path above a continuous path X , and how this applies to Gaussian processes. The interested reader is referred to [5,6] for further details.

2.1.1. Basic rough paths notions

For $s < t$ and $m \geq 1$, consider the simplex $\mathcal{S}_m([s, t]) = \{(u_1, \dots, u_m) \in [s, t]^m; u_1 < \dots < u_m\}$. We start by introducing the notion of increments, which turns out to be handy in the definition of a rough path.

Definition 2.1. Let $k \geq 1$. Then the space of $(k - 1)$ -increments, denoted by $\mathcal{C}_k([0, T], \mathbb{R}^d)$ or simply $\mathcal{C}_k(\mathbb{R}^d)$, is defined as

$$\mathcal{C}_k(\mathbb{R}^d) \equiv \left\{ g \in C(\mathcal{S}_k([0, T]); \mathbb{R}^n); \lim_{t_i \rightarrow t_{i+1}} g_{t_1 \dots t_k} = 0, \text{ for all } i \leq k - 1 \right\}.$$

In the sequel we will also often resort to a finite difference operator called δ , which acts on increments and is useful to split iterated integrals into simpler pieces.

Definition 2.2. Let $g \in \mathcal{C}_1(\mathbb{R}^d)$, $h \in \mathcal{C}_2(\mathbb{R}^d)$. Then for $(s, u, t) \in \mathcal{S}_3([0, T])$, we set

$$\delta g_{st} = g_t - g_s, \quad \text{and} \quad \delta h_{sut} = h_{st} - h_{su} - h_{ut}.$$

In order to define rough integrals, some minimal regularity assumptions on increments will have to be made. In particular, it will be convenient to measure the regularity of increments in \mathcal{C}_1 and \mathcal{C}_2 in terms of p -variation.

Definition 2.3. For $f \in \mathcal{C}_2(\mathbb{R}^d)$, $p > 1$ we set

$$\|f\|_{p\text{-var}} = \|f\|_{p\text{-var}; [0, T]} = \sup_{\mathcal{P} \subset [0, T]} \left(\sum_i |f_{t_i t_{i+1}}|^p \right)^{1/p},$$

where the supremum is taken over all subdivisions \mathcal{P} of $[0, T]$. The set of increments in $\mathcal{C}_2(\mathbb{R}^n)$ with finite p -variation is denoted by $\mathcal{C}_2^{p\text{-var}}(\mathbb{R}^d)$. For $f \in \mathcal{C}_1(\mathbb{R}^d)$, we denote $\|f\|_{p\text{-var}} = \|\delta f\|_{p\text{-var}}$.

We will also make an extensive use of Hölder norms, whose definition is recalled below:

Definition 2.4. We denote by $\mathcal{C}_2^\gamma(\mathbb{R}^d)$ the space of γ -Hölder functions on $[0, T]$. That is,

$$(2.1) \quad \mathcal{C}_2^\gamma(\mathbb{R}^d) = \left\{ f \in \mathcal{C}_2(\mathbb{R}^d) : \sup_{s,t \in [0,T]} \frac{|f_{st}|}{|t-s|^\gamma} < \infty \right\}.$$

We define $\mathcal{C}_1^\gamma(\mathbb{R}^d)$ the space of functions f such that $\delta f \in \mathcal{C}_2^\gamma(\mathbb{R}^d)$.

In the following we define the notion of control function:

Definition 2.5. A control function is a continuous map $\omega : \mathcal{S}_2([0, T]) \rightarrow [0, \infty)$ such that $\omega(s, s) = 0$ for $0 \leq s \leq T$ which satisfies the super-additivity, that is for all $s \leq t \leq u$ in $[0, T]$: $\omega(s, t) + \omega(t, u) \leq \omega(s, u)$.

A 2d control function is a continuous map $\omega : \mathcal{S}_2([0, T]) \times \mathcal{S}_2([0, T]) \rightarrow [0, \infty)$ which is zero on degenerate rectangles and satisfies $\omega_R(D) \geq \sum_{i=1}^n \omega_R(D_i)$ if $D_i : 0 \leq i \leq n$ are disjoint rectangles such that $D = \cup_i D_i$.

With these preliminary definitions in hand, we now define the notion of a rough path above a continuous p -variation path x with $p > 1$.

Definition 2.6. Let x be a continuous \mathbb{R}^d -valued p -variation path for some $p > 1$. We say that x gives rise to a geometric p -rough path if there exists a continuous path $(x_{st}^n, (s, t) \in \mathcal{S}_2([0, T]))$ with values in $(\mathbb{R}^d)^{\otimes n}$ for each $n \leq \lfloor p \rfloor$ such that $x_{st}^1 = \delta x_{st}$, and a control function ω_x such that

- (1) *Regularity:* For all $n \leq \lfloor p \rfloor$, x^n satisfies $|x_{st}^n| \leq \omega_x(s, t)^{n/p}$.
- (2) *Multiplicativity:* With δx^n as in Definition 2.2, we have

$$(2.2) \quad \delta x_{sut}^n = \sum_{n_1=1}^{n-1} x_{su}^{n_1} \otimes x_{ut}^{n-n_1}.$$

- (3) *Geometricity:* Let x^ε be a sequence of piecewise linear approximations of x . For any $n \leq \lfloor p \rfloor$ we assume that $x^{\varepsilon, n}$ converges in $\frac{p}{n}$ -variation norm to x^n , where $x_{st}^{\varepsilon, n}$ is defined for $(s, t) \in \Delta_2$ by

$$(2.3) \quad x_{st}^{\varepsilon, n} = \int_{(u_1, \dots, u_n) \in \mathcal{S}_n([s, t])} dX_{u_1}^\varepsilon \otimes \dots \otimes dx_{u_n}^\varepsilon.$$

In the sequel we will write \mathbf{x} for the rough path above x , that is

$$\mathbf{x}_{st} = (x_{st}^1, \dots, x_{st}^{\lfloor p \rfloor}), \quad (s, t) \in \mathcal{S}_2([0, T]).$$

One of the key success factors of the rough path theory is its ability to give a proper definition of stochastic calculus in very general contexts. Within this framework, the generic integrands in stochastic type integrals are so-called controlled paths, whose definition is recalled below. We first introduce some necessary notations about matrix products.

Notation 2.7. In order to avoid lengthy indices in our formulae throughout the paper, we will adopt the following convention for matrix products: two generic elements $v \in (\mathbb{R}^d)^{\otimes k}$ and $u \in \mathcal{L}((\mathbb{R}^d)^{\otimes k}, \mathbb{R}^m)$ will stand for families

$$v = \{v^{j_1 \dots j_k}; j_1, \dots, j_k \in \{1, \dots, d\}\},$$

$$u = \{u^{ij_1 \dots j_k}; i \in \{1, \dots, m\}, j_1, \dots, j_k \in \{1, \dots, d\}\},$$

where $v^{j_1 \dots j_k}$ and $u^{ij_1 \dots j_k}$ are real numbers. In this context the product uv is defined as an element of \mathbb{R}^m such that for $1 \leq i \leq m$ we have

$$(uv)^i := \sum_{j_1, \dots, j_k=1}^d u^{ij_1 \dots j_k} \times v^{j_1 \dots j_k}.$$

Similarly for $1 \leq k' \leq k$ and $w \in (\mathbb{R}^d)^{\otimes k'}$, we define uw as an element of $\mathcal{L}((\mathbb{R}^d)^{\otimes (k-k')}, \mathbb{R}^m)$ such that for $1 \leq i \leq m$ and $1 \leq j_{k'+1}, \dots, j_k \leq d$ we have

$$(uw)^{ij_{k'+1} \dots j_k} := \sum_{j_1, \dots, j_{k'}=1}^d u^{ij_1 \dots j_{k'} j_{k'+1} \dots j_k} \times w^{j_1 \dots j_{k'}}.$$

We can now state the definition of controlled process in the p -variation framework.

Definition 2.8. Let $\mathbf{x} = (x^1, \dots, x^{\lfloor p \rfloor})$ be a p -variation rough path as introduced in Definition 2.6. Let $y^0, \dots, y^{\ell-1}$ be continuous processes $y^k : [0, T] \rightarrow \mathcal{L}((\mathbb{R}^d)^{\otimes k}, \mathbb{R}^m)$ and define the remainder terms:

$$(2.4) \quad r_{st}^k = \delta y_{st}^k - y_s^{k+1} x_{st}^1 - \dots - y_s^{\ell-1} x_{st}^{\ell-k-1}$$

and $r_{st}^{\ell-1} = \delta y_{st}^{\ell-1}$, where we recall the notation δ given in Definition 2.2 and our Notation 2.7 on matrix products. If there is a control function ω_y such that

$$|\delta y_{st}^k| \leq \omega_y(s, t)^{1/p} \quad \text{and} \quad |r_{st}^k| \leq \omega_y(s, t)^{(\ell-k)/p}$$

for all $k = 0, 1, \dots, \ell - 1$, then we say that $\mathbf{y} = (y^0, \dots, y^{\ell-1})$ is a \mathbb{R}^m -valued path of order ℓ controlled by \mathbf{x} .

Remark 2.9. A controlled path has to be seen as a continuous path whose increments are ‘‘dominated’’ by the increments of x . Namely, y^0 is a continuous path taking values in \mathbb{R}^m such that the increments δy_{st}^0 as given in Definition 2.2 are given by

$$(2.5) \quad \delta y_{st}^0 = y_s^1 x_{st}^1 + \dots + y_s^{\ell-1} x_{st}^{\ell-1} + r_{st}^0.$$

The other relations in Definition 2.8 are imposed for algebraic sake.

Remark 2.10. In this paper, our main integration results will concern controlled paths. Hence it is worth recalling that this class of processes is rich enough. It includes for instance solutions of differential equations driven by x such as (1.1), as well as continuous paths of the form $g(x)$ for a smooth enough function g .

Remark 2.11. We will also use γ -Hölder norm versions of Definitions 2.6 and 2.8 in our discussion. We will omit these definitions for sake of conciseness. As an example, let us just mention that in a γ -Hölder version of (2.4) we would assume $y^k \in \mathcal{C}_1^\gamma$ and $r^k \in \mathcal{C}_2^{(\ell-k)\gamma}$.

The following proposition contains the classical result about integration of controlled processes with respect to a rough path, together with an approximation of the integral by enriched Riemann type sums.

Proposition 2.12. Let \mathbf{x} be a continuous p -variation rough path on $[0, T]$ and let \mathbf{y} be a \mathbb{R}^d -valued path of order $\ell = \lfloor p \rfloor$ controlled by \mathbf{x} as introduced in Definition 2.8. Consider a sequence of partitions of $[0, T]$ with mesh size $|\mathcal{P}| \rightarrow 0$. Then the following limit:

$$(2.6) \quad \lim_{|\mathcal{P}| \rightarrow 0} \sum_{k=0}^n y_{t_k}^0 x_{t_k t_{k+1}}^1 + y_{t_k}^1 x_{t_k t_{k+1}}^2 + \dots + y_{t_k}^{\ell-1} x_{t_k t_{k+1}}^\ell$$

exists almost surely. It is called rough integral of y with respect to \mathbf{x} and is denoted by $\int_0^T y_s d\mathbf{x}_s$.

One of the crucial ingredients in rough paths theory is the sewing lemma for integration. We label two discrete versions of this lemma, taken from [18], for further use. In the following we denote $\mathcal{S}_m(\llbracket s, t \rrbracket) = \{(u_1, \dots, u_m) \in \llbracket s, t \rrbracket^m; u_1 < \dots < u_m\}$, where $\llbracket s, t \rrbracket$ denotes the discrete interval related to a given partition of $[s, t]$ (see our forthcoming Notation 3.4). For notational sake, we just write \mathcal{S}_m for $\mathcal{S}_m(\llbracket 0, T \rrbracket)$.

Lemma 2.13. Consider a Banach space $(\mathcal{B}, \|\cdot\|)$ and $Q : \mathcal{S}_2 \rightarrow \mathcal{B}$. Recall that we set $\delta Q_{sut} = Q_{st} - Q_{su} - Q_{ut}$. Suppose that ω is a control on $\llbracket 0, T \rrbracket$. Moreover, assume that $Q_{t_k t_{k+1}} = 0$ for all $t_k \in \llbracket 0, T \rrbracket$ and that there exists a constant $\mu > 1$ such that

$$(2.7) \quad \|\delta Q_{sut}\| \leq \omega(s, t)^\mu$$

for all $(s, u, t) \in \mathcal{S}_3$. Then the following relation holds for all $(s, t) \in \mathcal{S}_2$:

$$\|Q_{st}\| \leq K_\mu \omega(s, t)^\mu, \quad \text{where } K_\mu = 2^\mu \sum_{l=1}^\infty l^{-\mu}.$$

The following lemma is a particular case of Lemma 2.13 with $\omega(s, t) = |t - s|$, which can be seen as the sewing lemma in Hölder norm.

Lemma 2.14. *Fix a constant $\mu > 1$. Let Q be as in Lemma 2.13, and we further assume that there exists a constant $C > 0$ such that*

$$|\delta Q_{sut}| \leq C \cdot |t - s|^\mu, \quad \text{for all } (s, u, t) \in \mathcal{S}_3.$$

Then for all $(s, t) \in \mathcal{S}_2$ we have

$$(2.8) \quad |Q_{st}| \leq CK_\mu |t - s|^\mu.$$

2.1.2. *Gaussian processes as rough paths*

Let us now turn to a more probabilistic setting for our computations. Namely we assume that $X_t = (X_t^1, \dots, X_t^d)$ is a continuous, centered Gaussian process with i.i.d. components, defined on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$. The covariance function of X is defined as follows

$$(2.9) \quad R(s, t) := E[X_s^j X_t^j],$$

where X^j is any of the components of X . We shall also resort to the following notation in the sequel

$$(2.10) \quad \sigma_t^2 := E[(X_t^j)^2], \quad \text{and} \quad \sigma_{st}^2 := E[(\delta X_{st}^j)^2].$$

The information concerning X used below is mostly encoded in the rectangular increments of the covariance function R , which are given for $s, t, u, v \in [0, T]$ by

$$(2.11) \quad R_{uv}^{st} := R(t, v) - R(t, u) - R(s, v) + R(s, u).$$

Notice that whenever the function R is given as a covariance function like in (2.9), the rectangular increments of R can also be written as

$$(2.12) \quad R_{uv}^{st} = E[(X_t^j - X_s^j)(X_v^j - X_u^j)].$$

Related to rectangular 2-d increments, the notion of 2-dimensional ρ -variation leads to an efficient way of constructing rough paths above a Gaussian process X . It will also feature prominently in our considerations, and thus we label its definition for further use.

Definition 2.15. For a general continuous function $R : [0, T]^2 \rightarrow \mathbb{R}$ and a parameter $\rho \geq 1$, we set

$$(2.13) \quad \|R\|_{\rho\text{-var}; [s,t] \times [u,v]} := \sup_{\substack{(t_i) \in \mathcal{P}([s,t]) \\ (t'_j) \in \mathcal{P}([u,v])}} \left(\sum_{i,j} |R_{t_i t_{i+1}}^{t'_j t'_{j+1}}|^\rho \right)^{\frac{1}{\rho}},$$

where $\mathcal{P}([s, t])$ denotes the set of all partitions of $[s, t]$ and where $R_{t_i t_{i+1}}^{t'_j t'_{j+1}}$ is defined in (2.11).

We also define the space of functions in the plane with finite 2-d ρ -variation:

Definition 2.16. Given a finite dimensional vector space E , we define $C^{\rho\text{-var}}([0, T]^2, E)$ to be the space of all functions $f : [0, T]^2 \rightarrow E$ such that $\|f\|_{\rho\text{-var}} < \infty$.

The standard assumption allowing to build a rough path above a generic Gaussian process concerns the ρ -variation of its covariance function. This is why we assume that the following hypothesis holds throughout the paper.

Hypothesis 2.17. *We assume that X is a centered continuous Gaussian process with covariance function R such that $\|R\|_{\rho\text{-var}} < \infty$ for $\rho \in [1, 2)$.*

Remark 2.18. Note that the ρ -variation norm $\|\cdot\|_{\rho\text{-var}}$ introduced in Definition 2.15 uses grid-like partitions. As pointed out in [7], those ρ -variations do not enjoy super-additivity properties. A standard way to circumvent this problem is to replace the ρ -variation of Definition 2.15 by the so-called controlled 2-d ρ -variation norm (see [7, Definition 1]), which we denote by $|\cdot|_{\rho\text{-var}}$. The norm $|\cdot|_{\rho\text{-var}}$ does satisfy sup-additivity properties (cf [7, Theorem 1(iii)]). Therefore, although we have not assumed $|R|_{\rho\text{-var}} < \infty$ in Hypothesis 2.17, we can consider a $\rho' = (\rho + \varepsilon) > \rho$ for ε arbitrarily small such that $|R|_{\rho'\text{-var}} < \infty$ (this is ensured by [7, relation (1.2)]). Then one is allowed to pick the control $\omega_R(D) = |R|_{\rho'\text{-var}}^{\rho'}(D)$. This kind of manipulation is routinely performed in e.g. [4,9].

With Remark 2.18 in mind and for notational sake, throughout the section we will skip the replacement of ρ by ρ' and pretend that the following is a consequence of Hypothesis 2.17.

Hypothesis 2.19. We assume that X is a centered continuous Gaussian process with covariance function R , ρ is a parameter lying in $[1, 2)$ and ω_R is a 2-d control (see Definition 2.5) such that for any rectangle D we have

$$\|R\|_{\rho\text{-var}, D} \leq \omega_R(D)^{1/\rho}.$$

The following result (stated e.g. in [6, Theorem 15.33]) relates the 2-d ρ -variation of R with the pathwise assumptions allowing to apply the abstract rough paths theory.

Proposition 2.20. Let $X = (X^1, \dots, X^d)$ be a continuous centered Gaussian process with i.i.d. components and covariance function R defined by (2.9). If R satisfies Hypothesis 2.17 then X gives raise to a geometric p -rough path according to Definition 2.6, provided $p > 2\rho$.

2.2. Wiener spaces associated to general Gaussian processes

In this section we consider again the continuous, centered Gaussian process X of Section 2.1. Recall that its covariance function R is defined by (2.9). We will describe the Cameron–Martin space assuming that we are in a real valued situation, the generalization to a \mathbb{R}^d -valued process being left to the patient reader.

The analysis of iterated integrals performed in Section 3 will be based on a Hilbert space \mathcal{H} allowing a proper definition of Wiener integrals as defined e.g. in [24]. Namely \mathcal{H} is defined to be the completion of the linear space of functions of the form

$$\mathcal{E} = \left\{ \sum_{i=1}^n a_i \mathbf{1}_{[0, t_i]} : a_i \in \mathbb{R}, t_i \in [0, T] \right\},$$

with respect to the inner product

$$(2.14) \quad \left\langle \sum_{i=1}^n a_i \mathbf{1}_{[0, t_i]}, \sum_{j=1}^m b_j \mathbf{1}_{[0, s_j]} \right\rangle_{\mathcal{H}} = \sum_{i=1}^n \sum_{j=1}^m a_i b_j R(t_i, s_j).$$

Remark 2.21. Consider the special case $X_0 = 0$, which means in particular that $R(0, 0) = 0$. Then, as suggested by (2.14), for any $h_1, h_2 \in \mathcal{H}$, we can infer that

$$(2.15) \quad \langle h_1, h_2 \rangle_{\mathcal{H}} = \int_0^T \int_0^T h_1(s) h_2(t) dR(s, t),$$

whenever the 2-d Young integral on the right-hand side is well-defined (one can refer e.g. to [6] for more details).

Since \mathcal{H} is the completion of \mathcal{E} with respect to the inner product defined by (2.14), it is isometric to the Hilbert space $H^1(X) \subseteq L^2(\Omega, \mathcal{F}, \mathbf{P})$ which is defined to be the $|\cdot|_{L^2(\Omega)}$ -closure of the set

$$\left\{ \sum_{i=1}^n a_i X_{t_i} : a_i \in \mathbb{R}, t_i \in [0, T], n \in \mathbb{N} \right\}.$$

In particular, we have that $|\mathbf{1}_{[0, t]}|_{\mathcal{H}} = |X_t|_{L^2(\Omega)}$. The isometry between \mathcal{H} and $H^1(X)$ is denoted by $X(h)$, and is called a Wiener integral.

Remark 2.22. As mentioned above in (2.14), the space \mathcal{H} is a closure of indicator functions. Hence it can be defined on any interval $[a, b] \subset [0, T]$. We denote by $\mathcal{H}([a, b])$ this restriction. For $[a, b] \subset [0, T]$, one can then check the following identity by a limiting procedure on simple functions

$$(2.16) \quad \langle f \mathbf{1}_{[a,b]}, g \mathbf{1}_{[a,b]} \rangle_{\mathcal{H}} = \langle f, g \rangle_{\mathcal{H}([a,b])}.$$

2.3. Malliavin calculus for Gaussian processes

In this section we review some basic aspects of Malliavin calculus. The reader is referred to [24] for further details.

As in Section 2.2, the family $X_t = (X_t^1, \dots, X_t^d)$ designates a continuous, centered Gaussian process with i.i.d. components, defined on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$. For sake of simplicity, we assume that \mathcal{F} is generated by $\{X_t; t \in [0, T]\}$. An \mathcal{F} -measurable real-valued random variable F is said to be cylindrical if it can be written, for some $m \geq 1$, as

$$F = f(X_{t_1}, \dots, X_{t_m}), \quad \text{for } 0 \leq t_1 < \dots < t_m \leq 1,$$

where $f : \mathbb{R}^m \rightarrow \mathbb{R}$ is a C_b^∞ function. The set of cylindrical random variables is denoted by S .

The Malliavin derivative is defined as follows: for $F \in S$, the derivative of F by

$$\mathbf{D}F = \sum_{i=1}^m \frac{\partial f}{\partial x_i}(X_{t_1}, \dots, X_{t_m}) \mathbf{1}_{[0, t_i]}.$$

More generally, we can introduce iterated derivatives. Namely, if $F \in S$, we set

$$\mathbf{D}^k F = \mathbf{D} \dots \mathbf{D}F.$$

For any $p \geq 1$, it can be checked that the operator \mathbf{D}^k is closable from S into $L^p(\Omega; \mathcal{H}^{\otimes k})$. We denote by $\mathbb{D}^{k,p}(\mathcal{H})$ the closure of the class of cylindrical random variables with respect to the norm

$$\|F\|_{k,p} = \left(E[|F|^p] + \sum_{j=1}^k E[\|\mathbf{D}^j F\|_{\mathcal{H}^{\otimes j}}^p] \right)^{\frac{1}{p}},$$

and we also set $\mathbb{D}^\infty(\mathcal{H}) = \bigcap_{p \geq 1} \bigcap_{k \geq 1} \mathbb{D}^{k,p}(\mathcal{H})$. The divergence operator δ^\diamond is then defined to be the adjoint operator of \mathbf{D} . Namely, for a process $u = \{u_t; t \in [0, T]\}$ in the domain of δ^\diamond we have

$$(2.17) \quad E[\delta^\diamond(u)F] = E[\langle \mathbf{D}F, u \rangle_{\mathcal{H}}],$$

for all $F \in \mathbb{D}^{1,2}$. Notice that if $u \in \mathbb{D}^{1,2}(\mathcal{H})$, then we also have $u \in \text{Dom}(\delta^\diamond)$. A typical elementary increment which can be represented thanks to the divergence operator is the following: for $(s, t) \in \mathcal{S}_2([0, T])$ and $1 \leq i \leq d$ we have

$$(2.18) \quad \delta X_{st}^i = \delta^\diamond(\mathbf{1}_{[s,t]} e_i),$$

where e_i denotes the i -th element of the canonical basis in \mathbb{R}^d .

We close this section by recalling the following results on Hermite polynomials. Those polynomials are defined by $H_0(x) = 1$, $H_1(x) = x$ and then recursively by $(n + 1)H_{n+1}(x) = xH_n(x) - H_{n-1}(x)$. They form an orthonormal basis of L^2 -functions with respect to the standard Gaussian measure on \mathbb{R} . For further use, we recall that

$$(2.19) \quad H_1(x) = x, \quad H_2(x) = \frac{1}{2}(x^2 - 1), \quad H_3(x) = \frac{1}{6}x^3 - \frac{1}{2}x.$$

We shall also invoke the following property of Gaussian vectors.

Proposition 2.23. Let X, Y be jointly normal random variables with $X, Y \sim \mathcal{N}(0, 1)$ and denote by H_n the n th Hermite polynomial. Then the following holds true:

$$(2.20) \quad E[H_n(X)H_m(Y)] = \begin{cases} 0 & \text{if } n \neq m, \\ \frac{1}{n!} (E[XY])^n & \text{if } n = m. \end{cases}$$

3. The trapezoid rule

This section is devoted to a complete statement and proof of the informal Theorem 1.2. We will first analyze some discrete sums in a finite chaos related to our rough path \mathbf{X} in Section 3.2, then move to some useful weighted sums in Section 3.3. Eventually the main part of our proof will be achieved in Section 3.4.

Throughout the section we consider a general centered Gaussian process X which satisfies Hypothesis 2.19. In particular, the covariance function R is defined by (2.9) and the variance of the increments δX_{st}^i is denoted by $\sigma^2(s, t)$ (see our notation (2.10)). As in Section 2, X admits a rough path lift \mathbf{X} . We also label a notation which will be useful for our future computations.

Notation 3.1. Let $[s, t] \times [u, v]$ be a generic rectangle in $[0, T]^2$. Consider a grid-like partition $\mathcal{P} = \{[t_k, t_{k+1}] \times [\tilde{t}_{k'}, \tilde{t}_{k'+1}]\}; s = t_0 < \dots < t_m = t, u = \tilde{t}_0 < \dots < \tilde{t}_n = v$. Then we set $D_{kk'} = [t_k, t_{k+1}] \times [\tilde{t}_{k'}, \tilde{t}_{k'+1}]$.

3.1. Two inequalities

We first derive an inequality on 2-d Young integrals which we make extensive use of. It is an elaboration of the Young–Loeve–Towghi inequality [26, Theorem 1.2]. Recall that for $y \in C([0, T]^2)$, y_{uv}^{st} denotes the rectangular increment of y over $[s, t] \times [u, v]$ defined in (2.11).

Lemma 3.2. Let $z \in C^{\rho\text{-var}}([0, T]^2, \mathbb{R}^d)$ and consider a function y sitting in the space $C^{\theta\text{-var}}([0, T]^2, \mathcal{L}(\mathbb{R}^d, \mathbb{R}^d))$ with $1/\rho + 1/\theta > 1$, where $C^{\rho\text{-var}}$ and $C^{\theta\text{-var}}$ are given in Definition 2.16. For some given $s < t < \sigma$ and $u < v < \eta$ we set $D = [s, \sigma] \times [u, \eta]$ and

$$(3.1) \quad \hat{y}_{rv} := \int_{[s,t] \times [u,v]} y_{ur'}^{sr'} dz_{rr'}.$$

Then the ρ -variation of \hat{y} on D can be bounded as follows:

$$(3.2) \quad \|\hat{y}\|_{\rho\text{-var}, D} \leq C \cdot \|y\|_{\theta\text{-var}; D} \cdot \|z\|_{\rho\text{-var}; D}.$$

Proof. We consider partitions $s = t_0 < \dots < t_m = \sigma$ and $u = \tilde{t}_0 < \dots < \tilde{t}_n = \eta$ of $[s, \sigma]$ and $[u, \eta]$, respectively. Recall from Notation 3.1 that we denote $D_{kk'} = [t_k, t_{k+1}] \times [\tilde{t}_{k'}, \tilde{t}_{k'+1}] \subset D$, and write

$$\hat{y}(D_{kk'}) = \hat{y}_{\tilde{t}_{k'}\tilde{t}_{k'+1}}^{t_k t_{k+1}} = \hat{y}_{t_{k+1}\tilde{t}_{k'+1}} - \hat{y}_{t_{k+1}\tilde{t}_{k'}} - \hat{y}_{t_k\tilde{t}_{k'+1}} + \hat{y}_{t_k\tilde{t}_{k'}}.$$

Using expression (3.1), one can decompose $\hat{y}(D_{kk'})$ according to our partition in the following way:

$$(3.3) \quad \begin{aligned} \hat{y}(D_{kk'}) &= \int_{D_{kk'}} y_{ur'}^{sr'} dz_{rr'} \\ &= \int_{D_{kk'}} y_{\tilde{t}_{k'}r'}^{t_k r'} dz_{rr'} + \int_{D_{kk'}} y_{\tilde{t}_{k'}r'}^{st_k} dz_{rr'} + \int_{D_{kk'}} y_{u\tilde{t}_{k'}}^{t_k r'} dz_{rr'} + \int_{D_{kk'}} y_{u\tilde{t}_{k'}}^{st_k} dz_{rr'} \\ &:= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

It should be noticed that the term I_1 above can be bounded directly thanks to the Young–Loeve–Towghi inequality [26, Theorem 1.2]. We get

$$(3.4) \quad |I_1| \leq C \|y\|_{\theta\text{-var}, D_{kk'}} \cdot \|z\|_{\rho\text{-var}, D_{kk'}}.$$

The term I_4 can also be treated easily. Indeed, we have

$$I_4 = y_{u\tilde{t}_{k'}}^{st_k} \cdot z_{\tilde{t}_{k'}\tilde{t}_{k'+1}}^{t_k t_{k+1}},$$

and thus

$$(3.5) \quad |I_4| \leq \|y\|_{\theta\text{-var}, [u, \tilde{t}_{k'}] \times [s, t_k]} \cdot \|z\|_{\rho\text{-var}, D_{kk'}}.$$

We now focus on the 1-d type integral I_2 in equation (3.3). In order to bound this term one can use the classical Young inequality [27] in order to get

$$\begin{aligned}
 (3.6) \quad |I_2| &= \left| \int_{[\tilde{t}_{k'}, \tilde{t}_{k'+1}]} y_{\tilde{t}_{k'}, r'}^{st_k} d(z_{t_{k+1}r'} - z_{t_k r'}) \right| \\
 &\leq C \|y_{\tilde{t}_{k'}}^{st_k}\|_{\theta\text{-var}, [\tilde{t}_{k'}, \tilde{t}_{k'+1}]} \cdot \|z_{t_{k+1}} - z_{t_k}\|_{\rho\text{-var}, [\tilde{t}_{k'}, \tilde{t}_{k'+1}]} \\
 &\leq C \|y\|_{\theta\text{-var}, [s, t_k] \times [\tilde{t}_{k'}, \tilde{t}_{k'+1}]} \cdot \|z\|_{\rho\text{-var}, D_{kk'}}.
 \end{aligned}$$

In the same way, we can also upper bound the term I_3 in (3.3) as

$$(3.7) \quad |I_3| \leq C \|y\|_{\theta\text{-var}, [u, \tilde{t}_{k'}] \times [t_k, t_{k+1}]} \cdot \|z\|_{\rho\text{-var}, D_{kk'}}.$$

Plugging (3.4)–(3.7) into (3.3), we have thus obtained that

$$\begin{aligned}
 |\hat{y}(D_{kk'})| &\leq C \{ \|y\|_{\theta\text{-var}, D_{kk'}} + \|y\|_{\theta\text{-var}, [s, t_k] \times [\tilde{t}_{k'}, \tilde{t}_{k'+1}]} \\
 &\quad + \|y\|_{\theta\text{-var}, [u, \tilde{t}_{k'}] \times [t_k, t_{k+1}]} + \|y\|_{\theta\text{-var}, [u, \tilde{t}_{k'}] \times [s, t_k]} \} \cdot \|z\|_{\rho\text{-var}, D_{kk'}}.
 \end{aligned}$$

Therefore by trivial monotonicity properties of θ -variations, we arrive at

$$(3.8) \quad |\hat{y}(D_{kk'})| \leq C \|y\|_{\theta\text{-var}, D} \cdot \|z\|_{\rho\text{-var}, D_{kk'}}.$$

As explained in Remark 2.18, we will skip the routine procedure of replacing ρ by ρ' in order to apply super-additivity relations to 2-dimensional ρ -variations. Hence summing relation (3.8) over k, k' and invoking super-additivity, we finally prove the desired inequality (3.2). \square

We close this section by giving a general convergence lemma for a sequence of stochastic processes. It is borrowed from [18, Lemma 3.5].

Lemma 3.3. *Let $\alpha, \beta > 0$. Let $\{z^n, n \in \mathbb{N}\}$ be a sequence of stochastic processes such that*

$$\|\delta z_{st}^n\|_{L^p(\Omega)} \leq C p n^{-\alpha} (t - s)^\beta,$$

for all $p \geq 1$, where K_p is a constant depending on p and where we recall the notation δ given in Definition 2.2. Then for $0 < \gamma < \beta$ and $\kappa > 0$, we can find an integrable random variable $G_{\gamma, \kappa}$ independent of n and admitting moments of any order, such that:

$$(3.9) \quad \|z^n\|_\gamma \leq G_{\gamma, \kappa} n^{-\alpha + \kappa}.$$

3.2. Upper-bounds for processes in a finite chaos

With the notions of Section 2 in hand, we now introduce a family of processes defined as sums of iterated integrals of X which appear naturally in the analysis of the approximation (1.9). We start by proving a bound on sums of Lévy area type processes which generalizes [14,15,18]. We first label a notation for further use.

Notation 3.4. Let $\mathcal{P} = \{0 = t_0 < \dots < t_n = T\}$ be a partition of $[0, T]$. Take $s, t \in [0, T]$. Then $\llbracket s, t \rrbracket := \{t_k \in \mathcal{P} : t_k \in [s, t]\}$. We denote $S_k(\llbracket s, t \rrbracket) = \{(t_1, \dots, t_k) : t_1, \dots, t_k \in \mathcal{P} \text{ and } s \leq t_1 \leq \dots \leq t_k \leq t\}$ as the discrete simplex.

The main bound involving Lévy area type objects is the following.

Lemma 3.5. *Suppose that Hypothesis 2.19 holds true for the \mathbb{R}^d -valued Gaussian process $X = (X^1, \dots, X^d)$ with covariance function R , 2-d control ω_R and $\rho \in [1, 2)$. Consider a partition $\{t_k : 0 \leq k \leq n\}$ of $[0, T]$ and define the process F on $\llbracket 0, T \rrbracket$ by*

$$(3.10) \quad F_t^{ij} = \begin{cases} \sum_{0 \leq t_k < t} X_{t_k t_{k+1}}^{2,ij} & \text{for } i \neq j, \\ \sum_{0 \leq t_k < t} X_{t_k t_{k+1}}^{2,ii} - E[X_{t_k t_{k+1}}^{2,ii}] & \text{for } i = j, \end{cases}$$

with the convention that $F_0^{ij} = 0$. Then for any $p \geq 1$ there exists a strictly positive constant $C = C_p$ such that for all $(s, t) \in \mathcal{S}_2$ and $0 \leq \varepsilon \leq 2 - \rho$ we have

$$(3.11) \quad \|\delta F_{st}^{ij}\|_p^2 \leq C \max_{k,k'} \omega_R(D_{kk'})^{\frac{\varepsilon}{\rho}} \cdot \omega_R([s, t]^2)^{\frac{2-\varepsilon}{\rho}},$$

where $\|\cdot\|_p$ denotes the $L^p(\Omega)$ norm and where the rectangle $D_{kk'}$ is defined in Notation 3.1.

Proof. By hypercontractivity for random variables in the second chaos (see [24, Theorem 1.4.1]), we just need to consider $p = 2$. Furthermore, when $i = j$ notice that

$$(3.12) \quad X_{t_k t_{k+1}}^{2,ii} = \int_{t_k}^{t_{k+1}} \int_{t_k}^{u_1} dX_{u_2}^i dX_{u_1}^i = \frac{1}{2}(\delta X_{t_k t_{k+1}}^i)^2,$$

where a complete justification of (3.12) is due to the Definition 2.6 and the fact that \mathbf{X} is assumed to be geometric. Therefore one can recast the definition of $\delta F_{t_k t_{k+1}}^{ii}$ in (3.10) as

$$(3.13) \quad \begin{aligned} \delta F_{t_k t_{k+1}}^{ii} &= X_{t_k t_{k+1}}^{2,ii} - E[X_{t_k t_{k+1}}^{2,ii}] = \frac{1}{2}(X_{t_k t_{k+1}}^{1,i})^2 - \frac{1}{2}E[(X_{t_k t_{k+1}}^{1,i})^2] \\ &= \sigma_{t_k t_{k+1}}^2 H_2\left(\frac{X_{t_k t_{k+1}}^{1,i}}{\sigma_{t_k t_{k+1}}}\right), \end{aligned}$$

where $H_2(x) = \frac{1}{2}(x^2 - 1)$ as recalled in (2.19) and where the notation σ_{st}^2 has been introduced in (2.10). Then putting together relations (3.10), (3.12) and (3.13) we have

$$\begin{aligned} \|\delta F_{st}^{ii}\|_2^2 &= E\left[\sum_k (X_{t_k t_{k+1}}^{2,ii} - E[X_{t_k t_{k+1}}^{2,ii}]) \sum_{k'} (X_{t_{k'} t_{k'+1}}^{2,ii} - E[X_{t_{k'} t_{k'+1}}^{2,ii}])\right] \\ &= E\left[\sum_k \sigma_{t_k t_{k+1}}^2 H_2\left(\frac{X_{t_k t_{k+1}}^{1,i}}{\sigma_{t_k t_{k+1}}}\right) \sum_{k'} \sigma_{t_{k'} t_{k'+1}}^2 H_2\left(\frac{X_{t_{k'} t_{k'+1}}^{1,i}}{\sigma_{t_{k'} t_{k'+1}}}\right)\right]. \end{aligned}$$

Next, expanding the double sum in k, k' above we get

$$(3.14) \quad \|\delta F_{st}^{ii}\|_2^2 = \sum_{k,k'} \sigma_{t_k, t_{k+1}}^2 \sigma_{t_{k'}, t_{k'+1}}^2 E\left[H_2\left(\frac{X_{t_k t_{k+1}}^{1,i}}{\sigma_{t_k, t_{k+1}}}\right) H_2\left(\frac{X_{t_{k'} t_{k'+1}}^{1,i}}{\sigma_{t_{k'}, t_{k'+1}}}\right)\right].$$

Since each $X_{t_k t_{k+1}}^{1,i}$ is a Gaussian random variable, we can now apply Proposition 2.23 with $X = \sigma_{t_k t_{k+1}}^{-1} X_{t_k t_{k+1}}^{1,i}$ and $Y = \sigma_{t_{k'} t_{k'+1}}^{-1} X_{t_{k'} t_{k'+1}}^{1,i}$. This yields

$$(3.15) \quad \begin{aligned} \|\delta F_{st}^{ii}\|_2^2 &= \frac{1}{2} \sum_{k,k'} \sigma_{t_k, t_{k+1}}^2 \sigma_{t_{k'}, t_{k'+1}}^2 \left(E\left[\frac{X_{t_k t_{k+1}}^{1,i} X_{t_{k'} t_{k'+1}}^{1,i}}{\sigma_{t_k, t_{k+1}} \sigma_{t_{k'}, t_{k'+1}}}\right]\right)^2 \\ &= \frac{1}{2} \sum_{k,k'} (R_{t_{k'} t_{k'+1}}^{t_k t_{k+1}})^2. \end{aligned}$$

Notice that the sum on the right hand side of (3.15) is a sum over rectangles $D_{k,k'} = [t_k, t_{k+1}] \times [t_{k'}, t_{k'+1}]$. Since we have $|R_{t_{k'} t_{k'+1}}^{t_k t_{k+1}}| \leq \omega_R(D_{kk'})^{1/\rho}$ thanks to Hypothesis 2.19, we obtain

$$(3.16) \quad \|\delta F_{st}^{ii}\|_2^2 \leq \frac{1}{2} \sum_{k,k'} \omega_R(D_{kk'})^{2/\rho}$$

$$(3.17) \quad \leq \frac{1}{2} \max_{k,k'} \omega_R(D_{kk'})^{\frac{\varepsilon}{\rho}} \cdot \sum_{k,k'} \omega_R(D_{kk'})^{\frac{2-\varepsilon}{\rho}}.$$

Now recall that in our statement we have chosen $0 \leq \varepsilon \leq 2 - \rho$, which yields $(2 - \varepsilon)/\rho \geq 1$. Hence invoking the super-additivity of ω_R and thus of $\omega_R^{(2-\varepsilon)/\rho}$, we arrive at

$$(3.18) \quad \|\delta F_{st}^{ii}\|_2^2 \leq \frac{1}{2} \max_{k,k'} \omega_R(D_{kk'})^{\frac{\varepsilon}{\rho}} \cdot \omega_R([s, t]^2)^{\frac{2-\varepsilon}{\rho}}.$$

Putting together inequality (3.18) and the aforementioned hypercontractivity argument, our claim (3.11) is proved for $i = j$.

Let us now handle the case $i \neq j$. Similarly to what we did in (3.14), we compute

$$(3.19) \quad \begin{aligned} \|\delta F_{st}^{ij}\|_2^2 &= \sum_{k,k'} E[X_{t_k t_{k+1}}^{2,ij} X_{t_{k'} t_{k'+1}}^{2,ij}] \\ &= \sum_{k,k'} E\left[\int_{t_k}^{t_{k+1}} X_{t_k}^{1,i} dX_r^{1,j} \int_{t_{k'}}^{t_{k'+1}} X_{t_{k'} r'}^{1,i} dX_{r'}^{1,j}\right]. \end{aligned}$$

In order to compute the right and side of (3.19) we proceed as in [6, p. 402]. Namely, we consider a Gaussian regularization X^ε of X whose rough path lift \mathbf{X}^ε also converges to \mathbf{X} . Let us write R^ε for the covariance function of the process X^ε and F^ε be the process F defined similarly to (3.10) for the process X^ε . A simple application of Fubini's theorem yields

$$(3.20) \quad \begin{aligned} \|\delta F_{st}^{\varepsilon,ij}\|_2^2 &= \sum_{k,k'} E\left[\int_{t_k}^{t_{k+1}} \int_{t_{k'}}^{t_{k'+1}} X_{t_k}^{\varepsilon,1,i} X_{t_{k'} r'}^{\varepsilon,1,i} dX_r^{\varepsilon,1,j} dX_{r'}^{\varepsilon,1,j}\right] \\ &= \sum_{k,k'} \int_{D_{k,k'}} R_{t_k t_{k'} r' r'}^{\varepsilon, t_k r} dR^\varepsilon(r, r'), \end{aligned}$$

where we recall that $D_{k,k'} = [t_k, t_{k+1}] \times [t_{k'}, t_{k'+1}]$. Taking limits in (3.20) as $\varepsilon \rightarrow 0$ we get

$$(3.21) \quad \|\delta F_{st}^{ij}\|_2^2 = \sum_{k,k'} \int_{D_{k,k'}} R_{t_k t_{k'} r' r'}^{t_k r} dR(r, r').$$

Then a direct application of inequality (3.2) yields

$$(3.22) \quad \|\delta F_{st}^{ij}\|_2^2 \leq C \sum_{k,k'} \|R\|_{\rho\text{-var}; D_{k,k'}}^2,$$

which in turn implies relation (3.16) thanks to Hypothesis 2.19. Starting from (3.22), we can thus conclude as we did for (3.17) and (3.18) in the case $i = j$. Our result (3.11) is now shown for the case $i \neq j$, which concludes our proof. \square

We now state an elaboration of Lemma 3.5 for third-order integrals.

Lemma 3.6. *Suppose that Hypothesis 2.19 holds for the Gaussian process $X = (X^1, \dots, X^d)$ with covariance function R , 2-d control ω_R and $\rho \in [1, 2)$. For $i, j, \ell \in \{1, \dots, d\}$, $(s, t) \in S_2$ and a generic partition $\{t_i; 0 \leq k \leq n\}$ of $[0, T]$ we denote*

$$(3.23) \quad \delta g_{st}^{ij\ell} = \sum_{s \leq t_k < t} X_{t_k t_{k+1}}^{3,ij\ell}.$$

Then for any $p \geq 2$ there exists a positive constant $C = C_{\rho,p}$ such that for all $(s, t) \in S_2$ we have

$$(3.24) \quad \|\delta g_{st}^{ij\ell}\|_{L^p(\Omega)}^2 \leq C \sum_{k,k'} \omega_R(D_{kk'})^{3/\rho} + C \sum_{k,k'} [\omega_R(D_{kk'}) \omega_R(D_{k'k'}) \omega_R(D_{kk'})]^{1/\rho},$$

where we recall our Notation 3.1 for $D_{kk'}$.

Proof. Along the same lines as Lemma 3.5, by hypercontractivity of random variables in the third chaos, we just need to consider $p = 2$. We focus on this case in the remainder of the proof. We split our considerations in several steps.

Step 1: equal indices. In this step, we show that (3.24) holds when the indices are equal: $i = j = \ell$. We first note that using geometricity similarly to what we have done in (3.12), we get $X_{t_k t_{k+1}}^{3,iii} = \frac{1}{6} (\delta X_{t_k t_{k+1}}^i)^3$. We now expand the cubic power in

terms of Hermite polynomials along the same lines as for (3.13). Namely, owing to (2.19) we recall that $x^3 = 6H_3(x) + 3x$. Therefore, renormalizing $(\delta X_{t_k t_{k+1}}^i)^3$ by $\sigma_{t_k t_{k+1}}$ (recall that σ_{st} is defined by (2.10)), we get

$$(3.25) \quad X_{t_k t_{k+1}}^{3,iii} = \sigma_{t_k t_{k+1}}^3 H_3(\sigma_{t_k t_{k+1}}^{-1} \delta X_{t_k t_{k+1}}^i) + \frac{1}{2} \sigma_{t_k t_{k+1}}^2 \delta X_{t_k t_{k+1}}^i.$$

Let us now go back to our expression (3.23) for $i = j = \ell$. By linearity of the expected value we have

$$(3.26) \quad \|\delta g_{st}^{iii}\|_2^2 = \sum_{k,k'} E[X_{t_k t_{k+1}}^{3,iii} X_{t_{k'} t_{k'+1}}^{3,iii}].$$

Plugging relation (3.25) into (3.26), invoking Proposition 2.23 and recalling that $\sigma_{t_k t_{k+1}}^2 = R_{t_k t_{k+1}}^{t_k t_{k+1}}$, we obtain the existence of a universal constant $C > 0$ such that:

$$(3.27) \quad \begin{aligned} \|\delta g_{st}^{iii}\|_2^2 &= \sum_{k,k'} (R_{t_k t_{k+1}}^{t_k t_{k+1}})^3 + \frac{1}{4} \sum_{k,k'} R_{t_k t_{k+1}}^{t_k t_{k+1}} R_{t_{k'} t_{k'+1}}^{t_{k'} t_{k'+1}} R_{t_k t_{k+1}}^{t_{k'} t_{k'+1}} \\ &\leq C \sum_{k,k'} \omega_R(D_{kk'})^{3/\rho} + C \sum_{k,k'} [\omega_R(D_{kk'}) \omega_R(D_{k'k'}) \omega_R(D_{kk'})]^{1/\rho}, \end{aligned}$$

where the last inequality stems from Hypothesis 2.19 and the fact that $|R_{uv}^{st}| \leq \omega_R([u, v] \times [s, t])^{1/\rho}$ and where we have used our Notation 3.1 for $D_{kk'}$. We have thus proved our claim (3.24) for $p = 2$ and $i = j = \ell$. As mentioned above, the general case $p \geq 2$ follows by hypercontractivity.

Step 2: three distinct indices. We turn to the proof of (3.24) for i, j, ℓ all distinct. To this aim, we first note that a regularization procedure similar to the one which lead to (3.21) yields

$$\begin{aligned} \|\delta g_{st}^{ij\ell}\|_2^2 &= \sum_{k,k'} E[X_{t_k t_{k+1}}^{3,ij\ell} X_{t_{k'} t_{k'+1}}^{3,ij\ell}] \\ &= \sum_{k,k'} E \left[\int_{t_k}^{t_{k+1}} \int_{t_k}^u X_{t_k, v}^{1,i} dX_v^{1,j} dX_u^{1,\ell} \int_{t_{k'}}^{t_{k'+1}} \int_{t_{k'}}^{u'} X_{t_{k'}, v'}^{1,i} dX_{v'}^{1,j} dX_{u'}^{1,\ell} \right] \\ &= \sum_{k,k'} E \left[\int_{t_k}^{t_{k+1}} \int_{t_{k'}}^{t_{k'+1}} \int_{t_k}^u \int_{t_{k'}}^{u'} X_{t_k, v}^{1,i} X_{t_{k'}, v'}^{1,i} dX_v^{1,j} dX_{v'}^{1,j} dX_u^{1,\ell} dX_{u'}^{1,\ell} \right], \end{aligned}$$

for all $(s, t) \in \mathcal{S}_2(\llbracket 0, T \rrbracket)$. Then using independence and Fubini's theorem (here again the standard regularization arguments are outlined in (3.21)) we get

$$(3.28) \quad \|\delta g_{st}^{ij\ell}\|_2^2 = \sum_{k,k'} \int_{t_k}^{t_{k+1}} \int_{t_{k'}}^{t_{k'+1}} \int_{t_k}^u \int_{t_{k'}}^{u'} R_{t_k v}^{t_{k'} v'} dR(v, v') dR(u, u').$$

Now applying twice inequality (3.2) to the right side of (3.28), we easily get

$$(3.29) \quad \|\delta g_{st}^{ij\ell}\|_2^2 \leq C \sum_{k,k'} \omega_R(D_{kk'})^{3/\rho}.$$

Our claim (3.24) is now proved for i, j, ℓ distinct.

Step 3: two distinct indices. We now turn to the case of two distinct indices in i, j, ℓ . In fact we should divide this case into 3 distinct subcases, namely $i = j \neq \ell$, $i \neq j = \ell$ and $i = \ell \neq j$.

Case $i = j \neq \ell$. Recalling Notation 3.1 for the intervals $[t_k, t_{k+1}] \times [\tilde{t}_{k'}, \tilde{t}_{k'+1}]$, we get

$$(3.30) \quad \begin{aligned} \|\delta g_{st}^{ii\ell}\|_2^2 &= \sum_{k,k'} E[X_{t_k t_{k+1}}^{3,ii\ell} X_{t_{k'} t_{k'+1}}^{3,ii\ell}] \\ &= \frac{1}{4} \sum_{k,k'} \int_{D_{kk'}} E[(X_{t_k u}^{1,i})^2 (X_{t_{k'} u'}^{1,i})^2] dR(u, u'). \end{aligned}$$

In order to evaluate the term $E[(X_{t_k u}^{1,i})^2(X_{t_{k'} u'}^{1,i})^2]$ above, we reproduce the steps leading from (3.13) to (3.15) in the proof of Lemma 3.5 (based on Hermite polynomial decompositions). This yields

$$(3.31) \quad \|\delta g_{st}^{ii\ell}\|_2^2 := I_1 + I_2,$$

where the terms I_1 and I_2 are defined by

$$(3.32) \quad I_1 = \frac{1}{4} \sum_{k,k'} \int_{D_{kk'}} R_{t_k u}^{t_k u} R_{t_{k'} u'}^{t_{k'} u'} dR(u, u') \quad \text{and} \quad I_2 = \frac{1}{4} \sum_{k,k'} \int_{D_{kk'}} (R_{t_k u}^{t_{k'} u'})^2 dR(u, u').$$

In the following we bound the two quantities I_1 and I_2 .

In order to bound the term I_2 in (3.32), let us set $\varphi(u, u') = (R_{t_k u}^{t_{k'} u'})^2$. We will first estimate the ρ -var norm of φ . To this aim, we decompose the rectangular increments of φ as follows

$$(3.33) \quad \varphi_{vu}^{v'u'} = R_{vu}^{v'u'} (R_{t_k u}^{t_{k'} v'} + R_{t_k v}^{t_{k'} u'}) + R_{vu}^{t_{k'} u'} (R_{t_k u}^{v'u'} + R_{t_k v}^{v'u'}).$$

From this decomposition it is readily checked using Hypothesis 2.19 that for $[v, u] \times [v', u'] \subset D_{kk'}$ we have

$$(3.34) \quad \begin{aligned} |\varphi_{vu}^{v'u'}| &\leq C \omega_R([v, u] \times [v', u'])^{1/\rho} \cdot \omega_R(D_{kk'})^{1/\rho} \\ &\quad + C \omega_R([v, u] \times [t_{k'}, t_{k'+1}])^{1/\rho} \cdot \omega_R([t_k, t_{k+1}] \times [v', u'])^{1/\rho}. \end{aligned}$$

This inequality can be used in order to evaluate the 2-d ρ -var norm of φ over the rectangle $D_{kk'}$. Indeed, let \mathcal{P} and \mathcal{P}' be partitions of $[t_k, t_{k+1}]$ and $[t_{k'}, t_{k'+1}]$, respectively. From (3.34) we have

$$\begin{aligned} \sum_{(v,u) \in \mathcal{P}, (v',u') \in \mathcal{P}'} |\varphi_{vu}^{v'u'}|^\rho &\leq \omega_R(D_{kk'}) \sum_{(v,u) \in \mathcal{P}, (v',u') \in \mathcal{P}'} \omega_R([v, u] \times [v', u']) \\ &\quad + \sum_{(v,u) \in \mathcal{P}} \omega_R([v, u] \times [t_{k'}, t_{k'+1}]) \cdot \sum_{(v',u') \in \mathcal{P}'} \omega_R([t_k, t_{k+1}] \times [v', u']), \end{aligned}$$

which, by the super-additivity of ω_R , easily yields

$$\sum_{(v,u) \in \mathcal{P}, (v',u') \in \mathcal{P}'} |\varphi_{vu}^{v'u'}|^\rho \leq \omega_R(D_{kk'})^2.$$

Since \mathcal{P} and \mathcal{P}' are generic partitions of $[t_k, t_{k+1}]$, this implies that

$$(3.35) \quad \|\varphi\|_{\rho\text{-var}; D_{kk'}} \leq \omega_R(D_{kk'})^{2/\rho}.$$

With relation (3.35) in hand, we can now establish a bound for I_2 . Indeed, recall that $\varphi(u, u') = (R_{t_k u}^{t_{k'} u'})^2$. In particular, we have $\varphi(t_k, u') = 0$ and $\varphi(u, t_{k'}) = 0$, and we get

$$\int_{D_{kk'}} (R_{t_k u}^{t_{k'} u'})^2 dR(u, u') = \int_{D_{kk'}} \varphi_{t_k u}^{t_{k'} u'} dR(u, u').$$

Therefore, plugging (3.35) in the definition (3.32) of I_2 and applying Lemma 3.2, we end up with

$$(3.36) \quad |I_2| \leq C \cdot \sum_{k,k'} \omega_R(D_{kk'})^{3/\rho}.$$

The estimation of I_1 can be done along the same lines as for I_2 . Namely, we define a function $\psi(u, u') = R_{t_k u}^{t_k u} R_{t_{k'} u'}^{t_{k'} u'}$. Then the rectangular increments of ψ can be decomposed as

$$\psi_{vu}^{v'u'} = (R_{t_k u}^{vu} + R_{t_k v}^{vu})(R_{t_{k'} u'}^{v'u'} + R_{t_{k'} v'}^{v'u'}).$$

from which we can deduce

$$(3.37) \quad |\psi_{vu}^{v'u'}| \leq C \cdot \omega_R([t_k, t_{k+1}] \times [v, u])^{1/\rho} \cdot \omega_R([t_{k'}, t_{k'+1}] \times [v', u'])^{1/\rho}.$$

Starting from (3.37) we can proceed as in (3.35)–(3.36). We arrive at

$$\|\psi\|_{\rho\text{-var}; D_{kk'}} \leq C \omega_R(D_{kk})^{1/\rho} \omega_R(D_{k'k'})^{1/\rho}.$$

Now applying Lemma 3.2 again we end up with

$$(3.38) \quad |I_1| \leq C \sum_{k,k'} \omega_R(D_{kk})^{1/\rho} \omega_R(D_{k'k'})^{1/\rho} \omega_R(D_{kk'})^{1/\rho}.$$

Let us conclude our estimates for this step: plugging (3.38) and (3.36) into (3.31) we have obtained

$$(3.39) \quad \|\delta g_{st}^{ii\ell}\|_2^2 \leq C \sum_{k,k'} [\omega_R(D_{kk'})^{3/\rho} + \omega_R(D_{kk})^{1/\rho} \omega_R(D_{k'k'})^{1/\rho} \omega_R(D_{kk'})^{1/\rho}].$$

Case $i = \ell \neq j$. We are now considering the general term

$$X_{t_k t_{k+1}}^{3,iji} = \int_{t_k < u < v < w < t_{k+1}} dX_u^i dX_v^j dX_w^i.$$

We first apply the shuffle identities for weakly geometric rough paths to get

$$X_{t_k t_{k+1}}^{3,iji} = X_{t_k t_{k+1}}^{1,i} \cdot X_{t_k t_{k+1}}^{2,ij} - 2X_{t_k t_{k+1}}^{3,ij}.$$

Plugging the above expression into our definition of g , we have obtained

$$(3.40) \quad \|\delta g_{st}^{iji}\|_2 \leq \left\| \sum_{s \leq t_k < t} X_{t_k t_{k+1}}^{1,i} \cdot X_{t_k t_{k+1}}^{2,ij} \right\|_2 + 2\|\delta g_{st}^{ijj}\|_2.$$

Note that the last term $2\|\delta g_{st}^{ijj}\|_2$ above is bounded by (3.39). In the following we thus focus on the estimate for the first term in the right hand side of (3.40). To this aim, we compute

$$\left\| \sum_{s \leq t_k < t} X_{t_k t_{k+1}}^{1,i} \cdot X_{t_k t_{k+1}}^{2,ij} \right\|_2^2 = \sum_{k,k'} \mathbb{E}[X_{t_k t_{k+1}}^{1,i} \cdot X_{t_k t_{k+1}}^{2,ij} \cdot X_{t_k' t_{k'+1}}^{1,i} \cdot X_{t_k' t_{k'+1}}^{2,ij}].$$

Then invoking the same regularization procedure as for (3.28), we get

$$\|X_{t_k t_{k+1}}^{1,i} \cdot X_{t_k t_{k+1}}^{2,ij}\|_2^2 = \sum_{k,k'} \int_{D_{kk'}} \mathbb{E}[X_{t_k t_{k+1}}^{1,i} \cdot X_{t_k u}^{1,i} \cdot X_{t_k' t_{k'+1}}^{1,i} \cdot X_{t_k' u'}^{1,i}] dR(u, u').$$

Along the same lines as for (3.31), we then obtain the decomposition

$$(3.41) \quad \left\| \sum_{s \leq t_k < t} X_{t_k t_{k+1}}^{1,i} \cdot X_{t_k t_{k+1}}^{2,ij} \right\|_2^2 =: I_3 + I_4 + I_5,$$

where the integrals I_3 , I_4 , I_5 are respectively defined by

$$(3.42) \quad I_3 = \sum_{k,k'} R_{t_k t_{k+1}}^{t_k' t_{k'+1}} \int_{D_{kk'}} R_{t_k u}^{t_k' u'} dR(u, u'), \quad I_4 = \sum_{k,k'} \int_{D_{kk'}} R_{t_k u}^{t_k' t_{k'+1}} R_{t_k' u'}^{t_k t_{k+1}} dR(u, u'),$$

$$I_5 = \sum_{k,k'} \int_{D_{kk'}} R_{t_k' u'}^{t_k' t_{k'+1}} R_{t_k u}^{t_k t_{k+1}} dR(u, u').$$

Let us bound the three terms above. First, a direct application of Lemma 3.2 yields

$$(3.43) \quad |I_3| \leq C \sum_{k,k'} \omega_R(D_{kk'})^{3/\rho}.$$

We turn to the estimate of I_4 and I_5 in (3.42). Towards this aim and similarly to what we did in order to get (3.36), set $\phi(u, u') = R_{t_k u}^{t_k' t_k'+1} R_{t_k' u'}^{t_k t_k+1}$. Then it is readily checked that the rectangular increments of ϕ are upper bounded by

$$|\phi_{uv}^{u'v'}| \leq |R_{uv}^{t_k' t_k'+1}| \cdot |R_{u'v'}^{t_k t_k+1}|.$$

Therefore, we can write

$$\sum_{(u,v) \in \mathcal{P}} \sum_{(u',v') \in \mathcal{P}'} |\phi_{uv}^{u'v'}|^\rho \leq \sum_{(u,v) \in \mathcal{P}} |R_{uv}^{t_k' t_k'+1}|^\rho \cdot \sum_{(u',v') \in \mathcal{P}'} |R_{u'v'}^{t_k t_k+1}|^\rho.$$

Note that by the definition of 2D-variation both summations on the right side above are bounded by $\omega_R(D_{kk'})$. We thus obtain

$$(3.44) \quad \|\phi\|_{\rho\text{-var}, D_{kk'}} \leq C \omega_R(D_{kk'})^{2/\rho}.$$

Now applying Lemma 3.2 to I_4 in (3.42) and using (3.44) we obtain the estimate

$$(3.45) \quad |I_4| \leq C \sum_{k,k'} \omega_R(D_{kk'})^{3/\rho}.$$

The estimation of I_5 can be obtained along the lines as for I_4 , and we obtain

$$(3.46) \quad |I_5| \leq C \sum_{k,k'} \omega_R(D_{kk'})^{1/\rho} \omega_R(D_{kk})^{1/\rho} \omega_R(D_{k'k'})^{1/\rho}.$$

To finish this step, let us gather our estimates for the case $i = \ell \neq j$. Plugging inequalities (3.43), (3.45) and (3.46) into (3.41), and then back into (3.40), we get a relation which mimics (3.39):

$$(3.47) \quad \|\delta g_{st}^{iji}\|_2^2 \leq C \sum_{k,k'} [\omega_R(D_{kk'})^{3/\rho} + \omega_R(D_{kk})^{1/\rho} \omega_R(D_{k'k'})^{1/\rho} \omega_R(D_{kk'})^{1/\rho}].$$

Case $i \neq j = \ell$. This situation is handled along the same lines as in the previous cases, and details are left to the reader. Namely, applying Fubini's theorem we have

$$X_{t_k t_{k+1}}^{3,iji} = \frac{1}{2} \int_{t_k}^{t_{k+1}} (X_{ut_{k+1}}^{1,j})^2 dX_u^{1,i}.$$

Using this relation we can write $\|\delta g_{st}^{iji}\|_2^2$ as in (3.30) with $X_{t_k u}^{1,i}$ and $X_{t_k' u'}^{1,i}$ replaced respectively by $X_{ut_{k+1}}^{1,i}$ and $X_{u't_{k'+1}}^{1,i}$. The estimation of $\|\delta g_{st}^{iji}\|_2^2$ is thus similar to the case when $i \neq j = \ell$ and we can show that the relation (3.39) also holds true for δg_{st}^{iji} .

Step 4: Conclusion. Let us summarize our considerations so far. Gathering the upper bounds (3.27), (3.29), (3.39), (3.40) and the case $i \neq j = \ell$, we have proved relation (3.24) for all possible values of $i, j, \ell \in \{1, \dots, d\}$ and $p = 2$. Recall again that the general case $p \geq 2$ is obtained by hypercontractivity, which finishes our proof. \square

Let g be the increment in relation (3.23). We now wish to obtain an upper bound for g similar to the bound (3.11) we have derived for F . This is the content of the next proposition.

Proposition 3.7. *Let X and g be as in Lemma 3.6. Let $\theta > 1$ be such that $\frac{1}{\theta} + \frac{1}{\rho} = 1$. Then there exists a constant $C > 0$ such that for all $i, j, \ell \in \{1, \dots, d\}$ and $(s, t) \in \mathcal{S}_2$ we have*

$$(3.48) \quad \|\delta g_{st}^{ij\ell}\|_p^2 \leq C \max_{k,k'} \omega_R(D_{kk'})^{3/\rho-1} \cdot \omega_R([s, t]^2) + C \max_k \omega_R(D_{kk})^{2(1/\rho-1/\theta)} \cdot \omega_R([s, t]^2)^{2/\theta+1/\rho}.$$

In particular, for ε such that $0 \leq \varepsilon \leq (3 - \rho) \wedge (2 - 2\rho/\theta)$ we have

$$(3.49) \quad \|\delta g_{st}^{ij\ell}\|_p^2 \leq C \max_{k,k'} \omega_R(D_{kk'})^{\frac{\varepsilon}{\rho}} \cdot \omega_R([s, t]^2)^{\frac{3-\varepsilon}{\rho}}.$$

Proof. We first observe that since $\rho < 2$ and $\frac{1}{\theta} + \frac{1}{\rho} = 1$, we have $\theta > 2 > \rho$. Applying Hölder's inequality to the second term on the right side of (3.24) yields

$$\left| \sum_{k,k'} [\omega_R(D_{kk})\omega_R(D_{k'k'})\omega_R(D_{kk'})]^{1/\rho} \right| \leq \left(\sum_{k,k'} [\omega_R(D_{kk})\omega_R(D_{k'k'})]^{2/\rho} \right)^{1/\theta} \cdot \left(\sum_{k,k'} \omega_R(D_{kk'}) \right)^{1/\rho}.$$

Hence thanks to an elementary algebraic manipulation and according to the definition of ρ -variation we get

$$\begin{aligned} \left| \sum_{k,k'} [\omega_R(D_{kk})\omega_R(D_{k'k'})\omega_R(D_{kk'})]^{1/\rho} \right| &\leq \left(\sum_k \omega_R(D_{kk})^{\theta/\rho} \right)^{2/\theta} \omega_R([s, t]^2)^{1/\rho} \\ &= \left(\sum_k \omega_R(D_{kk})^{\theta/\rho-1} \omega_R(D_{kk}) \right)^{2/\theta} \omega_R([s, t]^2)^{1/\rho}. \end{aligned}$$

Thus bounding the term $\omega_R(D_{kk})^{\theta/\rho-1}$ above by $\max_{k,k'} \omega_R(D_{kk})^{(\theta-\rho)/\rho}$ and owing to the super additive property of Hypothesis 2.19, we end up with

$$\begin{aligned} (3.50) \quad \left| \sum_{k,k'} [\omega_R(D_{kk})\omega_R(D_{k'k'})\omega_R(D_{kk'})]^{1/\rho} \right| &\leq \left(\max_k \omega_R(D_{kk})^{2(1/\rho-1/\theta)} \right) \cdot \omega_R([s, t]^2)^{2/\theta} \cdot \omega_R([s, t]^2)^{1/\rho} \\ &= \left(\max_k \omega_R(D_{kk})^{2(1/\rho-1/\theta)} \right) \cdot \omega_R([s, t]^2)^{2/\theta+1/\rho}. \end{aligned}$$

On the other hand, it is easy to see that

$$(3.51) \quad \sum_{k,k'} \omega_R(D_{kk'})^{3/\rho} \leq \max_{k,k'} \omega_R(D_{kk'})^{3/\rho-1} \cdot \omega_R([s, t]^2).$$

Gathering (3.50) and (3.51) in (3.24), this concludes relation (3.48). Relation (3.49) follows immediately from (3.48) and the fact that $\omega_R(D_{kk'}) \leq \omega_R([s, t]^2)$. \square

In the following, we turn to the estimate of another third-chaos functional.

Lemma 3.8. *Let X and F be as in Lemma 3.5. For $i, j, \ell = 1, \dots, d$ and $(s, t) \in \mathcal{S}_2$, we define the increment $h_{st}^{ij\ell}$ as:*

$$h_{st}^{ij\ell} = \sum_{s \leq t_k < t} X_{st_k}^{1,\ell} \delta F_{t_k t_{k+1}}^{ij}.$$

In addition, consider ε such that $0 \leq \varepsilon \leq 2 - \rho$. Then the following inequality holds true:

$$(3.52) \quad \|h_{st}^{ij\ell}\|_p^2 \leq 4(\omega_R([s, t]^2)^{\frac{3}{\rho}-\frac{2\varepsilon}{\rho}} + \omega_R([s, t]^2)^{\frac{3}{\rho}-\frac{\varepsilon}{\rho}}) \cdot \max_k \omega_R([t_k, t_{k+1}] \times [0, T])^{\frac{2\varepsilon}{\rho}}.$$

Proof. As in the proof of Lemma 3.6 we should distinguish cases according to possible equalities in the indices i, j, ℓ . We focus on the case $i = j$ in Step 1 to 3 below, and then deal with the case $i \neq j$ in Step 4.

Step 1: A decomposition of $\|h\|_2^2$. As mentioned above, let us first consider the case $i = j$ and find an estimate for $h_{st}^{ii\ell}$. We start by writing

$$(3.53) \quad E[|h_{st}^{ii\ell}|^2] = \sum_{k,k'} E[X_{st_k}^{1,\ell} X_{st_{k'}}^{1,\ell} \delta F_{t_k t_{k+1}}^{ii} \delta F_{t_{k'} t_{k'+1}}^{ii}].$$

In addition, recall from (2.18) and (3.13) that

$$(3.54) \quad \delta X_{st}^i = \delta^\diamond(\mathbf{1}_{[s,t]} e_i), \quad \text{and} \quad \delta F_{t_k t_{k+1}}^{ii} = \frac{1}{2} \sigma_{t_k t_{k+1}}^2 H_2 \left(\frac{X_{t_k t_{k+1}}^{1,i}}{\sigma_{t_k t_{k+1}}} \right).$$

Therefore, one can recast (3.53) as

$$(3.55) \quad E[|h_{st}^{ii\ell}|^2] = \frac{1}{4} \sum_{k,k'} \sigma_{t_k t_{k+1}}^2 \sigma_{t_{k'} t_{k'+1}}^2 E[\delta^\diamond(\mathbf{1}_{[s,t_k]} e_\ell) Z_{kk'}],$$

where the random variable $Z_{kk'}$ is defined by

$$(3.56) \quad Z_{kk'} = X_{st_k}^{1,\ell} H_2\left(\frac{X_{t_k t_{k+1}}^{1,i}}{\sigma_{t_k t_{k+1}}}\right) H_2\left(\frac{X_{t_{k'} t_{k'+1}}^{1,i}}{\sigma_{t_{k'} t_{k'+1}}}\right).$$

Hence resorting to the integration by parts formula (2.17), we get that:

$$E[|h_{st}^{ii\ell}|^2] = \frac{1}{4} \sum_{k,k'} \sigma_{t_k t_{k+1}}^2 \sigma_{t_{k'} t_{k'+1}}^2 E[\langle \mathbf{1}_{[s,t_k]} e_\ell, \mathbf{D} Z_{kk'} \rangle_{\mathcal{H}}],$$

where we recall that e_ℓ stands for the ℓ -th element of the canonical basis in \mathbb{R}^d . Computing the Malliavin derivative of $Z_{kk'}$ (and recalling that $H_2'(x) = x$), we let the reader check that we get the formula

$$(3.57) \quad E[|h_{st}^{ii\ell}|^2] = \frac{1}{4} \sum_{k,k'} (J_{kk'}^1 + J_{kk'}^2 + J_{kk'}^3),$$

where the terms $J_{kk'}^1, J_{kk'}^2, J_{kk'}^3$ are respectively defined by

$$(3.58) \quad \begin{aligned} J_{kk'}^1 &= E[X_{st_k}^{1,\ell} X_{st_{k'}}^{1,\ell}] \cdot E[\delta F_{t_k t_{k+1}}^{ii} \delta F_{t_{k'} t_{k'+1}}^{ii}], \\ J_{kk'}^2 &= \langle \mathbf{1}_{[s,t_k]}, \mathbf{1}_{[t_k, t_{k+1}]} \rangle_{\mathcal{H}} \cdot \langle \mathbf{1}_{[s,t_{k'}]}, \mathbf{1}_{[t_{k'}, t_{k'+1}]} \rangle_{\mathcal{H}} \cdot \langle \mathbf{1}_{[t_k, t_{k+1}]}, \mathbf{1}_{[t_{k'}, t_{k'+1}]} \rangle_{\mathcal{H}} \cdot \mathbf{1}_{\{i=\ell\}}, \\ J_{kk'}^3 &= \langle \mathbf{1}_{[s,t_k]}, \mathbf{1}_{[t_{k'}, t_{k'+1}]} \rangle_{\mathcal{H}} \cdot \langle \mathbf{1}_{[s,t_{k'}]}, \mathbf{1}_{[t_k, t_{k+1}]} \rangle_{\mathcal{H}} \cdot \langle \mathbf{1}_{[t_k, t_{k+1}]}, \mathbf{1}_{[t_{k'}, t_{k'+1}]} \rangle_{\mathcal{H}} \cdot \mathbf{1}_{\{i=\ell\}}. \end{aligned}$$

In the following, we show that the upper-bound in (3.52) holds for each $J_{kk'}^a, a = 1, 2, 3$, and therefore concludes the lemma.

Step 2: Estimate for $J_{kk'}^1$. In order to bound $J_{kk'}^1$, we use the definition (2.12) as well as Hypothesis 2.19 in order to get

$$|E[X_{st_k}^{1,\ell} X_{st_{k'}}^{1,\ell}]| = |R_{st_k}^{st_{k'}}| \leq \omega_R([s, t_k] \times [s, t_{k'}])^{1/\rho} \leq \omega_R([s, t]^2)^{1/\rho}.$$

Furthermore, invoking relations (3.13) and (3.15), we get

$$|E[\delta F_{t_k t_{k+1}}^{ii} \delta F_{t_{k'} t_{k'+1}}^{ii}]| = \frac{1}{2} (R_{t_{k'} t_{k'+1}}^{t_k t_{k+1}})^2 \leq \omega_R(D_{kk'})^{2/\rho}.$$

Hence resorting to the same arguments as in Lemma 3.5 in order to get inequality (3.17), we get that for any $\varepsilon \leq 2 - \rho$ we have:

$$(3.59) \quad \sum_{k,k'} |J_{kk'}^1| \leq \omega_R([s, t]^2)^{1/\rho} \cdot \sum_{k,k'} \omega_R(D_{kk'})^{2/\rho} \leq \max_k \omega_R(D_{kk'})^{\frac{\varepsilon}{\rho}} \cdot \omega_R([s, t]^2)^{\frac{3-\varepsilon}{\rho}}.$$

Otherwise stated, inequality (3.52) is satisfied for $\sum_{k,k'} |J_{kk'}^1|$.

Step 3: Estimate for $J_{kk'}^2$ and $J_{kk'}^3$. We turn to an upper bound of the term $J_{kk'}^2$ in (3.58). To this aim, consider $\theta > 2$ such that $\frac{1}{\theta} + \frac{1}{\rho} = 1$. Then applying Hölder's inequality to the summation in k, k' of (3.58), we get

$$\begin{aligned} \sum_{k,k'} J_{kk'}^2 &\leq \left(\sum_{k,k'} |\langle \mathbf{1}_{[s,t_k]}, \mathbf{1}_{[t_k, t_{k+1}]} \rangle_{\mathcal{H}} \langle \mathbf{1}_{[s,t_{k'}]}, \mathbf{1}_{[t_{k'}, t_{k'+1}]} \rangle_{\mathcal{H}}|^\theta \right)^{1/\theta} \\ &\quad \times \left(\sum_{k,k'} |\langle \mathbf{1}_{[t_k, t_{k+1}]}, \mathbf{1}_{[t_{k'}, t_{k'+1}]} \rangle_{\mathcal{H}}|^\rho \right)^{1/\rho}. \end{aligned}$$

Now we apply the estimate $\langle \mathbf{1}_{[u,v]}, \mathbf{1}_{[u',v']} \rangle_{\mathcal{H}} \leq \|R\|_{\rho\text{-var}, [u,v] \times [u',v']}$ to the three inner products in the above inequality. We arrive at

$$\begin{aligned}
 \sum_{k,k'} J_{kk'}^2 &\leq \left(\sum_{k,k'} \|R\|_{\rho\text{-var}, [t_k, t_{k+1}] \times [s, t]} \cdot \|R\|_{\rho\text{-var}, [t_{k'}, t_{k'+1}] \times [s, t]} \right)^{1/\theta} \cdot \left(\sum_{k,k'} \|R\|_{\rho\text{-var}, D_{kk'}}^\rho \right)^{1/\rho} \\
 (3.60) \quad &\leq \left(\sum_k \|R\|_{\rho\text{-var}, [t_k, t_{k+1}] \times [s, t]}^\theta \right)^{2/\theta} \cdot \omega_R([s, t]^2)^{1/\rho}.
 \end{aligned}$$

Similarly to what we have done in the proof of Proposition 3.7, we note that $\theta > 2 > \rho$. Hence combining Hypothesis 2.19 and the super-additivity of ω_R we get the following estimate for $0 < \varepsilon \leq 2 - \rho$:

$$(3.61) \quad \sum_{k,k'} J_{kk'}^2 \leq \max_k \omega_R([t_k, t_{k+1}] \times [s, t])^{\varepsilon/\rho} \cdot \max_{k'} \omega_R([t_{k'}, t_{k'+1}] \times [s, t])^{\varepsilon/\rho} \cdot \omega_R([s, t]^2)^{\frac{3-2\varepsilon}{\rho}}.$$

This implies that the upper-bound estimate (3.52) holds for $J_{kk'}^2$. It can also be shown that the same estimate holds for $J_{kk'}^3$. The proof is similar to that of $J_{kk'}^2$ and will be omitted. Combining (3.59) and (3.61), this completes the proof of (3.52) for $i = j$.

Step 4: The case $i \neq j$. We now turn to an estimate of $h_{st}^{ij\ell}$ when $i \neq j$. To this aim we first write an expression for $E[|h_{st}^{ij\ell}|^2]$ mimicking (3.53), with the important difference that the term $\delta F_{t_k t_{k+1}}^{ij}$ cannot be represented by Hermite polynomials as in (3.54). Note that when $i \neq j$ the integral $\int_s^t \delta X_{su}^i dX_u^j$ can be interpreted as a Wiener integral and thus it is also a Skorohod integral. Hence the equivalents for (3.55) and (3.56) whenever $i \neq j$ is

$$(3.62) \quad E[|h_{st}^{ij\ell}|^2] = \sum_{k,k'} E[\delta^\diamond(\mathbf{1}_{[s,t_k]} e_\ell) \tilde{Z}_{kk'}],$$

where

$$\tilde{Z}_{kk'} = X_{st_{k'}}^{1,\ell} \cdot \delta^\diamond(X_{t_k}^{1,i} \cdot \mathbf{1}_{[t_k, t_{k+1}]} e_j) \cdot \delta^\diamond(X_{t_{k'}}^{1,i} \cdot \mathbf{1}_{[t_{k'}, t_{k'+1}]} e_j).$$

Integrating relation (3.62) by parts similarly to (3.57), we end up with

$$E[|h_{st}^{ij\ell}|^2] = \sum_{k,k'} (\tilde{J}_{kk'}^1 + J_{kk'}^4),$$

where $\tilde{J}_{kk'}^1$ is given by

$$\tilde{J}_{kk'}^1 = \langle \mathbf{1}_{[s,t_k]}, \mathbf{1}_{[s,t_{k'}]} \rangle_{\mathcal{H}} E[\delta^\diamond(X_{t_k}^{1,i} \cdot \mathbf{1}_{[t_k, t_{k+1}]} e_j) \cdot \delta^\diamond(X_{t_{k'}}^{1,i} \cdot \mathbf{1}_{[t_{k'}, t_{k'+1}]} e_j)],$$

which can be bounded in a similar way as for $J_{kk'}^1$ in (3.58) and (3.59). As far as the term $J_{kk'}^4$ is concerned, it is defined by

$$\begin{aligned}
 (3.63) \quad J_{kk'}^4 &= \int_{D_{kk'}} (R_{st_k}^{t_{k'}u} R_{st_{k'}}^{t_k u'} + R_{st_k}^{t_k u'} R_{st_{k'}}^{t_{k'}u}) dR(u, u') \cdot \mathbf{1}_{\{i=\ell\}} \\
 &+ \int_{D_{kk'}} (R_{st_k}^{u t_{k'+1}} R_{st_{k'}}^{u' t_{k+1}} + R_{st_k}^{u' t_{k+1}} R_{st_{k'}}^{u t_{k'+1}}) dR(u, u') \cdot \mathbf{1}_{\{j=\ell\}}.
 \end{aligned}$$

In order to bound the expression above, we set $\phi(u, u') = R_{st_k}^{t_{k'}u} R_{st_{k'}}^{t_k u'}$. Then one of the terms in (3.63) is $\int_{D_{kk'}} \phi(u, u') dR(u, u')$. We wish to bound this term thanks to Lemma 3.2. To this aim, similarly to what we did in (3.33)–(3.34), we estimate the rectangular increments of ϕ . We get

$$|\phi_{uv}^{u'v'}| = |R_{st_k}^{uv} R_{st_{k'}}^{u'v'}| \leq \|R\|_{\rho\text{-var}, [s,t] \times [u,v]} \cdot \|R\|_{\rho\text{-var}, [s,t] \times [u',v']}.$$

Now we consider \mathcal{P} and \mathcal{P}' generic partitions of $[t_k, t_{k+1}]$ and $[t_{k'}, t_{k'+1}]$ respectively, as well as θ such that $\frac{1}{\theta} + \frac{1}{\rho} = 1$. Then we have

$$\begin{aligned} \sum_{[u,v] \times [u',v'] \in \mathcal{P} \times \mathcal{P}'} |\phi_{uv}^{u'v'}|^\theta &\leq \sum_{[u,v] \in \mathcal{P}} \|R\|_{\rho\text{-var}, [s,t] \times [u,v]}^\theta \cdot \sum_{[u',v'] \in \mathcal{P}'} \|R\|_{\rho\text{-var}, [s,t] \times [u',v']}^\theta \\ &\leq \|R\|_{\rho\text{-var}, [s,t] \times [t_k, t_{k+1}]}^\theta \cdot \|R\|_{\rho\text{-var}, [s,t] \times [t_{k'}, t_{k'+1}]}^\theta. \end{aligned}$$

Therefore, we obtain

$$(3.64) \quad \|\phi\|_{\theta\text{-var}, D_{kk'}} \leq \|R\|_{\rho\text{-var}, [s,t] \times [t_k, t_{k+1}]} \cdot \|R\|_{\rho\text{-var}, [s,t] \times [t_{k'}, t_{k'+1}]}.$$

Note that the estimate (3.64) of ϕ also holds for the other three functions in the right-hand side of (3.63), namely:

$$R_{st_k}^{t_k u'} R_{st_{k'}}^{t_{k'} u}, \quad R_{st_k}^{u' t_{k'+1}} R_{st_{k'}}^{u' t_{k+1}}, \quad R_{st_k}^{u' t_{k+1}} R_{st_{k'}}^{u' t_{k'+1}}.$$

The proof is similar and will be left to the reader. With (3.64) in hand, we can now invoke Lemma 3.2 for the right-hand side of relation (3.63). This yields the existence of a constant C such that

$$(3.65) \quad |J_{kk'}^4| \leq C \cdot \|R\|_{\rho\text{-var}, [s,t] \times [t_k, t_{k+1}]} \cdot \|R\|_{\rho\text{-var}, [s,t] \times [t_{k'}, t_{k'+1}]} \cdot \|R\|_{\rho\text{-var}, D_{kk'}}.$$

Starting from (3.65), we easily get an upper bound similar to (3.60) for $\sum_{k,k'} J_{kk'}^4$. Then we can proceed as in relation (3.61). We conclude that (3.52) holds for the case $i \neq j$. The proof is now complete. \square

3.3. Upper-bounds for weighted sums

In this section we give some estimates for weighted sums of the processes F and g defined in the previous subsection. These sums will be a part of our main terms in the analysis of the trapezoid rule.

Lemma 3.9. *Let X be a \mathbb{R}^d valued Gaussian process with covariance function R such that Hypothesis 2.17 holds with $\rho \in [1, 2)$, and therefore Hypothesis 2.19 is guaranteed by Remark 2.18 and subsequent comments. Let ω_R be the control in Hypothesis 2.19. Recall that the increment F is defined in (3.10) and fix a partition \mathcal{P} with mesh $|\mathcal{P}|$. We also consider a controlled process of order 2 according to Definition 2.8, which means in particular that the increments of y can be decomposed as*

$$(3.66) \quad y_{st} = y_s^1 X_{st}^1 + r_{st}, \quad s, t \in [0, T].$$

We call ω the control ω_y related to the increments of y in Definition 2.8, and recall that we have

$$(3.67) \quad |\delta y_{st}^1| \leq \omega(s, t)^{1/p}, \quad |r_{st}| \leq \omega(s, t)^{2/p}, \quad \text{for all } (s, t) \in \mathcal{S}_2([0, T])$$

almost surely. Eventually, we introduce below a parameter p such that $\frac{1}{p} = \frac{1-\varepsilon}{2\rho}$ for ε small enough. Then the following holds true:

(i) For every $M > 0$, we set $A_M = \{\omega(0, T) \leq M\}$. Then for all $(s, t) \in \mathcal{S}_2$ and $(i, j) \in \{1, \dots, d\}^2$ we have:

$$(3.68) \quad E \left[\mathbf{1}_{A_M} \cdot \left| \sum_{s \leq t_k < t} y_{t_k} \delta F_{t_k t_{k+1}}^{ij} \right| \right] \leq C \cdot \max_k \omega_R([t_k, t_{k+1}] \times [0, T])^{\frac{\varepsilon}{2\rho}}.$$

In particular,

$$\sum_{s \leq t_k < t} y_{t_k} \delta F_{t_k t_{k+1}}^{ij} \longrightarrow 0, \quad \text{in probability as } |\mathcal{P}| \rightarrow 0.$$

(ii) In case of Hölder continuous processes X and y , one can improve the convergence as follows. Namely suppose that $\delta y^1 \in \mathcal{C}^{1/p}$ and $r \in \mathcal{C}^{2/p}$ almost surely and that

$$(3.69) \quad \omega_R([s, t] \times [0, T]) \leq C|t - s|, \quad \text{for all } (s, t) \in \mathcal{S}_2([0, T]).$$

Furthermore, assume that the uniform partition $0 = t_0 < \dots < t_n = T$ over $[0, T]$ is considered. Then

$$(3.70) \quad \sum_{s \leq t_k < t} y_{st_k} \delta F_{t_k t_{k+1}}^{ij} \longrightarrow 0, \quad \text{almost surely as } n \rightarrow \infty.$$

Proof. *Step 1: A decomposition.* Consider $(s, t) \in \mathcal{S}_2$, where we recall that \mathcal{S}_2 stands for $\mathcal{S}_2([0, T])$. With the help of (3.66) we have the following decomposition

$$(3.71) \quad \sum_{s \leq t_k < t} y_{st_k} \delta F_{t_k t_{k+1}}^{ij} = y_s^1 \sum_{s \leq t_k < t} X_{st_k}^1 \delta F_{t_k t_{k+1}}^{ij} + \tilde{r}_{st},$$

where we denote

$$(3.72) \quad \tilde{r}_{st} = \sum_{s \leq t_k < t} r_{st_k} \delta F_{t_k t_{k+1}}^{ij}.$$

Step 2: Calculations for $\delta \tilde{r}$. In the following, in order to estimate \tilde{r} we first estimate $\delta \tilde{r}$. To this aim we observe that a simple computation yields $\delta r_{sut} = \delta y_{su}^1 X_{ut}^1$. Therefore, starting from (3.72) and using Definition 2.2 for the increment $\delta \tilde{r}$, some elementary calculations show that

$$(3.73) \quad \begin{aligned} \delta \tilde{r}_{sut} &= r_{su} \sum_{u \leq t_k < t} \delta F_{t_k t_{k+1}}^{ij} + \sum_{u \leq t_k < t} \delta r_{sut_k} \delta F_{t_k t_{k+1}}^{ij} \\ &= r_{su} \sum_{u \leq t_k < t} \delta F_{t_k t_{k+1}}^{ij} + \delta y_{su}^1 \sum_{u \leq t_k < t} X_{ut_k}^1 \delta F_{t_k t_{k+1}}^{ij}. \end{aligned}$$

Step 3: Moment estimates of $\delta \tilde{r}$ and \tilde{r} . We now hinge on relation (3.73) in order to upper bound \tilde{r} . We start by denoting

$$(3.74) \quad \begin{aligned} y^{1,M} &= \mathbf{1}_{A_M} \cdot y^1, & r^M &= \mathbf{1}_{A_M} \cdot r, & \tilde{r}^M &= \mathbf{1}_{A_M} \cdot \tilde{r}, \\ \delta \tilde{r}^M &= \mathbf{1}_{A_M} \cdot \delta \tilde{r}, & \omega^M(s, t) &= E[\mathbf{1}_{A_M} \cdot \omega(s, t)]. \end{aligned}$$

It is easy to see that ω^M is a control. By the inequalities in (3.67) we also have

$$(3.75) \quad E[|y_{su}^{1,M}|^p] \leq \omega^M(s, u), \quad \text{and} \quad E[|r_{su}^M|^{p/2}] \leq \omega^M(s, u).$$

We also recall that Hypothesis 2.19 holds with some 2-d control ω_R .

Next we multiply both sides of (3.73) by $\mathbf{1}_{A_M}$. According to our notation (3.74), we get

$$(3.76) \quad \delta \tilde{r}_{sut}^M = r_{su}^M \sum_{u \leq t_k < t} \delta F_{t_k t_{k+1}}^{ij} + \delta y_{su}^{1,M} \sum_{u \leq t_k < t} X_{ut_k}^1 \delta F_{t_k t_{k+1}}^{ij}.$$

We are now in a position to apply Hölder's inequality, Lemma 3.5, Lemma 3.8 and the upper bound (3.75) in order to get the existence of a constant $C > 0$ such that

$$(3.77) \quad \begin{aligned} E[|\delta \tilde{r}_{sut}^M|] &\leq C \omega^M(s, t)^{2/p} \cdot \omega_R([s, t]^2)^{\frac{1}{\rho} - \frac{\varepsilon}{2\rho}} \cdot \max_{k, k'} \omega_R(D_{kk'})^{\frac{\varepsilon}{2\rho}} \\ &\quad + C \omega^M(s, t)^{1/p} \cdot \omega_R([s, t]^2)^{\frac{3}{2\rho} - \frac{\varepsilon}{2\rho}} \cdot \max_k \omega_R([t_k, t_{k+1}] \times [0, T])^{\frac{\varepsilon}{\rho}}. \end{aligned}$$

Let us now discuss the exponents in (3.77). Indeed, recall that we have chosen p such that $\frac{1}{p} = \frac{1-\varepsilon}{2\rho}$. Therefore, owing to the fact that $\rho \in [1, 2)$, ε can be chosen small enough so that

$$(3.78) \quad v_{p,\rho} \equiv \left(\frac{2}{p} + \frac{1-\varepsilon/2}{\rho} \right) \wedge \left(\frac{1}{p} + \frac{3}{2\rho} - \frac{\varepsilon}{2\rho} \right) > 1.$$

In the sequel we pick a μ such that $1 < \mu < v_{p,\rho}$. With this notation in hand define a bivariate function $\tilde{\omega}$ by

$$\tilde{\omega}(s, t) = (\omega^M(s, t)^{2/p} \cdot \omega_R([s, t]^2)^{\frac{1}{\rho} - \frac{\varepsilon}{2\rho}})^{1/\mu} + (\omega^M(s, t)^{1/p} \cdot \omega_R([s, t]^2)^{\frac{3}{2\rho} - \frac{\varepsilon}{2\rho}})^{1/\mu}.$$

As a direct application of [7, Exercise 1.9 item (iii)], it is readily checked that $\tilde{\omega}$ is a control. In addition, one can recast (3.77) as

$$(3.79) \quad E(|\delta\tilde{r}_{sut}^M|) \leq C \max_k \omega_R([t_k, t_{k+1}] \times [0, T])^{\frac{\varepsilon}{2\rho}} \cdot \tilde{\omega}(s, t)^\mu,$$

where we recall that C is a constant that can change from line to line.

Summarizing our considerations for this step, we have obtained that \tilde{r} is an increment from \mathcal{S}_2 to the Banach space $\mathcal{B} = L^1(\Omega)$. Moreover, relation (3.72) easily entails $\tilde{r}_{t_\ell t_{\ell+1}}^M = 0$ for any point t_ℓ of the partition \mathcal{P} , and \tilde{r} satisfies (3.79). Therefore, a direct application of Lemma 2.13 on $\mathcal{B} = L^1(\Omega)$ yields

$$E(|\tilde{r}_{st}^M|) \leq 2K\mu \max_k \omega_R([t_k, t_{k+1}] \times [0, T])^{\frac{\varepsilon}{2\rho}} \cdot \tilde{\omega}(s, t)^\mu.$$

Plugging this estimate into (3.71) and combining it with (3.52), the proof of our claim (3.68) is now achieved.

Step 4: Path-wise estimates of F . We now turn to item (ii) in our lemma, assuming Hölder-continuity for y, y^1, r and considering uniform partitions of $[0, T]$ with $t_{k+1} - t_k = T/n$. In this context, condition (3.69) allows to write the upper-bound estimate of F (3.11) in Lemma 3.5 as:

$$\|\delta F_{st}^{ij}\|_q \leq C|t - s|^{\frac{1}{\rho} - \varepsilon} \cdot |\mathcal{P}|^\varepsilon = C|t - s|^{\frac{1}{\rho} - \varepsilon} \cdot n^{-\varepsilon}, \quad \text{for all } q > 1 \text{ and } (s, t) \in \mathcal{S}_2.$$

Applying Lemma 3.3 with $z^n = F, \beta = \frac{1}{\rho} - \varepsilon,$ and $\alpha = \varepsilon$ we obtain

$$(3.80) \quad |\delta F_{st}^{ij}| \leq G \cdot (t - s)^{\frac{1}{\rho} - 2\varepsilon} \cdot n^{-\varepsilon/2},$$

where G is a random variable admitting moments of all orders. In a similar way and with the help of Lemma 3.8, we can show that

$$(3.81) \quad \left| \sum_{s \leq t_k < t} X_{st_k}^1 F_{t_k t_{k+1}}^{ij} \right| \leq G \cdot |t - s|^{\frac{3}{\rho} - 2\varepsilon} \cdot n^{-\varepsilon/4}.$$

With those preliminaries in mind, we will upper bound the increment $\sum_{s \leq t_k < t} y_{t_k} \delta F_{t_k t_{k+1}}^{ij}$ thanks to relation (3.71). Namely in the right-hand side of (3.71) we have that almost surely

$$y_s^1 \sum_{s \leq t_k < t} X_{st_k}^1 \delta F_{t_k t_{k+1}}^{ij} \rightarrow 0, \quad \text{for all } (s, t) \in \mathcal{S}_2$$

thanks to (3.81). Therefore, in order to show (3.70) it remains to show the convergence of \tilde{r} .

Step 5: Path-wise estimates of $\delta\tilde{r}$ and \tilde{r} . In order to bound $\delta\tilde{r}$ in the Hölder case, we plug (3.80) and (3.81) into the expression (3.73) we have obtained for $\delta\tilde{r}$. We end up with

$$\begin{aligned} |\delta\tilde{r}_{sut}| &\leq G(\|r\|_{2/p} |t - s|^{\frac{2}{p} + \frac{1}{\rho} - 2\varepsilon} \cdot n^{-\varepsilon/2} + \|y^1\|_{1/p} |t - s|^{\frac{1}{p} + \frac{3}{\rho} - 2\varepsilon} \cdot n^{-\varepsilon/4}) \\ &\leq n^{-\varepsilon/4} G(\|r\|_{2/p} + \|y^1\|_{1/p}) \cdot |t - s|^\mu, \end{aligned}$$

where similarly to what we did in Step 3, we take $1 < \mu < \tilde{\nu}_{p,\rho}$ with

$$\tilde{\nu}_{p,\rho} = \left(\frac{2}{p} + \frac{1}{\rho} - 2\varepsilon \right) \wedge \left(\frac{1}{p} + \frac{3}{\rho} - 2\varepsilon \right).$$

Hence one can resort to the Hölder version of the sewing lemma contained in Lemma 2.14. We get

$$|\tilde{r}_{st}| \leq CG(\|r\|_{2/p} + \|y^1\|_{1/p}) \cdot n^{-\varepsilon/4} |t - s|^\mu,$$

from which we easily deduce

$$(3.82) \quad \lim_{n \rightarrow \infty} |\tilde{r}_{st}| = 0.$$

In conclusion, plugging (3.82) and (3.81) into (3.71) we have obtained relation (3.70). The proof is complete. □

We now handle some weighted sums of the increment X^3 which will feature in our trapezoid sums.

Lemma 3.10. *As in Lemma 3.9, we consider a Gaussian process X whose covariance R satisfies Hypothesis 2.17 with $\rho \in [1, 2)$. We call ω_R the control defined in Hypothesis 2.19. Let $X^3 = \{X_{st}^{3,ij\ell}; (s, t) \in \mathcal{S}_2([0, T]), i, j, \ell = 1, \dots, d\}$ be the third order element in the rough path above X and take a sequence of partitions \mathcal{P} with mesh $|\mathcal{P}|$. We also consider a continuous process y such that $y_0 = 0$. Then the following holds true:*

- (i) *Let us assume that the increments of y are dominated by a control ω over $[0, T]$. Namely we suppose that for all $(s, t) \in \mathcal{S}_2([0, T])$ we have $|\delta y_{st}| \leq \omega(s, t)^{1/p}$ almost surely, where p is such that $\frac{1}{p} = \frac{1-\varepsilon}{2\rho}$ as in Lemma 3.9. Consider a generic index $(i, j, \ell) \in \{1, \dots, d\}^3$, and recall that A_M is defined by $A_M = \{\omega(0, T) \leq M\}$ for $M > 0$. Then we have*

$$(3.83) \quad E \left[\mathbf{1}_{A_M} \cdot \left| \sum_{s \leq t_k < t} \delta y_{st_k} X_{t_k t_{k+1}}^{3,ij\ell} \right| \right] \leq C \cdot \max_{k,k'} \omega_R(D_{kk'})^{\frac{\varepsilon}{2\rho}}.$$

In particular,

$$(3.84) \quad \sum_{s \leq t_k < t} \delta y_{st_k} X_{t_k t_{k+1}}^{3,ij\ell} \longrightarrow 0, \quad \text{in probability as } |\mathcal{P}| \rightarrow 0.$$

- (ii) *If we are in a Hölder setting, namely $y \in \mathcal{C}^{1/p}$ and ω_R verifying (3.69), and if we also consider the uniform partition \mathcal{P} (see Lemma 3.9 item (ii)), then we get*

$$(3.85) \quad \sum_{s \leq t_k < t} \delta y_{st_k} X_{t_k t_{k+1}}^{3,ij\ell} \longrightarrow 0, \quad \text{almost surely as } n \rightarrow \infty.$$

Proof. The proof is very similar to what we did for Lemma 3.9. For sake of conciseness we will only outline some of the steps, focusing mainly on getting an equivalent of (3.77). Along the same lines as (3.74), we set

$$y_s^M = \mathbf{1}_{A_M} y_s, \quad r_{st}^M = \mathbf{1}_{A_M} \cdot \sum_{s \leq t_k < t} \delta y_{st_k} X_{t_k t_{k+1}}^{3,ij\ell}.$$

In this context we let the reader check that the equivalent of relation (3.76) becomes

$$\delta r_{sut}^M = \delta y_{su}^M \cdot \sum_{u \leq t_k < t} X_{t_k t_{k+1}}^{3,ij\ell}.$$

Hence applying Hölder’s inequality with p, q such that $\frac{1}{p} + \frac{1}{q} = 1$, invoking Proposition 3.7 and recalling that ω^M is introduced in relation (3.74), we get

$$(3.86) \quad E[|\delta r_{sut}^M|] \leq E[|\delta y_{su}^M|^p]^{1/p} \cdot \left\| \sum_{u \leq t_k < t} X_{t_k t_{k+1}}^{3,ij\ell} \right\|_q \\ \leq C \omega^M(s, u)^{1/p} \cdot \max_{k,k'} \omega_R(D_{kk'})^{\frac{\varepsilon}{2\rho}} \cdot \omega_R([s, t]^2)^{\frac{3-\varepsilon}{2\rho}},$$

for a positive constant C . Observe that (3.86) corresponds to (3.77) in the proof of Lemma 3.9. Also notice that we have chosen p (with a small enough ε) so that $\frac{1}{p} + \frac{3-\varepsilon}{2\rho} > 1$. Otherwise stated, condition (3.78) holds in the current context. Therefore one can prove our claims (3.83), (3.84) and (3.85) exactly as in Lemma 3.9. \square

Remark 3.11. Combining Lemma 3.8 with the above considerations, one can easily extend the conclusions of Lemma 3.10 to sums of the form

$$\sum_{s \leq t_k < t} \delta y_{st_k} X_{t_k t_{k+1}}^{2,ij} X_{t_k t_{k+1}}^{1,\ell}.$$

Details are omitted for sake of conciseness.

3.4. Convergence of the trapezoid rule

With the previous preliminary results in hand, we are now ready to give a complete statement and carry out the proof of our main Theorem 1.2.

Theorem 3.12. *Let X be a centered \mathbb{R}^d -valued Gaussian process on $[0, T]$ with covariance function R . Suppose that Hypothesis 2.17 holds true for $\rho \in [1, 2)$. Denote the rough path lift of X by $\mathbf{X} = (X^1, X^2, X^3)$. Next consider an \mathbb{R}^d -valued controlled process y of order 3, according to Definition 2.8. Specifically, there exist processes y^1, y^2, r^0, r^1 with regularities to be specified below and such that*

$$(3.87) \quad \delta y_{st} = y_s^1 X_{st}^1 + y_s^2 X_{st}^2 + r_{st}^0, \quad \delta y_{st}^1 = y_s^2 X_{st}^1 + r_{st}^1,$$

where we recall again our Notation 2.7 on matrix products. For a given partition of $[0, T]: \mathcal{P} = \{0 = t_0 < t_1 < \dots < t_{n+1} = T\}$ with mesh size $|\mathcal{P}|$, we define the trapezoid rule:

$$(3.88) \quad \text{tr-} \mathcal{J}_0^T(y, X) = \sum_{k=0}^n \frac{y_{t_k} + y_{t_{k+1}}}{2} \cdot X_{t_k t_{k+1}}^1,$$

where we denote $y_t = (y_t^1, \dots, y_t^d)$ and simply write $(y_{t_k} + y_{t_{k+1}}) \cdot X_{t_k t_{k+1}}^1$ for the inner product $\sum_{i=1}^d (y_{t_k}^i + y_{t_{k+1}}^i) X_{t_k t_{k+1}}^{1,i}$. Then the following holds true:

- (i) Assume that the framework of Definition 2.8 prevails and call ω the control function over $[0, T]$ such that for $\frac{1}{p}$ of the form $\frac{1-\varepsilon}{2\rho}$ we have

$$(3.89) \quad |r_{st}^0| \leq \omega(s, t)^{3/p}, \quad |\delta y_{st}^2| \leq \omega(s, t)^{1/p}, \quad |r_{st}^1| \leq \omega(s, t)^{2/p}.$$

Then as the mesh size of the partition $|\mathcal{P}|$ goes to 0, we have

$$(3.90) \quad \text{tr-} \mathcal{J}_0^T(y, X) \rightarrow \int_0^T y_s d\mathbf{X}_s \quad \text{in probability,}$$

where the right hand side above designates the rough integral of y against X as given in Proposition 2.12.

- (ii) Assume we are in a Hölder setting, that is ω_R verifies (3.69). Then recall that X generates a $\frac{1}{p}$ -Hölder rough path, and we also assume that $r^0 \in \mathcal{C}^{3/p}, r^1 \in \mathcal{C}^{2/p}$ and $y, y^1, y^2 \in \mathcal{C}^{1/p}$. Eventually, consider the uniform partition $0 = t_0 < \dots < t_n = T$ over $[0, T]$ with mesh T/n . Then the convergence in (3.90) holds almost surely.

Remark 3.13. As mentioned in Remark 1.4, our trapezoid sum (3.88) does not involve the derivatives y^1, y^2 of y . This is an important point, and we now elaborate on this aspect of our work.

- (i) The derivatives of a controlled process are usually non unique, unless \mathbf{X} is a truly rough path (see [5, Proposition 6.4] for a proof). However our main Hypothesis 2.17, as well as (3.69), are just regularity properties of the covariance function R . They do not imply that \mathbf{X} is a truly rough path, for which non degeneracy properties of R are required.
- (ii) Furthermore, it is possible to find examples for which the rough integral $\int_0^T y_s d\mathbf{X}_s$ does depend on the derivative process of y . See Example 3.14 below for a concrete case.
- (iii) A byproduct of Theorem 3.12 is thus the following: we have found that in a fairly general Gaussian context, the rough integral of a controlled process y does not depend on the derivatives y^1, y^2 . This side remark is an interesting result in its own right.

Example 3.14. Related to Remark 3.13, we now provide an example (kindly shared by Xi Geng [8]) of rough integral depending on the derivative of the integrand, based on a pure area process \mathbf{x} . Namely consider the canonical basis (e_1, e_2) of \mathbb{R}^2 and the $\mathbb{R}^2 \oplus (\mathbb{R}^2)^2$ -valued rough path \mathbf{x} defined by

$$(3.91) \quad \mathbf{x}_{st}^1 = 0, \quad \mathbf{x}_{st}^2 = (t - s)[e_1, e_2],$$

where $[e_1, e_2]$ stands for the free Lie bracket of e_1 and e_2 . It is easily checked that \mathbf{x} is a rough path according to Definition 2.6. Moreover \mathbf{x} is a $\frac{1}{2}$ -Hölder rough path, which means that it is a geometric α -Hölder rough path for all $\alpha < \frac{1}{2}$ (see [7] for a justification).

In this context, consider a controlled process $y = (y^0, y^1)$, with y^0, y^1 arbitrary but smooth. Since $\mathbf{x}^1 = 0$, we have $r^0 = \delta y^0 - y^1 \mathbf{x}^1 = \delta y^0$. Hence r^0 is smooth due to the fact that y^0 is smooth. This yields that y is a controlled process. Now owing to (2.6), using the notation of Proposition 2.12 and invoking (3.91) we have

$$\int_0^T y_s d\mathbf{x}_s = \lim_{|\mathcal{P}| \rightarrow 0} \sum_{0 \leq t_k < T} y_{t_k}^0 x_{t_k t_{k+1}}^1 + y_{t_k}^1 x_{t_k t_{k+1}}^2 = \left(\int_0^T y_s^1 ds \right) [e_1, e_2].$$

It is thus obvious that $\int_0^T y_s d\mathbf{x}_s$ depends on the derivative y^1 .

With those few remarks in hand, we can now turn to the proof of our main theorem.

Proof of Theorem 3.12. Proceeding with the implicit assumption that we sum over $t_k \in \mathcal{P}$, omitting the summation sign for notational convenience we get

$$(3.92) \quad \text{tr-} \mathcal{J}_0^T(y, X) = y_{t_k} \cdot X_{t_k t_{k+1}}^1 + \frac{1}{2} \delta y_{t_k t_{k+1}} \cdot X_{t_k t_{k+1}}^1.$$

Hence plugging the decomposition (3.87) into (3.92) and invoking Notation 2.7 on matrix products with $m = d$, we obtain

$$(3.93) \quad \text{tr-} \mathcal{J}_0^T(y, X) = y_{t_k} \cdot X_{t_k t_{k+1}}^1 + \frac{1}{2} (y_{t_k}^1 X_{t_k t_{k+1}}^1 + y_{t_k}^2 X_{t_k t_{k+1}}^2 + r_{t_k t_{k+1}}^0) \cdot X_{t_k t_{k+1}}^1.$$

Let us rearrange the right-hand side above by setting

$$\begin{aligned} I_1 &= y_{t_k} \cdot X_{t_k t_{k+1}}^1 + y_{t_k}^1 \cdot X_{t_k t_{k+1}}^2 + y_{t_k}^2 \cdot X_{t_k t_{k+1}}^3, \\ I_2 &= \frac{1}{2} y_{t_k}^1 X_{t_k t_{k+1}}^1 \cdot X_{t_k t_{k+1}}^1 - y_{t_k}^1 \cdot X_{t_k t_{k+1}}^2, \\ I_3 &= \frac{1}{2} y_{t_k}^2 X_{t_k t_{k+1}}^2 \cdot X_{t_k t_{k+1}}^1 - y_{t_k}^2 \cdot X_{t_k t_{k+1}}^3, \\ I_4 &= \frac{1}{2} r_{t_k t_{k+1}}^0 \cdot X_{t_k t_{k+1}}^1, \end{aligned}$$

where we have written $u \cdot v$ for inner products of vectors as well as matrices. Then one can recast (3.93) as

$$(3.94) \quad \text{tr-} \mathcal{J}_0^T(y, X) = I_1 + I_2 + I_3 + I_4.$$

Now we analyze the terms I_1, \dots, I_4 in (3.94) in order to prove (3.90). We will focus on the assumptions and conclusions of item (i), item (ii) being treated along the same lines.

First we observe that I_1 is exactly of the form (2.6), with $p < 4$ and thus $m = 3$. Hence a direct application of Proposition 2.12 yields the almost sure limit

$$I_1 \rightarrow \int_0^T y d\mathbf{X} \quad \text{as } |\mathcal{P}| \rightarrow 0.$$

Let us now analyze the term I_2 in (3.94). To this aim, an elementary examination of matrix indices reveals that

$$(3.95) \quad I_2 = y_{t_k}^1 \cdot \left(\frac{1}{2} X_{t_k t_{k+1}}^1 \otimes X_{t_k t_{k+1}}^1 - X_{t_k t_{k+1}}^2 \right).$$

Furthermore, since X is a geometric rough path, notice that a consequence of (2.2) is that for all $(s, t) \in \mathcal{S}_2([0, T])$ we have

$$\text{Sym}(X_{st}^2) = \frac{1}{2} X_{st}^1 \otimes X_{st}^1.$$

Hence one can write (3.95) as

$$I_2 = -y_{t_k}^1 \cdot \text{Antisym}(X_{t_k t_{k+1}}^2) = \frac{1}{2} \sum_{i,j=1}^d y_{t_k}^{1,ij} (X_{t_k t_{k+1}}^{2,ji} - X_{t_k t_{k+1}}^{2,ij})_{ij}.$$

Thanks to our definition (3.10) of the increment F , this becomes

$$(3.96) \quad I_2 = \frac{1}{2} \sum_{i,j=1}^d y_{t_k}^{1,ij} (\delta F_{t_k t_{k+1}}^{ji} - \delta F_{t_k t_{k+1}}^{ij}).$$

Hence owing to identity (3.96), and since we have assumed that the increments of y^1 are dominated by the control ω , Lemma 3.9 item (i) shows that

$$\lim_{|\mathcal{P}| \rightarrow 0} I_2 = 0, \quad \text{in probability.}$$

We now handle the sum I_3 in (3.94). To this aim, we resort to Lemma 3.10 item (i) for the terms $y_{t_k}^2 \cdot X_{t_k t_{k+1}}^3$ and to Remark 3.11 for the terms $y_{t_k}^2 X_{t_k t_{k+1}}^2 \cdot X_{t_k t_{k+1}}^1$. We end up with

$$\lim_{|\mathcal{P}| \rightarrow 0} I_3 = 0, \quad \text{in probability.}$$

In order to show the convergence in (3.90), it remains to show that $I_4 \rightarrow 0$ in probability. Towards this aim, recall from Definition 2.8 that the increment $r_{t_k t_{k+1}}^0$ is dominated by $\omega(t_k, t_{k+1})^{3/p}$. Hence for a small $\varepsilon > 0$ we get

$$(3.97) \quad \begin{aligned} |I_4| &= \frac{1}{2} \sum_{0 \leq t_k < T} |r_{t_k t_{k+1}}^0 \cdot X_{t_k t_{k+1}}^1| \leq \frac{1}{2} \sum_{0 \leq t_k < T} \omega(t_k, t_{k+1})^{3/p} \|X\|_{p\text{-var}, [t_k, t_{k+1}]} \\ &\leq \frac{1}{2} \max_k \omega(t_k, t_{k+1})^\varepsilon \cdot \sum_{0 \leq t_k < T} \omega(t_k, t_{k+1})^{3/p-\varepsilon} \|X\|_{p\text{-var}, [t_k, t_{k+1}]}. \end{aligned}$$

Now set

$$\tilde{\omega}(s, t) = \omega(s, t)^{3/p-\varepsilon} \cdot \|X\|_{p\text{-var}, [s, t]}, \quad (s, t) \in \mathcal{S}_2.$$

Using the same argument as for (3.79), it is easy to see that $\tilde{\omega}$ is a control. Therefore, by the super-additivity of $\tilde{\omega}$ we get

$$|I_4| \leq \frac{1}{2} \max_k \omega(t_k, t_{k+1})^\varepsilon \cdot \omega(0, T)^{3/p-\varepsilon} \cdot \|X\|_{p\text{-var}, [0, T]}.$$

Since $\max_k \omega(t_k, t_{k+1}) \rightarrow 0$ as $|\mathcal{P}| \rightarrow 0$ it follows that $I_4 \rightarrow 0$ almost surely. This completes the proof of (3.90). Moreover, recall that claim (ii) in our statement is obtained easily by adapting slightly the considerations above, similarly to what we have done in Lemma 3.9. This completes the proof of our theorem. \square

As mentioned in Theorem 1.2, typical examples of controlled processes are given by solutions of rough differential equations and processes of the form $y = f(X)$. Hence one can apply Theorem 3.12 in order to get a trapezoid rule (3.88) for $f(X)$. However, we would also like to consider Riemann sums which are closer to the ones handled in [1,12]. This is why we wish to consider sums of the form:

$$(3.98) \quad \text{m-}\mathcal{J}_0^T(f(X), X) := \sum_{k=0}^{n-1} f\left(\frac{X_{t_k} + X_{t_{k+1}}}{2}\right) \delta X_{t_k t_{k+1}}.$$

We now state a corollary of Theorem 3.12 giving the convergence of $\text{m-}\mathcal{J}_0^T(f(X), X)$ above.

Corollary 3.15. *Let X be as in Theorem 3.12. Consider function $f \in C_b^3(\mathbb{R}^d)$ and the midpoint rule $\text{m-}\mathcal{J}_0^T(f(X), X)$ defined by (3.98). Then we have*

$$(3.99) \quad \text{m-}\mathcal{J}_0^T(f(X), X) \rightarrow \int_0^T f(X_s) d\mathbf{X}_s$$

as the mesh size $|\mathcal{P}| \rightarrow 0$. As in Theorem 3.12, the convergence holds in probability if p -variation regularity is considered, and almost surely if Hölder continuity is assumed.

Proof. We first recall that for $a, b \in \mathbb{R}^d$ we have the following mean value identity

$$(3.100) \quad \frac{f(a) + f(b)}{2} - f\left(\frac{a+b}{2}\right) = \frac{1}{2} \partial^2 f(c) \left(\frac{b-a}{2} \otimes \frac{b-a}{2}\right),$$

where $c \in \mathbb{R}^d$ satisfies $c = a + \theta(b - a)$ for some $\theta \in [0, 1]$. Let us take the difference of (3.88) and (3.98) and then apply the mean value identity (3.100) with $a = X_{t_k}, b = X_{t_{k+1}}$. Then we obtain

$$(3.101) \quad \text{tr-}\mathcal{J}_0^T(f(X), X) - \text{m-}\mathcal{J}_0^T(f(X), X) = \frac{1}{8} \sum_{k=0}^{n-1} \partial^2 f(c) (\delta X_{t_k t_{k+1}} \otimes \delta X_{t_k t_{k+1}}) \cdot \delta X_{t_k t_{k+1}},$$

with $c = X_{t_k} + \theta \delta X_{t_k t_{k+1}}$. In order to prove (3.99), it suffices to show that the right-hand side of (3.101) converges to zero. To this aim, we observe that

$$(3.102) \quad \partial^2 f(c) = \partial^2 f(X_{t_k}) + \theta \partial^3 f(d) \delta X_{t_k t_{k+1}},$$

where $d = X_{t_k} + \lambda \delta X_{t_k t_{k+1}}$ for some $\lambda \in [0, 1]$. Substituting (3.102) into the right-hand side of (3.101) we obtain two terms. It is then easy to see that one of the two terms is in the form of $\sum_{0 \leq t_k < T} y_{t_k} X_{t_k t_{k+1}}^3$ with $y = \partial_2^2 f(X)$. It then follows from Lemma 3.10 that it converges to zero. The other term can be treated in a similar way as for I_4 in (3.97), which completes the proof. \square

3.5. Applications

In this section we will briefly list some important examples of Gaussian processes satisfying our standing Hypothesis 2.17. Notice that in the current paper we only request $V_\rho(R) < \infty$ with $\rho \in [1, 2)$, which is a weaker condition than in [4], and certainly weaker than in [9]. Hence all the examples listed in those two references also apply to our context. We just highlight some of them below.

- (i) The most obvious example is given by a fractional Brownian motion (fBm) B^H for which the covariance function R in (2.9) is given by

$$R^H(s, t) = \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H}).$$

Then R^H satisfies Hypothesis 2.17 whenever $H \in (\frac{1}{4}, 1)$, with $\omega_R([s, t]^2) = |t - s|$. One can also verify that (3.69) holds.

- (ii) If one considers a process X given as $X = B^{H_1} + B^{H_2}$ with $H_1, H_2 \in (\frac{1}{4}, 1)$ and two independent \mathbb{R}^d -valued fBms B^{H_1} and B^{H_2} , then one can also apply our main Theorem 1.2 to X , with $R(s, t) = R^{H_1}(s, t) + R^{H_2}(s, t)$.
- (iii) The bifractional Brownian motion, introduced in [13] is a centered Gaussian process whose covariance $R^{H,K}$ is given by

$$R^{H,K}(s, t) = \frac{1}{2^K} ((t^{2H} + s^{2H})^K - |t - s|^{2HK}).$$

This process generalizes fBm (obtained for $K = 1$), and fulfills our Hypothesis 2.17 whenever $HK \in (1/4, 1)$.

- (iv) We refer to [4] for a thorough exploration of random Fourier series, some of which yield a control such that $\omega_R([s, t]^2) \neq |t - s|$, but still satisfying Hypothesis 2.17.

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