EULER SCHEME FOR FRACTIONAL DELAY STOCHASTIC DIFFERENTIAL EQUATIONS BY ROUGH PATHS TECHNIQUES

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Abstract. In this note we study a discrete time approximation for the solution of a class of delayed stochastic differential equations driven by a fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$. In order to prove convergence we use rough paths techniques. Theoretical bounds are established and numerical simulations are displayed.

Mathematics Subject Classifications (2000): Primary 60H15; secondary 65C30.

Keywords: Fractional Brownian motion, stochastic differential equations, rough paths, discrete time approximation.

1. Introduction

The fractional Brownian motion with Hurst parameter $H \in (0, 1)$ is a centered Gaussian process $(X_t)_{t\in[0,1]}$ whose covariance function can be written as

$$E[X_t X_s] = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right).$$

The family of processes $\{X;\, H \in (0, 1)\}$ enjoys several nice properties:

- For $H = 1/2$, one recovers the classical Brownian motion.
- For any $H \in (0, 1)$, the paths of X are almost surely $(H - \rho)$-Hölder continuous for any arbitrarily small $\rho > 0$. Specifically, we have

$$|X_t - X_s| < F_0 |t-s|^{H-\rho} \quad \text{a.s.} \quad t, s \in [0, T], \tag{1.1}$$

- The covariance of the increments of X on intervals decays asymptotically as a negative power of the distance between the intervals.
- Fractional Brownian motion is the only finite-variance process which is self-similar (with index H) and has stationary increments.

These characteristics have converted the fractional Brownian family into the one of the most natural generalization of Brownian motion among the probability community, but also for practitioners, in the recent years.

At a theoretical level, it should be noticed that the martingale type techniques used for the construction of a stochastic calculus with respect to the usual Brownian motion $B^{1/2}$ cannot be invoked anymore when $H \neq 1/2$. However, when $H \in (1/2, 1)$ one can define

S. Torres was partially supported by the Project ECOS - CONICYT C15E05, REDES 150038, MATH-AMSUD 16-MATH-03 SIDRE Project and Fondecyt Grant 1171335.

S. Tindel is supported by the NSF grant DMS-1613163.

stochastic integrals and solutions to differential equations thanks to Young (see e.g. [11, 14] for an account on these techniques) or fractional calculus (as explained in [25, 31]) methods. The case $H \in (0, 1/2)$ is avoided in this article for sake of readability, but let us mention that one has to appeal to rough paths techniques (for which we refer to [11, 14] again) in order to solve stochastic equations in this situation.

As far as applications of differential systems are concerned, the wide range of contexts in which fBm driven models are used includes Biophysics [17, 27, 28], electrical engineering [7] and finance [4, 13, 15, 16, 29] situations. All those applications involve ordinary or Volterra type differential equations, but delayed systems can also be of huge importance. Indeed, to mention just a single biomedical example, bacteriophage systems are commonly described by delayed equations. Specifically, what we call bacteriophages are harmless viruses meant to attack bacteria involved in animal diseases according to a so-called lytic process. In short, the virus genetic material penetrates into the bacteria and uses the host replication mechanism to self-replicate. This lytic step induces a complex chain of reactions and takes about 30mn to be completed, while treatments are usually measured in hours. Thus, mathematical modelings of the treatment naturally involve delayed equations, as assessed by the recent articles [1, 3, 2, 26]. While the aforementioned references are concerned either with deterministic or Brownian driven equations for sake of simplicity, let us stress the fact that there are experimental evidences that fBm models should also be dealt with in this context.

At a more theoretical level, random delay systems can also be seen a first approximation of stochastic infinite dimensional differential equations such as stochastic PDEs. This point of view is developed at length e.g in [22]. Since stochastic PDEs are notoriously hard to handle, it is worth trying o first understand better the behavior delay equations driven by fractional Brownian motions. This gives another appeal to the study of these systems.

With those motivations in mind, let us proceed to the mathematical description of the model we are dealing with. Namely, we consider the following stochastic delay differential equation driven by a fractional Brownian motion X (FSDDE) with Hurst parameter $H > 1/2$,

$$\begin{aligned} dY_t &= b(Y_t)dt + \sigma(Y_{t-r}, Y_t)dX_t^H, \quad t \in [0,T] \\ Y_s &= \phi(s) \quad s \in [-r, 0] \end{aligned} \tag{1.2}$$

where ϕ is a Hölder continuous function on $[-r, 0]$ and r is a positive time delay. As a solution to this equation we shall define a process $\{X_t, t \in [-r, T]\}$ satisfying

$$\begin{aligned} &Y_t = \phi(0) + \int_0^t b(Y_s)ds + \int_0^t \sigma(Y_{s-r}, Y_s)dX_s, \quad t \in [0,T] \\ &Y_t = \phi(t), \qquad t \in [-r, 0], \end{aligned} \tag{1.3}$$

where the integral with respect to fractional Brownian motion is the generalized Riemann-Stieltjes integral introduced by Young, expressed in the formalism given by [14]. It is worth mentioning that this kind of equation has been first introduced in [24, 19] (see also [9] and [10] for a diffusion coefficient of the form $\sigma(Y_{t-r})$).

In this context, the current article focuses on a sequence of discrete time approximations Y^n of the solution Y to the FSDDE (1.3), on a compact interval $[0, T]$. In a natural way, this scheme will be based on a regular partition of $[0,T]$ with mesh proportional to $1/n$ and we will pursue two dual objectives:

(1) At a theoretical level, we will show a strong convergence result for the sequence Y^n. Namely, under suitable assumptions on the coefficients of (1.3), we shall see that almost surely, the difference $\sup_{t\in[0,T]} |Y_t^n - Y_t|$ is of order $n^{-(2H-1-\rho)}$ for ρ arbitrarily small. This means that one can reach the same rate of convergence as in the non-delayed case, which was first obtained in [21, 23]. Moreover, comparing our result to other studies concerning numerical schemes for pathwise equations [8, 12, 19], let us highlight the fact that we are dealing here with coefficients b, σ with linear growth (as opposed to bounded coefficients). This covers more realistic situations in terms of modeling, but is also a non negligible part of the technical problems we have to face in the sequel. We believe that our scheme is the first one covering the case of coefficients with linear growth, even in cases with no delay.

(2) We then illustrate our theoretical results by simulations in order to depict the typical path of a delayed differential equation driven by fractional Brownian motion. In particular, the reader will observe the influence of the Hurst parameter H in terms of regularity of the path and convergence of the numerical scheme.

This paper is organized as follows. Section 2 is devoted to some preliminaries related to rough paths theory and conditions for existence and uniqueness for the fractional delay equation (1.2). In section 3 we define our discrete Euler scheme for the solution of FSDDE (1.3) and we study its rate of convergence. Finally, in section 4 some numerical examples are given.

Notation. Let $\pi : 0 = t_0 < t_1 < \cdots < t_n = T$ be a partition on $[0,T]$. Take $s,t \in [0,T]$. We denote by $[\![s,t]\!]$ the discrete interval that consists of t_k's such that $t_k \in [s,t]$. Let I be either the interval $[s,t]$ or the discrete interval $[\![s,t]\!]$. We denote by $\mathcal{S}_k(I)$ the simplex $\{s \le t_1 < \cdots < t_k \le t\}$. Throughout our computations, C_x designates a constant which depends on some Hölder norm of the signal x. The value of this kind of constant can change from line to line.

2. Preliminaries

This section is devoted to some preliminary considerations about Young integration, as well as delay equations driven by a Hölder noisy signal.

2.1. Hölder continuous paths. In this subsection, we introduce some basic concepts of Young integration theory, in both discrete and continuous settings. Let $\gamma \ge \frac{1}{2}$, and call $T > 0$ a fixed finite time horizon. The following notation will prevail until the end of the paper: for a vector space V and two functions $f \in C([0,T], V)$ and $g \in C(\mathcal{S}_2([0,T]), V)$ we set

$$\delta f_{st} = f_t - f_s, \quad \text{and} \quad \delta g_{sut} = g_{st} - g_{su} - g_{ut}. \tag{2.1}$$

We start with the definition of some Hölder semi-norms: consider here a path $x \in C([0,T], \mathbb{R}^m)$. Then we set

$$\|x\|_{[s,t],\gamma} := \sup_{(u,v)\in\mathcal{S}_2([s,t])} \frac{|\delta x_{uv}|}{|v-u|^\gamma}. \tag{2.2}$$

We denote by $C^\gamma([s,t];\mathbb{R}^m)$, or $C^\gamma([s,t])$ in a shorter form, the space of continuous functions taking values in $\mathbb{R}^m$ such that $\|x\|_{[s,t],\gamma}$ is finite. In the same way, if $g \in C(\mathcal{S}_2([0,T]),V)$ and $h \in C(\mathcal{S}_3([0,T]),V)$, we define:

$$\|g\|_{[s,t],\gamma} := \sup_{(u,v)\in\mathcal{S}_2([s,t])} \frac{|g_{uv}|}{|v-u|^\gamma}, \quad \text{and} \quad \|h\|_\mu = \sup_{(s,u,t)\in\mathcal{S}_3([0,T])} \frac{|h_{sut}|}{|t-s|^\mu}. \tag{2.3}$$

We also notice that the Hölder norms in (2.2) and (2.3) can be defined for functions on the discrete grid $[\![0,T]\!]$. The corresponding spaces will be denoted by $C^\mu(\mathcal{S}_k([\![0,T]\!]))$.

For Hölder continuous paths with Hölder exponent greater that $1/2$, the classical Young (or generalized Stieltjes) integral can be defined in the following way (see [30]).

Proposition 2.1. *Let f and g be two real-valued γ-Hölder functions on $[0,T]$, with $\gamma > \frac{1}{2}$. Then for $(s,t) \in \mathcal{S}_2([0,T])$ the integral*

$$\int_s^t f_r \, dg_r$$

is defined as a limit of Riemann sums. In addition, it can be bounded as follows:

$$\left| \int_s^t f_r \, dg_r \right| \le C_\gamma \left(|f_s||\delta g_{st}| + \|f\|_{\gamma,[s,t]} \|g\|_{\gamma,[s,t]} |t-s|^{2\gamma} \right).$$

Notice that equation (1.3), as well as all our noisy integrals, will be interpreted in the Young sense in the remainder of the paper.

We now state a lemma which will be crucial in the analysis of our numerical scheme (see [20] for a proof). It allows to get estimates of a function $f \in C_2^\mu$ in terms of δf, which might be a simpler object.

Lemma 2.2. *Let f be a function defined on $[\![0,T]\!]$ and $\mu > 1$. Then the following inequalities hold true:*

(i) *Whenever $f_{t_i t_{i+1}} = 0$ for all $0 \le i < n$ we have*

$$\|f\|_\mu \le C_\mu \|\delta f\|_\mu .$$

(ii) *In the general case where the quantities $f_{t_i t_{i+1}}$ do not all vanish, we obtain:*

$$\|f\|_\mu \le C_\mu \|\delta f\|_\mu + \sup\left\{ |f_{t_i t_{i+1}}|;\, 0 \le i < n \right\}.$$

2.2. Stochastic delay equation. Recall that our aim is to get some convergence results for the fractional delay equation (1.3). However, observe that one can also solve recursively equation (1.3) in the following way:

(i) First on $[0,r]$ we solve the equation:

$$Y_t = Y_0 + \int_0^t b(Y_s) ds + \int_0^t \sigma(h_s^{(0)}, Y_s) dX_s,$$

where $Y_0 = \phi(0)$ and $h_s^{(0)} = Y_{s-r} = \phi(s-r)$.

(ii) If we assume that the equation is solved on $[(j-1)r, jr]$, equation (1.3) on $[jr,(j+1)r]$ becomes:

$$Y_t = Y_{jr} + \int_{jr}^t b(Y_s) ds + \int_{jr}^t \sigma(h_s^{(j)}, Y_s) dX_s, \tag{2.4}$$

where $h_r^{(j)} = Y_{s-r}$ and Y_{jr} is considered as an initial condition.

In any of those cases $h^{(j)}$ can be though of as an input of the system, as opposed to an unknown. We are thus reduced to the discretization of the following equation on $[0,T]$ (we shall take $T = r$ for our application to delay equations):

$$Y_t = Y_0 + \int_0^t \sigma(h_s, Y_s) dX_s \quad \text{for} \quad t \in [0,T], \tag{2.5}$$

where $X = (X_t)_{t\geq 0}$ is a fractional Brownian motion with Hurst parameter $H > 1/2$ and h could be any $\mathbb{R}^m$-valued process whose trajectories are in C^γ for some $\gamma \in (1/2, 1)$, as defined in (2.2). Here again, the stochastic integral in equation (2.5) is understood thanks to Proposition 2.1.

Once our delay equation (1.3) has been reduced to (2.5), let us give the set of hypothesis we wish to consider for the coefficients:

Hypothesis 2.3. *In equation (2.5), consider a measurable coefficient $\sigma : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$, which is twice differentiable in x, y Moreover we assume:*

(1) *The path h is a given element of C^γ for some $\gamma \in (1/2, 1)$.*
(2) *There exist a constant $C_\sigma > 0$ and $C_\sigma^1 > 0$ such that:*

$$|\sigma(x_1, y_1) - \sigma(x_2, y_2)| \leq C_\sigma \{|x_2 - x_1| + |y_2 - y_1|\}, \quad \forall x_1, x_2, y_1, y_2 \in \mathbb{R}^m.$$

(3) *The function σ also satisfies the bound:*

$$|\sigma(x, y)| \leq C_\sigma^1 (1 + |x| + |y|), \quad \forall x, y \in \mathbb{R}^m.$$

As mentioned in the introduction, we point out that, in order to cover relevant cases in terms of applications, we only assume a linear growth for σ in Hypothesis 2.3. This induces some technical difficulties with respect to the standard bounded coefficient case. Nevertheless, one can prove the following existence and uniqueness result (see [25] for a proof based on fractional calculus).

Theorem 2.4. *Consider a fBm X with Hurst parameter $H \in (1/2, 1)$. Let σ and h functions that satisfy Hypothesis 2.3. Then equation (2.5) admits a unique solution $Y \in C^\gamma([0, r])$ for any $\frac{1}{2} < \gamma < H$. Furthermore, the increments of this solution Y can be decomposed as follows for all $(s,t) \in \mathcal{S}_2([0,T])$:*

$$\delta Y_{st} = \sigma(h_s, Y_s)\, \delta X_{st} + R_{st}^Y, \tag{2.6}$$

where the increment R^Y satisfies $|R_{st}^Y| \leq C_x |t-s|^{2\gamma}$ for an almost surely finite constant C_x depending only on $\|X\|_\gamma$.

3. The Euler scheme

In this section we define properly and analyze the Euler scheme related to equation (1.3) rewritten as (2.5). In order to alleviate notations, we will consider that the function σ, the process h and Y are $\mathbb{R}$-valued. It should be noticed that the extension of our considerations to multidimensional situations is just a matter of adding indices, which is left to the patient reader.

3.1. A priori bounds on the scheme. For a given $T > 0$, let $[\![0,T]\!]_n$ be an equidistant partition of the time interval $[0,T]$ with step size T/n, i.e.

$$[\![0,T]\!]_n = \{0 = t_0 < t_1 < \cdots < t_n = T\}, \quad \text{where} \quad t_j = \frac{jT}{n}.$$

If $s,t \in [0,T]$ with $s < t$, we denote

$$[\![s,t]\!] := [s,t] \cap [\![0,T]\!]_n$$

According to our recursive expression for equation (2.4), the numerical approximation considered here can be defined in the following way:

Definition 3.1. *The Euler scheme related to Equation (2.5) is a sequence $\{Y^n;\, n \geq 1\}$ where $Y^n = \{Y^n_{t_i};\, t_i \in [\![0,T]\!]\}$ is defined recursively by:*

$$Y^n_{t_{i+1}} = Y^n_{t_i} + \sigma(h^n_{t_i}, Y^n_{t_i})\delta X_{t_i t_{i+1}}, \tag{3.1}$$

where the process h^n is an approximation of h satisfying the following hypothesis:

$$|h^n_t - h_t| \leq C^1_h \frac{1}{n^{2\gamma-1-\varepsilon}}, \tag{3.2}$$

as well as the uniform bounds

$$\delta h^n_{st} \leq C^2_h \, |t-s|^\gamma, \quad \text{and} \quad |\delta(h - h^n)_{st}| \leq C^2_h \frac{|t-s|^\gamma}{n^{2\gamma-1-\varepsilon}}. \tag{3.3}$$

In our definition (3.1) we also assume Y_0 to be a fixed constant. In the sequel we will set $C^3_h = C^1_h + C^2_h$.

Remark 3.2. In case of a delay equation, on each interval of the form $[lr,(l+1)r]$ one can take h^n as the Euler approximation $Y^n|_{[(l-1)r,lr]}$ of equation (1.3) on the interval $[(l-1)r, lr]$. As we shall see, $Y^n|_{[(l-1)r,lr]}$ fulfills the assumptions (3.2) and (3.3).

Let us now collect some basic information about the approximation Y^n. According to relation (3.1) we have that, for any point t_i of the uniform partition:

$$\delta Y^n_{t_i t_{i+1}} = \sigma(h^n_{t_i}, Y^n_{t_i})\delta X_{t_i t_{i+1}} \tag{3.4}$$

We can extend this relation to any couple of points (s,t) in $\mathcal{S}_2([\![0,T]\!]_n)$ by writing:

$$\delta Y^n_{st} = \sigma(h^n_s, Y^n_s)\delta X_{st} + R^n_{st}, \tag{3.5}$$

where we have just set:

$$R^n_{st} := \delta Y^n_{st} - \sigma(h^n_s, Y^n_s)\delta X_{st}. \tag{3.6}$$

With this definition, we have that $R^n_{t_i t_{i+1}} = 0$ according to (3.4).

The dicrete increment R^n introduced in (3.6) is expected to be 2γ-Hölder continuous. One of the key ingredients in the analysis of our scheme is the study of $\|R^n\|_{2\gamma}$. To this aim, notice that by Lemma 2.2 it is enough to study $\|\delta R^n\|_{2\gamma}$. The following lemma is a first step in this direction.

Lemma 3.3. *Assume that the coefficients of equation (2.5) fulfill Hypothesis 2.3, and let R^n be the remainder defined by (3.6). Then, for $(s,u,t) \in \mathcal{S}_3([\![0,T]\!]_n)$, the increment δR^n satisfies the following inequality:*

$$|\delta R^n_{sut}| \leq C_\sigma \left[C^2_h |s-u|^\gamma + |\delta Y^n_{su}| \right] \|X\|_\gamma \, |t-s|^\gamma, \tag{3.7}$$

where C_h^2 is defined by relation (3.3).

Proof. Let us apply the operator δ on both sides of relation (3.6). Invoking the fact that $\delta\delta Y^n = 0$ and $\delta\delta X = 0$, we get:

$$\begin{aligned}|\delta R^n_{sut}| &= |\delta\delta Y^n_{sut} - \delta[\sigma(h^n_s, Y^n_s)\delta X]_{sut}| \\ &= |\delta\sigma(h^n, Y^n)_{su}\delta X_{ut} - \sigma(h^n_s, Y^n_s)\delta(\delta X)_{sut}| \\ &= |\delta\sigma(h^n, Y^n)_{su}\delta X_{ut}|. \end{aligned} \tag{3.8}$$

We now resort to the linear bound on σ given in Hypothesis 2.3, which yields

$$\begin{aligned}|\delta R^n_{sut}| &\le C_\sigma \left[|\delta h^n_{su}| + |\delta Y^n_{su}|\right] \|X\|_\gamma |t-u|^\gamma \\ &\le C_\sigma \left[C_h^2|s-u|^\gamma + |\delta Y^n_{su}|\right] \|X\|_\gamma |t-s|^\gamma,\end{aligned}$$

where the last inequality comes from our assumption (3.3). $\square$

According to equation (3.7), the study of δR^n_{sut} can be reduced to an upper bound on $|\delta Y^n_{su}|$ for $s<u<t$. The following lemma delivers this bound.

Lemma 3.4. *Let Hypothesis 2.3 as well as the upper bounds (3.2) and (3.3) prevail. Let Y^n be the Euler approximation of equation (2.5). Then for all $s,t \in [\![0,T]\!]_n$ the following bound holds true:*

$$|\delta Y^n_{st}| \le \hat{C}\,|t-s|^\gamma, \tag{3.9}$$

where the constant $\hat{C}$ satisfies $\hat{C} \le c_1 \exp(c_2(1+\|X\|_\gamma^{1/\gamma}))$ for two strictly positive constants c_1, c_2.

Proof. We will prove by induction that, for all $l = 1, 2, \dots n$ and $s,t \in [\![0,T]\!]_n$ such that $0 \le s < t \le t_l$ we have:

$$|\delta Y^n_{st}| \le C_2|t-s|^\gamma. \tag{3.10}$$

This proof will be divided in several steps.

Step 1: case $\mathbf{l = 1}$. For $l=1$, owing to the Hölder continuity of the process X and the linear growth of σ, we have

$$\begin{aligned}|\delta Y^n_{0t_1}| &= |\sigma(h^n_0, Y^n_0)||\delta X_{0t_1}| \\ &\le C^1_\sigma(1+|h^n_0|+|Y^n_0|)\,\|X\|_\gamma\, t_1^\gamma.\end{aligned} \tag{3.11}$$

This yields relation (3.10) for $l=1$.

Step 2: Upper bound for $\mathbf{R^n}$. Consider an index $l = 1,\dots,n$. Our induction assumption is that (3.10) is true $0 \le s < t \le t_l$. We shall now propagate the induction, that is prove that the inequality is true for its successor, $l+1$. We will thus study (3.10) for $s,t \in [\![0,T]\!]_n$ with $0 \le s < t \le t_{l+1}$. Furthermore, if $0 \le s < t \le t_l$ inequality (3.10) is trivially satisfied thanks to our induction hypothesis. Let us now focus on the case $0 \le s < t = t_{l+1}$. Namely, consider $s<u<t$ with $t = t_{l+1}$. Since $|u-s| < |t-s|$, owing to (3.7) and our induction hypothesis we get that,

$$\begin{aligned}|\delta R^n_{sut}| &\le C_\sigma\left[C_h^2|u-s|^\gamma + |\delta Y^n_{su}|\right]\|X\|_\gamma |t-s|^\gamma \\ &\le C_\sigma\left[(C_h^2+C_2)|u-s|^\gamma\right]\|X\|_\gamma |t-s|^\gamma \\ &\le C_\sigma(C_h^2+C_2)\,\|X\|_\gamma\,|t-s|^{2\gamma}.\end{aligned} \tag{3.12}$$

Since inequality (3.12) is also obviously true if $t < t_{l+1}$, we can apply Lemma 2.2 over $[\![0, t_{l+1}]\!]$ in order to obtain:

$$\|R^n\|_{[s,t],\gamma} \le C_\gamma C_\sigma (C_h^2 + C_2) \|X\|_\gamma |t-s|^\gamma,$$

or otherwise stated, for $0 \le u \le v \le t_{l+1}$:

$$|R^n_{uv}| \le C_\gamma C_\sigma (C_h^2 + C_2) \|X\|_\gamma |v-u|^{2\gamma}. \tag{3.13}$$

Step 3: Local upper bound for $\delta \mathbf{Y^n}$. Recall that the dynamics of Y^n is governed by equation (3.5). We will start by bounding uniformly the quantity Y^n_t, which is obtained thanks to an anlysis of the increment δY^n_{0t}. Namely, invoking the dynamics (3.5), we get:

$$\begin{aligned}
|\delta Y^n_{0t}| &\le |\sigma(h^n_0, Y^n_0)| |\delta X_{0t}| + |R^n_{0t}| \\
&\le C^1_\sigma (1 + |h^n_0| + |Y^n_0|) |\delta X_{0t}| + |R^n_{0t}| \\
&\le C^1_\sigma (1 + |h^n_0| + |Y^n_0|) \|X\|_\gamma |t|^\gamma + |R^n_{0t}| \\
&\le C^1_\sigma \left(1 + |h^n_0| + |Y^n_0| + C_\gamma C_\sigma (C_h^2 + C_2)|t|^\gamma\right) \|X\|_\gamma |t|^\gamma, \qquad (3.14)
\end{aligned}$$

where the last inequality is due to (3.13). We thus end up with the following inequality on $[\![0, t_{l+1}]\!]$:

$$\begin{aligned}
|Y^n_t| &\le |Y^n_0| + \left(C^1_\sigma (1 + |h^n_0| + |Y^n_0|) + C_\gamma C_\sigma (C_h^2 + C_2)|t|^\gamma\right) \|X\|_\gamma |t|^\gamma \\
&\equiv K(Y^n_0, C_2, \sigma, h^n_0, X, t). \qquad (3.15)
\end{aligned}$$

Let us now handle the case of a general increment δY^n_{st}. We first resort to (3.5) and Hypothesis 2.3 in order to get

$$|\delta Y^n_{st}| \le |\sigma(h^n_s, Y^n_s)| |\delta X_{st}| + |R^n_{st}| \le C^1_\sigma (1 + |h^n_s| + |Y^n_s|) |\delta X_{st}| + |R^n_{st}|.$$

Now plug inequalities (3.3), (3.13) and(3.15), which yields:

$$\begin{aligned}
&|\delta Y^n_{st}| \le C^1_\sigma \left(1 + |h^n_0| + C_h^2 |s|^\gamma + K(Y^n_0, C_2, \sigma, h^n_0, X, s)\right) |\delta X_{st}| + |R^n_{st}| \\
&\le C^1_\sigma \left(1 + |h^n_0| + C_h^2 |s|^\gamma + K(Y^n_0, C_2, \sigma, h^n_0, X, s)\right) \|X\|_\gamma |t-s|^\gamma + |R^n_{st}| \\
&\le \left(C^1_\sigma (1 + |h^n_0| + C_h^2 |s|^\gamma + K(Y^n_0, C_2, \sigma, h^n_0, X, s)) + C_\gamma C_\sigma (C_h^2 + C_2)|t-s|^\gamma\right) \|X\|_\gamma |t-s|^\gamma.
\end{aligned}$$

Gathering all the terms above, we thus obtain:

$$\begin{aligned}
|\delta Y^n_{st}| \le \|X\|_\gamma |t-s|^\gamma \Big\{ & C^1_\sigma \Big(1 + |h^n_0| + |Y^n_0| + C_h^2 |s|^\gamma + C^1_\sigma \|X\|_\gamma |s|^\gamma (1 + |h^n_0| + |Y^n_0|) \\
& + C_\gamma C_\sigma C_h^2 \|X\|_\gamma |s|^{2\gamma}\Big) + C_\gamma C_\sigma C_h^2 |t-s|^\gamma + C_2 \left(C^1_\sigma C_\gamma C_\sigma \|X\|_\gamma |s|^{2\gamma} + C_\gamma C_\sigma |t-s|^\gamma\right)\Big\}.
\end{aligned} \tag{3.16}$$

We shall now perform the steps allowing to go from an inequality of the form (3.16) to our claim (3.10). We thus consider a small enough time τ. In the interval $[\![0, t_{l+1}]\!] \cap [0, \tau]$ we can recast (3.16) as:

$$\|\delta Y^n\|_\gamma \le C_3 \|X\|_\gamma + (C_4 + C_5 C_2) \|X\|_\gamma |\tau|^\gamma \tag{3.17}$$

where the constants C_3, C_4 and C_5 are respectively given by $C_3 = C^1_\sigma (1 + |h^n_0| + |Y^n_0|)$, $C_4 = (1 + C_\gamma C_\sigma) C_h^2 + \|X\|_\gamma (C^1_\sigma (1 + |h^n_0| + |Y^n_0|) + (C_\gamma C_\sigma) C_h^2 |T|^\gamma)$ and $C_5 = C_\gamma C_\sigma (C^1_\sigma \|X\|_\gamma |T|^\gamma + 1)$.

In order to get some stability in our Hölder norms estimates, let us assume that the quantity $b(\tau) \equiv C_5 \|X\|_\gamma \tau^\gamma$ satisfies $b(\tau) < 1/2$. Otherwise stated, assume that τ verifies:

$$\tau = \frac{1}{(2C_5(\|X\|_\gamma + 1))^{1/\gamma}}. \tag{3.18}$$

Moreover, in inequality (3.10), let us choose the constant C_2 such that

$$C_2 \geq 2\left(C_3 \|X\|_\gamma + C_4 \|X\|_\gamma |T|^\gamma\right). \tag{3.19}$$

Then plugging the values (3.18) and (3.19) into (3.17) we get the following bound on $[\![0, t_{l+1}]\!] \cap [0, \tau]$:

$$\|\delta Y^n\|_\gamma \leq C_2 \tag{3.20}$$

In conclusion, in $[\![0, t_{l+1}]\!] \cap [0, \tau]$ we have propagated our induction claim (3.10).

Step 4: Global upper bound for $\delta \mathbf{Y^n}$. Up to now we have performed our computations on $[0, \tau]$ only, where τ is defined by (3.18). The computations on any interval of the form $[k\tau, (k+1)\tau]$ would be exactly the same, except for the fact that the initial value Y_0^n has to be updated to $Y_{k\tau}$. Then recall that our quantities of interest are C_2 in inequality (3.20) and $K(Y_0^n, C_2, \sigma, h_0^n, X, t)$ in relation (3.15). We will keep track of those constants below. Let us also highlight the fact that the small time step τ given by (3.18) does not depend on the initial condition Y_0, which means that it can be considered as a given constant in the remainder of the proof.

Let us now see how to update the initial data $Y_{k\tau}^n$ in each sub-interval. To this aim we notice that a straightforward generalization of (3.14) gives:

$$|Y_{(k+1)\tau}^n| \leq A|Y_{k\tau}^n| + B, \tag{3.21}$$

where

$$A = 1 + C_\sigma^1 \|X\|_\gamma \tau^\gamma, \quad \text{and} \quad B = C_\sigma^1 (1 + |h_0^n|) \|X\|_\gamma \tau^\gamma + C_\gamma C_\sigma \left(C_2 + C_h^2\right) \|X\|_\gamma |\tau|^{2\gamma}.$$

In addition, starting from (3.21), an easy induction procedure yields:

$$|Y_{k\tau}^n| \leq A^k |Y_0^n| + \frac{A^k - 1}{A - 1} B. \tag{3.22}$$

In order to get a global bound for Y^n, we are now reduced to compute the number of intervals $[k\tau, (k+1)\tau]$ necessary to cover $[0, T]$. Calling $n_{\max}$ this number and resorting to expression (3.18), it is readily checked that:

$$n_{\max} = \frac{T}{\tau} = T\left(2C_5(1 + \|X\|_\gamma)\right)^{\frac{1}{\gamma}}.$$

Plugging this expression into (3.22) we end up with the following bound valid for all $k \leq n_{\max}$:

$$|Y_{k\tau}^n| \leq c_1(1 + |h_0^n| + |Y_0^n|) \exp\left(c_2(1 + \|X\|_\gamma^{1/\gamma})\right).$$

Taking into account relation (3.20), we let the patient reader check that this proves relation (3.9). $\square$

3.2. Convergence of the scheme. We are now ready to state the main result of this section, which gives a theoretical bound on the speed of convergence for Y^n.

Theorem 3.5. *Let Y^n be the Euler scheme given in Definition 3.1, and let Y be the solution to equation (2.5). We set $Z^n = Y - Y^n$. Then for all $0 < \varepsilon < 2\gamma - 1$ and $0 \le s < t \le T$ we have the following bound:*

$$|Z_t^n| \le \frac{C_{1,x}}{n^{2\gamma-1-\epsilon}}, \quad \text{where} \quad C_{1,x} \le c_1 \exp(c_2(1 + \|x\|_\gamma^{1/\gamma})), \tag{3.23}$$

for two strictly positive constants c_1, c_2 depending on σ and ε. Furthermore, the increments of Z^n also satisfy a bound of the form:

$$|\delta Z_{st}^n| \le \frac{C_{3,x}|t-s|^\gamma}{n^{2\gamma-1-\epsilon}}, \quad \text{where} \quad C_{3,x} \le c_3 \exp(c_4(1 + \|x\|_\gamma^{1/\gamma})), \tag{3.24}$$

for two strictly positive constants c_3, c_4.

Proof. Recall that Y^n is defined recursively as:

$$\delta Y_{t_i t_{i+1}}^n = \sigma(h_{t_i}^n, Y_{t_i}^n)\delta X_{t_i t_{i+1}} \tag{3.25}$$

while the increments of the solution Y to equation (2.5) can be expressed as:

$$\delta Y_{t_i t_{i+1}} = \sigma(h_{t_i}, Y_{t_i})\delta X_{t_i t_{i+1}} + R_{t_i t_{i+1}}^Y, \tag{3.26}$$

for a remainder R^Y such that $\|R^Y\|_{2\gamma} < \infty$. We now divide the proof in several steps.

Step 1: Dynamics for $\mathbf{Z}^n$. Thanks to a Taylor type expansion, one can linearize the increment of Z^n between two successive partition points as follows:

$$\begin{aligned} \delta Z_{t_i t_{i+1}}^n &= \left(\sigma(h_{t_i}, Y_{t_i}) - \sigma(h_{t_i}^n, Y_{t_i}^n)\right)\delta X_{t_i t_{i+1}} + R_{t_i, t_{i+1}}^Y \\ &=: (\sigma_{t_i}^1(h_{t_i} - h_{t_i}^n) + \sigma_{t_i}^2 Z_{t_i}^n)\delta X_{t_i t_{i+1}} + R_{t_i t_{i+1}}^Y, \end{aligned} \tag{3.27}$$

where σ^1 and σ^2 are respectively given by

$$\sigma_s^1 = \int_0^1 \partial_1\sigma\left(\lambda h_s + (1-\lambda)h_s^n, Y_s\right)d\lambda, \qquad \sigma_s^2 = \int_0^1 \partial_2\sigma\left(h_s^n, \lambda Y_s^n + (1-\lambda)Y_s\right)d\lambda. \tag{3.28}$$

Observe that both σ^1 and σ^2 are bounded functions.

Starting from (3.27), we now follow some of the ideas of Lemma 3.4. Namely we assume in general that for s, t in the discrete simplex $\mathcal{S}_2(⟦0,\tau⟧)$, the increment δZ_{st}^n can be decomposed as follows:

$$\delta Z_{st}^n = (\sigma_s^1(h_s - h_s^n) + \sigma_s^2 Z_s^n)\delta X_{st} + R_{st}^n \tag{3.29}$$

where we have just set:

$$R_{st}^n = \delta Z_{st}^n - (\sigma_s^1(h_s - h_s^n) + \sigma_s^2 Z_s^n)\delta X_{st}. \tag{3.30}$$

Notice that the increment R_{st}^n depends on Y as well as Y^n through the paths σ^1, σ^2. Also observe that for all $0 \le i \le n-1$ we have $R_{t_i t_{i+1}}^n = R_{t_i t_{i+1}}^Y$. In addition, similarly to what we have done in Lemma 3.3, the increment δR^n can be expressed as:

$$\delta R_{sut}^n = \delta\sigma_{su}^1(h_u - h_u^n)\delta X_{ut} + \sigma_s^1\delta(h-h^n)_{su}\delta X_{ut} + \delta\sigma_{su}^2 Z_u^n \delta X_{ut} + \sigma_s^2 \delta Z_{su}^n \delta X_{ut}. \tag{3.31}$$

We will now formulate an induction assumption for Z^n on the interval $[\![0, t_q]\!] \cap [0, \tau]$, for a small enough time constant τ to be determined later on:

$$|Z_0^n| \le \frac{d_1}{n^{2\gamma-1-\varepsilon}}, \qquad |Z_s^n| \le \frac{C_1}{n^{2\gamma-1-\varepsilon}}, \quad \text{and} \quad |\delta Z_{su}^n| \le \frac{C_2}{n^{2\gamma-1-\varepsilon}}|u-s|^{\gamma}, \tag{3.32}$$

where C_1, C_2, d_1 are three positive constants such that $d_1 \le C_1$ and where we have picked ε such that $0 < \varepsilon < 2\gamma - 1$. We shall dedicate our main efforts to the propagation of this induction hypothesis (which is easily verified for $q = 1$).

Step 2: Upper bound for $\mathbf{R}^n$. As in the proof of Lemma 3.4, we mainly focus our attention on a bound for R_{st}^n with $0 \le s < t = t_q$ with the additional condition $t_q \le \tau$. Then introducing a parameter $\varepsilon > 0$ and plugging relations (3.3) and (3.32) into (3.31) we end up with:

$$\begin{aligned}
|\delta R_{sut}^n| &\le \Big\{ (C_h^3 + C_1) \max\{\|\sigma^1\|_\gamma, \|\sigma^2\|_\gamma\} + 2C_h^1 \|\sigma^1\|_\infty + C_2 \|\sigma^2\|_\infty \Big\} \frac{\|X\|_\gamma (t-s)^{2\gamma}}{n^{2\gamma-1-\varepsilon}} \\
&\le \Big\{ (C_h^3 + C_1) \max\{\|\sigma^1\|_\gamma, \|\sigma^2\|_\gamma\} + 2C_h^1 \|\sigma^1\|_\infty + C_2 \|\sigma^2\|_\infty \Big\} \frac{\|X\|_\gamma}{n^{2\gamma-1-\varepsilon}} (t-s)^{1+\varepsilon} \tau^{2\gamma-1-\varepsilon},
\end{aligned}$$

where σ^1, σ^2 have been defined by (3.28). Hence we easily deduce:

$$|\delta R_{sut}^n| \le \frac{C_3}{n^{2\gamma-1-\varepsilon}} (t-s)^{1+\epsilon},$$

with a constant C_3 defined by:

$$\begin{aligned}
C_3 &= \Big\{ (C_h^3 + C_1) \max\{\|\sigma^1\|_\gamma, \|\sigma^2\|_\gamma\} + 2C_h^1 \|\sigma^1\|_\infty + C_2 \|\sigma^2\|_\infty \Big\} \|X\|_\gamma \tau^{2\gamma-1-\varepsilon} \\
&= \big\{ (C_h^3 + C_1) M_{\sigma,\gamma} + 2C_h^1 \|\sigma^1\|_\infty + C_2 \|\sigma^2\|_\infty \big\} \|X\|_\gamma \tau^{2\gamma-1-\varepsilon},
\end{aligned} \tag{3.33}$$

and where $M_{\sigma,\gamma} \equiv \max(\|\sigma^1\|_\gamma, \|\sigma^2\|_\gamma)$. We also recall that $C_h^3 = C_h^1 + C_h^2$. Moreover, since Y is governed by equation (2.6), observe that:

$$R_{t_i t_{i+1}}^n = R_{t_i t_{i+1}}^Y, \quad \text{and thus} \quad |R_{t_i t_{i+1}}^n| \le C_x (t_{i+1} - t_i)^{2\gamma}. \tag{3.34}$$

Therefore, since $t_{i+1} - t_i = \frac{T}{n}$ and choosing the parameter ε as in equation (3.32), we end up with:

$$\frac{|R_{t_i t_{i+1}}^n|}{|t_{i+1} - t_i|^{1+\varepsilon}} \le \frac{C_x}{n^{2\gamma-1-\varepsilon}}$$

Hence, owing to Lemma 2.2 item (ii) and inequality (3.34), this yields:

$$\|R^n\|_{[\![0,\tau]\!],1+\varepsilon} \le \frac{C_{1+\varepsilon} C_3 + C_x}{n^{2\gamma-1-\varepsilon}} = \frac{C_4}{n^{2\gamma-1-\varepsilon}}, \tag{3.35}$$

where we recall that the Hölder norms of the form $\|R^n\|_{[\![0,\tau]\!],1+\varepsilon}$ are defined by (2.2). Summarizing, we have obtained the global bound (3.35) on the norm of R^n in $[\![0, \tau]\!]$.

Step 3: Upper bound for the increments of $\mathbf{Z}^n$. We now plug our global bound (3.35) and our standing assumption (3.32) into (3.29), and we get the following bound for δZ_{st}^n, where we recall that we consider a couple of points $(s, t) \in \mathcal{S}_2([\![0, \tau]\!])$ with $t = t_q$:

$$\begin{aligned}
|\delta Z_{st}^n| &\le M_{\sigma,\gamma} \|X\|_\gamma \frac{C_h^1 + C_1}{n^{2\gamma-1-\varepsilon}} |t-s|^\gamma + \frac{C_4}{n^{2\gamma-1-\epsilon}} |t-s|^{1+\varepsilon} \\
&\le \frac{M_{\sigma,\gamma} \|X\|_\gamma (C_h^1 + C_1) + C_4 \tau^{1+\varepsilon-\gamma}}{n^{2\gamma-1-\varepsilon}} |t-s|^\gamma \le \frac{C_5}{n^{2\gamma-1-\varepsilon}} |t-s|^\gamma,
\end{aligned} \tag{3.36}$$

where we have set $C_5 = M_{\sigma,\gamma} \|X\|_\gamma (C_h^1 + C_1) + C_4\tau^{1+\varepsilon-\gamma}$.

Let us write the constant C_5 we have just defined in a more explicit manner. Indeed, specifying the values of each constant $C_1, \dots, C_4$ we obtain:

$$\begin{aligned}
C_5 &= M_{\sigma,\gamma} \|X\|_\gamma (C_h^3 + C_1) + C_4\tau^{1+\varepsilon-\gamma} \\
&= M_{\sigma,\gamma} \|X\|_\gamma (C_h^3 + C_1) + (C_{1+\varepsilon}C_3 + C_x)\tau^{1+\varepsilon-\gamma} \\
&= M_{\sigma,\gamma} \|X\|_\gamma (C_h^3 + C_1) \\
&\qquad + \Big(C_{1+\varepsilon}\left\{(C_h^3 + C_1)M_{\sigma,\gamma} + 2C_h^1 \left\|\sigma^1\right\|_\infty + C_2\left\|\sigma^2\right\|_\infty\right\}\|X\|_\gamma \tau^{2\gamma-1-\varepsilon} + C_x\Big)\tau^{1+\varepsilon-\gamma} \\
&= M_{\sigma,\gamma} \|X\|_\gamma (C_h^3 + C_1) + C_{1+\varepsilon}\left((C_h^1 + C_1)M_{\sigma,\gamma} + 2C_h^1\left\|\sigma^1\right\|_\infty + C_2\left\|\sigma^2\right\|_\infty\right)\|X\|_\gamma \tau^\gamma \\
&\qquad + C_x\tau^{1+\varepsilon-\gamma}
\end{aligned} \tag{3.37}$$

Now in order to propagate the induction hypothesis (3.32), we need to choose our parameter τ such that $C_5 \le C_2$. Going back to identity (3.37), we will first choose τ such that:

$$\tau^\gamma < \frac{\alpha_1}{\|X\|_\gamma}, \quad \text{with} \quad \alpha_1 \le \frac{1}{2C_{1+\varepsilon}\left\|\sigma^2\right\|_\infty}. \tag{3.38}$$

Then a sufficient condition in order to achieve $C_5 \le C_2$ is

$$\begin{aligned}
\frac{C_2}{2} \ge{}& M_{\sigma,\gamma}\|X\|_\gamma (C_h^3 + C_1) \\
&+ C_{1+\varepsilon}\left((C_h^3 + C_1)M_{\sigma,\gamma} + 2C_h^1\left\|\sigma^1\right\|_\infty\right)\|X\|_\gamma \tau^\gamma + C_x\tau^{1+\varepsilon-\gamma}.
\end{aligned} \tag{3.39}$$

If we further assume that $\tau \le 1$, one can recast (3.39) as:

$$C_2 \ge 2\left\{M_{\sigma,\gamma}\|X\|_\gamma (C_h^3 + C_1) + C_{1+\varepsilon}\left((C_h^3 + C_1)M_{\sigma,\gamma} + 2C_h^1\left\|\sigma^1\right\|_\infty\right)\|X\|_\gamma + C_x\right\}. \tag{3.40}$$

We thus choose $C_2 = 2\{M_{\sigma,\gamma}\|X\|_\gamma (C_h^3+C_1)+C_{1+\varepsilon}((C_h^3+C_1)M_{\sigma,\gamma}+2C_h^1\|\sigma^1\|_\infty)\|X\|_\gamma+C_x\}$. In conclusion, under the assumptions (3.38) and $\tau \le 1$, we have propagated (3.32) as far as the increments δZ_{st} are concerned.

Step 4: Upper bound for $\mathbf{Z}^n$. We still have to propagate our assumption on Z_s^n in (3.32). We thus consider again a time parameter τ and $t \le \tau$. Furthermore, we simplify the notation in (3.40) and remark that we can take C_2 of the form

$$C_2 = \alpha_1(1 + C_1)(1 + \|X\|_\gamma),$$

for a large enough constant α_1. Now according to the relation on Z_0^n and δZ^n imposed by (3.32), we have:

$$|Z_t^n| \le |Z_0^n| + |\delta Z_{0t}^n| \le \frac{\alpha_2}{n^{2\gamma-1-\varepsilon}}, \quad \text{with} \quad \alpha_2 = d_1 + \alpha_1(1 + C_1)(1 + \|X\|_\gamma)\tau^\gamma. \tag{3.41}$$

With this expression of α_2 in hand, if we further assume $C_1 \ge 4d_1$ and $d_1 > \frac{1}{8}$, it is readily checked that

$$\alpha_2 \le C_1, \quad \text{as soon as} \quad \tau^\gamma \le \frac{\alpha_3}{1 + \|X\|_\gamma} \quad \text{with} \quad \alpha_3 = \frac{1}{4\alpha_1}. \tag{3.42}$$

Step 5: Global bounds. Gathering the conditions (3.38) and (3.42), we have obtained that there exists $\alpha_4 > 0$ such that if

$$\tau = \frac{\alpha_4}{(1 + \|X\|_\gamma)^{\frac{1}{\gamma}}}, \tag{3.43}$$

then relation (3.32) holds on $[0, \tau]$, with

$$C_1 = 4d_1, \quad d_1 > \frac{1}{8} \quad \text{and} \quad C_2 = \alpha_1(1 + C_1)(1 + \|X\|_\gamma). \tag{3.44}$$

We now get bounds on successive intervals called I_j, denoted by $I_j = [\tau_j, \tau_{j+1}]$. In particular we take $\tau_0 = 0$ and $\tau_1 = \tau$ as defined in (3.43). In order to estimate the length $|I_j| = \tau_{j+1} - \tau_j$ we remark that all the computations of the previous steps are valid except for the fact that the initial value Z_0^n has to be updated to $Z_{j\tau}^n$. We thus start our induction procedure on I_j by assuming

$$|Z_0^n| \le \frac{d_{1,j}}{n^{2\gamma-1-\varepsilon}}, \qquad |Z_s^n| \le \frac{C_{1,j}}{n^{2\gamma-1-\varepsilon}}, \quad \text{and} \quad |\delta Z_{su}^n| \le \frac{C_{2,j}}{n^{2\gamma-1-\varepsilon}}|u-s|^\gamma. \tag{3.45}$$

Then, owing to the same computations as in Step 4, we end up with a generalization of (3.43) and (3.44) as follows:

$$C_{1,j+1} = 4\,C_{1,j}, \qquad C_{2,j} = \alpha_1(1 + C_{1,j})(1 + \|X\|_\gamma), \qquad \tau_{j+1} - \tau_j = \frac{\alpha_4}{(1 + \|X\|_\gamma)^{\frac{1}{\gamma}}}. \tag{3.46}$$

In particular we note that $\tau_{j+1} - \tau_j$ is constant, as well as the relation linking $C_{2,j}$ and $C_{1,j}$. In addition, iterating (3.46) we obviously get

$$C_{1,j} = C_0\,4^j. \tag{3.47}$$

We can now bound the number of iterations needed in order to fill the interval $[0, T]$. Indeed, according to (3.46) it is enough to take $j \ge \frac{(1+\|X\|_\gamma)^{1/\gamma} T}{\alpha_4}$ in order to have $\tau_j \ge T$. Plugging this value into (3.46) and (3.47) and then reporting into equation (3.32) stated for each $[\tau_j, \tau_{j+1}]$, our claims (3.23) and (3.24) are now proved. □

4. Numerical examples

This section is devoted to an illustration of our numerical scheme with simulations. We shall focus on the particular FSDDE given by

$$\begin{aligned} dY_t &= aY_t dt + (b_1 Y_t + b_2 Y_{t-1})dX_t, \quad t \in [0, 1.5] \\ Y_s &= 1 + s \quad s \in [-1, 0], \end{aligned} \tag{4.1}$$

where $X = (X_t)_{t\ge 0}$ is a fractional Brownian motion with Hurst parameter $H > 1/2$.

Since this linear equation can be solved explicitly, comparisons of our approximation with the real solution will be easy. For example, if $b_1 = 0$, one may consider equation (4.1) as a deterministic linear equation perturbed by a delayed fractional noise.

Let us start by examining the Euler approximation introduced in Section 3. In Figure 1 and Figure 2 we show the exact solution (red line) to the deterministic equation $dY_t = aY_t dt$ when the coefficient a satisfies $a = -2$, together with four sample paths for the FSDDE (4.1) with $b_1 = 0$, $b_2 = 0.7$ and $H = 0.75$ (Figure 1), $H = 0.9$ (Figure 2). The Euler approximation of the solution to equation (4.1) was done with $n = 10000$ steps. These pictures show the perturbation of the deterministic equation $dY_t = aY_t dt$ under the influence of the noisy term, according to the value of the parameter b_2 in equation (4.1).

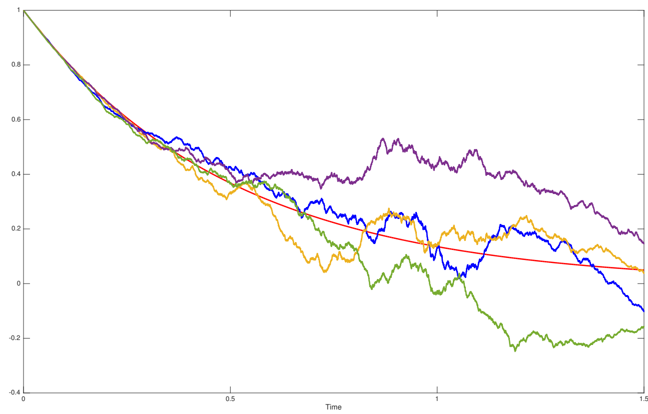

FIGURE 1. Sample path of the deterministic exact solution $dY_t = aY_t dt$ with $a = -2$ (red line) and four paths of the approximated solution to (4.1) by our Euler scheme, with $b_1 = 0, b_2 = 0.7$ and $H = 0.75$.

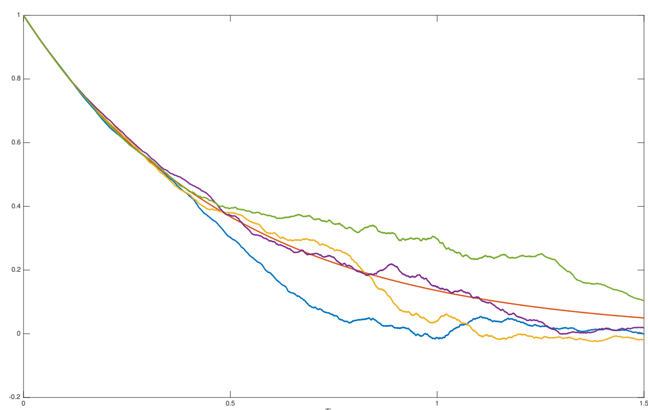

FIGURE 2. Sample path of the deterministic exact solution $dY_t = aY_t dt$ with $a = -2$ (red line) and four paths of the approximated solution to (4.1) by our Euler scheme, with $b_1 = 0, b_2 = 0.7$ and $H = 0.9$.

As far as convergence rate of our approximation is concerned, we have made two different experiments:

(i) First we have simulated 1000 sample paths of (4.1) with $a = -2$, $b_1 = 0, b_2 = 0.7$ and $H = 0.85$ according to our approximation and compared the result with the true value of the solution. We have found that the tail of the error $|Y_T - Y_T^n|$ at the terminal time $T = 1.5$ seems to behave roughly as the absolute value of a Gaussian random variable.

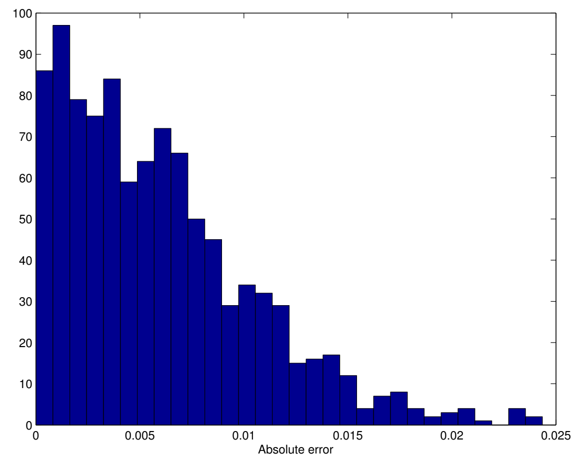

FIGURE 3. Histogram for the absolute value of the error obtained on 1000 paths of equation (4.1) for $H = 0.85$.

(ii) Figure 4 shows the rate of convergence evaluated at the terminal time $T = 1.5$. Namely, for different values of the Hurst parameter H we simulate $m = 1000$ paths of Y and Y^n. Then for each path we compute the absolute value $|Y_T - Y_T^n|$ and we take the mean of those errors. We observe that the averaged error becomes smaller when H is close to 1, which is consistent with Theorem 3.5.

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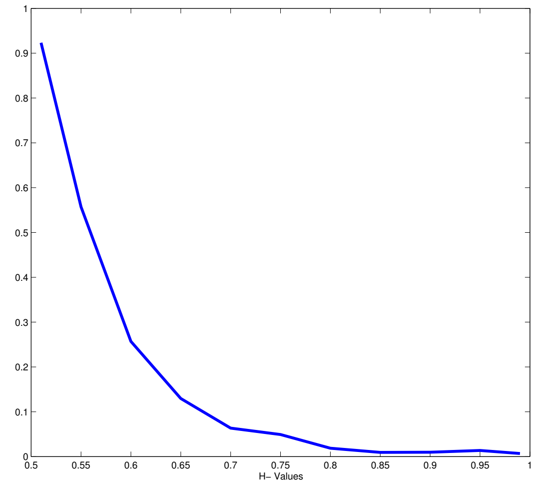

Figure 4. Decay of the absolute error for the Euler scheme in terms of H.

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