Determinants

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Differential equations and linear algebra - MA 262

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Pearson Collections

Outline

Outline

Particular cases

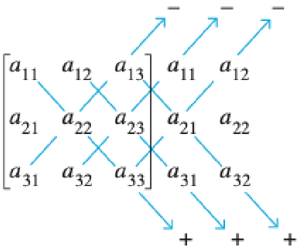
1×1 matrix:

$$A = [a_{11}] \implies \det(A) = a_{11}$$

2×2 matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \implies \det(A) = a_{11}a_{22} - a_{12}a_{21}$$

3×3 matrix:



Remarks

Generalization: The determinant is defined for any $n \times n$ matrix
$\hookrightarrow$ Combinatorics involved

Motivation: In general

$$\det(A) \neq 0 \quad \Longleftrightarrow \quad A \text{ is invertible}$$

Notation:

$$\det(A) \equiv |A|$$

Examples

2×2 matrix:

$$\begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix} = -1$$

3×3 matrix:

$$\begin{vmatrix} 1 & 3 & -4 \\ 2 & 5 & -1 \\ 1 & 0 & 6 \end{vmatrix} = 11$$

Recursive method: strategy

Fact:

The determinant computation requires $n!$ operations

Aim:

Reduce the order of a determinant by an expansion

Vocabulary:

First we have to introduce the notions of

- Minor
- Cofactor

Minors of a matrix

Definition 1.

Let A be a $n \times n$ matrix. Then

$A_{ij} =$
det(matrix obtained by deleting ith row and jth column of A)

The quantity A_{ij} is called minor of a_{ij}.

Example:

$$A = \begin{bmatrix} 1 & 3 & -4 \\ 2 & 5 & -1 \\ 1 & 0 & 6 \end{bmatrix} \implies A_{12} = \begin{vmatrix} 2 & -1 \\ 1 & 6 \end{vmatrix} = 13$$

Cofactors of a matrix

Definition 2.

Let A be a $n \times n$ matrix. Then

$$C_{ij} = (-1)^{i+j} A_{ij}$$

The quantity C_{ij} is called cofactor of a_{ij}.

Example:

$$A = \begin{bmatrix} 1 & 3 & -4 \\ 2 & 5 & -1 \\ 1 & 0 & 6 \end{bmatrix} \implies C_{12} = -M_{12} = -13$$

Remark: Alternate signs assignment for C_{ij}

Cofactor expansion

Theorem 3.

Let

- A be a $n \times n$ matrix.

Then

1. One can expand the determinant along the i-th row:

$$\det(A) = \sum_{k=1}^{n} a_{ik} C_{ik}$$

2. One can expand the determinant along the j-th column:

$$\det(A) = \sum_{k=1}^{n} a_{kj} C_{kj}$$

Example of application

Rule:

To simplify computations, choose row or column with 0's

Example:

Here we expand along the 3rd row

$$\begin{vmatrix} 1 & 3 & -4 \\ 2 & 5 & -1 \\ 1 & 0 & 6 \end{vmatrix} = \begin{vmatrix} 3 & -4 \\ 5 & -1 \end{vmatrix} + 6 \begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix} = 11$$

Outline

Introduction

Problem with determinants:

- For a $n \times n$, matrix, they require $n!$ operations
- This is computationally too demanding

Aim of this section:

- See properties in order to shorten computation time

Determinants of triangular matrices

Theorem 4.

Let

- A be an upper or lower triangular matrix.
- $n \equiv$ size of A.

Then

$$\det(A) = a_{11}\, a_{22} \cdots a_{nn} = \prod_{i=1}^{n} a_{ii}$$

Example:

$$\begin{vmatrix} 1 & 3 & -4 \\ 0 & 5 & -1 \\ 0 & 0 & 6 \end{vmatrix} = 30$$

Elementary row operations and determinants

Effect of elementary row operations:

If A is a $n \times n$ matrix, then

1. Let B be the matrix obtained by permuting 2 rows of A. Then
$$\det(B) = -\det(A)$$
2. Let B obtained by multiplying 1 row of A by $k \in \mathbb{R}$. Then
$$\det(B) = k \det(A)$$
3. Let B obtained by adding $k\times$ a row of A to a different row of A. Then
$$\det(B) = \det(A)$$

Example of application

3×3 matrix:

$$\begin{vmatrix} 1 & 3 & -4 \\ 2 & 5 & -1 \\ 1 & 0 & 6 \end{vmatrix} \overset{A_{12}(-2),\, A_{13}(-1)}{=} \begin{vmatrix} 1 & 3 & -4 \\ 0 & -1 & 7 \\ 0 & -3 & 10 \end{vmatrix}$$

$$\overset{M_2(-1),\, M_3(-1)}{=} (-1)^2 \begin{vmatrix} 1 & 3 & -4 \\ 0 & 1 & -7 \\ 0 & 3 & -10 \end{vmatrix} \overset{A_{23}(-3)}{=} \begin{vmatrix} 1 & 3 & -4 \\ 0 & 1 & -7 \\ 0 & 0 & 11 \end{vmatrix} = 11$$

Remark:

This technique is really useful for $n \geq 4$

Further properties of determinants

Some more properties:

4. We have
$$\det(A^T) = \det(A)$$
5. If A has a column of 0's, then
$$\det(A) = 0$$
6. If 2 rows or columns of A are the same, then
$$\det(A) = 0$$
7. For two matrices A and B, we have
$$\det(A\,B) = \det(A)\,\det(B)$$

Application of Property 4

Example:

When further simplifications are available for columns

$$\begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 5 \\ -1 & 3 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ 0 & 5 & 2 \end{vmatrix} \overset{A_{23}(-5)}{=} \begin{vmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & -13 \end{vmatrix} = -13$$

Outline

Cramer's rule

Theorem 5.

Consider a $n \times n$ matrix A, a vector $\mathbf{b}$ and the system

$$A\mathbf{x} = \mathbf{b}. \tag{1}$$

For $1 \le k \le n$ set ($\mathbf{b}$ inserted at column k):

$$A_k(\mathbf{b}) = \begin{bmatrix} a_{11} & a_{12} & \dots & b_1 & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & b_2 & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ a_{n1} & a_{n2} & \dots & b_n & \dots & a_{nn} \end{bmatrix}$$

Then if $\det(A) \neq 0$ the solution of (1) is given by

$$x_k = \frac{\det(A_k(\mathbf{b}))}{\det(A)}$$

Example

System:

$$\begin{array}{rrrl} 3x_1 & +2x_2 & -x_3 & = 4 \\ x_1 & +x_2 & -5x_3 & = -3 \\ -2x_1 & -x_2 & +4x_3 & = 0 \end{array}$$

Determinants:

$$\det(A) = \begin{vmatrix} 3 & 2 & -1 \\ 1 & 1 & -5 \\ -2 & -1 & 4 \end{vmatrix} = 8, \qquad \det(A_1(\mathbf{b})) = \begin{vmatrix} 4 & 2 & -1 \\ -3 & 1 & -5 \\ 0 & -1 & 4 \end{vmatrix} = 17$$

Solution:

$$x_1 = \frac{17}{8}$$

Cofactors of a matrix

Definition 6.

Let A be a $n \times n$ matrix. Then

$$C_{ij} = (-1)^{i+j} A_{ij}$$

The quantity C_{ij} is called cofactor of a_{ij}.

Example:

$$A = \begin{bmatrix} 1 & 3 & -4 \\ 2 & 5 & -1 \\ 1 & 0 & 6 \end{bmatrix} \implies C_{12} = -M_{12} = -13$$

Remark: Alternate signs assignment for C_{ij}

Adjoint matrix

Definition 7.

Let A be a $n \times n$ matrix. Then

- Matrix of cofactors:
  Obtained by replacing each term of A by its cofactor
  Denoted by M_C
- Adjoint matrix: Denoted by $\text{adj}(A)$ and defined as

$$\text{adj}(A) = M_C^T$$

The adjoint method

Theorem 8.

Let A be a $n \times n$ matrix. Assume:

$$\det(A) \neq 0.$$

Then

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A).$$

Remark: Along the same lines we have

$$A \text{ invertible} \quad \Longleftrightarrow \quad \det(A) \neq 0$$

Example

Matrix:

$$A = \begin{bmatrix} 2 & 0 & -3 \\ -1 & 5 & 4 \\ 3 & -2 & 0 \end{bmatrix}$$

Cofactor and adjoint matrix:

$$M_C = \begin{bmatrix} 8 & 12 & -13 \\ 6 & 9 & 4 \\ 15 & -5 & 10 \end{bmatrix}, \qquad \mathrm{adj}(A) = \begin{bmatrix} 8 & 6 & 15 \\ 12 & 9 & -5 \\ -13 & 4 & 10 \end{bmatrix}$$

Inverse: $\det(A) = 55$ and thus

$$A^{-1} = \frac{1}{55} \begin{bmatrix} 8 & 6 & 15 \\ 12 & 9 & -5 \\ -13 & 4 & 10 \end{bmatrix}$$

Determinant as area or volume

Theorem 9.

Let A be a 2×2 or 3×3 matrix. Then

(1) If A is a 2×2 matrix we have

$$\det(A) = \text{ area of parallelogram given by columns of } A$$

(2) If A is a 3×3 matrix we have

$$\det(A) = \text{ volume of parallepiped given by columns of } A$$

Example of area

Aim: Compute area of parallelogram given by

$$(-2,-2), \qquad (0,3), \qquad (4,-1), \qquad (6,4)$$

Translation: We translate by $(2,2)$ to get a vertex at $\mathbf{0}$

$$(0,0), \qquad (2,5), \qquad (6,1), \qquad (8,6)$$

Area:

$$\text{Area} = \left| \begin{vmatrix} 2 & 6 \\ 5 & 1 \end{vmatrix} \right| = 28$$

Area and linear transformation in $\mathbb{R}^2$

Theorem 10.

Let

- $T : \mathbb{R}^2 \to \mathbb{R}^2$ linear transformation
- A matrix of T
- S parallelogram in $\mathbb{R}^2$

Then we have

$$\text{Area}\,(T(S)) = |\det(A)|\ \text{Area}\,(S)$$

Area and linear transformation in $\mathbb{R}^3$

Theorem 11.

Let

- $T : \mathbb{R}^3 \to \mathbb{R}^3$ linear transformation
- A matrix of T
- S parallepiped in $\mathbb{R}^3$

Then we have

$$\text{Volume}\,(T(S)) = |\det(A)|\ \text{Volume}\,(S)$$

Application (1)

Aim: Find area of region E delimited by ellipse

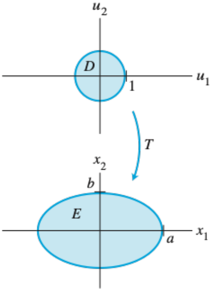
$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$$

Strategy: Let $D =$ unit disk in $\mathbb{R}^2$. We write

$$E = T(D), \quad \text{with} \quad A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

Application

Illustration:



Area:

$$\text{Area}\,(E) = \text{Area}\,(T(D)) = |\det(A)|\ \text{Area}\,(D) = \pi ab$$