Higher order differential equations

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Differential equations and linear algebra - MA 262

Taken from *Differential equations and linear algebra*
Edwards, Penney, Calvis

Outline

Outline

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Second order equation

Second order linear differential equation:

$$ay'' + b\,y' + c\,y = f(t) \tag{1}$$

Second order homogeneous linear differential equation:

$$ay'' + b\,y' + c\,y = 0$$

Natural type of solution: Of the form

$$\exp(\alpha\, t)$$

Existence and uniqueness

Theorem 1.

General homogeneous linear equation:

$$a y'' + b\, y' + c\, y = 0 \tag{2}$$

Initial condition:

$$y(t_0) = Y_0, \quad y'(t_0) = Y_1. \tag{3}$$

Hypothesis:

- a, b, c, t_0, Y_0, Y_1 are real numbers.
- $a \neq 0$.

Conclusion:
There exists a unique function y satisfying (2)-(3) on $\mathbb{R}$.

Solutions as a vector space

Theorem 2.

General homogeneous linear equation:

$$a y'' + b\, y' + c\, y = 0 \tag{4}$$

Results:

1. The set of solutions to (4) is a

   Vector space of dimension 2

2. If y_1, y_2 are two linearly independent solutions of (4)
   $\hookrightarrow$ The general solution can be written as

$$y = c_1 y_1 + c_2 y_2$$

Criterion for independence

Theorem 3.

General homogeneous linear equation:

$$ay'' + b\,y' + c\,y = 0 \tag{5}$$

Results:

1. Let y_1, y_2 be solutions of (5). Then
$$y_1, y_2 \text{ are linearly independent}$$
$$\Longleftrightarrow$$
$$y_1(\tau)y_2'(\tau) - y_2(\tau)y_1'(\tau) \neq 0 \text{ for a given } \tau \in \mathbb{R}.$$

2. The condition above can also be written as
$$y_1, y_2 \text{ are linearly independent}$$
$$\Longleftrightarrow$$
$$W[y_1, y_2](\tau) \neq 0 \text{ for a given } \tau \in \mathbb{R}.$$

Wronskian

Definition 4.

Let

- y_1, y_2 two differentiable functions

Then

$$W[y_1, y_2](t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$$

Remark: With the expression of determinant we get

$$W[y_1, y_2](t) = y_1(t)y_2'(t) - y_2(t)y_1'(t)$$

Example

Equation:

$$y'' + y' - 6y = 0$$

Exponential solutions: We find two solutions

$$y_1 = e^{2x}, \qquad y_2 = e^{-3x}$$

Wronskian:

$$W[y_1, y_2](x) = -5e^{-x} \neq 0$$

Conclusion: General solution of the form

$$y = c_1 y_1 + c_2 y_2$$

Outline

Auxiliary equation

Proposition 5.

Equation considered: for $a, b, c \in \mathbb{R}$,

$$ay'' + by' + cy = 0. \tag{6}$$

Auxiliary equation:

$$ar^2 + br + c = 0. \tag{7}$$

Facts about the auxiliary equation:

- Equation (7) admits two roots r_1, r_2
- Those two roots can be repeated or complex valued

Construction of solutions

Equation: Homogeneous with constant coefficients (6).

Roots of characteristic polynomial: r_1, r_2.

Rules to find solutions: separate 3 cases,

1. If $r_1, r_2 \in \mathbb{R}$ non repeated root,

$$y_1 = \exp(r_1 t), \; y_2 = \exp(r_2 t) \quad \text{solutions to equation (6).}$$

2. If $r_1 = \alpha + \imath\beta$ and $r_2 = \alpha - \imath\beta$ conjugate complex roots,

$$y_1 = \exp(\alpha t)\cos(\beta t), \; y_2 = \exp(\alpha t)\sin(\beta t)$$

solutions to equation (6).

Construction of solutions (2)

Rules to find solutions (ctd): separate 3 cases,

3. If $r \in \mathbb{R}$ repeated root,

$$y_1 = \exp(r\,t)\,, \;\; y_2 = t\exp(r\,t) \quad \text{solutions to equation (6).}$$

Remark:

1. All the solutions y_1, y_2 above are linearly independent
2. Solutions y_1, y_2 above are called fundamental solutions of (6)

Example with simple roots

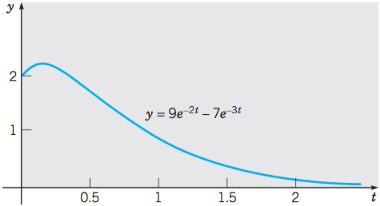
Equation considered:

$$y'' + 5y' + 6y = 0 \quad y(0) = 2, \quad y'(0) = 3. \tag{8}$$

Solution: given by

$$y = 9\exp(-2t) - 7\exp(-3t)$$

Graph of solution:



Example with complex roots

Equation:

$$y'' + y' + 9.25y = 0. \tag{9}$$

Characteristic equation:

$$r^2 + r + 9.25 = 0.$$

Roots of characteristic equation:

$$r_1 = -\frac{1}{2} + 3\imath, \qquad r_2 = -\frac{1}{2} - 3\imath$$

Real valued fundamental solutions:

$$y_1 = e^{-\frac{t}{2}} \cos(3t), \qquad y_2 = e^{-\frac{t}{2}} \sin(3t).$$

Example with complex roots (2)

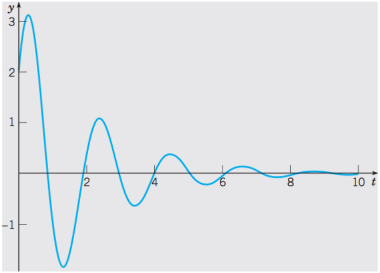
Initial value problem: equation (9) with

$$y(0) = 2, \quad \text{and} \quad y'(0) = 8.$$

Solution:

$$y = e^{-\frac{t}{2}} \left[2\cos(3t) + 3\sin(3t) \right].$$

Graph: decaying oscillations



Example with double root

Equation:

$$y'' - y' + 0.25y = 0. \tag{10}$$

Roots of characteristic equation:

$$r = \frac{1}{2}$$

Fundamental solutions:

$$y_1 = e^{\frac{t}{2}}, \qquad y_2 = te^{\frac{t}{2}}.$$

Example with double root (2)

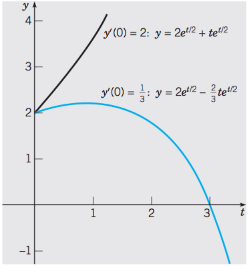
Initial value problem: equation (10) with

$$y(0) = 2, \quad \text{and} \quad y'(0) = \frac{1}{3}.$$

Solution:

$$y = \left(2 - \frac{2}{3}t\right) e^{\frac{t}{2}}.$$

Graph:



Example with double root (3)

Modification of initial value: equation (10) with

$$y(0) = 2, \quad \text{and} \quad y'(0) = 2.$$

Solution:

$$y = (2 + t)\, e^{\frac{t}{2}}.$$

Question:

Separation between increasing and decreasing behavior of y
$\hookrightarrow$ according to value of $y'(0)$.

Outline

Equation of order n

Linear differential equation of order n: in non standard form

$$a_n(x)y^{(n)} + a_{n-1}(x)\,y^{(n-1)} + \cdots + a_1(x)\,y' + a_0(x)\,y = b(x)$$

Linear differential equation of order n: in standard form

$$y^{(n)} + p_1\,y^{(n-1)} + p_2\,y^{(n-2)} + \cdots + p_{n-1}\,y' + p_n\,y = g(x) \qquad (11)$$

Existence and uniqueness

Theorem 6.

General linear equation:

$$y^{(n)} + p_1\, y^{(n-1)} + p_2\, y^{(n-2)} + \cdots + p_{n-1}\, y' + p_n\, y = g(x) \tag{12}$$

Initial condition:

$$y(x_0) = \gamma_0, \quad y'(x_0) = \gamma_1, \quad \ldots \quad , \quad y^{(n-1)}(x_0) = \gamma_{n-1}. \tag{13}$$

Hypothesis:

- $x_0 \in I$, where $I = (\alpha, \beta)$.
- $p_1, \ldots, p_n$ and g continuous on I.

Conclusion:

There exists a unique function y satisfying (12)-(13) on I.

Example of maximal interval

Equation considered:

$$x(x-1)y''' - 3xy'' + 6x^2y' - \cos(x)y = (x+5)^{1/2}$$
$$y(-2) = 2, \quad y'(-2) = 1, \quad y''(-2) = -1.$$

Equivalent form:

$$y''' - \frac{3x}{x(x-1)}y'' + \frac{6x^2}{x(x-1)}y' - \frac{\cos(x)}{x(x-1)} = \frac{(x+5)^{1/2}}{x(x-1)}.$$

Application of Theorem 6:

- p_1, p_2, p_3 continuous on $(-\infty, 0) \cup (0,1) \cup (1,\infty)$
- g continuous on $(-5, 0) \cup (0,1) \cup (1,\infty)$
- $-2 \in (-5, 0)$

We thus get unique solution on $(-5, 0)$

Wronskian

Definition 7.

Let

- $\{f_1, \ldots, f_k\}$ be a family of functions in $C^{k-1}(I)$.

The Wronskian of these functions is defined by

$$W[f_1, \ldots, f_k](x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_k(x) \\ f_1'(x) & f_2'(x) & \cdots & f_k'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(k-1)}(x) & f_2^{(k-1)}(x) & \cdots & f_k^{(k-1)}(x) \end{vmatrix}$$

Homogeneous equations

Theorem 8.

General homogeneous linear equation:

$$y^{(n)} + p_1\, y^{(n-1)} + p_2\, y^{(n-2)} + \cdots + p_{n-1}\, y' + p_n\, y = 0 \qquad (14)$$

Results:

1. The set of solutions of (14) is a vect. space of dimension n
2. Let $y_1, \ldots, y_n$ be solutions of (14). Then

$$y_1, \ldots, y_n \text{ are linearly independent}$$
$$\Longleftrightarrow$$
$$W[y_1, \ldots, y_n](x_0) \neq 0 \text{ for a given } x_0 \in I.$$

Example

Equation:

$$y'' + y' - 6y = 0$$

Exponential solutions: We find two solutions

$$y_1 = e^{2x}, \qquad y_2 = e^{-3x}$$

Wronskian:

$$W[y_1, y_2](x) = -5e^{-x} \neq 0$$

Conclusion: General solution of the form

$$y = c_1 y_1 + c_2 y_2$$

Another example of Wronskian

Equation:

$$(D-1)(D-2)(D+3)y = 0$$

Exponential solutions: We find 3 solutions

$$y_1 = e^x, \qquad y_2 = e^{2x}, \qquad y_3 = e^{-3x}$$

Wronskian:

$$W[y_1, y_2, y_3](x) = 20 \neq 0$$

Conclusion: General solution of the form

$$y = c_1 y_1 + c_2 y_2 + c_3 y_3$$

Non homogeneous equation

Theorem 9.

General linear equation:

$$Ly = F(x). \tag{15}$$

Hypothesis: We have found

- $y_1, \ldots, y_n$ solutions of $Ly = 0$
- A particular solution y_p of $Ly = F$

Conclusion:
The general solution of equation (15) is

$$y = y_p + c_1\, y_1 + c_2\, y_2 + \cdots + c_n\, y_n$$

Example

Equation:

$$y'' + y' - 6y = 8e^{5x} \tag{16}$$

General solution of the homogeneous system: We have seen that

$$y = c_1 e^{2x} + c_2 e^{-3x}$$

Particular solution: One can check that y_p solves (16) with

$$y_p(x) = \frac{1}{3} e^{5x}$$

Conclusion:

General solution for the non homogenous system of the form

$$y = c_1 e^{2x} + c_2 e^{-3x} + \frac{1}{3} e^{5x}$$

Outline

Homogeneous equation with constant coefficients

Equation considered: for $a_0, \ldots, a_n \in \mathbb{R}$,

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} \cdots + a_0 y = 0. \tag{17}$$

Auxiliary polynomial:

$$P(r) = a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0.$$

Facts about P:

1. P has n roots (real, complex or repeated) $r_1, \ldots, r_n$.
2. P factorizes as: $P(r) = a_n (r - r_1) \cdots (r - r_n)$.

Construction of solutions

Equation: Homogeneous with constant coefficients (17).

Roots of characteristic polynomial: $r_1, \ldots, r_n$.

Rules to find solutions: separate 4 cases,

1. If $r_j \in \mathbb{R}$ non repeated root,

$$\exp(r_j x) \quad \text{solution to equation (17).}$$

2. If $r_j = a + \imath b$ and $r_{j+1} = a - \imath b$ conjugate complex roots,

$$\exp(ax)\cos(bx), \; \exp(ax)\sin(bx) \quad \text{solutions to equation (17).}$$

Construction of solutions (2)

Rules to find solutions (ctd): separate 4 cases,

3. If $r_j \in \mathbb{R}$ repeated root of order s,

$$\exp\left(r_j x\right), x \exp\left(r_j x\right), \ldots, x^{s-1} \exp\left(r_j x\right)$$

   are solutions to equation (17).

4. If $r_j = a \pm \imath b \in \mathbb{C}$ repeated roots of order s,

$$\begin{array}{l} \exp(ax)\cos(bx), x\exp(ax)\cos(bx), \ldots, x^{s-1}\exp(ax)\cos(bx) \\ \exp(ax)\sin(bx), x\exp(ax)\sin(bx), \ldots, x^{s-1}\exp(ax)\sin(bx) \end{array}$$

are solutions to equation (17).

Remark: All the solutions above are linearly independent

Example with complex roots

Equation:

$$y^{(3)} + y'' + 3y' - 5y = 0$$

Auxiliary polynomial:

$$P(r) = r^3 + r^2 + 3r - 5 = (r-1)(r^2 + 2r + 5)$$

Roots:

- $r = 1$
- $r = -1 + 2\imath$
- $r = -1 - 2\imath$

Example with complex roots (2)

Roots:

- $r = 1$
- $r = -1 + 2\imath$
- $r = -1 - 2\imath$

General solution: of the form

$$y = c_1 e^{x} + c_2 e^{-x} \cos(2x) + c_3 e^{-x} \sin(2x)$$

Example with multiple roots

Equation:

$$D^3(D-2)^2(D^2+1)^2 y = 0$$

Auxiliary polynomial:

$$P(r) = r^3(r-2)^2(r^2+1)^2$$

Roots:

- $r = 0$, multiplicity 3
- $r = 2$, multiplicity 2
- $r = \pm\imath$, multiplicity 2

Example with multiple roots (2)

Fundamental solutions:

- $y_1 = 1$
- $y_2 = x$
- $y_3 = x^2$
- $y_4 = e^{2x}$
- $y_5 = xe^{2x}$
- $y_6 = \cos(x)$
- $y_7 = x\cos(x)$
- $y_8 = \sin(x)$
- $y_9 = x\sin(x)$

General solution:

$$y = \sum_{j=1}^{9} C_j\, y_j$$

Example with computations in $\mathbb{C}$ (1)

Equation:

$$y^{(4)} + 4y = 0$$

Auxiliary polynomial:

$$P(r) = r^4 + 4$$

Roots:

- $r = \sqrt{2\imath}$
- $r = -\sqrt{2\imath}$
- $r = \sqrt{-2\imath}$
- $r = -\sqrt{-2\imath}$

Question:
How to express those roots in $\mathbb{C}$?

Example with computations in $\mathbb{C}$ (2)

Applying Euler's formula: We get

$$\sqrt{2\imath} = 1 + \imath, \qquad \sqrt{-2\imath} = -1 + \imath$$

Expression for the roots:

$$1 \pm \imath, \qquad -1 \pm \imath$$

General solution:

$$y = e^{x}\left(c_1 \cos(x) + c_2 \sin(x)\right) + e^{-x}\left(c_3 \cos(x) + c_4 \sin(x)\right)$$

Example with repeated roots in $\mathbb{C}$ (1)

Equation:

$$\left(D^2 + 6D + 13\right)^2 y = 0$$

Example with repeated roots in $\mathbb{C}$ (2)

Auxiliary polynomial:

$$P(r) = \left(r^2 + 6r + 13\right)^2 = \left((r+3)^2 + 4\right)^2$$

Roots:

- $r = -3 + 2\imath$, multiplicity 2
- $r = -3 - 2\imath$, multiplicity 2

General solution:

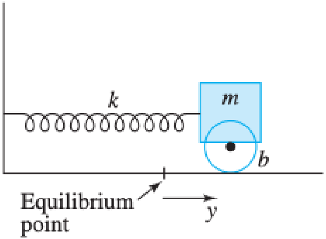
$$y = e^{-3x}\left(c_1 \cos(2x) + c_2 \sin(2x)\right) + x e^{-3x}\left(c_3 \cos(2x) + c_4 \sin(2x)\right)$$

Outline

Mass on a spring

Situation: Mass *m* a horizontal spring, and either

- Initial displacement.
- Initial velocity



Mass on a spring dynamics

Forces: if $y \equiv$ displacement from equilibrium position,

1. Spring: $F_s = -ky$.
2. Resistive or damping: $F_d = -by'$.
3. Applied external force F (in this course we'll take $F = 0$).

Remark:
Expressions for F_s and F_d are approximate.

Equation for dynamics

Newton's law:

$$my'' + by' + ky = 0. \tag{18}$$

Solution: With initial condition $y(0), y'(0)$
$\hookrightarrow$ according to Theorem 9 there exists a unique solution to (18).

Undamped free vibration

Particular situation in (18):

$$F = 0, \quad \text{and} \quad b = 0.$$

Resulting equation:

$$my'' + ky = 0$$

Standard form of the equation:

$$y'' + \omega^2 y = 0, \quad \text{with} \quad \omega = \left(\frac{k}{m}\right)^{1/2} \tag{19}$$

Undamped free vibration (2)

General solution of (19):

$$y = c_1 \cos(\omega t) + c_2 \sin(\omega t), \quad \text{with} \quad \omega = \left(\frac{k}{m}\right)^{1/2}$$

Other expression for solution:

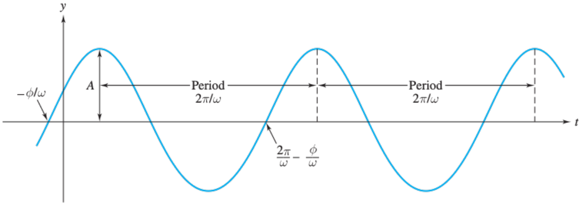
$$y = A \cos\left(\omega t - \phi\right), \quad \text{where} \quad A = \left(c_1^2 + c_2^2\right)^{1/2}, \quad \tan(\phi) = \frac{c_2}{c_1}.$$

Undamped free vibration (3)

Vocabulary: we call

- ω: natural frequency (does not depend on initial condition).
- A: amplitude of motion (does depend on initial condition).
- ϕ: phase.

Period of motion: $T = 2\pi \left(\frac{m}{k}\right)^{1/2}$
$\hookrightarrow$ Larger mass vibrates more slowly.



Hooke's law

Hooke: England's Leonardo

- Scientist
- Mathematician
- Architect
- Mechanical engineer
- 1st person to observe cells

Hooke's law, first version: The true Theory of Elasticity or Springs, and a particular Explication thereof in several Subjects in which it is to be found: And the way of computing the velocity of Bodies moved by them.

ceiiinosssttuv

Hooke's law (2)

Hooke's law, first version:

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Hooke's law, second version:

Ut tensio sic vis

Hooke's law, translation:

As extension, so is force

Hooke's law, modern translation:

Force is directly proportional to extension

Example of undamped vibration (1)

Description of the experiment:

- Mass $m = \frac{1}{2}$ attached to a spring
- By applying a force $F = 100\,N$, spring is stretched $d = 2$ meters
- Initial position 1 m, initial velocity -5m/s

Problem:

Find an equation for the spring motion

Example of undamped vibration (2)

Constant k: According to Hook's law,

$$k = \frac{F}{d} = 50$$

Equation for the motion:

$$y'' + 100y = 0$$

General solution:

$$y = c_1 \cos(10t) + c_2 \sin(10t)$$

Example of undamped vibration (3)

Initial value problem: With $y(0) = 1$ and $y'(0) = -5$,

$$y = \cos(10t) - \frac{1}{2}\sin(10t)$$

Amplitude-phase form:

$$y = \frac{\sqrt{5}}{2}\cos\left(10t + 0.46\right),$$

where we have computed

$$0.46 = \tan^{-1}\left(\frac{1}{2}\right)$$

Damped free vibrations

Equation:

$$my'' + by' + ky = 0.$$

Roots:

$$\begin{aligned} r_1,\ r_2 &= \frac{b}{2m}\left[-1 \pm \left(1 - \frac{4km}{b^2}\right)^{1/2}\right] \\ &= -\frac{b}{2m} \pm \frac{(b^2 - 4km)^{1/2}}{2m}. \end{aligned}$$

Remark:
$\mathcal{R}(r_1)$, $\mathcal{R}(r_2) < 0$, thus exponentially decreasing amplitude

Damped free vibrations (2)

3 cases:

1. If $b^2 - 4km > 0$, then (overdamped case):

$$y = c_1 \exp(r_1 t) + c_2 \exp(r_2 t).$$

2. If $b^2 - 4km = 0$, then (critical damping):

$$y = [c_1 + c_2 t] \exp\left(-\frac{bt}{2m}\right).$$

3. If $b^2 - 4km < 0$, then (underdamped case):

$$y = [c_1 \cos(\beta t) + c_2 \sin(\beta t)] \exp\left(-\frac{bt}{2m}\right), \tag{20}$$

where $\beta = \frac{(4km - b^2)^{1/2}}{2m} > 0$.

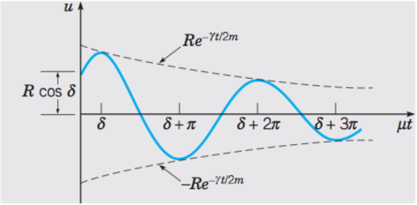
Underdamped case

Case under consideration:

If b small, we have $b^2 - 4km < 0 \implies$ motion governed by (20).

Expression for y:

$$y = A \exp\left(-\frac{bt}{2m}\right) \cos\left(\beta t - \phi\right).$$



Underdamped (2)

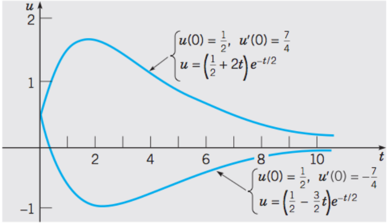
Quasi-period: when $b^2 - 4km < 0$, given by T. We have

$$T = \frac{2\pi}{\beta} = \frac{4\pi m}{(4km - b^2)^{1/2}}$$

Conclusion:
Small damping $\Longrightarrow$ smaller period for oscillations.

Critical and over damping

Critically damped case: when $b^2 - 4km = 0$
$\hookrightarrow$ mass passes through equilibrium at most once.



Overdamped case: when $b^2 - 4km > 0$
$\hookrightarrow$ mass passes through equilibrium at most once.

Outline

Outline

Solving non homogeneous equations (repeated)

Theorem 10.

General linear equation:

$$Ly = F(x). \tag{21}$$

Hypothesis: We have found

- $y_1, \ldots, y_n$ solutions of $Ly = 0$
- A particular solution y_p of $Ly = F$

Conclusion:

The general solution of equation (21) is

$$y = y_p + c_1\, y_1 + c_2\, y_2 + \cdots + c_n\, y_n$$

Method of undetermined coefficients

Nonhomogeneous linear equation with constant coefficients:

$$a_n y^{(n)} + \cdots + a_0 y = f(x).$$

Aim: Find a particular solution y_p to the equation.

Table of possible guess: restricted to a limited number of cases,

Function f	Guess
$\alpha \exp(ax)$	$A \exp(ax)$
$\alpha \sin(\omega x) + \beta \cos(\omega x)$	$A \sin(\omega x) + B \cos(\omega x)$
$\alpha_n x^n + \cdots + \alpha_0$	$A_n x^n + \cdots + A_0$
$(\alpha_n x^n + \cdots + \alpha_0) \exp(ax)$	$(A_n x^n + \cdots + A_0) \exp(ax)$
$(\alpha \sin(\omega x) + \beta \cos(\omega x)) \exp(ax)$	$(A \sin(\omega x) + B \cos(\omega x)) \exp(ax)$

Example of application

Equation:

$$y'' - 3y' - 4y = 2\sin(t) \tag{22}$$

Guess for particular solution:

$$y_p(t) = A\sin(t) + B\cos(t)$$

Equation for A, B: plugging into (22) we get

$$-5A + 3B = 2, \quad \text{and} \quad -3A - 5B = 0.$$

Particular solution:

$$y_p(t) = -\frac{5}{17}\sin(t) + \frac{3}{17}\cos(t)$$

Example of application (2)

Homogeneous equation:

$$y'' - 3y' - 4y = 0$$

Solution of homogeneous equation:

$$y = c_1 e^{-t} + c_2 e^{4t}.$$

General solution of nonhomogeneous equation (22):

$$y = c_1 e^{-t} + c_2 e^{4t} - \frac{5}{17}\sin(t) + \frac{3}{17}\cos(t).$$

Second example of application

Equation:

$$y'' - 3y' - 4y = -8e^t \cos(2t) \tag{23}$$

Guess for particular solution:

$$y_p(t) = Ae^t \cos(2t) + Be^t \sin(2t)$$

Equation for A, B: plugging into (23) we get

$$10A + 2B = 8, \quad \text{and} \quad 2A - 10B = 0.$$

Particular solution:

$$y_p(t) = \frac{10}{13} e^t \cos(2t) + \frac{2}{13} e^t \sin(2t)$$

Second example of application (2)

Homogeneous equation:

$$y'' - 3y' - 4y = 0$$

Solution of homogeneous equation:

$$y = c_1 e^{-t} + c_2 e^{4t}.$$

General solution of nonhomogeneous equation (23):

$$y = c_1 e^{-t} + c_2 e^{4t} + \frac{10}{13}\cos(2t) + \frac{2}{13}\sin(2t).$$

Elaboration of the guess: real valued root

Proposition 11.

Equation:

$$a_n y^{(n)} + \cdots + a_0 y = (\gamma_m x^m + \cdots + \gamma_1 x + \gamma_0)\, e^{rx}$$

Situation:

- Auxiliary Polynomial: $P(r) = a_n r^n + \cdots + a_0$
- $r \in \mathbb{R}$ root of P with multiplicity s

Particular solution: of the form

$$y_p(x) = x^s \, (A_m x^m + \cdots + A_1 x + A_0)\, e^{rx}.$$

Elaboration of the guess: complex valued root

Proposition 12.

Equation:

$$a_n y^{(n)} + \cdots + a_0 y = (\gamma_m x^m + \cdots + \gamma_1 x + \gamma_0)\, e^{\alpha x} \cos(\beta x)$$

Situation:

- Auxiliary Polynomial: $P(r) = a_n r^n + \cdots + a_0$
- $\alpha + \imath\beta \in \mathbb{C}$ root of P with multiplicity s

Particular solution: of the form

$$\begin{aligned} y_p(x) &= x^s\,(A_m x^m + \cdots + A_1 x + A_0)\, e^{\alpha x} \cos(\beta x) \\ &+ x^s\,(B_m x^m + \cdots + B_1 x + B_0)\, e^{\alpha x} \sin(\beta x). \end{aligned}$$

Elaboration of the guess: complex valued root (2)

Proposition 13.

Equation:

$$a_n y^{(n)} + \cdots + a_0 y = (\gamma_m x^m + \cdots + \gamma_1 x + \gamma_0)\, e^{\alpha x} \sin(\beta x)$$

Situation:

- Auxiliary Polynomial: $P(r) = a_n r^n + \cdots + a_0$
- $\alpha + \imath\beta \in \mathbb{C}$ root of P with multiplicity s

Particular solution: of the form

$$\begin{aligned} y_p(x) &= x^s\,(A_m x^m + \cdots + A_1 x + A_0)\, e^{\alpha x} \cos(\beta x) \\ &+ x^s\,(B_m x^m + \cdots + B_1 x + B_0)\, e^{\alpha x} \sin(\beta x). \end{aligned}$$

Example of application

Equation:

$$y^{(3)} - 3y^{(2)} + 3y' - y = 4e^t.$$

Auxiliary polynomial:

$$P(r) = (r-1)^3 \quad \Longrightarrow \quad s = 3.$$

Solution to homogeneous equation: for $c_1, c_2, c_3 \in \mathbb{R}$,

$$y_c = c_1 e^t + c_2 t e^t + c_3 t^2 e^t.$$

Example of application (2)

Guess for particular solution:

$$y_p(t) = At^3 e^t$$

General solution:

$$y = c_1 e^t + c_2 t e^t + c_3 t^2 e^t + \frac{2}{3} t^3 e^t.$$

Example with superposition

Equation:

$$y^{(3)} - 4y' = t + 3\cos(t) + e^{-2t}.$$

Characteristic polynomial:

$$P(r) = r(r-2)(r+2).$$

Solution to homogeneous equation: for $c_1, c_2, c_3 \in \mathbb{R}$,

$$y_c = c_1 + c_2 e^{2t} + c_3 e^{-2t}.$$

Example with superposition (2)

Sub-equation 1:

$$y^{(3)} - 4y' = t.$$

Guess for particular solution 1:

$$y_{p,1}(t) = t(A_1 t + A_0) \quad \Longrightarrow \quad A_1 = -\frac{1}{8},\ A_0 = 0.$$

Sub-equation 2:

$$y^{(3)} - 4y' = 3\cos(t).$$

Guess for particular solution 2:

$$y_{p,2}(t) = B\cos(t) + C\sin(t) \quad \Longrightarrow \quad B = 0,\ C = -\frac{3}{5}.$$

Example with superposition (3)

Sub-equation 3:

$$y^{(3)} - 4y' = e^{-2t}.$$

Guess for particular solution 3:

$$y_{p,3}(t) = Dte^{-2t} \quad \Longrightarrow \quad D = \frac{1}{8}.$$

General solution:

$$y = c_1 + c_2 e^{2t} + c_3 e^{-2t} - \frac{t^2}{8} - \frac{3}{5}\sin(t) + \frac{t}{8}e^{-2t}.$$

Outline

Objective

Limitations of the undetermined coefficients method:

1. Only applies to equations with constant coefficients
2. Non homogeneous term f of a specific form (see table)

Aim in this section:

- Assume we know fund. sol. for a 2nd order linear equation
- General method to compute y_p

Variation of parameters method (2nd order)

Theorem 14.

Equation considered:

$$ay'' + by' + cy = f$$

Hypothesis:
We know two linearly independent solutions y_1, y_2 of

$$ay'' + by' + cy = 0.$$

Conclusion: A particular solution y_p is given by

$$y_p = u_1 y_1 + u_2 y_2,$$

where u_1, u_2 satisfy

$$\begin{cases} y_1 u_1' + y_2 u_2' &= 0 \\ y_1' u_1' + y_2' u_2' &= \frac{f}{a} \end{cases}$$

Variation of parameters: application

Equation:

$$y'' + y = \sec(x), \qquad x > 0 \tag{24}$$

Fundamental solutions of homogeneous equation:

$$y_1(x) = \cos(x), \qquad y_2(x) = \sin(x).$$

Basic form of y_p:

$$y_p(x) = u_1(x)\,\cos(x) + u_2(x)\,x\,\sin(x)$$

Variation of parameters: application (2)

System for u_1', u_2':

$$\begin{cases} \cos(x)u_1' + \sin(x)u_2' & = 0 \\ -\sin(x)u_1' + \cos(x)u_2' & = \sec(x) \end{cases}$$

Solution to the system: By Cramer's rule we get

$$u_1'(x) = -\sin(x)\sec(x), \qquad u_2'(x) = \cos(x)\sec(x) = 1,$$

Expression for u_1, u_2: A direct integration yields

$$u_1(x) = \ln\left(|\cos(x)|\right), \qquad u_2(x) = x.$$

Variation of parameters: application (3)

Expression for y_p: We get

$$
\begin{aligned}
y_p(x) &= u_1(x)\ \cos(x) + u_2(x)\ \sin(x) \\
&= \cos(x) \ln(|\cos(x)|) + x\ \sin(x)
\end{aligned}
$$

General solution to (24):

$$y(x) = c_1\ \cos(x) + c_2\ \sin(x) + \cos(x) \ln(|\cos(x)|) + x\ \sin(x).$$

Variation of parameters: 2nd application

Equation:

$$y'' + 4y' + 4y = e^{-2x} \ln(x), \qquad x > 0 \tag{25}$$

Fundamental solutions of homogeneous equation:

$$y_1(x) = e^{-2x}, \qquad y_2(x) = x\, e^{-2x}.$$

Basic form of y_p:

$$y_p(x) = u_1(x)\, e^{-2x} + u_2(x)\, x\, e^{-2x}$$

Variation of parameters: 2nd application (2)

System for u_1', u_2':

$$\begin{cases} e^{-2x} u_1' + x\, e^{-2x} u_2' & = 0 \\ -2e^{-2x} u_1' + (1-2x) e^{-2x} u_2' & = e^{-2x} \ln(x) \end{cases}$$

Solution to the system: By Cramer's rule we get

$$u_1'(x) = -x \ln(x), \qquad u_2'(x) = \ln(x),$$

Expression for u_1, u_2: Integrating by parts we get

$$u_1(x) = \frac{1}{4} x^2 \left(1 - 2\ln(x)\right), \qquad u_2(x) = x(\ln(x) - 1),$$

Variation of parameters: 2nd application (3)

Expression for y_p: We get

$$
\begin{aligned}
y_p(x) &= u_1(x)\, e^{-2x} + u_2(x)\, x\, e^{-2x} \\
&= \frac{1}{4} x^2 \left(2\ln(x) - 3\right) e^{-2x}
\end{aligned}
$$

General solution to (25):

$$
y(x) = e^{-2x}\left[c_1 + c_2 x + \frac{1}{4} x^2 \left(2\ln(x) - 3\right)\right].
$$

George Green

Some facts about Green:

- Lifespan: 1793-1841, in England
- Self taught in Math, originally a baker
- Mathematician, Physicist
- 1st mathematical theory of electromagnetism
- Went to college when he was 40
- Died 1 year later (alcoholism?)