Linear equations in linear algebra

Samy Tindel

Purdue University

Differential equations and linear algebra - MA 262

Taken from *Differential equations and linear algebra*
Pearson Collections

Outline

1. Systems of linear equations
2. Row reduction and echelon form
3. Vector equations
4. Matrix equation $A\mathbf{x} = \mathbf{b}$
5. Solution sets of linear systems
6. Linear dependence
7. Introduction to linear transformations
8. The matrix of a linear transformation

Outline

1. Systems of linear equations
2. Row reduction and echelon form
3. Vector equations
4. Matrix equation $A\mathbf{x} = \mathbf{b}$
5. Solution sets of linear systems
6. Linear dependence
7. Introduction to linear transformations
8. The matrix of a linear transformation

Systems of linear equations

General form of a $m \times n$ linear system:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \tag{1}$$

System coefficients: The real numbers a_{ij}

System constants: The real numbers b_i

Homogeneous system: When $b_i = 0$ for all i

Example of linear system

Linear system in $\mathbb{R}^3$:

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ x_2 - x_3 &= 2 \\ x_2 + x_3 &= 6 \end{aligned}$$

Unique solution by substitution:

$$x_1 = -5, \quad x_2 = 4, \quad x_3 = 2$$

Notation $\mathbb{R}^n$

Definition of $\mathbb{R}^n$:

Set of ordered n-uples of real numbers $(x_1, \ldots, x_n)$

Matrix notation:

An element of $\mathbb{R}^n$ can be seen as a row or column n-vector

$$(x_1, \ldots, x_n) \quad \longleftrightarrow \quad \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \longleftrightarrow \quad [x_1, \ldots, x_n]$$

Related matrices

Matrix of coefficients: For the system (1), given by

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

Augmented matrix: For the system (1), given by

$$A^\sharp = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} & b_m \end{bmatrix}$$

Vector formulation

Example of system in $\mathbb{R}^3$:

$$\begin{array}{llll} x_1 & +3x_2 & -4x_3 & = 1 \\ 2x_1 & +5x_2 & -x_3 & = 5 \\ x_1 & & +6x_3 & = 3 \end{array}$$

Related matrices:

$$A = \begin{bmatrix} 1 & 3 & -4 \\ 2 & 5 & -1 \\ 1 & 0 & 6 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix}, \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Augmented matrix:

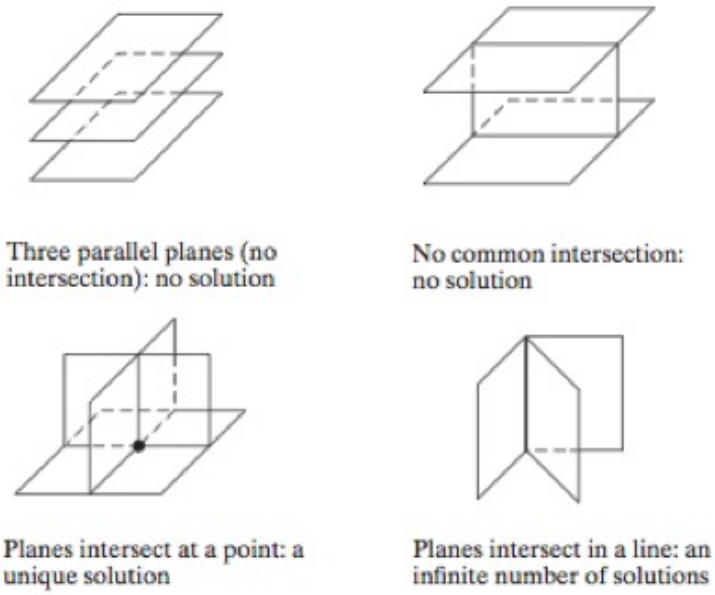
$$A^\sharp = \begin{bmatrix} 1 & 3 & -4 & 1 \\ 2 & 5 & -1 & 5 \\ 1 & 0 & 6 & 3 \end{bmatrix}$$

Linear systems in $\mathbb{R}^3$

General system in $\mathbb{R}^3$:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}$$

Geometrical interpretation: Intersection of 3 planes in $\mathbb{R}^3$



Sets of solutions

Possible sets of solutions: In the general $\mathbb{R}^n$ case we can have

- No solution to a linear system
- A unique solution
- An infinite number of solutions

Definition 1.

Consider a linear system given by (1). Then

1. If there is at least one solution the system is consistent
2. If there is no solution the system is inconsistent

Outline

1. Systems of linear equations
2. Row reduction and echelon form
3. Vector equations
4. Matrix equation $A\mathbf{x} = \mathbf{b}$
5. Solution sets of linear systems
6. Linear dependence
7. Introduction to linear transformations
8. The matrix of a linear transformation

Elementary row operations

Operations leaving the system unchanged:

1. Permute equations
2. Multiply a row by a nonzero constant
3. Add a multiple of one equation to another equation

Example of system:

$$\begin{array}{llll} x_1 & +3x_2 & -4x_3 & = 1 \\ 2x_1 & +5x_2 & -x_3 & = 5 \\ x_1 & & +6x_3 & = 3 \end{array}$$

Example of elementary row operations

Example of system:

$$\begin{array}{llll} x_1 & +3x_2 & -4x_3 & = 1 \\ 2x_1 & +5x_2 & -x_3 & = 5 \\ x_1 & & +6x_3 & = 3 \end{array}$$

Permute R_1 and R_2: Denoted by P_{12}

$$\begin{array}{llll} 2x_1 & +5x_2 & -x_3 & = 5 \\ x_1 & +3x_2 & -4x_3 & = 1 \\ x_1 & & +6x_3 & = 3 \end{array}$$

Example of elementary row operations (2)

Example of system:

$$\begin{array}{rrrl} x_1 & +3x_2 & -4x_3 & = 1 \\ 2x_1 & +5x_2 & -x_3 & = 5 \\ x_1 & & +6x_3 & = 3 \end{array}$$

Multiply R_2 by -2: Denoted by $M_2(-2)$

$$\begin{array}{rrrl} 2x_1 & +5x_2 & -x_3 & = 5 \\ -4x_1 & -10x_2 & +2x_3 & = -10 \\ x_1 & & +6x_3 & = 3 \end{array}$$

Example of elementary row operations (3)

Example of system:

$$\begin{array}{llll} x_1 & +3x_2 & -4x_3 & = 1 \\ 2x_1 & +5x_2 & -x_3 & = 5 \\ x_1 & & +6x_3 & = 3 \end{array}$$

Add $2R_3$ to R_1: Denoted by $A_{31}(2)$

$$\begin{array}{llll} 3x_1 & +3x_2 & +8x_3 & = 7 \\ 2x_1 & +5x_2 & -x_3 & = 5 \\ x_1 & & +6x_3 & = 3 \end{array}$$

Elementary operations in matrix form

Example of system:

$$\begin{array}{rrrl} x_1 & +3x_2 & -4x_3 & = 1 \\ 2x_1 & +5x_2 & -x_3 & = 5 \\ x_1 & & +6x_3 & = 3 \end{array}$$

Adding $2R_3$ to R_1:

$$\begin{bmatrix} 1 & 3 & -4 & 1 \\ 2 & 5 & -1 & 5 \\ 1 & 0 & 6 & 3 \end{bmatrix} \overset{A_{31}(2)}{\sim} \begin{bmatrix} 3 & 3 & 8 & 7 \\ 2 & 5 & -1 & 5 \\ 1 & 0 & 6 & 3 \end{bmatrix}$$

Row-echelon matrices

Definition 2.

A $m \times n$ matrix is row-echelon whenever

1. If there are rows consisting only of 0's
   $\hookrightarrow$ They are at the bottom of the matrix
2. 1st nonzero entries of each row have a triangular shape
   $\hookrightarrow$ Called leading entries
3. All entries in a column below a leading entry are 0's

Reduced row-echelon matrices

Definition 3.

A reduced row-echelon matrix is a matrix A such that

1. A is row-echelon
2. All leading entries are $= 1$
3. Any column with a leading 1 has zeros everywhere else

Example of row-echelon matrix

Row-echelon matrix:

$$\begin{bmatrix} 1 & 1 & -1 & 4 \\ 0 & 1 & -3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Related system:

$$\begin{array}{llll} x_1 & +x_2 & -x_3 & = 4 \\ & +x_2 & -3x_3 & = 5 \\ & & x_3 & = 2 \end{array}$$

Solution of the system: very easy thanks to a back substitution

$$x_3 = 2, \qquad x_2 = 11, \qquad x_1 = -5$$

Strategy to solve general systems:

$\hookrightarrow$ Reduction to a row-echelon system

Reduction to a row-echelon system

Algorithm:

1. Start with a $m \times n$ matrix A. If $A = 0$, go to step 7.
2. Pivot column: leftmost nonzero column.
   Pivot position: topmost position in the pivot column.
3. Use elementary row operations to put 1 in the pivot position.
4. Use elementary row operations to put zeros below pivot position.
5. If all rows below pivot are 0, go to step 7.
6. Otherwise apply steps 2 to 5 to the rows below pivot position.
7. The matrix is row-echelon.

Example of reduction

First operations:

$$\begin{bmatrix} 3 & 2 & -5 & 2 \\ 1 & 1 & -1 & 1 \\ 1 & 0 & -3 & 4 \end{bmatrix} \underset{\sim}{P_{12}} \begin{bmatrix} 1 & 1 & -1 & 1 \\ 3 & 2 & -5 & 2 \\ 1 & 0 & -3 & 4 \end{bmatrix} \underset{\sim}{A_{12}(-3), A_{13}(-1)} \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & -1 & -2 & -1 \\ 0 & -1 & -2 & 3 \end{bmatrix}$$

Iteration:

$$\underset{\sim}{M_2(-1)} \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & -1 & -2 & 3 \end{bmatrix} \underset{\sim}{A_{23}(1)} \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix} \underset{\sim}{M_3(1/4)} \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Reduced row-echelon matrices

Definition 4.

A reduced row-echelon matrix is a matrix A such that

1. A is row-echelon
2. All leading entries are $= 1$
3. Any column with a leading 1 has zeros everywhere else

Example of reduced row-echelon

Example:

$$\begin{bmatrix} 1 & 9 & 26 \\ 0 & 14 & 28 \\ 0 & -14 & -28 \end{bmatrix} \overset{M_2(1/14)}{\sim} \begin{bmatrix} 1 & 9 & 26 \\ 0 & 1 & 2 \\ 0 & -14 & -28 \end{bmatrix} \overset{A_{21}(-9),\, A_{23}(14)}{\sim} \begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Remark: One can also

1. Obtain the row-echelon form
2. Get the reduced row-echelon working upward and to the left

Basic idea to solve linear systems

Aim:

Solve a linear system of equations

Strategy:

1. Reduce the augmented matrix to row-echelon
2. Solve the system backward thanks to the row-echelon form

Example of system (1)

System:

$$\begin{array}{rrrl} 3x_1 & -2x_2 & +2x_3 & = 9 \\ x_1 & -2x_2 & +x_3 & = 5 \\ 2x_1 & -x_2 & -2x_3 & = -1 \end{array}$$

Augmented matrix:

$$A^\sharp = \begin{bmatrix} 3 & -2 & 2 & 9 \\ 1 & -2 & 1 & 5 \\ 2 & -1 & -2 & -1 \end{bmatrix}$$

Example of system (2)

Row-echelon form of the augmented matrix:

$$A^\sharp \sim \begin{bmatrix} 1 & -2 & 1 & 5 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

System corresponding to the augmented matrix:

$$\begin{array}{rrrl} x_1 & -2x_2 & +x_3 & = 5 \\ & x_2 & +3x_3 & = 5 \\ & & x_3 & = 2 \end{array}$$

Solution set:

$$S = \{(1, -1, 2)\}$$

Solving with reduced row-echelon

System:

$$\begin{array}{rrrl} x_1 & +2x_2 & -x_3 & = 1 \\ 2x_1 & +5x_2 & -x_3 & = 3 \\ x_1 & +3x_2 & +2x_3 & = 6 \end{array}$$

Augmented matrix:

$$A^\sharp = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 5 & -1 & 3 \\ 1 & 3 & 2 & 6 \end{bmatrix}$$

Solving with reduced row-echelon (2)

Reduced row-echelon form of the augmented matrix:

$$A^\sharp \sim \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

System corresponding to the augmented matrix:

$$\begin{aligned} x_1 \qquad\qquad &= 5 \\ x_2 \qquad &= -1 \\ x_3 &= 2 \end{aligned}$$

Solution set:

$$S = \{(5, -1, 2)\}$$

Comparison between reduced and non reduced

Pros and cons:

- For Gauss-Jordan, the final backward system is easier to solve
- The reduced row-echelon form is costly in terms of computations
- Overall, Gauss is more efficient than Gauss-Jordan for systems

Main interest of Gauss-Jordan:

- One can compute the inverse of a matrix

Ex. of system with ∞ number of solutions

System:

$$\begin{array}{rrrrl} x_1 & -2x_2 & +2x_3 & -x_4 & = 3 \\ 3x_1 & +x_2 & +6x_3 & +11x_4 & = 16 \\ 2x_1 & -x_2 & +4x_3 & +4x_4 & = 9 \end{array} \tag{2}$$

Row-echelon form of the augmented matrix:

$$A^\sharp \sim \begin{bmatrix} 1 & -2 & 2 & -1 & 3 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Ex. of system with ∞ number of solutions (2)

Consistency: From row-echelon form we have

- 4 variables
- 2 leading entries
- No equation of the form $1 = 0$

Therefore we have a

$\hookrightarrow$ Consistent system with infinite number of solutions

Ex. of system with ∞ number of solutions (3)

Rule for systems with ∞ number of solutions:

- Choose as free variables those variables that **do not** correspond to a leading 1 in row-echelon form of $A^\sharp$

Application to system (2):

- Free variables: $x_3 = s$ and $x_4 = t$

Solution set:

$$\begin{aligned} S &= \{(5-2s-3t,\ 1-2t,\ s,\ t);\ s,t \in \mathbb{R}\} \\ &= \{(5,1,0,0) + s\,(-2,0,1,0) + t\,(-3,-2,0,1);\ s,t \in \mathbb{R}\} \end{aligned}$$

Outline

1. Systems of linear equations
2. Row reduction and echelon form
3. Vector equations
4. Matrix equation $A\mathbf{x} = \mathbf{b}$
5. Solution sets of linear systems
6. Linear dependence
7. Introduction to linear transformations
8. The matrix of a linear transformation

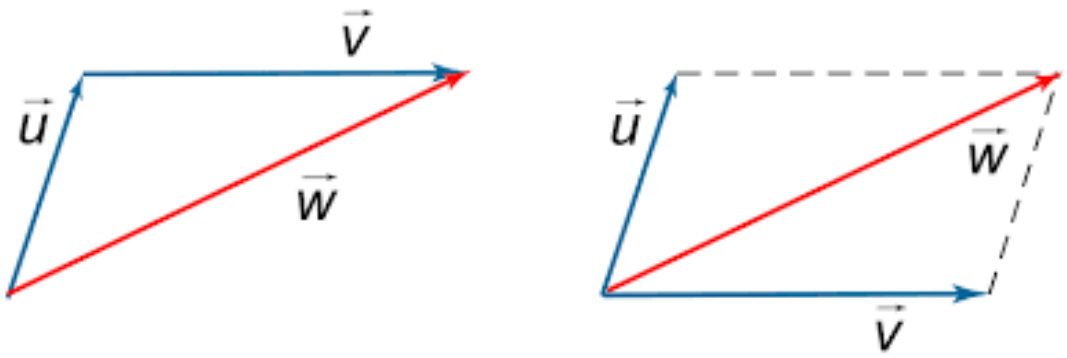
Vectors

Fact: A tuple

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

can be geometrically interpreted as a vector

Illustration of an addition:



Algebraic operations on vectors (1)

1. Commutativity of addition: For all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$,

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$$

2. Associativity of addition: For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$,

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$$

3. Existence of a zero vector: There exists $\mathbf{0} \in \mathbb{R}^n$ such that

$$\mathbf{v} + \mathbf{0} = \mathbf{v}$$

4. Existence of additive inverses in $\mathbb{R}^n$:
For all $\mathbf{v} \in \mathbb{R}^n$, there exists $-\mathbf{v} \in \mathbb{R}^n$ such that

$$\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$$

Algebraic operations on vectors (2)

5. Unit property: For all $\mathbf{v} \in \mathbb{R}^n$, we have

$$1\,\mathbf{v} = \mathbf{v}$$

6. Associativity of scalar multiplication:

$$(rs)\mathbf{v} = r(s\,\mathbf{v})$$

7. Distributivity 1:

$$r(\mathbf{u} + \mathbf{v}) = r\,\mathbf{u} + r\,\mathbf{v}$$

8. Distributivity 2:

$$(r + s)\mathbf{v} = r\,\mathbf{v} + s\,\mathbf{v}$$

Linear combinations

Definition 5.

Let

- $\mathbf{v}_1, \ldots, \mathbf{v}_p$ family of vectors in $\mathbb{R}^n$.
- $c_1, \ldots, c_p$ family of scalars in $\mathbb{R}$

Then a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ with weights $c_1, \ldots, c_p$ is given by

$$\mathbf{y} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p.$$

Example in $\mathbb{R}^2$

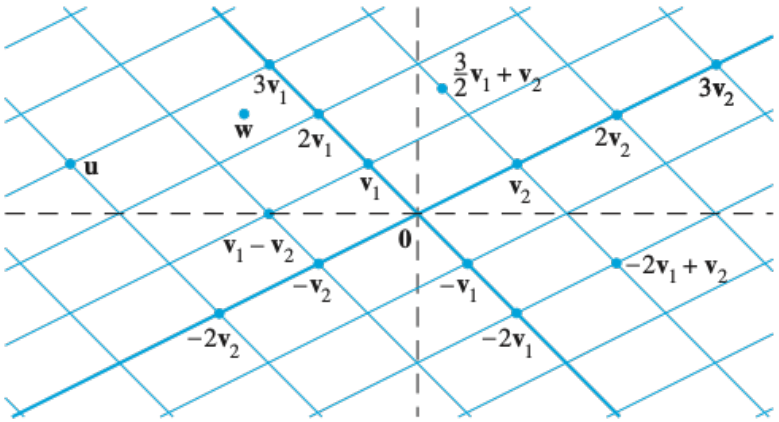
Example: Consider the vectors of $\mathbb{R}^2$:

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Then

$$2\mathbf{v}_1 - \mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

Set of linear combinations:



Spanning subset of $\mathbb{R}^n$

Definition 6.

Let

- $\mathbf{v}_1, \ldots, \mathbf{v}_p$ family of vectors in $\mathbb{R}^n$.

Consider the collection of $\mathbf{x} \in \mathbb{R}^n$ which can be written as a linear combination

$$\mathbf{x} = c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k,$$

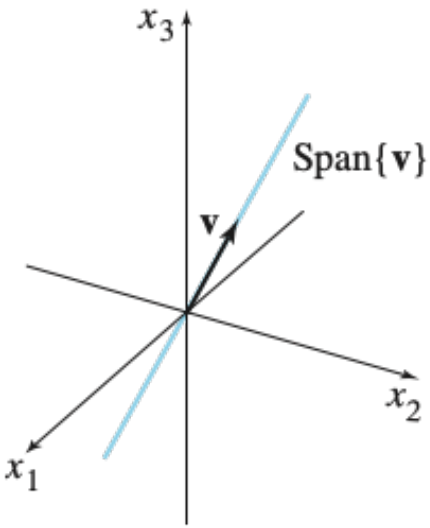
we say that

$$\mathbf{x} \in \mathrm{Span}\left\{\mathbf{v}_1, \ldots, \mathbf{v}_p\right\}$$

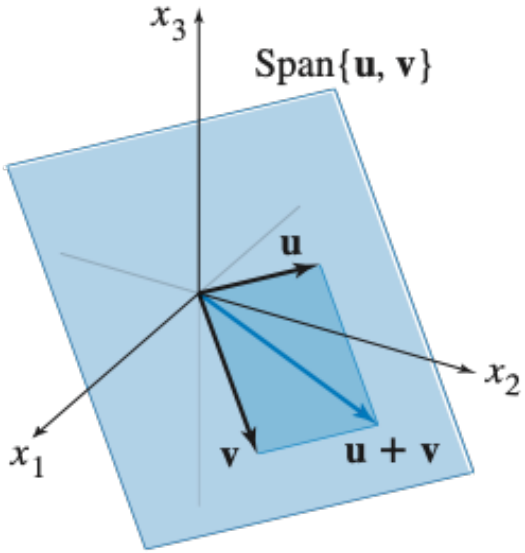
Example in $\mathbb{R}^3$

Illustration of spans:

$\mathrm{Span}\{\mathbf{v}\}$



$\mathrm{Span}\{\mathbf{u}, \mathbf{v}\}$



Outline

1. Systems of linear equations
2. Row reduction and echelon form
3. Vector equations
4. Matrix equation $A\mathbf{x} = \mathbf{b}$
5. Solution sets of linear systems
6. Linear dependence
7. Introduction to linear transformations
8. The matrix of a linear transformation

Matrix

Definition 7.

A $m \times n$ matrix is a rectangular array of numbers with m rows and n columns

Example of 2×3 matrix:

$$A = \begin{bmatrix} \frac{3}{2} & \frac{2}{3} & \frac{1}{5} \\ 0 & \frac{5}{4} & -\frac{3}{7} \end{bmatrix}$$

Index notation

Recall: we have

$$A = \begin{bmatrix} \frac{3}{2} & \frac{2}{3} & \frac{1}{5} \\ 0 & \frac{5}{4} & -\frac{3}{7} \end{bmatrix}$$

Index notation: For the matrix A we have

$$a_{12} = \frac{2}{3}, \qquad a_{21} = 0, \qquad a_{23} = -\frac{3}{7}$$

Index notation for a $m \times n$ matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

Vectors

A row 3-vector:

$$a = \begin{bmatrix} \frac{2}{3} & -\frac{1}{5} & \frac{4}{7} \end{bmatrix}$$

A column 5-vector:

$$b = \begin{bmatrix} 1 \\ 4 \\ \pi \\ -67 \\ 3 \end{bmatrix}$$

Multiplication as a linear combination

Definition 8.

Let

- A a $m \times n$ matrix
- $\mathbf{x} \in \mathbb{R}^n$

Then

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n$$

Example

Matrices:

$$A = \begin{bmatrix} 1 & -2 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Product:

$$A\mathbf{x} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

Rules for multiplications

Rules to follow:

$$A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v} \qquad \text{Distributive law}$$

$$A(c\mathbf{u}) = c(A\mathbf{u}) \qquad \text{Associative law}$$

$$A\mathbf{0} = \mathbf{0} \qquad \text{Absorbing state}$$

Vector formulation

Example of system in $\mathbb{R}^3$:

$$\begin{array}{llll} x_1 & +3x_2 & -4x_3 & = 1 \\ 2x_1 & +5x_2 & -x_3 & = 5 \\ x_1 & & +6x_3 & = 3 \end{array}$$

Related matrices:

$$A = \begin{bmatrix} 1 & 3 & -4 \\ 2 & 5 & -1 \\ 1 & 0 & 6 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix}, \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Vector formulation of the system:

$$Ax = \mathbf{b}$$

Existence of solutions

Theorem 9.

Let

- A a $m \times n$ matrix

Then the following statements are equivalent

1. For each $\mathbf{b} \in \mathbb{R}^m$, the system $Ax = \mathbf{b}$ has a solution
2. Each $\mathbf{b} \in \mathbb{R}^m$ is a linear combination of columns of A
3. The columns of A span $\mathbb{R}^m$
4. A has a pivot in every row

Example

Matrix of a generalized system:

$$A^\sharp = \begin{bmatrix} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{bmatrix}$$

Row-echelon form:

$$A^\sharp \sim \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 0 & 0 & b_1 - \frac{1}{2}b_2 + b_3 \end{bmatrix}$$

Conclusion:

- There is a solution only if $b_1 - \frac{1}{2}b_2 + b_3 = 0$
- This is due to the fact that we have a pivot on 2 rows only
- The columns of A don't span $\mathbb{R}^3$

Outline

1. Systems of linear equations
2. Row reduction and echelon form
3. Vector equations
4. Matrix equation $A\mathbf{x} = \mathbf{b}$
5. Solution sets of linear systems
6. Linear dependence
7. Introduction to linear transformations
8. The matrix of a linear transformation

Homogeneous systems

Proposition 10.

For a $m \times n$ matrix A, consider the system

$$A\mathbf{x} = \mathbf{0}.$$

Then we have

1. The system is always consistent
   $\hookrightarrow$ with trivial solution $\mathbf{x} = \mathbf{0}$.
2. The system has a nontrivial solution
   $\hookrightarrow$ iff the equation has at least one free variable
   $\hookrightarrow$ iff $\exists$ a row with no pivot in row-echelon form of A

Example of homogeneous system

Matrix:

$$A = \begin{bmatrix} 0 & 2 & 3 \\ 0 & 1 & -1 \\ 0 & 3 & 7 \end{bmatrix}$$

System:

$$A\mathbf{x} = \mathbf{0}.$$

Augmented matrix:

$$A^\sharp = \begin{bmatrix} 0 & 2 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & 7 & 0 \end{bmatrix}$$

Example of homogeneous system (2)

Reduced echelon form of $A^\sharp$:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Number of solutions: No pivot in 3rd row
$\hookrightarrow$ infinite number of solutions

Set of solutions:

$$S = \{(s, 0, 0);\ s \in \mathbb{R}\}\,.$$

2nd example of homogeneous system

Matrix:

$$A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix}$$

System:

$$A\mathbf{x} = \mathbf{0}.$$

Augmented matrix:

$$A^\sharp = \begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix}$$

2nd example of homogeneous system (2)

Reduced echelon form of $A^\sharp$:

$$\begin{bmatrix} 1 & 0 & -\frac{4}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Number of solutions: No pivot in 3rd row
$\hookrightarrow$ infinite number of solutions

Set of solutions: for $t \in \mathbb{R}$, of the form

$$\mathbf{x} = t \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$$

Inhomogeneous systems

Theorem 11.

Assume

- Equation $A\mathbf{x} = \mathbf{b}$ is consistent
- $\mathbf{p}$ is a particular solution of $A\mathbf{x} = \mathbf{b}$

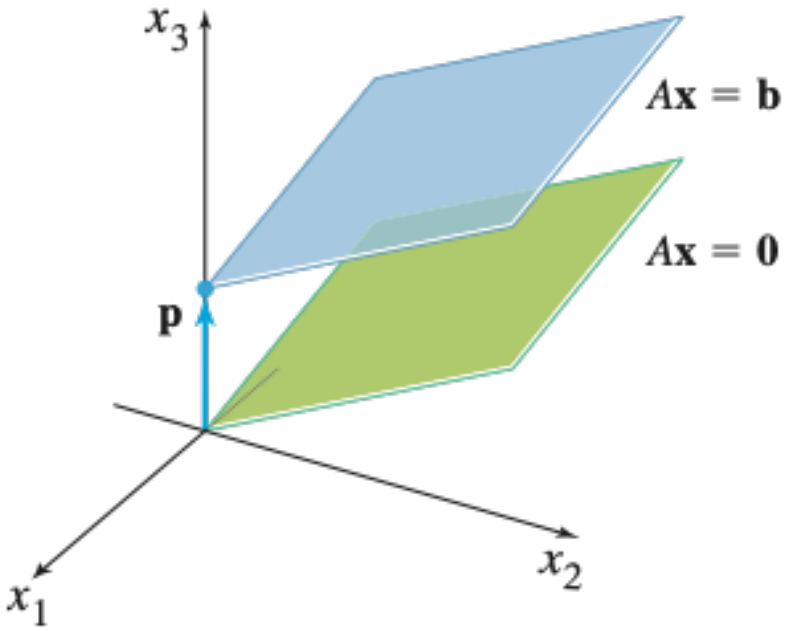
Then the general solution of $A\mathbf{x} = \mathbf{b}$ can be written as

$$\mathbf{w} = \mathbf{p} + \mathbf{v}_h,$$

where $\mathbf{v}_h$ is the general solution of $A\mathbf{x} = \mathbf{0}$

General solution for inhomogeneous systems

Solution of $A\mathbf{x} = \mathbf{b}$ vs solution of $A\mathbf{x} = \mathbf{0}$:



Example of inhomogeneous system

Matrices:

$$A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$$

System:

$$A\mathbf{x} = \mathbf{b}.$$

Augmented matrix:

$$A^\sharp = \begin{bmatrix} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{bmatrix}$$

Example of inhomogeneous system (2)

Reduced echelon form of $A^\sharp$:

$$\begin{bmatrix} 1 & 0 & -\frac{4}{3} & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Number of solutions:

No pivot in 3rd row, no equation of the form $0 = 1$

$\hookrightarrow$ infinite number of solutions

Set of solutions: of the form

$$\mathbf{x} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$$

Outline

1. Systems of linear equations
2. Row reduction and echelon form
3. Vector equations
4. Matrix equation $A\mathbf{x} = \mathbf{b}$
5. Solution sets of linear systems
6. Linear dependence
7. Introduction to linear transformations
8. The matrix of a linear transformation

Definition

Definition 12.

Let $\mathbf{v}_1, \ldots, \mathbf{v}_p$ be a family of vectors in $\mathbb{R}^n$. Then

1. If there exist $c_1, \ldots, c_p$ not all zero such that

$$c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = 0,$$

   we say that $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly dependent

2. If we have

$$c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = 0 \quad \Longrightarrow \quad c_1 = c_2 = \cdots = c_p = 0,$$

   we say that $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly independent

Examples

Simple examples:

- The family $\{\mathbf{v}\}$ is linearly dependent iff $\mathbf{v} = \mathbf{0}$
- The family $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent iff $\mathbf{v}_2 = c\,\mathbf{v}_1$
- If $\mathbf{0}$ is an element of $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$
  $\hookrightarrow$ then $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are linearly dependent
- If $p > n$ and $\mathbf{v}_1, \ldots, \mathbf{v}_p \in \mathbb{R}^n$
  $\hookrightarrow$ then $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are linearly dependent

Example in $\mathbb{R}^3$

Family of vectors: We consider

$$\mathbf{v}_1 = (1, 2, -1), \quad \mathbf{v}_2 = (2, -1, 1), \quad \mathbf{v}_3 = (8, 1, 1)$$

Then $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent

Proof: The system

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$$

can be written as:

$$\begin{array}{rrrl} c_1 & +2c_2 & +8c_3 & = 0 \\ 2c_1 & -c_2 & +c_3 & = 0 \\ -c_1 & +c_2 & +c_3 & = 0 \end{array}$$

Example in $\mathbb{R}^3$ (2)

Proof (ctd): System in row-echelon form

$$\begin{bmatrix} 1 & 2 & 8 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

One row is $\mathbf{0}$, so that we have linear dependence

Explicit linear dependence: We solve for $\mathbf{c}$

$$c_3 = t, \quad c_2 = -3t, \quad c_1 = -2t$$

Then choosing (arbitrary choice) $t = 1$ we get

$$-2\mathbf{v}_1 - 3\mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$$

Linear dependence and systems

Proposition 13.

Let

- $\mathbf{v}_1, \ldots, \mathbf{v}_p$ be a family of vectors in $\mathbb{R}^n$.
- $A = [\mathbf{v}_1, \ldots, \mathbf{v}_p]$ defined as a $M_{n,p}(\mathbb{R})$ matrix

Then

- The vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are linearly dependent iff the system $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution

Example of application

Family of vectors: We consider

$$\mathbf{v}_1 = (1, 2, -1), \quad \mathbf{v}_2 = (2, -1, 1), \quad \mathbf{v}_3 = (8, 1, 1)$$

Recall: We have obtained a row-echelon form

$$\begin{bmatrix} 1 & 2 & 8 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

One row is $\mathbf{0}$, so that the system $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution

Conclusion:

$\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent

Outline

1. Systems of linear equations
2. Row reduction and echelon form
3. Vector equations
4. Matrix equation $A\mathbf{x} = \mathbf{b}$
5. Solution sets of linear systems
6. Linear dependence
7. Introduction to linear transformations
8. The matrix of a linear transformation

Definition of mapping

Definition 14.

Let

- n and m two integers

Then:

1. A mapping is a rule which assigns to each vector $\mathbf{v} \in \mathbb{R}^n$ a vector $\mathbf{w} = T(\mathbf{v}) \in \mathbb{R}^m$
2. Notation: $T : \mathbb{R}^n \to \mathbb{R}^m$

Linear transformation

Definition 15.

Let

- n and m two integers
- A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$

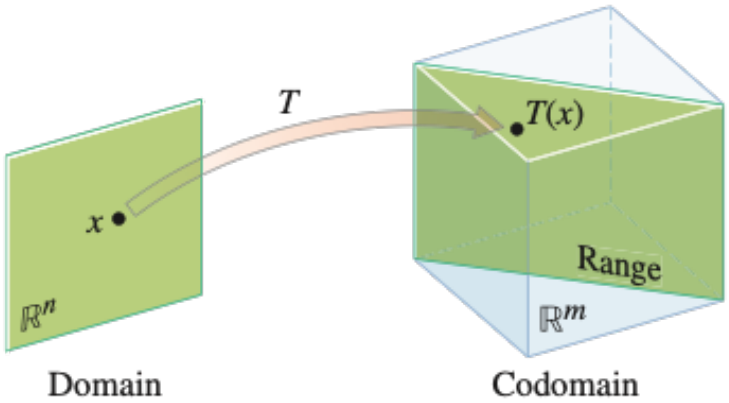
Then T is a linear transformation if it satisfies:

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
2. $T(c\,\mathbf{v}) = c\,T(\mathbf{v})$

Domain and codomain

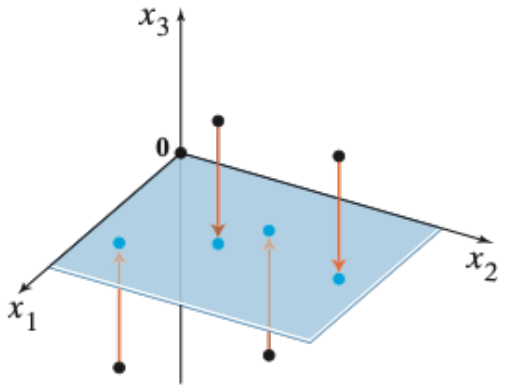
Vocabulary: For $T : \mathbb{R}^n \to \mathbb{R}^m$ linear

- $\mathbb{R}^n$ is called domain of T
- $\mathbb{R}^m$ is called codomain of T
- $T(\mathbb{R}^n) \subset \mathbb{R}^m$ is called range of T

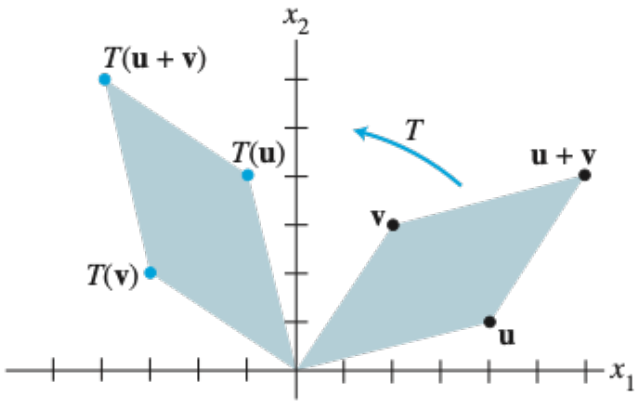


Examples of linear transformations

Projection: Domain $= \mathbb{R}^3$, Codomain $= \mathbb{R}^3$, Range $=$ Plane



Rotation: Domain $= \mathbb{R}^2$, Codomain $= \mathbb{R}^2$, Range $= \mathbb{R}^2$



Outline

1. Systems of linear equations
2. Row reduction and echelon form
3. Vector equations
4. Matrix equation $A\mathbf{x} = \mathbf{b}$
5. Solution sets of linear systems
6. Linear dependence
7. Introduction to linear transformations
8. The matrix of a linear transformation

Matrix form

Theorem 16.

Let

- $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation
- $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ canonical basis of $\mathbb{R}^n$

Then T is described by the matrix transformation

$$T(\mathbf{x}) = A\mathbf{x},$$

where A is the matrix defined by

$$A = [T(\mathbf{e}_1), T(\mathbf{e}_2), \ldots, T(\mathbf{e}_n)]$$

Example of matrix form

Linear transformation: Defined as $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that

$$T(\mathbf{e}_1) = T(1,0) = (2,3,-1), \qquad T(\mathbf{e}_2) = T(0,1) = (5,-4,7)$$

Matrix form: $T(x) = A\,\mathbf{x}$ with

$$A = [T(\mathbf{e}_1), T(\mathbf{e}_2)] = \begin{bmatrix} 2 & 5 \\ 3 & -4 \\ -1 & 7 \end{bmatrix}$$

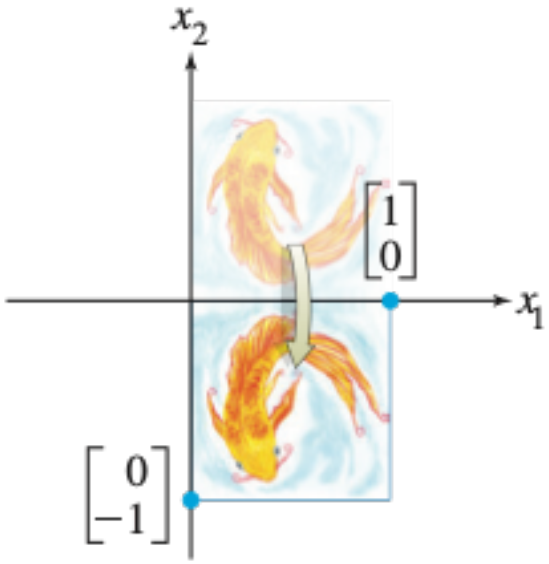
General expression:

$$T(x) = A\,\mathbf{x} = \begin{bmatrix} 2x_1 + 5x_2 \\ 3x_1 - 4x_2 \\ -x_1 + 7x_2 \end{bmatrix}$$

Examples of transformations in $\mathbb{R}^2$

Reflection on x_1-axis:

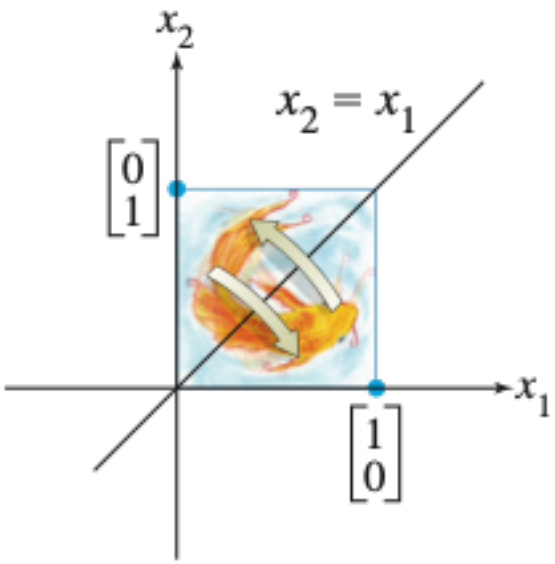
$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$



Examples of transformations in $\mathbb{R}^2$ (2)

Reflection on line $x_2 = x_1$:

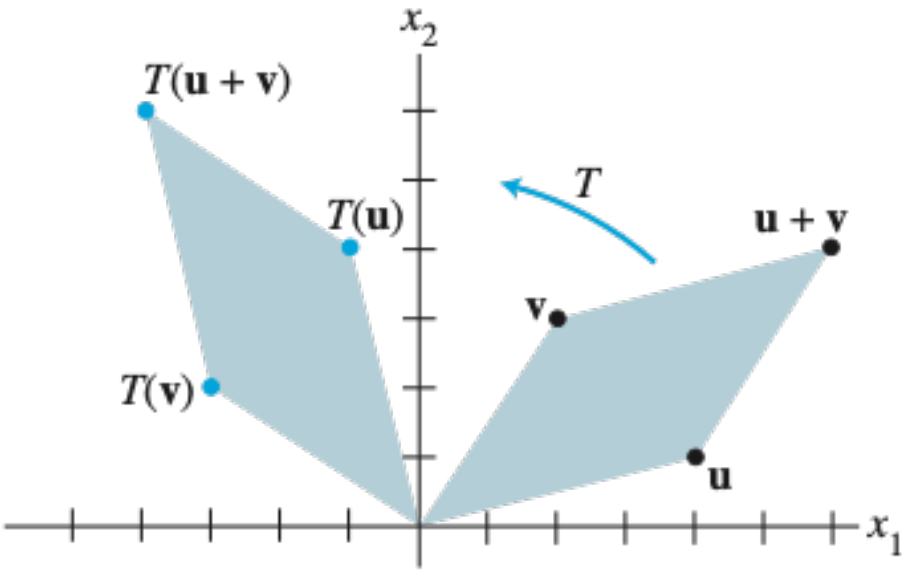
$$A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$



Examples of transformations in $\mathbb{R}^2$ (3)

Rotation:

$$A_3 = \begin{bmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{bmatrix}$$



Composition of transformations in $\mathbb{R}^2$

Reflection on x_1 then reflection on $x_2 = x_1$:

$$A = A_2\, A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Resulting transformation:

Rotation with $\varphi = \frac{\pi}{2}$

Properties of T according to its matrix

Theorem 17.

Let

- $T : \mathbb{R}^n \to \mathbb{R}^m$ linear transformation
- A its standard matrix form

Then

1. T is onto
   iff columns of A span $\mathbb{R}^m$
   iff there is a pivot in every row of A (Thm 9)
2. T is one-to-one
   iff columns of A are linearly independent
   iff $\mathbf{0}$ is the unique solution of $A\mathbf{x} = \mathbf{0}$ (Prop 13)

Example from $\mathbb{R}^2$ to $\mathbb{R}^3$

Transformation: For $\mathbf{x} \in \mathbb{R}^2$

$$T(\mathbf{x}) = \begin{bmatrix} 3x_1 + x_2 \\ 5x_1 + 7x_2 \\ x_1 + 3x_2 \end{bmatrix}$$

Matrix:

$$A = \begin{bmatrix} 3 & 1 \\ 5 & 7 \\ 1 & 3 \end{bmatrix}$$

Properties of T:

1. Only two columns in $A \Longrightarrow$ is not onto in $\mathbb{R}^3$
2. Columns of A are independent $\Longrightarrow$ T is one-to-one