Linear systems and matrices

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Differential equations and linear algebra - MA 262

Taken from *Differential equations and linear algebra*
Edwards, Penney, Calvis

Outline

Outline

Systems of linear equations

General form of a $m \times n$ linear system:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \tag{1}$$

System coefficients: The real numbers a_{ij}

System constants: The real numbers b_i

Homogeneous system: When $b_i = 0$ for all i

Example of linear system

Linear system in $\mathbb{R}^3$:

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ x_2 - x_3 &= 2 \\ x_2 + x_3 &= 6 \end{aligned}$$

Unique solution by substitution:

$$x_1 = -5, \quad x_2 = 4, \quad x_3 = 2$$

Notation $\mathbb{R}^n$

Definition of $\mathbb{R}^n$:

Set of ordered n-uples of real numbers $(x_1, \ldots, x_n)$

Matrix notation:

An element of $\mathbb{R}^n$ can be seen as a row or column n-vector

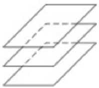
$$(x_1, \ldots, x_n) \quad \longleftrightarrow \quad \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \longleftrightarrow \quad [x_1, \ldots, x_n]$$

Linear systems in $\mathbb{R}^3$

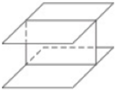
General system in $\mathbb{R}^3$:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3
\end{aligned}
$$

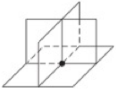
Geometrical interpretation: Intersection of 3 planes in $\mathbb{R}^3$



Three parallel planes (no intersection): no solution



No common intersection: no solution



Planes intersect at a point: a unique solution

Planes intersect in a line: an infinite number of solutions

Sets of solutions

Possible sets of solutions: In the general $\mathbb{R}^n$ case we can have

- No solution to a linear system
- A unique solution
- An infinite number of solutions

Definition 1.

Consider a linear system given by (1). Then

1. If there is at least one solution the system is consistent
2. If there is no solution the system is inconsistent

Related matrices

Matrix of coefficients: For the system (1), given by

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

Augmented matrix: For the system (1), given by

$$A^\sharp = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} & b_m \end{bmatrix}$$

Vector formulation (1)

Example of system in $\mathbb{R}^3$:

$$\begin{array}{rrrl} x_1 & +3x_2 & -4x_3 & = 1 \\ 2x_1 & +5x_2 & -x_3 & = 5 \\ x_1 & & +6x_3 & = 3 \end{array}$$

Vector formulation (2)

Related matrices:

$$A = \begin{bmatrix} 1 & 3 & -4 \\ 2 & 5 & -1 \\ 1 & 0 & 6 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix}, \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Vector formulation of the system:

$$Ax = b$$

Systems of differential equations (1)

Example of system in $\mathbb{R}^2$

$$
\begin{aligned}
x_1' &= 3x_1 + \sin(t)x_2 + e^t \\
x_2' &= 7tx_1 + t^2 x_2 - 4e^{-t}
\end{aligned}
$$

Systems of differential equations (2)

Related matrices:

$$A(t) = \begin{bmatrix} 3 & \sin(t) \\ 7t & t^2 \end{bmatrix}, \qquad \mathbf{b}(t) = \begin{bmatrix} e^t \\ -4e^{-t} \end{bmatrix}, \qquad \mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Vector formulation of the system:

$$\mathbf{x}'(t) = A(t)\,\mathbf{x}(t) + \mathbf{b}(t)$$

Second order differential equations and systems (1)

A family of functions: For $A, B \in \mathbb{R}$, set

$$y(x) = Ae^{3x} + Be^{-3x}$$

Equation solved by y: It can be shown that

$$y'' - 9y = 0$$

Initial data:

$$y(0) = 7, \qquad y'(0) = 9$$

Question:
Find A and B according to the initial data

Second order differential equations and systems (2)

Related system

$$\begin{aligned} A \quad &+B &= 7 \\ 3A \quad &-3B &= 9 \end{aligned}$$

Solution:

$$A = 5, \qquad B = 2$$

Solution to initial value problem:

$$y(x) = 5e^{3x} + 2e^{-3x}$$

Outline

Elementary row operations

Operations leaving the system unchanged:

1. Permute equations
2. Multiply a row by a nonzero constant
3. Add a multiple of one equation to another equation

Example of system:

$$
\begin{array}{rrrl}
x_1 & +3x_2 & -4x_3 & = 1 \\
2x_1 & +5x_2 & -x_3 & = 5 \\
x_1 & & +6x_3 & = 3
\end{array}
$$

Example of elementary row operations

Example of system:

$$\begin{array}{rrrl} x_1 & +3x_2 & -4x_3 & = 1 \\ 2x_1 & +5x_2 & -x_3 & = 5 \\ x_1 & & +6x_3 & = 3 \end{array}$$

Permute R_1 and R_2: Denoted by P_{12}

$$\begin{array}{rrrl} 2x_1 & +5x_2 & -x_3 & = 5 \\ x_1 & +3x_2 & -4x_3 & = 1 \\ x_1 & & +6x_3 & = 3 \end{array}$$

Example of elementary row operations (2)

Example of system:

$$\begin{array}{cccc} x_1 & +3x_2 & -4x_3 & =1 \\ 2x_1 & +5x_2 & -x_3 & =5 \\ x_1 & & +6x_3 & =3 \end{array}$$

Multiply R_2 by -2: Denoted by $M_2(-2)$

$$\begin{array}{cccc} 2x_1 & +5x_2 & -x_3 & =5 \\ -4x_1 & -10x_2 & +2x_3 & =-10 \\ x_1 & & +6x_3 & =3 \end{array}$$

Example of elementary row operations (3)

Example of system:

$$\begin{array}{rrrl} x_1 & +3x_2 & -4x_3 & = 1 \\ 2x_1 & +5x_2 & -x_3 & = 5 \\ x_1 & & +6x_3 & = 3 \end{array}$$

Add $2R_3$ to R_1: Denoted by $A_{31}(2)$

$$\begin{array}{rrrl} 3x_1 & +3x_2 & +8x_3 & = 7 \\ 2x_1 & +5x_2 & -x_3 & = 5 \\ x_1 & & +6x_3 & = 3 \end{array}$$

Elementary operations in matrix form

Example of system:

$$\begin{array}{rrrl} x_1 & +3x_2 & -4x_3 & = 1 \\ 2x_1 & +5x_2 & -x_3 & = 5 \\ x_1 & & +6x_3 & = 3 \end{array}$$

Adding $2R_3$ to R_1:

$$\begin{bmatrix} 1 & 3 & -4 & 1 \\ 2 & 5 & -1 & 5 \\ 1 & 0 & 6 & 3 \end{bmatrix} \underset{\sim}{A_{31}(2)} \begin{bmatrix} 3 & 3 & 8 & 7 \\ 2 & 5 & -1 & 5 \\ 1 & 0 & 6 & 3 \end{bmatrix}$$

Row-echelon matrices

Definition 2.

A $m \times n$ matrix is row-echelon whenever

1. If there are rows consisting only of 0's
   $\hookrightarrow$ They are at the bottom of the matrix
2. 1st nonzero entries of each row have a triangular shape
   $\hookrightarrow$ Called leading entries
3. All entries in a column below a leading entry are 0's

Reduced row-echelon matrices

Definition 3.

A reduced row-echelon matrix is a matrix A such that

1. A is row-echelon
2. All leading entries are $= 1$
3. Any column with a leading 1 has zeros everywhere else

Example of row-echelon matrix (1)

Row-echelon matrix:

$$\begin{bmatrix} 1 & 1 & -1 & 4 \\ 0 & 1 & -3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Example of row-echelon matrix (2)

Related system:

$$\begin{aligned} x_1 \quad +x_2 \quad -x_3 \quad &= 4 \\ +x_2 \quad -3x_3 \quad &= 5 \\ x_3 \quad &= 2 \end{aligned}$$

Solution of the system: very easy thanks to a back substitution

$$x_3 = 2, \qquad x_2 = 11, \qquad x_1 = -5$$

Strategy to solve general systems:
$\hookrightarrow$ Reduction to a row-echelon system

Reduction to a row-echelon system

Algorithm:

1. Start with a $m \times n$ matrix A. If $A = 0$, go to step 7.
2. Pivot column: leftmost nonzero column.
   Pivot position: topmost position in the pivot column.
3. Use elementary row operations to put 1 in the pivot position.
4. Use elementary row operations to put zeros below pivot position.
5. If all rows below pivot are 0, go to step 7.
6. Otherwise apply steps 2 to 5 to the rows below pivot position.
7. The matrix is row-echelon.

Example of reduction

First operations:

$$\begin{bmatrix} 3 & 2 & -5 & 2 \\ 1 & 1 & -1 & 1 \\ 1 & 0 & -3 & 4 \end{bmatrix} \underset{\sim}{P_{12}} \begin{bmatrix} 1 & 1 & -1 & 1 \\ 3 & 2 & -5 & 2 \\ 1 & 0 & -3 & 4 \end{bmatrix} \underset{\sim}{A_{12}(-3), A_{13}(-1)} \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & -1 & -2 & -1 \\ 0 & -1 & -2 & 3 \end{bmatrix}$$

Iteration:

$$\underset{\sim}{M_2(-1)} \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & -1 & -2 & 3 \end{bmatrix} \underset{\sim}{A_{23}(1)} \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix} \underset{\sim}{M_3(1/4)} \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Basic idea to solve linear systems

Aim:

Solve a linear system of equations

Strategy:

1. Reduce the augmented matrix to row-echelon
2. Solve the system backward thanks to the row-echelon form

Example of system (1)

System:

$$\begin{array}{rrrr} 3x_1 & -2x_2 & +2x_3 & = 9 \\ x_1 & -2x_2 & +x_3 & = 5 \\ 2x_1 & -x_2 & -2x_3 & = -1 \end{array}$$

Augmented matrix:

$$A^\sharp = \begin{bmatrix} 3 & -2 & 2 & 9 \\ 1 & -2 & 1 & 5 \\ 2 & -1 & -2 & -1 \end{bmatrix}$$

Example of system (2)

Row-echelon form of the augmented matrix:

$$A^\sharp \sim \begin{bmatrix} 1 & -2 & 1 & 5 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

System corresponding to the augmented matrix:

$$\begin{array}{cccl} x_1 & -2x_2 & +x_3 & = 5 \\ & x_2 & +3x_3 & = 5 \\ & & x_3 & = 2 \end{array}$$

Solution set:

$$S = \{(1, -1, 2)\}$$

Ex. of system with ∞ number of solutions

System:

$$
\begin{array}{rrrrl}
x_1 & -2x_2 & +2x_3 & -x_4 & = 3 \\
3x_1 & +x_2 & +6x_3 & +11x_4 & = 16 \\
2x_1 & -x_2 & +4x_3 & +4x_4 & = 9
\end{array}
\tag{2}
$$

Row-echelon form of the augmented matrix:

$$
A^\sharp \sim \begin{bmatrix} 1 & -2 & 2 & -1 & 3 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
$$

Ex. of system with ∞ number of solutions (2)

Consistency: From row-echelon form we have

- 4 variables
- 2 leading entries
- No equation of the form $1 = 0$

Therefore we have a
$\hookrightarrow$ Consistent system with infinite number of solutions

Ex. of system with ∞ number of solutions (3)

Rule for systems with ∞ number of solutions:

- Choose as free variables those variables
  that **do not** correspond to a leading 1 in row-echelon form of $A^\sharp$

Application to system (2):

- Free variables: $x_3 = s$ and $x_4 = t$

Solution set:

$$\begin{aligned} S &= \{(5-2s-3t,\, 1-2t,\, s,\, t);\; s,t\in\mathbb{R}\} \\ &= \{(5,1,0,0)+s\,(-2,0,1,0)+t\,(-3,-2,0,1);\; s,t\in\mathbb{R}\} \end{aligned}$$

Outline

Reduced row-echelon matrices

Definition 4.

A reduced row-echelon matrix is a matrix A such that

1. A is row-echelon
2. All leading entries are $= 1$
3. Any column with a leading 1 has zeros everywhere else

Example of reduced row-echelon (1)

Matrix:

$$\begin{bmatrix} 1 & 9 & 26 \\ 0 & 14 & 28 \\ 0 & -14 & -28 \end{bmatrix}$$

Example of reduced row-echelon (2)

Reduced row-echelon form:

$$\begin{bmatrix} 1 & 9 & 26 \\ 0 & 14 & 28 \\ 0 & -14 & -28 \end{bmatrix} \overset{M_2(1/14)}{\sim} \begin{bmatrix} 1 & 9 & 26 \\ 0 & {\color{red}1} & 2 \\ 0 & -14 & -28 \end{bmatrix} \overset{A_{21}(-9),\, A_{23}(14)}{\sim} \begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Remark: One can also

1. Obtain the row-echelon form
2. Get the reduced row-echelon working upward and to the left

Example of system (1)

System:

$$\begin{array}{rrrr} 3x_1 & -2x_2 & +2x_3 & = 9 \\ x_1 & -2x_2 & +x_3 & = 5 \\ 2x_1 & -x_2 & -2x_3 & = -1 \end{array}$$

Augmented matrix:

$$A^\sharp = \begin{bmatrix} 3 & -2 & 2 & 9 \\ 1 & -2 & 1 & 5 \\ 2 & -1 & -2 & -1 \end{bmatrix}$$

Example of system (2)

Row-echelon form of the augmented matrix:

$$A^\sharp \sim \begin{bmatrix} 1 & -2 & 1 & 5 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

System corresponding to the augmented matrix:

$$\begin{array}{rrrl} x_1 & -2x_2 & +x_3 & = 5 \\ & x_2 & +3x_3 & = 5 \\ & & x_3 & = 2 \end{array}$$

Solution set:

$$S = \{(1, -1, 2)\}$$

Solving with reduced row-echelon

System:

$$\begin{array}{cccc} x_1 & +2x_2 & -x_3 & = 1 \\ 2x_1 & +5x_2 & -x_3 & = 3 \\ x_1 & +3x_2 & +2x_3 & = 6 \end{array}$$

Augmented matrix:

$$A^\sharp = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 5 & -1 & 3 \\ 1 & 3 & 2 & 6 \end{bmatrix}$$

Solving with reduced row-echelon (2)

Reduced row-echelon form of the augmented matrix:

$$A^\sharp \sim \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

System corresponding to the augmented matrix:

$$\begin{aligned} x_1 \qquad\qquad &= 5 \\ x_2 \qquad &= -1 \\ x_3 &= 2 \end{aligned}$$

Solution set:

$$S = \{(5, -1, 2)\}$$

Comparison between reduced and non reduced

Pros and cons:

- For Gauss-Jordan, the final backward system is easier to solve
- The reduced row-echelon form is costly in terms of computations
- Overall, Gauss is more efficient than Gauss-Jordan for systems

Main interest of Gauss-Jordan:

- One can compute the inverse of a matrix

Homogeneous systems

Proposition 5.

For a $m \times n$ matrix A, consider the system

$$A\mathbf{x} = \mathbf{0}.$$

Then we have

1. The system is always consistent
   $\hookrightarrow$ with trivial solution $\mathbf{x} = \mathbf{0}$.
2. If $m < n$, the system has ∞ number of solutions.
3. If $m = n$, then trivial solution $\mathbf{x} = \mathbf{0}$ is the unique solution iff A is row-equivalent to I_n

Example of homogeneous system

Matrix:

$$A = \begin{bmatrix} 0 & 2 & 3 \\ 0 & 1 & -1 \\ 0 & 3 & 7 \end{bmatrix}$$

System:

$$A\mathbf{x} = \mathbf{0}.$$

Example of homogeneous system (2)

Reduced echelon form of A:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Number of solutions: We have $m = n = 3$ and A not equivalent to I_3
$\hookrightarrow$ infinite number of solutions

Set of solutions:

$$S = \{(s, 0, 0);\ s \in \mathbb{R}\}.$$

Outline

Matrix

Definition 6.

A $m \times n$ matrix is a rectangular array of numbers with m rows and n columns

Example of 2×3 matrix:

$$A = \begin{bmatrix} \frac{3}{2} & \frac{2}{3} & \frac{1}{5} \\ 0 & \frac{5}{4} & -\frac{3}{7} \end{bmatrix}$$

Index notation

Recall: we have

$$A = \begin{bmatrix} \frac{3}{2} & \frac{2}{3} & \frac{1}{5} \\ 0 & \frac{5}{4} & -\frac{3}{7} \end{bmatrix}$$

Index notation: For the matrix A we have

$$a_{12} = \frac{2}{3}, \qquad a_{21} = 0, \qquad a_{23} = -\frac{3}{7}$$

Index notation for a $m \times n$ matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

Vectors

A row 3-vector:

$$a = \begin{bmatrix} \frac{2}{3} & -\frac{1}{5} & \frac{4}{7} \end{bmatrix}$$

A column 5-vector:

$$b = \begin{bmatrix} 1 \\ 4 \\ \pi \\ -67 \\ 3 \end{bmatrix}$$

Transpose

Definition 7.

Let

- $A = (a_{ij})$ be a $m \times n$ matrix

Then A^T is the matrix defined by

$$a_{ij}^T = a_{ji}$$

Example of transpose

Matrix:

$$A = \begin{bmatrix} \frac{3}{2} & \frac{2}{3} & \frac{1}{5} \\ 0 & \frac{5}{4} & -\frac{3}{7} \end{bmatrix},$$

Transpose:

$$A^T = \begin{bmatrix} \frac{3}{2} & 0 \\ \frac{2}{3} & \frac{5}{4} \\ \frac{1}{5} & -\frac{3}{7} \end{bmatrix}$$

Square matrices

Definition 8.

Let

- $A = (a_{ij})$ be a $m \times n$ matrix

If $m = n$, then A is a square matrix

Diagonal of a square matrix: Elements a_{ii}. An example is

$$A = \begin{bmatrix} 1 & 7 & 4 \\ 2 & 9 & 0 \\ 8 & 5 & 5 \end{bmatrix}$$

Square matrices (2)

Diagonal matrix:

$$\mathrm{Diag}(1,2,3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Symmetric matrix: such that $A^T = A$. Example given by

$$A = \begin{bmatrix} 1 & 7 & 4 \\ 7 & 9 & 0 \\ 4 & 0 & 5 \end{bmatrix}$$

Skew-symmetric matrix: such that $A^T = -A$. Example given by

$$A = \begin{bmatrix} 0 & 7 & 4 \\ -7 & 0 & 0 \\ -4 & 0 & 0 \end{bmatrix}$$

Matrix function

Definition 9.

A $m \times n$ matrix function is a matrix whose elements are functions of a variable t.

Example: A 2×3 matrix-valued function

$$A = \begin{bmatrix} t^2 & \cos(t) & 3t - 2 \\ \ln(t) & e^{-5t} & t\sin(t) \end{bmatrix}$$

Elementary operations on matrices

Addition:

$$A + B = (a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij})$$

Scalar multiplication: for $\alpha \in \mathbb{R}$

$$\alpha A = \alpha(a_{ij}) = (\alpha a_{ij})$$

Multiplication: If A is $m \times n$ and B is $n \times p$, then

$$C = AB \quad \Longrightarrow \quad c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Rules for multiplications

Identity: I_n is a $n \times n$ matrix defined by

$$I_n = \mathrm{Diag}(1, \ldots, 1)$$

Rules to follow:

$$\begin{aligned} A(B+C) &= AB + AC && \text{Distributive law} \\ A(BC) &= (AB)C && \text{Associative law} \\ A\,0 = 0\,A &= 0 && \text{Absorbing state} \\ A\,\mathrm{Id} = \mathrm{Id}\,A &= A && \text{Identity element} \end{aligned}$$

Rule **not** to follow:

- $AB \neq BA$ in general.

Example of elementary operations (1)

Matrices:

$$A = \begin{bmatrix} 1 & -2 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$$

Example of elementary operations (2)

Matrices:

$$A = \begin{bmatrix} 1 & -2 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$$

Sum and scalar multiplication:

$$A + B = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}, \qquad 2A = \begin{bmatrix} 2 & -4 \\ 0 & 4 \end{bmatrix}$$

Products:

$$AB = \begin{bmatrix} 0 & 3 \\ 2 & -2 \end{bmatrix} \quad \text{and} \quad BA = \begin{bmatrix} 2 & -2 \\ 1 & -4 \end{bmatrix}$$

Dot product

Definition 10.

Let

- a and b two column n-vectors

Then $a \cdot b$ is the number defined by

$$a \cdot b = a^T b = \sum_{k=1}^{n} a_k \, b_k$$

Example of dot product

Vectors: We consider

$$a = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$$

Dot product: We get

$$a \cdot b = -7$$

Properties of the transpose

Theorem 11.

Let

- A, C be two $m \times n$ matrices
- B be a $n \times p$ matrix

Then

1. $(A^T)^T = A$
2. $(A + C)^T = A^T + C^T$
3. $(AB)^T = B^T A^T$

Triangular matrices

Example of upper triangular matrix:

$$U = \begin{bmatrix} 1 & 7 & 4 \\ 0 & 9 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Example of lower triangular matrix:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -7 & 9 & 0 \\ -4 & 0 & 5 \end{bmatrix}$$

Triangular matrices (2)

Triangular matrices and products:

1. The product of two upper trg. mat. is an upper trg. mat.
2. The product of two lower trg. mat. is a lower trg. mat.

Example:

$$\begin{bmatrix} 1 & 7 & 4 \\ 0 & 9 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & -1 & -4 \\ 0 & 9 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 62 & 16 \\ 0 & 81 & 0 \\ 0 & 0 & 25 \end{bmatrix}$$

Outline

Definition of inverse

Problem: Consider a $n \times n$ matrix A. We wish to find B such that

$$A\,B = I_n, \quad \text{and} \quad B\,A = I_n \tag{3}$$

Definition 12.

If B satisfies (3), we set

$$B = A^{-1}.$$

A^{-1} is called the inverse of A.

Remark on notation:

- A^{-1} does **not** mean $\frac{1}{A}$
- $\frac{1}{A}$ has no meaning unless $n = 1$, i.e $A \in \mathbb{R}$

Computation for a 2-d case

Theorem 13.

Let $A \in \mathbb{R}^{2\times 2}$ of the form

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then

1. A is invertible iff its determinant is non zero

$$\det(A) \equiv ad - bc \neq 0$$

2. If A is invertible we have

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example

Matrices:

$$A = \begin{bmatrix} 1 & -2 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$$

Inverse:

$$A^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & \frac{1}{2} \end{bmatrix}, \qquad B^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$$

Relation with previous notions

Inverse and systems: If A is invertible, then

$$A\mathbf{x} = \mathbf{b} \quad \Longleftrightarrow \quad \mathbf{x} = A^{-1}\mathbf{b}$$

Inverse and rank: A is invertible iff

$$A \text{ is row-equivalent to } I_n$$

Gauss-Jordan technique (for 3×3 matrices)

Method:

1. Form an augmented matrix of the form

$$A^\sharp = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{bmatrix}$$

2. Use the Gauss-Jordan reduction technique, which yields

$$A^\sharp \sim \begin{bmatrix} 1 & 0 & 0 & b_{11} & b_{12} & b_{13} \\ 0 & 1 & 0 & b_{21} & b_{22} & b_{23} \\ 0 & 0 & 1 & b_{31} & b_{32} & b_{33} \end{bmatrix} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \equiv B$$

3. Then $B = A^{-1}$

2-d example

Matrix:

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

Augmented matrix:

$$A^{\sharp} = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

Gauss-Jordan reduced form:

$$A^{\sharp} \sim \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

Inverse:

$$A^{-1} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

2-d example (2)

System:

$$\begin{aligned} x_1 \quad &+2x_2 &= 3 \\ x_1 \quad &+x_2 &= 2 \end{aligned}$$

Related matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Solution:

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Solution set:

$$S = \{(1, 1)\}$$

Properties of the inverse

Proposition 14.

Let

- A and B invertible $n \times n$ matrices

Then

1. $(A^{-1})^{-1} = A$
2. $(AB)^{-1} = B^{-1}A^{-1}$
3. $(A^{T})^{-1} = (A^{-1})^{T}$

Checking property 2 (1)

Example of matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \quad AB = \begin{bmatrix} 7 & 3 \\ 5 & 2 \end{bmatrix}$$

Checking property 2 (2)

Inverses:

$$A^{-1} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}, \quad B^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$

Rule #2:

$$(AB)^{-1} = B^{-1}A^{-1} = \begin{bmatrix} -2 & 3 \\ 5 & -7 \end{bmatrix}$$

Outline

Outline

Particular cases

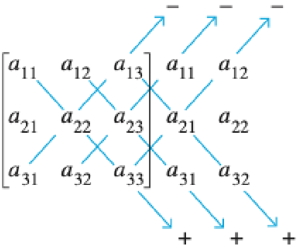
1×1 matrix:

$$A = [a_{11}] \quad \Longrightarrow \quad \det(A) = a_{11}$$

2×2 matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \Longrightarrow \quad \det(A) = a_{11}a_{22} - a_{12}a_{21}$$

3×3 matrix:



Remarks

Generalization: The determinant is defined for any $n \times n$ matrix
$\hookrightarrow$ Combinatorics involved

Motivation: In general

$$\det(A) \neq 0 \quad \Longleftrightarrow \quad A \text{ is invertible}$$

Notation:

$$\det(A) \equiv |A|$$

Examples

2×2 matrix:

$$\begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix} = -1$$

3×3 matrix:

$$\begin{vmatrix} 1 & 3 & -4 \\ 2 & 5 & -1 \\ 1 & 0 & 6 \end{vmatrix} = 11$$

Recursive method: strategy

Fact:

The determinant computation requires $n!$ operations

Aim:

Reduce the order of a determinant by an expansion

Vocabulary:

First we have to introduce the notions of

- Minor
- Cofactor

Minors of a matrix

Definition 15.

Let A be a $n \times n$ matrix. Then

$A_{ij} =$
$\det(\text{matrix obtained by deleting } i\text{th row and } j\text{th column of } A)$

The quantity A_{ij} is called minor of a_{ij}.

Example of minor

Example:

$$A = \begin{bmatrix} 1 & 3 & -4 \\ 2 & 5 & -1 \\ 1 & 0 & 6 \end{bmatrix} \quad \Longrightarrow \quad A_{12} = \begin{vmatrix} 2 & -1 \\ 1 & 6 \end{vmatrix} = 13$$

Cofactors of a matrix

Definition 16.

Let A be a $n \times n$ matrix. Then

$$C_{ij} = (-1)^{i+j} A_{ij}$$

The quantity C_{ij} is called cofactor of a_{ij}.

Example of cofactor

Example:

$$A = \begin{bmatrix} 1 & 3 & -4 \\ 2 & 5 & -1 \\ 1 & 0 & 6 \end{bmatrix} \quad \Longrightarrow \quad C_{12} = -M_{12} = -13$$

Remark: Alternate signs assignment for C_{ij}

Cofactor expansion

Theorem 17.

Let

- A be a $n \times n$ matrix.

Then

1. One can expand the determinant along the i-th row:

$$\det(A) = \sum_{k=1}^{n} a_{ik} C_{ik}$$

2. One can expand the determinant along the j-th column:

$$\det(A) = \sum_{k=1}^{n} a_{kj} C_{kj}$$

Example of application

Rule:

To simplify computations, choose row or column with 0's

Example:

Here we expand along the 3rd row

$$\begin{vmatrix} 1 & 3 & -4 \\ 2 & 5 & -1 \\ 1 & 0 & 6 \end{vmatrix} = \begin{vmatrix} 3 & -4 \\ 5 & -1 \end{vmatrix} + 6 \begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix} = 11$$

Outline

Introduction

Problem with determinants:

- For a $n \times n$, matrix, they require $n!$ operations
- This is computationally too demanding

Aim of this section:

- See properties in order to shorten computation time

Determinants of triangular matrices

Theorem 18.

Let

- A be an upper or lower triangular matrix.
- $n \equiv$ size of A.

Then

$$\det(A) = a_{11}\, a_{22} \cdots a_{nn} = \prod_{i=1}^{n} a_{ii}$$

Example of triangular matrix

Example:

$$\begin{vmatrix} 1 & 3 & -4 \\ 0 & 5 & -1 \\ 0 & 0 & 6 \end{vmatrix} = 30$$

Elementary row operations and determinants

Effect of elementary row operations:

If A is a $n \times n$ matrix, then

1. Let B be the matrix obtained by permuting 2 rows of A.
   Then
   $$\det(B) = -\det(A)$$
2. Let B obtained by multiplying 1 row of A by $k \in \mathbb{R}$.
   Then
   $$\det(B) = k \det(A)$$
3. Let B obtained by adding $k\times$ a row of A to a different row of A.
   Then
   $$\det(B) = \det(A)$$

Example of application

3×3 matrix:

$$\begin{vmatrix} 1 & 3 & -4 \\ 2 & 5 & -1 \\ 1 & 0 & 6 \end{vmatrix} \overset{A_{12}(-2),\, A_{13}(-1)}{=} \begin{vmatrix} 1 & 3 & -4 \\ 0 & -1 & 7 \\ 0 & -3 & 10 \end{vmatrix}$$

$$\overset{M_2(-1),\, M_3(-1)}{=} (-1)^2 \begin{vmatrix} 1 & 3 & -4 \\ 0 & 1 & -7 \\ 0 & 3 & -10 \end{vmatrix} \overset{A_{23}(-3)}{=} \begin{vmatrix} 1 & 3 & -4 \\ 0 & 1 & -7 \\ 0 & 0 & 11 \end{vmatrix} = 11$$

Remark:

This technique is really useful for $n \geq 4$

Further properties of determinants

Some more properties:

4. We have
$$\det(A^T) = \det(A)$$
5. If A has a column of 0's, then
$$\det(A) = 0$$
6. If 2 rows or columns of A are the same, then
$$\det(A) = 0$$
7. For two matrices A and B, we have
$$\det(A\,B) = \det(A)\,\det(B)$$

Application of Property 4

Example:

When further simplifications are available for columns

$$\begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 5 \\ -1 & 3 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ 0 & 5 & 2 \end{vmatrix} \overset{A_{23}(-5)}{=} \begin{vmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & -13 \end{vmatrix} = -13$$

Outline

Cramer's rule

Theorem 19.

Consider a $n \times n$ matrix A, a vector $\mathbf{b}$ and the system

$$A\mathbf{x} = \mathbf{b}. \tag{4}$$

For $1 \leq k \leq n$ set ($\mathbf{b}$ inserted at column k):

$$A_k(\mathbf{b}) = \begin{bmatrix} a_{11} & a_{12} & \dots & b_1 & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & b_2 & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ a_{n1} & a_{n2} & \dots & b_n & \dots & a_{nn} \end{bmatrix}$$

Then if $\det(A) \neq 0$ the solution of (4) is given by

$$x_k = \frac{\det(A_k(\mathbf{b}))}{\det(A)}$$

Example

System:

$$\begin{array}{cccc} 3x_1 & +2x_2 & -x_3 & = 4 \\ x_1 & +x_2 & -5x_3 & = -3 \\ -2x_1 & -x_2 & +4x_3 & = 0 \end{array}$$

Determinants:

$$\det(A) = \begin{vmatrix} 3 & 2 & -1 \\ 1 & 1 & -5 \\ -2 & -1 & 4 \end{vmatrix} = 8, \qquad \det(A_1(\mathbf{b})) = \begin{vmatrix} 4 & 2 & -1 \\ -3 & 1 & -5 \\ 0 & -1 & 4 \end{vmatrix} = 17$$

Solution:

$$x_1 = \frac{17}{8}$$

Cofactors of a matrix (reloaded)

Definition 20.

Let A be a $n \times n$ matrix. Then

$$C_{ij} = (-1)^{i+j} A_{ij}$$

The quantity C_{ij} is called cofactor of a_{ij}.

Example:

$$A = \begin{bmatrix} 1 & 3 & -4 \\ 2 & 5 & -1 \\ 1 & 0 & 6 \end{bmatrix} \quad \Longrightarrow \quad C_{12} = -M_{12} = -13$$

Remark: Alternate signs assignment for C_{ij}

Adjoint matrix

Definition 21.

Let A be a $n \times n$ matrix. Then

- Matrix of cofactors:
  Obtained by replacing each term of A by its cofactor
  Denoted by M_C
- Adjoint matrix: Denoted by $\mathrm{adj}(A)$ and defined as

$$\mathrm{adj}(A) = M_C^T$$

The adjoint method

Theorem 22.

Let A be a $n \times n$ matrix. Assume:

$$\det(A) \neq 0.$$

Then

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A).$$

Remark: Along the same lines we have

$$A \text{ invertible} \quad \Longleftrightarrow \quad \det(A) \neq 0$$

Example

Matrix:

$$A = \begin{bmatrix} 2 & 0 & -3 \\ -1 & 5 & 4 \\ 3 & -2 & 0 \end{bmatrix}$$

Cofactor and adjoint matrix:

$$M_C = \begin{bmatrix} 8 & 12 & -13 \\ 6 & 9 & 4 \\ 15 & -5 & 10 \end{bmatrix}, \qquad \operatorname{adj}(A) = \begin{bmatrix} 8 & 6 & 15 \\ 12 & 9 & -5 \\ -13 & 4 & 10 \end{bmatrix}$$

Inverse: $\det(A) = 55$ and thus

$$A^{-1} = \frac{1}{55} \begin{bmatrix} 8 & 6 & 15 \\ 12 & 9 & -5 \\ -13 & 4 & 10 \end{bmatrix}$$

Determinant as area or volume

Theorem 23.

Let A be a 2×2 or 3×3 matrix. Then

(1) If A is a 2×2 matrix we have

$$\det(A) = \text{ area of parallelogram given by columns of } A$$

(2) If A is a 3×3 matrix we have

$$\det(A) = \text{ volume of parallepiped given by columns of } A$$

Example of area

Aim: Compute area of parallelogram given by

$$(-2,-2), \qquad (0,3), \qquad (4,-1), \qquad (6,4)$$

Translation: We translate by $(2,2)$ to get a vertex at $\mathbf{0}$

$$(0,0), \qquad (2,5), \qquad (6,1), \qquad (8,6)$$

Area:

$$\text{Area} = \left| \begin{vmatrix} 2 & 6 \\ 5 & 1 \end{vmatrix} \right| = 28$$

Area and linear transformation in $\mathbb{R}^2$

Theorem 24.

Let

- $T : \mathbb{R}^2 \to \mathbb{R}^2$ linear transformation
- A matrix of T
- S parallelogram in $\mathbb{R}^2$

Then we have

$$\text{Area}\,(T(S)) = |\det(A)|\ \text{Area}\,(S)$$

Area and linear transformation in $\mathbb{R}^3$

Theorem 25.

Let

- $T : \mathbb{R}^3 \to \mathbb{R}^3$ linear transformation
- A matrix of T
- S parallepiped in $\mathbb{R}^3$

Then we have

$$\text{Volume}\,(T(S)) = |\det(A)|\ \text{Volume}\,(S)$$

Application (1)

Aim: Find area of region E delimited by ellipse

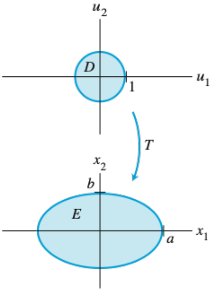
$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$$

Strategy: Let $D =$ unit disk in $\mathbb{R}^2$. We write

$$E = T(D), \quad \text{with} \quad A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

Application

Illustration:



Area:

$$\text{Area}\,(E) = \text{Area}\,(T(D)) = |\det(A)|\ \text{Area}\,(D) = \pi ab$$