Mathematical models and numerical methods

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Differential equations and linear algebra - MA 262

Taken from *Differential equations and linear algebra*
Edwards, Penney, Calvis

Outline

Outline

1. Population models
2. Equilibrium solutions and stability
3. Numerical approximation: Euler's method

Malthusian growth

Hypothesis:
Rate of change proportional to value of population

Equation: for $k \in \mathbb{R}$ and $P_0 \geq 0$,

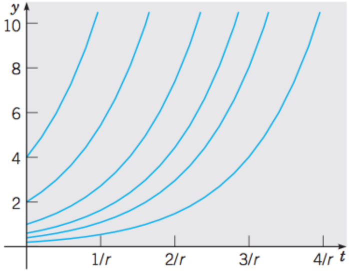
$$\frac{dP}{dt} = k\,P, \qquad P(0) = P_0$$

Solution:

$$P = P_0 \exp(kt)$$

Exponential growth (2)

Integral curves:



Limitation of model:

- Cannot be valid for large time t.

Logistic population model

Basic idea:

- Growth rate decreases when population increases.

Model:

$$\frac{dP}{dt} = r\left(1 - \frac{P}{C}\right)P, \tag{1}$$

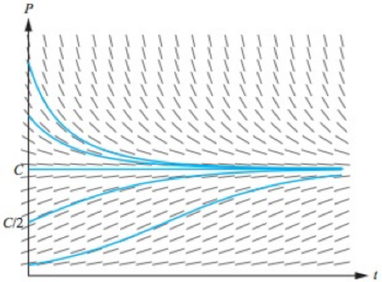
where

- $r \equiv$ reproduction rate
- $C \equiv$ carrying capacity

Logistic model: qualitative study

Information from slope field:

- Equilibrium at $P = C$
- If $P < C$ then $t \mapsto P$ increasing
- If $P > C$ then $t \mapsto P$ decreasing
- Possibility of convexity analysis



Logistic model: solution

First observation: Equation (1) is separable

Integration: Integrating on both sides of (1) we get

$$\ln\left(\left|\frac{P}{C-P}\right|\right) = rt + c_1$$

which can be solved as:

$$P(t) = \frac{c_2 C}{c_2 + e^{-rt}}$$

Initial value problem: If P_0 is given we obtain

$$P(t) = \frac{C\,P_0}{P_0 + (C - P_0)e^{-rt}}$$

Information obtained from the resolution

Asymptotic behavior:

$$\lim_{t\to\infty} P(t) = C$$

Prediction: If

- Logistic model is accurate
- P_0, r and C are known

Then we know the value of P at any time t

Outline

Introduction

General form of autonomous equations:

$$\frac{dy}{dt} = f(y) \tag{2}$$

Solving autonomous equations:

This is a special case of separable equation

Aim of the section:

1. Information on equation (2) with graphical methods
2. Applications to population growth models

Logistic growth

Hypothesis:

- Growth rate depends on population
- Related equation: $\frac{dy}{dt} = h(y)y$

Specifications for h:

- $h(y) \simeq r > 0$ for small values of y
- $y \mapsto h(y)$ decreases for larger values of y
- $h(y) < 0$ for large values of y

Possibility: $h(y) = r - ay$

Verhulst equation: for $r, K > 0$

$$\frac{dy}{dt} = f(y), \quad \text{with} \quad f(y) = r\left(1 - \frac{y}{K}\right)y \tag{3}$$

Logistic growth (2)

Vocabulary:

- r: Intrinsic growth.
- K: Saturation level or carrying capacity.
- Solutions to $f(y) = 0$: critical points.

Equilibrium solutions:

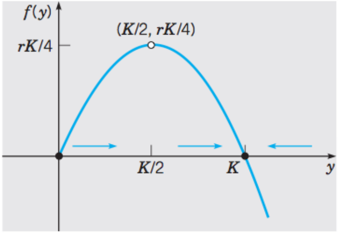
- Defined as $y \equiv \ell$, where ℓ critical point
- Here 2 equilibrium: $y = 0$ and $y = K$
- If we have:
  - $y(0) = 0$ or $y(0) = K$
  - y satisfies (3),

  then y stays constant

Logistic growth (3)

Graphical interpretation 1:

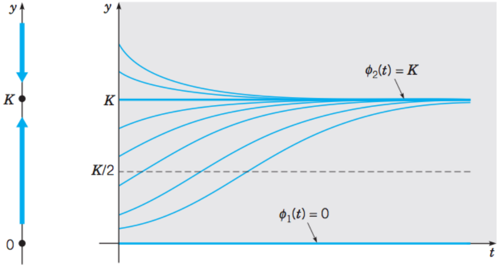
- Draw $y \mapsto f(y)$.
- Here f parabola, intercepts $(0, 0)$ and $(K, 0)$.
- We have $\frac{dy}{dt} > 0$ if $y \in (0, K)$
- We have $\frac{dy}{dt} < 0$ if $y > K$
- Vocabulary: y-axis is called phase line



Logistic growth (4)

Graphical interpretation 2: behavior of $t \mapsto y(t)$

- Draw line $y = 0$ and $y = K$
- Other curves:
  - Increasing if $y < K$
  - Decreasing if $y > K$
  - Flattens out as $y \to 0$ or $y \to K$
- Curves do not intersect
- Possibility of a convexity/concavity analysis (threshold $\frac{K}{2}$)



Logistic growth (5)

Stable and unstable equilibrium:

1. We have seen (phase diagram):
   - y increases if $y < K$
   - y decreases if $y > K$

   Thus K stable equilibrium
2. We have seen (phase diagram):
   - y increases as long as $y > 0$ (and $y < K$)

   Thus 0 unstable equilibrium

Remark:
See also the notion of semi-stable equilibrium

Logistic growth (6)

Solving the equation: Equation (3) can be written as

$$\left[\frac{1}{y} + \frac{1/K}{1 - y/K}\right] dy = r\,dt$$

Solution is given by:

$$y = \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}}$$

Equilibrium revisited: For all $y_0 > 0$ we have:

$$\lim_{t \to \infty} y(t) = K.$$

Thus K stable equilibrium.

Critical threshold example

Equation considered: for $r, T > 0$

$$\frac{dy}{dt} = f(y), \quad \text{with} \quad f(y) = -r\left(1 - \frac{y}{T}\right) y \tag{4}$$

Critical points:

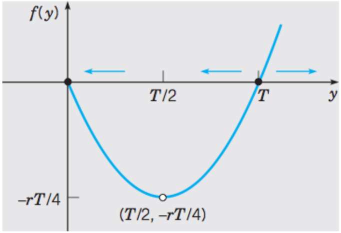
$$f(y) = 0 \quad \Longleftrightarrow \quad y = 0 \quad \text{or} \quad y = T$$

This corresponds to 2 equilibrium.

Critical threshold example (2)

Graphical interpretation 1:

- Here f parabola, intercepts $(0, 0)$ and $(T, 0)$.
- We have $\frac{dy}{dt} < 0$ if $y \in (0, T)$
- We have $\frac{dy}{dt} > 0$ if $y > T$



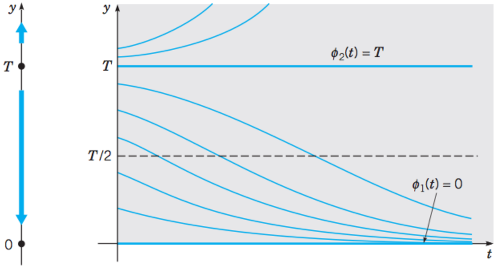
Conclusion for equilibrium:

- T unstable equilibrium
- 0 stable equilibrium

Critical threshold example (3)

Graphical interpretation 2: behavior of $t \mapsto y(t)$

- Draw line $y = 0$ and $y = T$
- Other curves:
  - Increasing if $y > T$
  - Decreasing if $y < T$
  - Flattens out as $y \to 0$
- Curves do not intersect
- Possibility of a convexity/concavity analysis (threshold $\frac{T}{2}$)



Critical threshold example (4)

Solving the equation: Like for equation (3) we get

$$y(t) = \frac{y_0 T}{y_0 + (T - y_0)e^{rt}} \tag{5}$$

Limiting behavior: according to (5),

1. If $0 < y_0 < T$, we have $\lim_{t\to\infty} y(t) = 0$.
2. If $y_0 > T$, we have $\lim_{t\to t^*} y(t) = \infty$, where

$$t^* = \frac{1}{r} \ln\left(\frac{y_0}{y_0 - T}\right)$$

This behavior could not be inferred from graphic representation.

A problem taken from Edward's book (1)

Equation:

$$y' = y^2 - 5y + 4$$

Problem:

Classify the equilibrium points

A problem taken from Edward's book (2)

Recasting the equation:

$$y' = (y-1)(y-4)$$

Classifying the equilibriums: We have

- 1 stable equilibrium
- 4 unstable equilibrium

Another problem taken from Edward's book (1)

Equation:

$$y' = (y-2)^2$$

Problem:

Classify the equilibrium points

Another problem taken from Edward's book (2)

Recall:

$$y' = (y-2)^2$$

Classifying the equilibriums: We have

- 2 semi-stable equilibrium

Outline

Euler

Euler: the best mathematician ever

- Modern math notation
- Series expansions
- Number theory
- Graph theory
- More than 30,000 pages of math
- Lived in Switzerland, Russia, Prussia

Approximations of first order equations: why?

Generic first order equation: Of the form

$$\frac{dy}{dt} = f(t, y), \qquad y(t_0) = y_0 \tag{6}$$

General facts about (6):

1. If f is continuous, equation can be solved in neighborhood of t_0.
2. Solution y cannot be computed explicitly.

Conclusion:

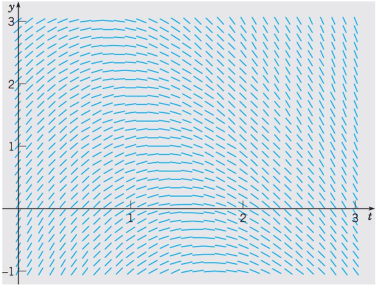
We need approximations in order to understand behavior of y.

Starting from direction fields

Equation considered:

$$\frac{dy}{dt} = 3 - 2t - 0.5y \tag{7}$$

Direction fields for (7):



Basic idea: Linking the tangent lines on the graph
$\hookrightarrow$ we get an approximation of solution.

Questions about approximation methods

Basic issues:

1. Method to link tangent lines.
2. Do we get an approximation of real solution?
3. Rate of convergence for approximation.

First steps of approximation (1)

Equation considered: equation (6), that is

$$\frac{dy}{dt} = f(t, y), \qquad y(t_0) = y_0.$$

First steps of approximation (2)

Approximation near t_0:

- Solution passes through (t_0, y_0)
- Slope at (t_0, y_0) is $f(t_0, y_0)$
- Consider t_1 close to t_0

Then linear approximation of $y(t_1)$ is given by:

$$y_1 = y_0 + f(t_0, y_0)\,(t_1 - t_0)\,.$$

First steps of approximation (3)

Approximation near t_1:

- Solution passes through $(t_1, y(t_1))$
- Problem: we don't know the exact value of $y(t_1)$
- We approximate $y(t_1)$ by y_1
- Approximate slope at (t_1, y_1) is given by $f(t_1, y_1)$
- Consider t_2 close to t_1

Then linear approximation of $y(t_2)$ is given by:

$$y_2 = y_1 + f(t_1, y_1)\,(t_2 - t_1)\,.$$

Euler scheme

Proposition 1.

Equation considered: equation (6), that is

$$\frac{dy}{dt} = f(t, y), \qquad y(t_0) = y_0.$$

Hypothesis: constant step in time,

$$t_{n+1} - t_n = h.$$

Notation: $f_n = f(t_n, y_n)$, $\hat{y}$ = Euler's approximation.

Conclusion: Recursive formula for Euler's scheme,

$$\begin{aligned} y_{n+1} &= y_n + f_n\, h \\ \hat{y}(t) &= y_n + f_n\,(t - t_n)\,, \quad \text{for } t \in [t_0 + nh, t_0 + (n+1)h) \end{aligned}$$

Example of Euler scheme (1)

Equation considered: back to equation (7), that is

$$\frac{dy}{dt} = \overbrace{3 - 2t - 0.5y}^{f(t,y)}, \qquad y(0) = 1$$

Euler scheme:

With $h = 0.2$

Example of Euler scheme (2)

Exact solution: we find

$$y = \phi(t) = 14 - 4t - 13 \exp\left(-\frac{t}{2}\right)$$

Euler scheme, step 1: with $h = 0.2$ we have

- $f_0 = f(0, 1) = 2.5$
- $\hat{y}(t) = 1 + 2.5t$ for $t \in (0, 0.2)$
- $y_1 = 1.5$

Example of Euler scheme (3)

Euler scheme, step 2: with $h = 0.2$ we have

- $f_1 = f(0.2, 1.5) = 1.85$
- $\hat{y}(t) = 1.5 + 1.85(t - 0.2)$ for $t \in (0.2, 0.4)$
- $y_2 = 1.87$

Numerical results:

t	Exact	Euler with $h = 0.2$	Tangent line
0.0	1.00000	1.00000	$y = 1 + 2.5t$
0.2	1.43711	1.50000	$y = 1.13 + 1.85t$
0.4	1.75650	1.87000	$y = 1.364 + 1.265t$
0.6	1.96936	2.12300	$y = 1.6799 + 0.7385t$
0.8	2.08584	2.27070	$y = 2.05898 + 0.26465t$
1.0	2.11510	2.32363	

Remark: about 10% error at $t = 1$

$\hookrightarrow$ Approximation not accurate enough, smaller h needed.

Example of Euler scheme (3)

Numerical results with varying h:

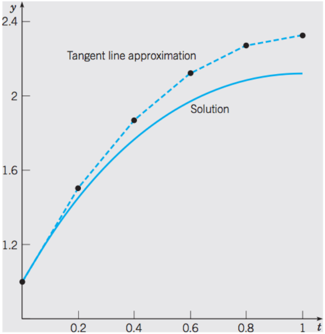
t	Exact	$h = 0.1$	$h = 0.05$	$h = 0.025$	$h = 0.01$
0.0	1.0000	1.0000	1.0000	1.0000	1.0000
1.0	2.1151	2.2164	2.1651	2.1399	2.1250
2.0	1.2176	1.3397	1.2780	1.2476	1.2295
3.0	−0.9007	−0.7903	−0.8459	−0.8734	−0.8898
4.0	−3.7594	−3.6707	−3.7152	−3.7373	−3.7506
5.0	−7.0671	−7.0003	−7.0337	−7.0504	−7.0604

Comments:

- Error decreases with time step.
- Error could possibly be of order h.

Example of Euler scheme (4)

Graphical comparison for $h = 0.2$:



Remark: $\hat{y} \geq y$

$\hookrightarrow$ Due to the fact that y concave $\Longrightarrow$ tangent above graph

Euler scheme for fast increasing solution

Equation considered:

$$\frac{dy}{dt} = 4 - t + 2y, \qquad y(0) = 1 \tag{8}$$

Exact solution: we find

$$y = \phi(t) = -\frac{7}{4} - \frac{t}{2} + \frac{11}{4} \exp(2t)$$

Thus exponential growth for y.

Euler scheme for fast increasing solution (2)

Numerical results with varying h:

t	Exact	$h = 0.1$	$h = 0.05$	$h = 0.025$	$h = 0.01$
0.0	1.000000	1.000000	1.000000	1.000000	1.000000
1.0	19.06990	15.77728	17.25062	18.10997	18.67278
2.0	149.3949	104.6784	123.7130	135.5440	143.5835
3.0	1109.179	652.5349	837.0745	959.2580	1045.395
4.0	8197.884	4042.122	5633.351	6755.175	7575.577
5.0	60573.53	25026.95	37897.43	47555.35	54881.32

Comments:

- Error still decreases with h
- Worse performance than for (7).

Explanation of difference:

- For (7) all solutions converge to $\phi(t) = 14 - 14t$
  $\hookrightarrow$ successive errors are not propagating
- For (8) solutions diverge exponentially
  $\hookrightarrow$ strong propagation of successive errors