

Hom. eq. with constant coeff

①

Recall: For an equation
of the form

$$a y'' + b y' + c y = 0,$$

we tried to find solution of
the form

$$y = e^{rx}$$

Then r was a root for

$$P(r) = ar^2 + br + c$$

New situation.

$$a_n y^{(n)} + \cdots + a_0 y = 0$$

We still assume $y = e^{rx}$.

Then r root for $P(r) = a_n r^n + \cdots + a_0$

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A difference between 2nd order
and n-th order diff eq

- For 2nd order diff eq, a root can only be repeated twice (2 roots only) if it is real valued. A complex valued root cannot be repeated
- For n-th, $n > 2$, a real valued root can be repeated up to n times. A complex valued root can also be repeated $\left[\frac{n}{2}\right]$ times

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Third order eq

$$y''' + y'' + 3y' - 5y = 0$$

Polynomial

$$P(r) = r^3 + r^2 + 3r - 5$$

Trivial root: $r = 1$. Thus

$$P(r) = (r-1)(r^2 + ar + 5)$$

Then $a-1 = 1 \Rightarrow a = 2$ and

$$P(r) = (r-1)(r^2 + 2r + 5)$$

complete the square: roots: $-1 \pm 2i$

$$\begin{aligned} P(r) &= (r-1) \underbrace{(r^2 + 4r + 4 + 1)}_{(r+2)^2 + 1} \\ &= (r-1) (r - (-1+2i)) \\ &\quad (r - (-1-2i)) \end{aligned}$$

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$$\text{Roots : } R_1 = 1$$

$$R_2 = -1 + 2i$$

$$R_3 = -1 - 2i \quad (= \bar{R}_2)$$

Fundamental sol. According to
the recipe,

$$y_1 = e^x$$

$$y_2 = e^{-x} \cos(2x)$$

$$y_3 = e^{-x} \sin(2x)$$

General solution :

$$y = c_1 e^x + c_2 e^{-x} \cos(2x) + c_3 e^{-x} \sin(2x)$$

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Rmk In order to get a unique sol, we should specify an initial condition of the form

$$y(0) = a \quad y'(0) = b \quad y''(0) = c$$

Then we should solve a 3×3 system in order to find

$$c_1, c_2, c_3$$

⑥

Notation

$$(D-1)(D+2)y$$

Interpretation

- (i) D stands for "differentiation"
Namely $Dy \equiv y'$
- (ii) $(D-1)(D+2)$ should be expanded like a polynomial

Thus

$$(D-1)(D+2)y$$

$$= (D^2 + D - 2)y$$

$$= y'' + y' - 2y$$

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Eq

$$D^3 (D-2)^2 \cdot (D^2+1)^2 y = 0$$

Order of this eq: 9

Advantage of the D-fam:
the polynomial is (almost) factorized

$$\begin{aligned} P(r) &= r^3 (r-2)^2 (r^2+1)^2 \\ &= r^3 (r-2)^2 (r-i)^2 (r+i)^2 \end{aligned}$$

<u>Root</u>	<u>Multiplicity</u>
0	3
2	2
$\pm i$	2

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Fundamental solution

$$y_1 = 1, \quad y_2 = x, \quad y_3 = x^2$$

$$y_4 = e^{2x} \quad y_5 = x e^{2x}$$

$$y_6 = \cos(x) \quad y_7 = x \cos(x)$$

$$y_8 = \sin(x) \quad y_9 = x \sin(x)$$

General solution

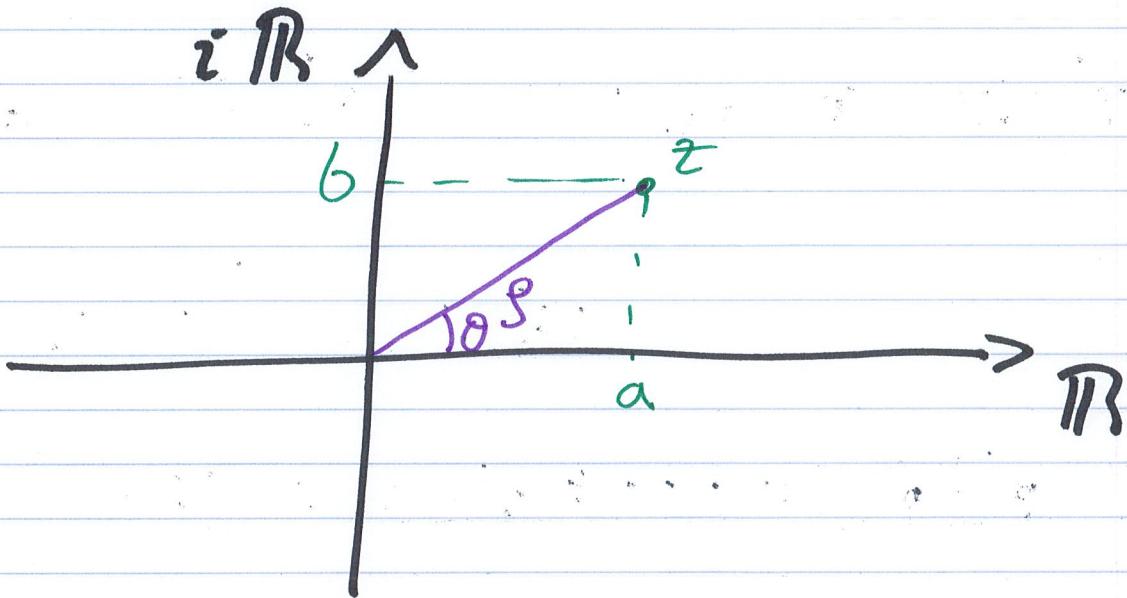
$$y = c_1 y_1 + c_2 y_2 + \dots + c_9 y_9$$

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Euler's representation of \mathbb{C}

A complex number z is usually written as

$$z = a + i b \quad (\text{Cartesian})$$



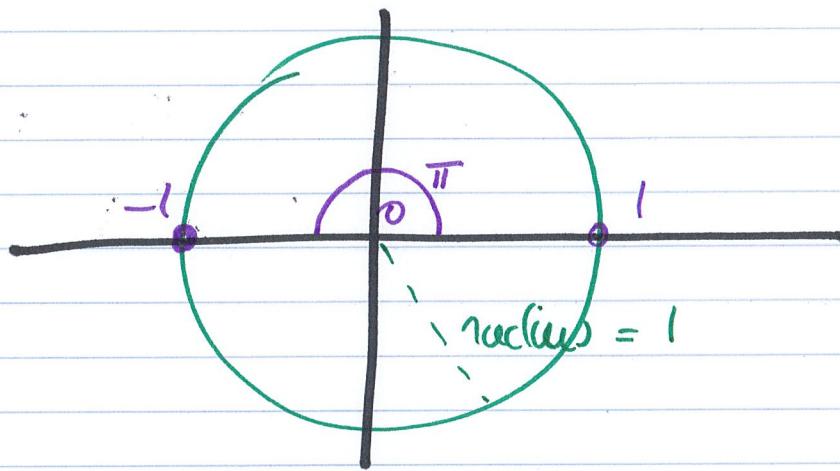
Euler's representation is related to polar coordinates

$$z = r e^{i(\theta + 2\pi k)} \quad k = 0, 1, 2, \dots$$

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Euler's formula

$$e^{i\pi} = -1$$



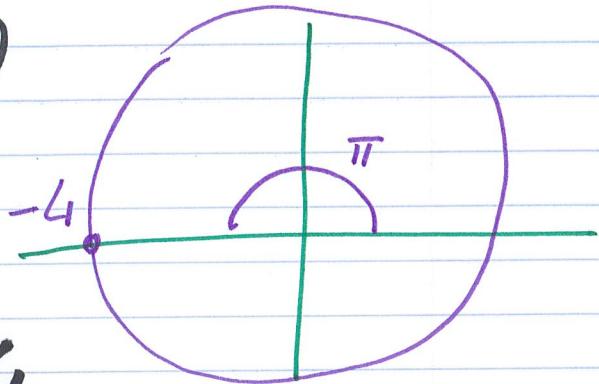
(11)

Eq

$$y^{(4)} + 4y = 0$$

Polynomial

$$P(r) = r^4 + 4$$



$$P(r) = 0 \Leftrightarrow r^4 = -4$$

Write -4 using Euler

$$-4 = 4 e^{i(\pi + k2\pi)}$$

$$\text{Thus } r^4 = -4 e^{i(\pi + k2\pi)}$$

$$\Leftrightarrow r^4 = 4 e^{i(\frac{\pi}{4} + k\frac{\pi}{2})}$$

$$\Leftrightarrow r = \sqrt[4]{4} e^{i(\frac{\pi}{4} + k\frac{\pi}{2})}$$

with $k = 0, 1, 2, 3$

$$e^{i\theta} = \cos(\theta) + i \sin(\theta)$$

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Summarize: The 4 roots are

$$\rho = \sqrt{2} e^{i\left(\frac{\pi}{4} + k\frac{\pi}{2}\right)} \quad k=0,1,2,3$$

Thus

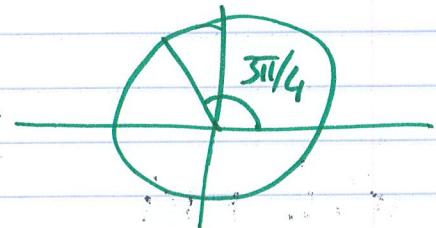


$$\rho_1 = \sqrt{2} e^{i\frac{\pi}{4}}$$

$$= \sqrt{2} \left(\cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right) \right)$$

$$= \sqrt{2} \left(\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right)$$

$$= 1 + i$$



$$\rho_2 = \sqrt{2} e^{i\frac{3\pi}{4}}$$

$$= \sqrt{2} \left(\cos\left(\frac{3\pi}{4}\right) + i \sin\left(\frac{3\pi}{4}\right) \right)$$

$$= \sqrt{2} \left(-\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right)$$

$$= -1 + i$$

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Other roots

$$\lambda_3 = \bar{\lambda}_2 = -1 - i$$

$$\lambda_4 = \bar{\lambda}_1 = -1 + i \quad | - i$$

Fund solutions

$$y_1 = e^x \cos(x) \quad \text{with } \lambda_1 = 1+i$$

$$y_2 = e^x \sin(x)$$

$$y_3 = e^{-x} \cos(x) \quad \text{with } \lambda_3 = -1+i$$

$$y_4 = e^{-x} \sin(x)$$