Outline

Equation $\quad y' = f(t, y)$

Ex: $\quad y' = 3y - 5t$

Question $\quad$ Before computing anything about y, do we know that the solution exists?

Existence and uniqueness result

Theorem 1.

General nonlinear equation:

$$y' = f(t, y), \qquad y(t_0) = y_0 \in \mathbb{R}. \tag{3}$$

Partial derivative

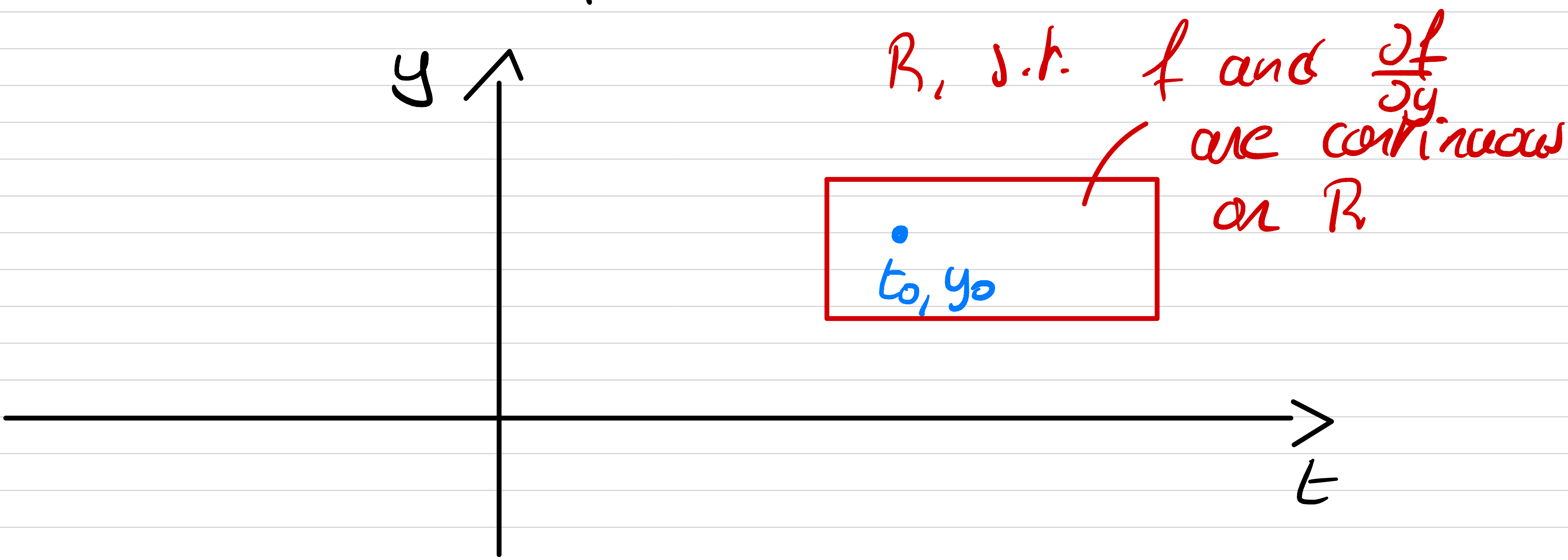
Hypothesis:

- $(t_0, y_0) \in R$, where $R = (\alpha, \beta) \times (\gamma, \delta)$.
- f and $\boxed{\frac{\partial f}{\partial y}}$ continuous on R.

Conclusion:

One can find $h > 0$ such that there exists a unique function y
$\hookrightarrow$ satisfying equation (3) on $(t_0 - h,\ t_0 + h)$.

Visual version of Thm 1



Conclusion: unique solution on $(t_0 - h, t_0 + h)$

Example of existence and uniqueness (1)

Equation considered:

$t_0 = 0$

$$\underbrace{y' = 3x\, y^{1/3}}_{f(x,y)}, \quad \text{and} \quad y(0) = a.$$

$\hookrightarrow y_0$

Basic facts about continuity in $\mathbb{R}^2$

(i) Polynomials are continuous

Ex : $f(x,y) = xy^5 - 27x^8y^3$

(ii) Exp, sin, trig functions are continuous

Ex: $f(x,y) = \exp(x^2y^3 - y^8)$

$f(x,y) = \cos(x^4 - yx^2)$

(iii) If P, Q are polynomials, then

$f = \dfrac{P}{Q}$ continuous if $Q \neq 0$

Ex: $f(x,y) = \dfrac{xy^5 - x^8y}{(y-1)^2}$

continuous if $y \neq 1$

Equation $\quad y' = \underbrace{3x\,y^{1/3}}_{f(x,y)}$ $\qquad y(\underset{t_0}{0}) = \underset{y_0}{a}$

Continuity of f $\quad$ f is a product of continuous function $\Rightarrow$ f continuous on $\mathbb{R}^2$

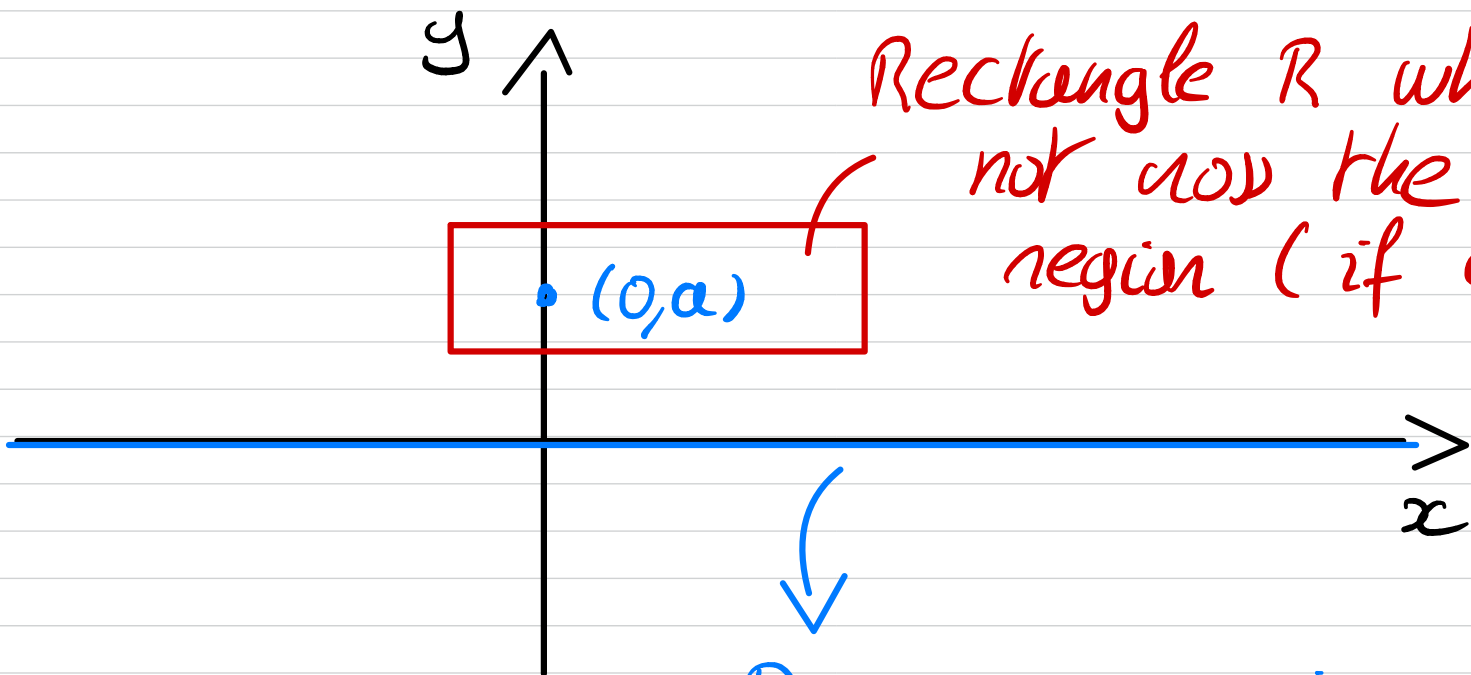
Continuity of $\frac{\partial f}{\partial y}$ $\quad$ In order to compute $\frac{\partial f}{\partial y}$, we freeze x and we differentiate wrt y.

$$f(x,y) = 3x\,y^{1/3} \Rightarrow \frac{\partial f}{\partial y} = 3x \times \frac{1}{3} y^{-2/3}$$

$$= x\,y^{-2/3} = \frac{x}{y^{2/3}}$$

$\frac{\partial f}{\partial y}$ is continuous <u>if $y \neq 0$</u>

Eq $\quad y' = 3xy^{1/3} \qquad y(\overset{x_0}{0}) = a$



Rectangle R which does not cross the dangerous region (if $a \neq 0$)

Dangerous region

Conclusion: if $a \neq 0$, unique solution on $(-h, h)$

If $a = 0$, we cannot conclude

Example of existence and uniqueness (2)

Application of Theorem 1: we have

$$f(x,y) = 3x\,y^{1/3}, \qquad \frac{\partial f}{\partial y}(x,y) = xy^{-2/3}$$

Therefore if $a \neq 0$:

1. There exists rectangle R such that
   - $(0,a) \in R$
   - f and $\frac{\partial f}{\partial y}$ continuous on R
2. According to Theorem 1 there is unique solution on interval $(-h,h)$, with $h > 0$

Second example of existence and uniqueness (1)

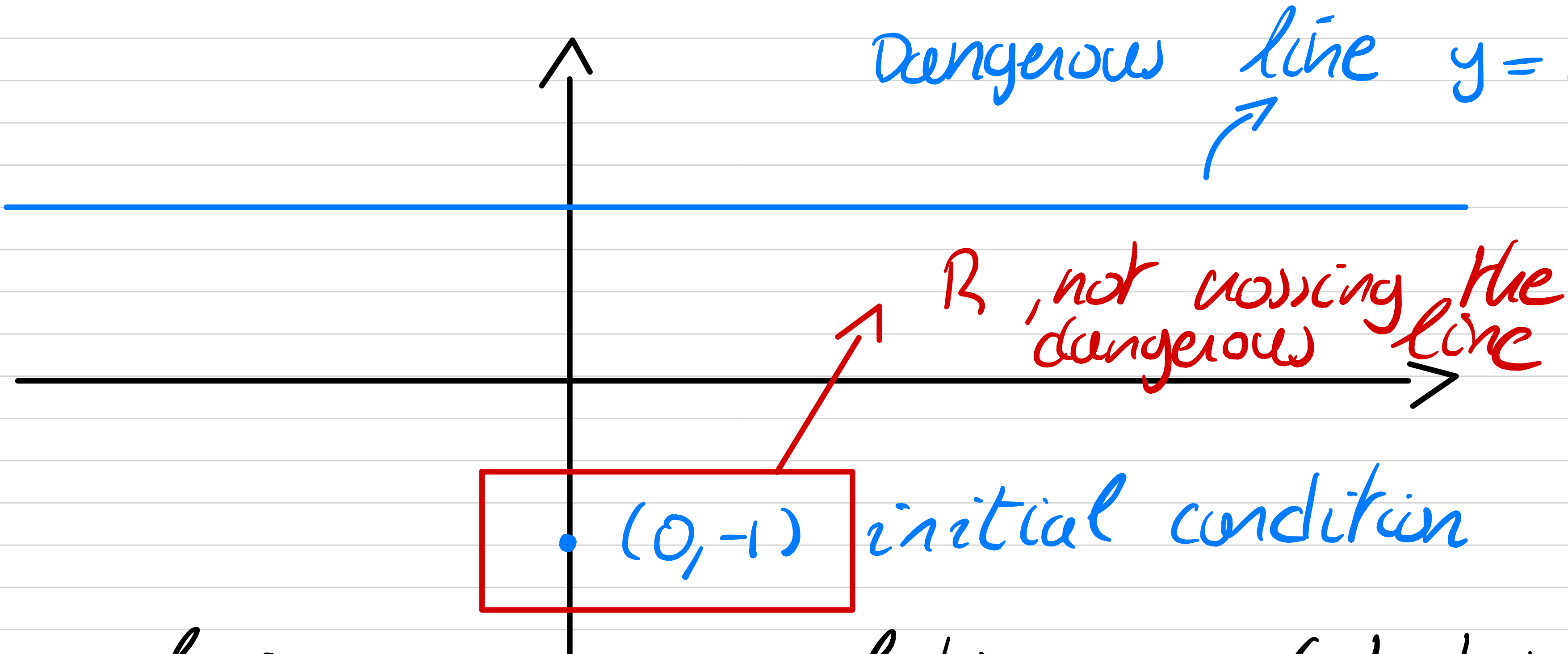
Equation considered:

$$y' = \frac{3x^2 + 4x + 2}{2(y-1)}, \quad \text{and} \quad y(0) = -1.$$

Eq: $$y' = \overbrace{\frac{3x^2+4x+2}{2(y-1)}}^{f(x,y)} \qquad y(0) = -1$$

Then f is continuous if $y \neq 1$

Moreover $$\frac{\partial f}{\partial y} = -\frac{3x^2+4x+2}{2(y-1)^2}$$ is continuous if $y \neq 1$



Conclusion: unique solution on $(-h, h)$

Rmk we have $y = 1 - \left((x+2)(\underbrace{x^2+2}_{>0}) \right)^{\frac{1}{2}}$

>0 if $x > -2$

Domain of def for y: $(-2, \infty)$

Thm 1 was only giving interval of def

$= (-h, h)$

Second example of existence and uniqueness (2)

Application of Theorem 1: we have

$$f(x,y) = \frac{3x^2+4x+2}{2(y-1)}, \qquad \frac{\partial f}{\partial y}(x,y) = -\frac{3x^2+4x+2}{2(y-1)^2}$$

Therefore:

1. There exists rectangle R such that
   - $(0,-1) \in R$
   - f and $\frac{\partial f}{\partial y}$ continuous on R
2. According to Theorem 1 there is unique solution on interval $(-h,h)$, with $h>0$

Second example of existence and uniqueness (3)

Comparison with explicit solution: We will see that

$$y = 1 - \left((x+2)(x^2+2)\right)^{1/2}$$

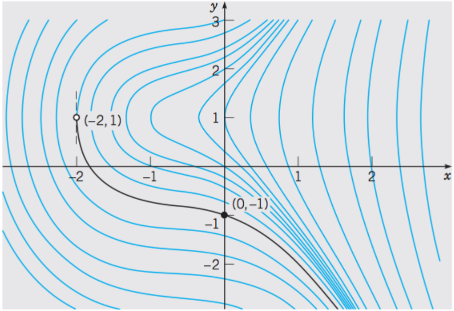
Interval of definition: $x \in (-2, \infty)$
$\hookrightarrow$ much larger than predicted by Theorem 1

Changing initial condition: consider $y(0) = 1$, on line $y = 1$. Then:

1. Theorem 1: nothing about possible solutions
2. Direct integration:
   - We find $y = 1 \pm (x^3 + 2x^2 + 2x)^{1/2}$
   - 2 possible solutions defined for $x > 0$

Second example of existence and uniqueness (3)

Interval of definition on integral curves:



Comments:

- Interval of definition delimited by vertical tangents

Example with non-uniqueness (1)

Equation considered:

$$y' = y^{1/3}, \quad \text{and} \quad y(0) = 0.$$

Application of Theorem 1: $f(y) = y^{1/3}$. Hence,

- $f : \mathbb{R} \to \mathbb{R}$ continuous on $\mathbb{R}$, differentiable on $\mathbb{R}^*$
- Theorem 1: gives existence, not uniqueness

Solving the problem: Separable equation, thus

- General solution: for $c \in \mathbb{R}$, $y = \left[\frac{2}{3}(t + c)\right]^{3/2}$
- With initial condition $y(0) = 0$,

$$y = \left(\frac{2t}{3}\right)^{3/2}$$

Example with non-uniqueness (2)

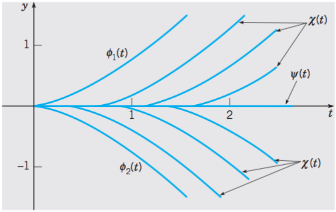
3 solutions to the equation:

$$\phi_1(t) = \left(\frac{2t}{3}\right)^{3/2}, \quad \phi_2(t) = -\left(\frac{2t}{3}\right)^{3/2}, \quad \psi(t) = 0.$$

Family of solutions: For any $t_0 \geq 0$,

$$\chi(t) = \chi_{t_0}(t) = \begin{cases} 0 & \text{for } 0 \leq t < t_0 \\ \pm\left(\frac{2(t-t_0)}{3}\right)^{3/2} & \text{for } t \geq t_0 \end{cases}$$

Integral curves:



Slope field for a gravity equation (1)

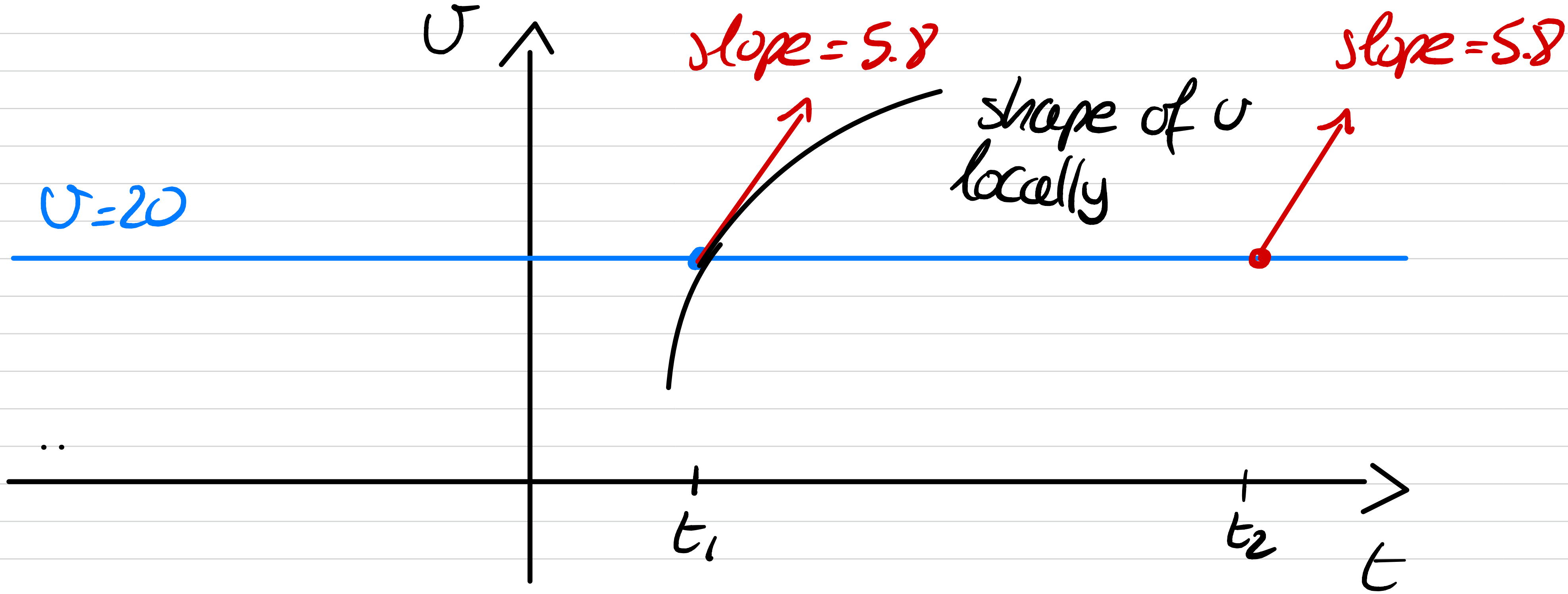
Other way to write the equation: $v' = 9.8 - \frac{v}{5}$

Gravity equation with friction

$$\frac{dv}{dt} = 9.8 - \frac{v}{5} \tag{4}$$

Idea: If the value of v is known, then the value of v' is also known

Ex: If $v = 20$, then $v' = 9.8 - \frac{20}{5} = 5.8$



<u>Next step:</u> Draw many of these point-slopes, in order to see how the solution behaves

Conclusions from the graph

(i) Special value: if we start from v s.t

$$v' = 9.8 - \frac{v}{5} = 0 \iff v = 49,$$

then $v' = 0$ forever $\Rightarrow$ flat curve

$\qquad\qquad > \quad 49 \qquad\qquad\qquad\qquad\qquad\qquad < 0$

(ii) If $v < 49$, then $v' = 9.8 - \frac{v}{5} > 0$

decreasing

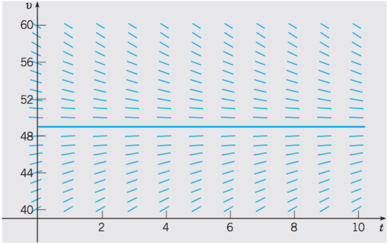
$\Rightarrow v$ increasing

(iii) $\lim_{t \to \infty} v(t) = 49$ (equilibrium)

Slope field for a gravity equation (2)

Meaning of the graph:

$\hookrightarrow$ Values of $\frac{dv}{dt}$ according to values of v



Slope field for a gravity equation (3)

What can be seen on the graph:

- Critical value: $v_c = 49\text{ms}^{-1}$, solution to $9.8 - \frac{v}{5} = 0$
- If $v < v_c$: positive slope
- If $v > v_c$: negative slope