

Outline

- 1 Differential equations and mathematical models
- 2 Integrals as general and particular solutions
- 3 Slope fields and solution curves
- 4 Separable equations and applications
- 5 Linear equations
- 6 Substitution methods and exact equations**
 - Homogeneous equations
 - Bernoulli equations
 - Exact differential equations
 - Reducible second order differential equations
- 7 Chapter review

Objective

General 2nd order differential equation:

$$\frac{d^2y}{dx^2} = F\left(x, y, \frac{dy}{dx}\right) \rightarrow \begin{array}{l} \text{separate} \\ \text{chapter for} \\ \text{higher order} \\ \text{diff eq.} \end{array}$$

indep. variable (with arrow pointing to x)

Aim: See cases of 2nd order differential equations
↪ which can be solved with 1st order techniques

2nd order eq. with missing dependent variable

Case 1: Equation of the form

$$\frac{d^2y}{dx^2} = F\left(x, \frac{dy}{dx}\right) \rightarrow \begin{array}{l} \text{No dependence} \\ \text{on } y \end{array}$$

Method for case 1: the function $v = y'$ solves

$$\frac{dv}{dx} = F(x, v).$$

Then compute $y = \int v(x) dx$.

General eq $y'' = F(x, y')$

Method Set $y' = v$
Then $y'' = v'$

and eq becomes

$$v' = F(x, v)$$

First order
diff eq.

Example (1)

Equation:

$$\frac{d^2y}{dx^2} = \frac{1}{x} \left(\frac{dy}{dx} + x^2 \cos(x) \right), \quad x > 0.$$

Eq $y'' = \frac{1}{x} (y' + x^2 \cos(x))$

$$\Leftrightarrow y'' = \underbrace{\frac{1}{x} y' + x \cos(x)}_{F(x, y')}$$

Let $y' = u$. We get

$$u' = \frac{1}{x} u + x \cos(x) \rightarrow \text{linear eq}$$

$$\Leftrightarrow u' - \frac{1}{x} u = x \cos(x)$$

Integrating factor $\mu = e^{-\int \frac{1}{x} dx} = e^{-\ln(x)} = \frac{1}{x}$

General solution $u = x \sin(x) + c, x$

Solving for y

$$v = y', \text{ so}$$

$$y = \int v \, dx$$

$$= \int (x \sin(x) + c_1 x) \, dx$$

ibp

$$-x \cos(x) + \sin(x) + c_1 x^2 + c_2$$

with $c_1, c_2 \in \mathbb{R}$

Rmk

This is a second order diff eq

\Rightarrow The general solution depends
on 2 parameters

Example (2)

Change of variable: $v = y'$ solves the linear equation

$$v' - x^{-1}v = x \cos(x)$$

Integrating factor:

$$I(x) = \exp\left(\int x^{-1} dx\right) = x^{-1}$$

Solving for v :

$$v = x \sin(x) + cx$$

Solving for y :

$$y = \int v = -x \cos(x) + \sin(x) + c_1 x^2 + c_2$$

2nd order eq. with missing independent variable

Case 2: Equation of the form

$$\frac{d^2 y}{dx^2} = F\left(y, \frac{dy}{dx}\right) \rightarrow \text{Variable } x \text{ is missing}$$

Method for case 2: We set $v = \frac{dy}{dx}$. Then observe that

$$\frac{d^2 y}{dx^2} = \frac{dv}{dx} = \frac{dv}{dy} \frac{dy}{dx} = v \frac{dv}{dy}$$

Thus v solves the 1st order equation

$$v \frac{dv}{dy} = F(y, v).$$

General eq $y'' = F(y, y')$

We let $v = \frac{dy}{dx}$. We assume $v = v(y)$
Then

$$\text{lhs} = y'' = \frac{d^2 y}{dx^2} = \frac{dv}{dx} \stackrel{\text{chain rule}}{=} \frac{dv}{dy} \underbrace{\frac{dy}{dx}}_v = v \frac{dv}{dy}$$

Thus eq becomes

$$v \frac{dv}{dy} = F(y, v) \rightarrow \text{1st order diff eq.}$$

Eq $y'' = -\frac{2}{1-y} (y')^2$ $F(y, y')$

Set $v = \frac{dy}{dx}$. The eq. becomes

$$v \frac{dv}{dy} = -\frac{2}{1-y} v^2 \rightarrow \text{1st order diff eq. for } v=v(y)$$

$$\times \frac{1}{v^2} \Leftrightarrow \frac{1}{v} dv = -\frac{2}{1-y} dy \rightarrow \text{separable}$$

General solution for v & y

$$v = C_1 (1-y)^2$$

$$\Leftrightarrow \frac{dy}{dx} = C_1 (1-y)^2 \rightarrow \text{separable eq}$$

$$\Rightarrow \boxed{y = \frac{C_1 x + (C_2 - 1)}{C_1 x + C_2}}$$

Example(1)

Equation:

$$\frac{d^2y}{dx^2} = -\frac{2}{1-y} \left(\frac{dy}{dx} \right)^2.$$

Example (2)

Change of variable: $v = y'$ solves the 1st order separable equation

$$v \frac{dv}{dy} = -\frac{2}{1-y} v^2$$

Solving for v :

$$v(y) = c_1(1-y)^2$$

Example (3)

Separable equation for y :

$$\frac{dy}{dx} = c_1(1 - y)^2$$

Solving for y :

$$y = \frac{c_1x + (c_2 - 1)}{c_1x + c_2}$$