Vector spaces

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Differential equations and linear algebra - MA 262

Taken from *Differential equations and linear algebra*
Edwards, Penney, Calvis

Outline

Outline

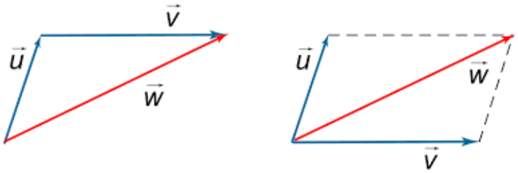
Vectors in $\mathbb{R}^3$

Fact: A tuple

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

can be geometrically interpreted as a vector

Illustration of an addition:



Some operations on vectors in $\mathbb{R}^3$

Algebraic operations:

- $1\,\mathbf{x} = \mathbf{x}$
- $(st)\mathbf{x} = s(t\,\mathbf{x})$
- $r(\mathbf{x} + \mathbf{y}) = r\,\mathbf{x} + r\,\mathbf{y}$
- $(s + t)\mathbf{x} = s\,\mathbf{x} + t\,\mathbf{x}$

Generalization:

1. Solutions of differential equations exhibit the same kind of structure
2. We need a more abstract concept
   $\hookrightarrow$ Vector spaces

Vector space definition

Proposition 1.

The space $\mathbb{R}^3$ is such that

- An addition and scalar multiplication are defined on $\mathbb{R}^3$
- Those operations satisfy conditions 1–10 below

The space $\mathbb{R}^3$ is an example of vector space.

Conditions 1 to 5

1. Closure under addition:
   If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, then $\mathbf{u} + \mathbf{v} \in \mathbb{R}^3$
2. Commutativity of addition: For all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$,
   $$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$$
3. Associativity of addition: For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$,
   $$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$$
4. Existence of a zero vector: There exists $\mathbf{0} \in \mathbb{R}^3$ such that
   $$\mathbf{v} + \mathbf{0} = \mathbf{v}$$
5. Existence of additive inverses in $\mathbb{R}^3$:
   For all $\mathbf{v} \in \mathbb{R}^3$, there exists $-\mathbf{v} \in \mathbb{R}^3$ such that
   $$\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$$

Conditions 6 to 10

6. Closure under scalar multiplication:
   If $\mathbf{u} \in \mathbb{R}^3$ and $k \in \mathbb{R}$, then $k\,\mathbf{u} \in \mathbb{R}^3$
7. Distributivity 1:
$$r(\mathbf{u} + \mathbf{v}) = r\,\mathbf{u} + r\,\mathbf{v}$$
8. Distributivity 2:
$$(r + s)\mathbf{v} = r\,\mathbf{v} + s\,\mathbf{v}$$
9. Associativity of scalar multiplication:
$$(rs)\mathbf{v} = r(s\,\mathbf{v})$$
10. Unit property: For all $\mathbf{v} \in \mathbb{R}^3$, we have $1\,\mathbf{v} = \mathbf{v}$

Linear dependence of 2 vectors

Definition 2.

Let $\mathbf{v}_1, \mathbf{v}_2$ be two vectors in $\mathbb{R}^3$. Then

1. If there exist c_1, c_2 not all zero such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0},$$

   we say that $\{\mathbf{v}_1, \mathbf{v}_2\}$ are linearly dependent

2. If we have

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0} \quad \Longrightarrow \quad c_1 = c_2 = 0,$$

   we say that $\{\mathbf{v}_1, \mathbf{v}_2\}$ are linearly independent

Example of linear dependence/independence

Example of linear dependence: The vectors

$$\mathbf{u} = (3, -2) \quad \text{and} \quad \mathbf{v} = (-6, 4)$$

are linearly dependent

Example of linear independence: The vectors

$$\mathbf{u} = (3, -2) \quad \text{and} \quad \mathbf{v} = (5, -7)$$

are linearly independent

Linear dependence of 3 vectors

Definition 3.

Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be 3 vectors in $\mathbb{R}^3$. Then

1. If there exist c_1, c_2, c_3 not all zero such that
$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0},$$
we say that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are linearly dependent
2. If we have
$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0} \quad \Longrightarrow \quad c_1 = c_2 = c_3 = 0,$$
we say that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are linearly independent

Example of linear dependence/independence

Example of linear dependence: The vectors

$$\mathbf{u} = (1, 2, -3), \quad \mathbf{v} = (3, 1, -2) \quad \text{and} \quad \mathbf{w} = (5, -5, 6)$$

are linearly dependent, with

$$4\mathbf{u} - 3\mathbf{v} + \mathbf{w} = \mathbf{0}$$

Example of linear independence: The vectors

$$\mathbf{u} = (1, 2, -3), \quad \mathbf{v} = (3, 1, -2) \quad \text{and} \quad \mathbf{w} = (5, -5, 0)$$

are linearly independent

Criterion for linear dependence of 3 vectors

Proposition 4.

Let

$$\mathbf{v}_1, \ \mathbf{v}_2, \ \mathbf{v}_3$$

be 3 vectors in $\mathbb{R}^3$.

Then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are linearly independent iff

$$\det(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \neq 0$$

Linear dependence with determinants

Example of linear dependence: The vectors

$$\mathbf{u} = (1, 2, -3), \quad \mathbf{v} = (3, 1, -2) \quad \text{and} \quad \mathbf{w} = (5, -5, 6)$$

are such that

$$\det(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = 0$$

Example of linear independence: The vectors

$$\mathbf{u} = (1, 2, -3), \quad \mathbf{v} = (3, 1, -2) \quad \text{and} \quad \mathbf{w} = (5, -5, 0)$$

are such that

$$\det(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = 30 \neq 0$$

Outline

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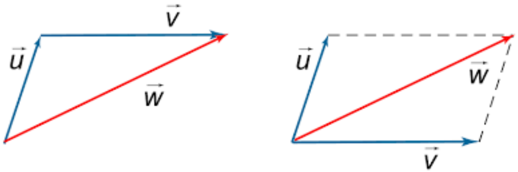

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4. Existence of a zero vector: There exists $\mathbf{0} \in \mathbb{R}^n$ such that
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   If $\mathbf{u} \in \mathbb{R}^n$ and $k \in \mathbb{R}$, then $k\,\mathbf{u} \in \mathbb{R}^n$
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10. Unit property: For all $\mathbf{v} \in \mathbb{R}^n$, we have $1\,\mathbf{v} = \mathbf{v}$

Examples of vector spaces

Examples:

- Scalar vector spaces: $\mathbb{R}$ or $\mathbb{C}$
- Vectors as usual: $\mathbb{R}^n$ or $\mathbb{C}^n$
- 2×2 matrices (check assumptions 1–10)

Notations for most common vector spaces:

- $M_{m\times n}(\mathbb{R}) \equiv$ vector space of $m \times n$ real matrices
- $M_n(\mathbb{R}) \equiv$ vector space of square $n \times n$ real matrices
- $\mathbb{P}_n \equiv$ space of polynomials of degree at most n (check)
- $C(I) \equiv$ continuous functions on an interval I

Subspace

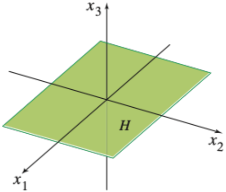
Definition 6.

Let S such that

- S is a nonempty subset of a vector space V

Then

$$S \text{ is a subspace}$$
$$\Longleftrightarrow$$
$$\mathbf{0} \in S,\ S \text{ is closed under addition and scalar multiplication}$$



Example of subspace (1)

Homogeneous linear system:

In $\mathbb{R}^3$, the set S of solutions of system (1) is a subspace.

$$\begin{array}{rrrl} x_1 & +2x_2 & -x_3 & = 0 \\ 2x_1 & +5x_2 & -4x_3 & = 0 \end{array} \tag{1}$$

Example of subspace (2)

Proof: The row-echelon form of the system (1) is

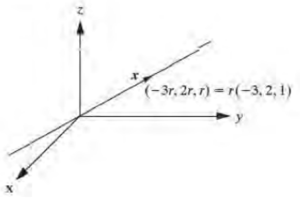
$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 \end{bmatrix}$$

Thus the set of solutions is

$$S = \left\{ \mathbf{x} \in \mathbb{R}^3; \, \mathbf{x} = (-3r, 2r, r), \text{ where } r \in \mathbb{R} \right\}.$$

One then proves stability by $+$ and scalar $\times$

Geometric interpretation: S is a line (intersection of 2 planes) in $\mathbb{R}^3$



Counter-example of subspace (1)

Line which does not pass through $\mathbf{0}$: The set

$$S = \left\{ \mathbf{x} \in \mathbb{R}^2;\ \mathbf{x} = (r, -3r+1), \text{ where } r \in \mathbb{R} \right\}.$$

is **not** a subspace of $\mathbb{R}^2$

Counter-example of subspace (2)

Proof that S is not a subspace: We have

$$\mathbf{x} \equiv (0, 1) \in S, \qquad \mathbf{y} \equiv (1, -2) \in S,$$

but

$$\mathbf{x} + \mathbf{y} = (1, -1) \notin S$$

Other examples of subspaces

Examples:

1. In $M_n(\mathbb{R})$
   $\hookrightarrow$ The set $S = \{A \in M_n(\mathbb{R});\ A^T = A\}$ is a subspace

2. In $C([a,b])$
   $\hookrightarrow$ The set $S = \{f \in C([a,b]);\ f(a) = 0\}$ is a subspace

3. In $C([a,b])$
   $\hookrightarrow$ The set $C^k([a,b])$ is a subspace

Subspace spanned by a set

Definition 7.

Let $\mathbf{v}_1, \ldots, \mathbf{v}_p \in V$. We define

$$\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\} \equiv \text{Set of linear combinations of } \mathbf{v}_1, \ldots, \mathbf{v}_p$$

Theorem 8.

Let $\mathbf{v}_1, \ldots, \mathbf{v}_p \in V$. Then

$$\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\} \text{ is a subspace of } V$$

Homogeneous linear systems

Theorem 9.

Let

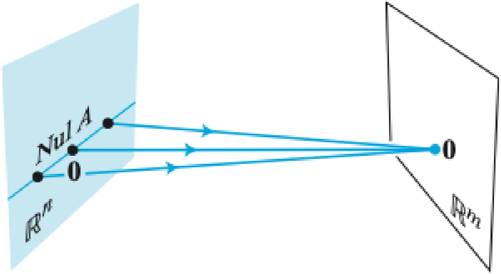
- A be a $m \times n$ matrix
- System $A\mathbf{x} = \mathbf{0}$
- $S \equiv$ set of solutions of $A\mathbf{x} = \mathbf{0}$

Then:

1. S is a subspace of $\mathbb{R}^n$.
2. S is called null space of A or solution space of A
3. Notation: $\mathrm{Null}(A)$.

Nullspace (2)

Illustration: if $A : \mathbb{R}^n \to \mathbb{R}^m$



Example of Nullspace

Matrix:

$$A^{\sharp} = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 & 0 \\ 1 & -2 & 2 & 3 & -1 & 0 \\ 2 & -4 & 5 & 8 & -4 & 0 \end{bmatrix}$$

We wish to describe the set of $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{0}$

Reduced echelon form:

$$A^{\sharp} \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Example of Nullspace (2)

Reduced echelon form:

$$A^\sharp \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Free variables: x_2, x_4 and x_5

Description of Nullspace:

$$\mathrm{Null}(A) = \{r\mathbf{u} + s\mathbf{v} + t\mathbf{w};\ r, s, t \in \mathbb{R}\} = \mathrm{Span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\},$$

with

$$\mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{w} = \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

Outline

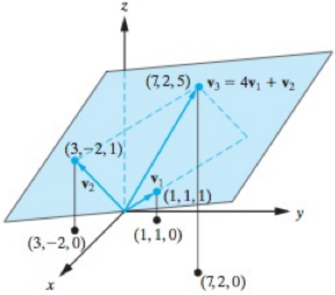
Aim of the current and following sections

Basic questions:

- There are many different spanning sets for a space or subspace
- Is there a *best* choice among those spanning sets?
- Is there a minimal number for the cardinal of a spanning set?

Concept to answer those questions:

$\hookrightarrow$ Linear dependence



Definition

Definition 10.

Let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be a family of vectors in V. Then

1. If there exist $c_1, \ldots, c_k$ not all zero such that
$$c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k = 0,$$
we say that $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly dependent
2. If we have
$$c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k = 0 \quad \Longrightarrow \quad c_1 = c_2 = \cdots = c_k = 0,$$
we say that $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly independent

Examples

Simple examples:

- The family $\{\mathbf{v}\}$ is linearly dependent iff $\mathbf{v} = \mathbf{0}$
- The family $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent iff $\mathbf{v}_2 = c\,\mathbf{v}_1$
- If $\mathbf{0}$ is an element of $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$
  $\hookrightarrow$ then $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly dependent

Example in $\mathbb{R}^3$

Family of vectors: We consider

$$\mathbf{v}_1 = (1, 2, -1), \quad \mathbf{v}_2 = (2, -1, 1), \quad \mathbf{v}_3 = (8, 1, 1)$$

Then $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent

Proof: The system

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$$

can be written as:

$$\begin{array}{llll} c_1 & +2c_2 & +8c_3 & = 0 \\ 2c_1 & -c_2 & +c_3 & = 0 \\ -c_1 & +c_2 & +c_3 & = 0 \end{array}$$

Example in $\mathbb{R}^3$ (2)

Proof (ctd): System in row-echelon form

$$\begin{bmatrix} 1 & 2 & 8 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

One row is $\mathbf{0}$, so that we have linear dependence

Explicit linear dependence: We solve for $\mathbf{c}$

$$c_3 = t, \quad c_2 = -3t, \quad c_1 = -2t$$

Then choosing (arbitrary choice) $t = 1$ we get

$$-2\mathbf{v}_1 - 3\mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$$

Example with polynomials (1)

Family in $\mathbb{P}_1$: Consider

$$\mathbf{p}_1(t) = 1, \qquad \mathbf{p}_2(t) = t, \qquad \mathbf{p}_3(t) = 4 - t$$

Linear dependence: We have that

$$\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\} \text{ are linearly dependent}$$

Example with polynomials (2)

Proof of the linear dependence: We have

$$\mathbf{p}_3 = 4\mathbf{p}_1 - \mathbf{p}_2$$

Thus

$$\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\} \text{ are linearly dependent}$$

Independence of n vectors in $\mathbb{R}^n$

Theorem 11.

Let

- $\mathbf{v}_1, \ldots, \mathbf{v}_n$ vectors in $\mathbb{R}^n$
- Form the matrix $A = [\mathbf{v}_1, \cdots, \mathbf{v}_n]$

Then

$$\mathbf{v}_1, \ldots, \mathbf{v}_n \text{ linearly independent iff } \det(A) \neq 0$$

Example in $\mathbb{R}^3$ reloaded

Family of vectors: We consider

$$\mathbf{v}_1 = (1, 2, -1), \quad \mathbf{v}_2 = (2, -1, 1), \quad \mathbf{v}_3 = (8, 1, 1)$$

Then $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent

Example in $\mathbb{R}^3$ reloaded (2)

Determinant: We have

$$\det(A) = \begin{vmatrix} 1 & 2 & 8 \\ 2 & -1 & 1 \\ -1 & 1 & 1 \end{vmatrix} = 0$$

Conclusion:

$\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent

Outline

Outline

Definition of basis

Definition 12.

Let

- V vector space
- $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be a family of vectors in V.

The family $\mathcal{B}$ is called a basis if

(a) The vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are linearly independent

(b) The vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ span V

Canonical basis of $\mathbb{R}^3$

Claim: Let

$$\mathbf{e}_1 = (1, 0, 0), \quad \mathbf{e}_2 = (0, 1, 0), \quad \mathbf{e}_3 = (0, 0, 1)$$

Then $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ forms a basis of $\mathbb{R}^3$.

Canonical basis of $\mathbb{R}^3$ (2)

Proof of (a): We have

$$\det(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = 1.$$

Hence

$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are linearly independent

Canonical basis of $\mathbb{R}^3$ (3)

Proof of (b): If $\mathbf{v} = (v_1, v_2, v_3)$, then

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3$$

Hence

$$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \text{ span } V$$

Generalization: One can easily find canonical bases for

- $\mathbb{R}^n$
- $M_{m,n}(\mathbb{R})$
- $\mathbb{P}_n$

Spanning set theorem

Theorem 13.

Let

- $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ set in V
- $H = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$

Then

1. If v_k is a linear combination of the remaining vectors in S, we have
$$\text{Span}(S) = \text{Span}\left(\{\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}, \mathbf{v}_{k+1}, \ldots, \mathbf{v}_p\}\right)$$
2. If $H \neq \{\mathbf{0}\}$, some subset of S is a basis for H

Application (1)

Family S: In $\mathbb{R}^3$ we consider $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ with

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} 6 \\ 16 \\ -5 \end{bmatrix}$$

Problem:

Find a basis for H

Application (2)

Linear dependence: We have

$$\mathbf{v}_3 = 5\mathbf{v}_1 + 3\mathbf{v}_2$$

Conclusion: Since $\mathbf{v}_1, \mathbf{v}_2$ are independent

$$\{\mathbf{v}_1, \mathbf{v}_2\} \text{ is a basis for } H$$

Basis of a column space

Theorem 14.

Let A be a $m \times n$ matrix. Then

A basis of $\mathrm{Col}(A)$
is given by
set of column vectors of A corresponding to leading 1's
in any row-echelon form of A

Application of Theorem 14

Example: Consider

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 7 & 8 & 9 & 10 \end{bmatrix}$$

Row-echelon form of A: We have

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis for Col(A):

$$\{(1, 4, 7);\ (2, 5, 8)\}$$

Outline

Number of vectors in a basis

Theorem 15.

Let V be a vector space such that

- V has a basis $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ with n elements

Then

1. Any set of more than n vectors is linearly dependent
2. All bases of V have n vectors

Definition 16.

In the context of Theorem 15, we call

$$n \equiv \text{dimension of } V$$

Application of Theorem 15

Vector space: $\mathbb{R}^3$

Canonical basis:

$$\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} = \{(1,0,0); (0,1,0); (0,0,1)\}$$

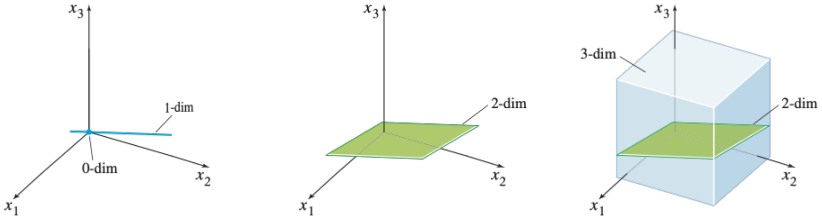
Dimension: $= 3$

Examples of linearly dependent family in $\mathbb{R}^3$:
Any family of 4 or more vectors, such as

$$\{(1,2,3); (1,7,4); (1,-1,5); (2,-1,5)\}$$

Subspaces of $\mathbb{R}^3$ and their dimensions

Typical examples of 0,1 and 2-dim subspaces of $\mathbb{R}^3$:



Application of Theorem 15 to polynomials

Vector space: $\mathbb{P}_2$

Canonical basis:

$$\mathcal{B} = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\} = \left\{1, t, t^2\right\}$$

Dimension: $= 3$

Examples of linearly dependent family in $\mathbb{R}^3$:
Any family of 4 or more polynomials, such as

$$\left\{1 + 2t + t^2; 1 + 7t + 4t^2; 1 - t + 5t^2; 2 - t + 5t^2\right\}$$

Simplified criterion for basis

Theorem 17.

Consider

- V a vector space with $\dim[V] = n$
- $\mathbf{v}_1, \ldots, \mathbf{v}_n$ a family of n linearly independent vectors in V

Then $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ is a basis of V

Application of Theorem 17

Claim: The family

$$\mathbf{p}_1(x) = 1 + x, \quad \mathbf{p}_2(x) = 2 - 2x + x^2, \quad \mathbf{p}_3(x) = 1 + x^2$$

is a basis for $\mathbb{P}_2$.

Application of Theorem 17 (2)

Proof of the claim: We know that $\dim[\mathbb{P}_2] = 3$. Moreover

$$\det([\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3]) = -3 \neq 0.$$

Thus $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ are linearly independent
$\hookrightarrow$ according to Theorem 17, $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is a basis for $\mathbb{P}_2$.

Outline

Homogeneous linear systems (repeated)

Theorem 18.

Let

- A be a $m \times n$ matrix
- System $A\mathbf{x} = \mathbf{0}$
- $S \equiv$ set of solutions of $A\mathbf{x} = \mathbf{0}$

Then:

1. S is a subspace of $\mathbb{R}^n$.
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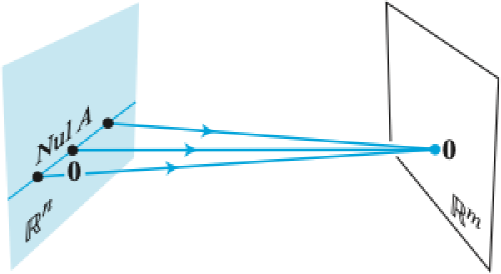
3. Notation: $\mathrm{Null}(A)$.

Nullspace (2)

Illustration: if $A : \mathbb{R}^n \to \mathbb{R}^m$



Example of Nullspace

Matrix:

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

We wish to describe the set of $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{0}$

Reduced echelon form:

$$A^{\sharp} \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Example of Nullspace (2)

Reduced echelon form:

$$A^\sharp \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Free variables: x_2, x_4 and x_5

Description of Nullspace:

$$\mathrm{Null}(A) = \{r\mathbf{u} + s\mathbf{v} + t\mathbf{w};\ r, s, t \in \mathbb{R}\} = \mathrm{Span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\},$$

with

$$\mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{w} = \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

Definition of column space

Definition 19.

Let A be a $m \times n$ matrix. The following holds true:

- The columns of A are vectors in $\mathbb{R}^m$

Then we define a subset of $\mathbb{R}^m$:

$$\mathrm{Col}(A) = \mathrm{Span}\,\{\text{Columns of } A\}\,.$$

Note: $\mathrm{Col}(A)$ is a subspace of $\mathbb{R}^m$

Example of column space

Example: Consider

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 7 & 8 & 9 & 10 \end{bmatrix}$$

Then

$$\mathrm{Col}(A) = \mathrm{Span}\left\{(1,4,7),\ (2,5,8),\ (3,6,9),\ (4,7,10)\right\}.$$

Comparison Null(A) vs Col(A)

Situation: Consider a $m \times n$ matrix

$$A \in \mathbb{R}^{m,n}$$

Facts about Null(A):

1. Null(A) subspace of $\mathbb{R}^n$
2. Null(A) obtained by nontrivial computations

Facts about Col(A):

1. Col(A) subspace of $\mathbb{R}^m$
2. Col(A) immediately obtained as a Span

Dimension of Null(A) and Col(A)

Proposition 20.

Let A be a $m \times n$ matrix. Then

1. $\dim(\mathrm{Null}(A))$ = number of free variables in eq. $A\mathbf{x} = \mathbf{0}$
2. $\dim(\mathrm{Col}(A))$ = number of pivot columns in A

Application of Proposition 20

Matrix: Consider

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 7 & 8 & 9 & 10 \end{bmatrix}$$

Row-echelon form of $A^\sharp$: We have

$$[A, \mathbf{0}] \sim \begin{bmatrix} 1 & 2 & 3 & 4 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Conclusion: We have

$$\dim(\mathrm{Null}(A)) = 2, \qquad \dim(\mathrm{Col}(A)) = 2$$

Definition of row space

Definition: Let A be a $m \times n$ matrix. The following holds true:

- The rows of A are vectors in $\mathbb{R}^n$

Then we set:

$$\mathrm{Row}(A) = \mathrm{Span}\left\{\text{Rows of } A\right\}.$$

Example: Consider

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 7 & 8 & 9 & 10 \end{bmatrix}$$

Then

$$\mathrm{Row}(A) = \mathrm{Span}\left\{(1,2,3,4),\ (4,5,6,7),\ (7,8,9,10)\right\}.$$

Basis of a row space

Theorem 21.

Let A be a $m \times n$ matrix. Then

A basis of $\text{Row}(A)$
is given by
set of nonzero row vectors in the row-echelon form of A

Application

Matrix: Consider

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 7 & 8 & 9 & 10 \end{bmatrix}$$

Row-echelon form of A: We find

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis for Row(A):

$$\{(1,2,3,4);\ (0,1,2,3)\}$$

Definition of column space (again)

Definition: Let A be a $m \times n$ matrix. The following holds true:

- The columns of A are vectors in $\mathbb{R}^m$

Then we set:

$$\mathrm{Col}(A) = \mathrm{Span}\left\{\text{Columns of } A\right\}.$$

Example: Consider

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 7 & 8 & 9 & 10 \end{bmatrix}$$

Then

$$\mathrm{Col}(A) = \mathrm{Span}\left\{(1, 4, 7),\ (2, 5, 8),\ (3, 6, 9),\ (4, 7, 10)\right\}.$$

Basis of a column space

Theorem 22.

Let A be a $m \times n$ matrix. Then

A basis of $\mathrm{Col}(A)$
is given by
set of column vectors of A corresponding to leading 1's
in any row-echelon form of A

Summary for bases of Col, Row, Null

Theorem 23.

Let A be a $m \times n$ matrix. Then

- Basis for $\mathrm{Col}(A)$:
  Set of column vectors corresponding to leading 1's in row-echelon form of A
- Basis for $\mathrm{Row}(A)$:
  Set of non 0 row vectors in row-echelon form of A
- Basis for $\mathrm{Null}(A)$:
  Solving $A\mathbf{x} = \mathbf{0}$

Summary for dimensions of Col, Row, Null

Theorem 24.

Let A be a $m \times n$ matrix. Then

- Dimension of $\mathrm{Col}(A)$:
  $\#$ of pivot columns in A
- Dimension of $\mathrm{Row}(A)$:
  $\#$ of pivot columns in A (same as $\dim(\mathrm{Col}(A))$)
- Dimension of $\mathrm{Null}(A)$:
  $\#$ of free variables in $A\mathbf{x} = \mathbf{0}$

Rank

Definition 25.

Let A be a $m \times n$ matrix. Then we define

$$\text{rank}(A) \equiv \dim(\text{Col}(A))$$

Application

Matrix: Consider

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 7 & 8 & 9 & 10 \end{bmatrix}$$

Row-echelon form of A: We have found

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis for Col(A):

$$\{(1, 4, 7);\ (2, 5, 8)\} \quad \Longrightarrow \quad \text{rank}(A) = 2$$

The rank theorem

Theorem 26.

Let A be a $m \times n$ matrix. Then we have

$$\operatorname{rank}(A) + \dim(\operatorname{Null}(A)) = n$$

Moreover

$$\operatorname{rank}(A) = \dim(\operatorname{Col}(A)) = \dim(\operatorname{Row}(A))$$

Application of Theorem 26

Matrix: Consider

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 7 & 8 & 9 & 10 \end{bmatrix}$$

Basis for Row(A):

$$\{(1, 2, 3, 4);\ (0, 1, 2, 3)\}$$

Basis for Col(A):

$$\{(1, 4, 7);\ (2, 5, 8)\}$$

Verification of Theorem 26: We have

$$\dim(\mathrm{Row}(A)) = \dim(\mathrm{Col}(A)) = 2 \equiv \mathrm{rank}(A)$$

Application of Theorem 26 (2)

Row-echelon form of A: We find

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Null(A): We have found

$$\dim(\text{Null}(A)) = 2$$

Verification of Theorem 26: Recall that $A \in \mathbb{R}^{3\times 4}$. We get

$$\text{rank}(A) + \dim(\text{Null}(A)) = 2 + 2 = 4$$