MIDTERM EXAM 1

VERSION 1

Name: ______________________ **Section:** __________

1. You must use a #2 pencil on the mark-sense sheet (answer sheet).

2. Write 01 in the TEST/QUIZ NUMBER boxes and blacken in the appropriate spaces below.

3. On the mark-sense sheet, fill in the instructor's name and the course number.

4. Fill in your NAME and 10 digits PURDUE ID NUMBER, and blacken in the appropriate spaces.

5. Fill in the SECTION NUMBER boxes, which is 064.

6. Sign the mark-sense sheet.

7. Fill in your name on the question sheet above.

8. There are 8 questions, each worth an equal amount of points. Blacken in your choice of the correct answer in the spaces provided for questions 1-8. Do all your work on the question sheets. Turn in both the mark-sense sheets and the question sheets when you are finished.

9. Show your work on the question sheets. Although no partial credit will be given, any disputes about grades or grading will be settled by examining your written work on the question sheets.

10. NO CALCULATORS, BOOKS, OR PAPERS ARE ALLOWED. Use the back of the test pages for scrap paper. Do not cheat. Everyone caught cheating will lose their exam and will be reported to the Dean of Students.

Date: October 6, 2016.

Problem 1. If $y = y(x)$ is the solution to

$$\frac{dy}{dx} = \frac{2x}{y + x^2 y}, \qquad y(0) = -2$$

then $y(1) =$

A. $[2\ln(2) + 4]^{\frac{1}{2}}$

B. $-[2\ln(4) + 1]^{\frac{1}{2}}$

C. $[3\ln(2) + 2]^{\frac{1}{3}}$

D. $-[2\ln(2) + 4]^{\frac{1}{2}}$

E. $[2\ln(3) + 2]^{\frac{1}{4}}$

Separable equation

$$y y' = \frac{2x}{1+x^2}$$

$$\Leftrightarrow \quad \frac{1}{2} y^2 = \ln(1+x^2) + c$$

If $y(0) = -2$ then $c = 2$ and

$$y^2 = 2\ln(1+x^2) + 4$$

$$y = -(2\ln(1+x^2) + 4)^{\frac{1}{2}}$$

($-$ determined with initial data)

$$y(1) = -(2\ln(2) + 4)^{\frac{1}{2}}$$

Problem 2. The solution of the equation

$$\underbrace{\exp(x)\sin(y) - 2y\sin(x)}_{M} + [\underbrace{\exp(x)\cos(y) + 2\cos(x)}_{N}]\, y' = 0$$

is given implicitly by

A. $\exp(x)\sin(y) + 2y\cos(x) = c$
B. $\exp(x)\cos(y) + 2y\sin(x) = c$
C. $2\exp(x)\cos(y) + y\sin(x) = c$
D. $\exp(x)\sin(y) + y\cos(x) = c$
E. $\exp(x)\cos(y) + y\sin(x) = c$

- We have

$$\partial_y M = e^x \cos y - 2\sin x = \partial_x N$$

$\Rightarrow$ exact equation

- $\psi(x,y) = \int M\,dx = e^x \sin(y) + 2y\cos x + h(y)$

Then

$$\partial_y \psi = N \Leftrightarrow e^x \cos y + 2\cos x + h'(y) = e^x \cos y + 2\cos x$$

$\Leftrightarrow h'(y) = 0$

Thus solution given by

$$e^x \sin y + 2y\cos x = c$$

Problem 3. We approximate the equation

$$y' = 3 + t - y, \qquad y(0) = 1$$

with Euler's method. If we choose a step size $h = .1$, then we obtain $(t_2, y_2) =$

A. (.2, 1.14)
B. (.1, 1.47)
C. (.2, 1.22)
D. (.2, 1.39)
E. (.2, 1.87)

Scheme: $y_{n+1} = (3 + 0.1n - y_n) \times 0.1 + y_n$

$y_1 = (3 - 1) \times 0.1 + 1 = 1.2$

$y_2 = (3 + 0.1 - 1.2) \times 0.1 + 1.2 = 1.39$

Problem 4. Consider the equation

$$y'' + 2y' + 2y = 0$$

The general solution is given by:

A. $[c_1 \cos(t) + c_2 \sin(t)]e^{t}$
B. $[c_1 \cos(2t) + c_2 \sin(2t)]e^{-2t}$
C. $[c_1 \cos(t) + c_2 \sin(t)]e^{-2t}$
D. $[c_1 \cos(2t) + c_2 \sin(2t)]e^{-t}$
E. $[c_1 \cos(t) + c_2 \sin(t)]e^{-t}$

Characteristic equation:

$r^2+2r+2=0$

Roots: $r_1 = -1+i$ $\quad r_2 = -1-i$

Problem 5. If $y = y(x)$ is the solution to

$$y'' + y' - 2y = 0, \qquad y(0) = 1, \quad y'(0) = 1$$

then $y(1) =$

A. $2e^{-2} + e$
B. e
C. $3e^{-2} - e$
D. $4e^{-2} + e$
E. $2e^{-2}$

$y_1 = e^{-2x}$ $\qquad y_2 = e^{x}$

$W[y_1, y_2](x) = \begin{vmatrix} e^{-2x} & e^{x} \\ -2e^{-2x} & e^{x} \end{vmatrix} = 3e^{-x}$

$C_1 = \frac{1}{3}\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$ $\qquad C_2 = \frac{1}{3}\begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} = 1$

$\Rightarrow \quad y = y_2 = e^{x}$

$y(1) = e$

Problem 6. Consider the equation:

$$2xy\,dy - (x^2 + 3y^2)\,dx = 0.$$

Its general solution is given by

A. $y^2 = cx^3 - x^2$
B. $y^3 = cx^2 - x^3$
C. $y^3 = cx^3 - x^2$
D. $y^2 = 2x^3 - cx^2$
E. $y^3 - 2x^3 = c\,x$

Eq $\Leftrightarrow$ $y' = \dfrac{x^2 + 3y^2}{2xy} = \dfrac{1 + 3(y/x)^2}{2\,y/x}$ CV: $y = vx$

We get $\dfrac{2vv'}{1+v^2} = \dfrac{1}{x}$ -> separable equation

$\Leftrightarrow$ $1 + v^2 = C_2\, x$

$\Leftrightarrow$ $\dfrac{y^2}{x^2} = C_2 x - 1$

$\Leftrightarrow$ $y^2 = C_2 x^3 - x^2$

Problem 7. A tank contains 2 m^3 of water and 20 g of salt. Water containing a salt concentration of 2 g of salt per m^3 of water flows into the tank at a rate of 2 m^3/min, and the mixture in the tank flows out at the same rate. We call $Q(t)$ the quantity of salt at time t in the tank. In order to have $Q(t) \leq 6$ g, we have to wait for

A. 6 mn

B. $2\ln(6)$ mn

C. $3\ln(2)$ mn

D. $2\ln(8)$ mn

E. $\ln(8)$ mn

$Q_{in} = C_{in} \cdot r = 4$

$Q_{out} = C_{out} \cdot r = 2 \frac{Q}{V} = Q$

Equation: $Q' = 4 - Q \quad , Q(0) = 2$

With integrating factor we find

$Q(t) = 4 + 16 e^{-t}$

Then $Q(t) \leq 6 \iff 16 e^{-t} \leq 2$

$\iff t \geq \ln(8)$

Note: C is also valid

Problem 8. Which of the following is the solution of

$$y' + y = 5\sin(2t), \qquad y(0) = 0.$$

A. $-2\cos(2t) + \sin(2t) + 2e^{-t}$
B. $2\cos(t) + \sin(t) - 2e^{-2t}$
C. $2\cos(2t) + 3\sin(2t) - 2e^{-t}$
D. $3\cos(2t) + 2\sin(2t) - 3e^{-t}$
E. $-2\cos(2t) - \sin(2t) + 2e^{-t}$

Integrating factor : $\mu = e^t$

Eq $\Leftrightarrow$ $(ye^t)' = 5e^t\sin(2t)$

A priori form: $\int 5e^t\sin(2t) = [a\cos(2t) + b\sin(2t)]e^t$

One finds $a = -2$, $b = 1$

Thus $\int 5e^t\sin(2t) = (-2\cos(2t) + \sin(2t))e^{-t}$

and $y = -2\cos(2t) + \sin(2t) + c_1 e^{-t}$

If $y(0) = 0$, then $c_1 = 2$ and

$y = -2\cos(2t) + \sin(2t) + 2e^{-t}$

Problem 9. Consider the differential equation:

$$\frac{dy}{dt} = (y-2)^2\,(y-1)\,(y-3)\,, \qquad -\infty < y_0 < \infty$$

The equilibrium points for the equation can be classified as

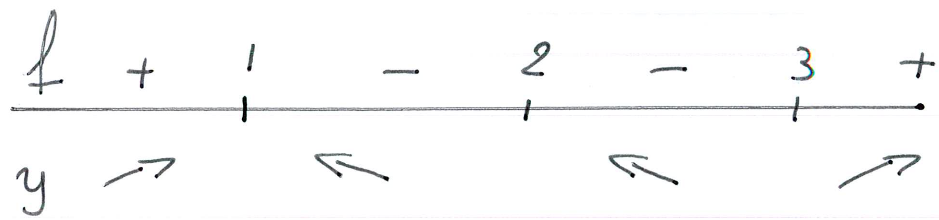
A. semi-stable: $1, 2$; stable: 3

B. stable: 1; unstable: 2

C. stable: 1; semi-stable: 2; unstable: 3

D. semi-stable: 1; unstable: 2; stable: 3

E. stable: $1, 3$; semi-stable: 2



Problem 10. Consider the following initial value problem:

$$\left(t^2 - 4t + 3\right) y' + ty = \ln(t), \qquad y(1.5) = \pi.$$

The maximal interval on which this equation admits a unique solution is

A. $(1, 3)$
B. $(1, \pi)$
C. $(0, 3)$
D. $(\frac{\pi}{2}, \pi)$
E. $(1.5, 3)$

$$Eq \Leftrightarrow \quad y' + \frac{t}{(t-1)(t-3)} y = \frac{\ln(t)}{(t-1)(t-3)}$$

$\Rightarrow$ Maximal interval around 1.5 is $(1,3)$