MA 26600
Fall 2016

Differential equations
Purdue University

MIDTERM EXAM 2

VERSION 1

Name: ______________________________ **Section:** __________

1. You must use a #2 pencil on the mark-sense sheet (answer sheet).

2. Write 01 in the TEST/QUIZ NUMBER boxes and blacken in the appropriate spaces below.

3. On the mark-sense sheet, fill in the instructor's name and the course number.

4. Fill in your NAME and 10 digits PURDUE ID NUMBER, and blacken in the appropriate spaces.

5. Fill in the SECTION NUMBER boxes, which is 064.

6. Sign the mark-sense sheet.

7. Fill in your name on the question sheet above.

8. There are 10 questions, each worth an equal amount of points. Blacken in your choice of the correct answer in the spaces provided for questions 1-8. Do all your work on the question sheets. Turn in both the mark-sense sheets and the question sheets when you are finished.

9. Show your work on the question sheets. Although no partial credit will be given, any disputes about grades or grading will be settled by examining your written work on the question sheets.

10. NO CALCULATORS, BOOKS, OR PAPERS ARE ALLOWED. Use the back of the test pages for scrap paper. Do not cheat. Everyone caught cheating will lose their exam and will be reported to the Dean of Students.

Date: November 17, 2016.

Problem 1. Consider the differential equation:

$$4y'' - 4y' + 5y = 0, \qquad y\left(\frac{\pi}{2}\right) = 0,\ y'\left(\frac{\pi}{2}\right) = 1$$

The unique solution is given by:

A. $-2e^{\frac{\pi}{4}}\cos(t)\,e^{\frac{t}{2}}$

B. $-e^{\frac{\pi}{2}}\sin(t)\,e^{t}$

C. $e^{-\frac{\pi}{4}}\cos(t)\,e^{\frac{t}{2}}$

D. $-e^{-\frac{\pi}{4}}\cos(t)\,e^{\frac{t}{2}}$

E. $\frac{1}{2}e^{-\frac{\pi}{4}}\cos(2t)\,e^{\frac{t}{2}}$

$Z(r) = 4r^2 - 4r + 5$ roots : $r = \frac{1}{2} \pm i$

Hence $y_1 = e^{\frac{t}{2}}\cos(t)$ $y_2 = e^{\frac{t}{2}}\sin(t)$

Wronskian :

$$W(t) = \begin{vmatrix} e^{\frac{t}{2}}\cos(t) & e^{\frac{t}{2}}\sin(t) \\ e^{\frac{t}{2}}\left(\frac{1}{2}\cos(t) - \sin(t)\right) & e^{\frac{t}{2}}\left(\cos(t) + \frac{1}{2}\sin(t)\right) \end{vmatrix}$$

$$W\left(\frac{\pi}{2}\right) = \begin{vmatrix} e^{\frac{\pi}{4}} & e^{\frac{\pi}{4}} \\ -e^{\frac{\pi}{4}} & \frac{1}{2}e^{\frac{\pi}{4}} \end{vmatrix} = e^{\frac{\pi}{2}}$$

Therefore $c_1 = e^{-\frac{\pi}{2}}\begin{vmatrix} 0 & e^{\frac{\pi}{4}} \\ 1 & \frac{1}{2}e^{\frac{\pi}{4}} \end{vmatrix} = -e^{-\frac{\pi}{4}}$ $c_2 = \begin{vmatrix} 0 & 0 \\ -1 & 1 \end{vmatrix} e^{-\frac{\pi}{4}} = 0$

We get $y = -e^{-\frac{\pi}{4}} e^{\frac{t}{2}} \cos(t)$ D

Problem 2. Consider the initial value problem parametrized by $\alpha \in \mathbb{R}$:

$$y'' - 4y' + 4y = 0, \qquad y(0) = \alpha,\ y'(0) = -1.$$

We have $\lim_{t\to\infty} y(t) = -\infty$ whenever α belongs to

A. $(+\frac{1}{2}, \infty)$
B. $(-\frac{1}{2}, \infty)$
C. $(-\infty, \frac{1}{2})$
D. $(1, \infty)$
E. $(-\infty, -1)$

$Z(r) = r^2 - 4r + 4 = (r-2)^2$

Hence $y_1 = e^{2t}$ $\quad y_2 = t e^{2t}$

Wronskian:

$$W(t) = \begin{vmatrix} e^{2t} & t e^{2t} \\ 2e^{2t} & (2t+1)e^{2t} \end{vmatrix}$$

$$W(0) = \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = 1$$

Therefore $c_1 = \begin{vmatrix} \alpha & 0 \\ -1 & 1 \end{vmatrix} = \alpha$

$c_2 = \begin{vmatrix} 1 & \alpha \\ 2 & -1 \end{vmatrix} = -(2\alpha + 1)$

We get $y = \alpha e^{2t} \overbrace{- (2\alpha+1) t e^{2t}}^{\text{dominant term}}$

and $\lim_{t\to\infty} y(t) = -\infty \iff 2\alpha + 1 > 0 \iff \alpha > -\frac{1}{2}$

B

Problem 3. Consider the following equation on $(0, \infty)$:

$$(x-1)y'' - 2xy' + (x+1)y = 0.$$

We know that one fundamental solution is given by $y_1 = e^x$. Then the second fundamental solution is

A. $y_2 = (x-1)^3\, e^x$
B. $y_2 = (x-1)^2\, e^x$
C. $y_2 = (x+1)^2\, e^x$
D. $y_2 = (x+1)^3\, e^x$
E. $y_2 = x\, e^x$

We set $y_2 = v e^x$. Then

$\times(x+1) \quad y_2 = v e^x$

$\times(-2x) \quad y_2' = v e^x + v' e^x$

$\times(x-1) \quad y_2'' = v e^x + 2v' e^x + v'' e^x$

Thus $(x+1)\, y_2 - 2x\, y_2' + (x-1)\, y_2''$

$= e^x \left((x-1) v'' - 2v' \right)$

and y_2 solves the equation iff $(x-1)v'' - 2v' = 0$

We set $w = v'$. Then $(x-1) w' = 2w$. A solution to this equation is $w = (x-1)^2$. Since $v = \int w$, we get $v = \frac{1}{3}(x-1)^3$ and we can choose

$y_2 = (x-1)^3\, e^x$ $\quad$ A

Problem 4. For the equation:

$$y^{(4)} - 3y^{(3)} + 2y^{(2)} = 4t - e^t + 3e^{3t}$$

the particular solution Y has the following form

A. $Y = at^2 + bt + cte^t + de^{3t}$

B. $Y = at^3 + ce^t + de^{3t}$

C. $Y = at^2 + bt + cte^t + de^{-3t}$

D. $Y = at^3 + bt^2 + cte^t + de^{3t}$

E. $Y = at^3 + bt^2 + ce^t + dte^{3t}$

$Z(r) = r^4 - 3r^3 + 2r^2 = r^2(r^2 - 3r + 2) = r^2(r-1)(r-2)$

Since 0 is a double root and 1 is a simple root, our guess for Y is D

Problem 5. The longest interval in which the following initial value problem:

$$(5-t)y'' + (t-4)y' + 2y = \ln(t), \qquad y(2) = 8,\ y'(2) = -8.$$

is certain to have a unique twice-differentiable solution is given by

A. $(0, \infty)$
B. $(2, 4)$
C. $(0, 4)$
D. $(0, 5)$
E. $(-\infty, 4)$

Write the equation under the form

$$y'' + \frac{t-4}{5-t} y' + \frac{2}{5-t} y = \frac{\ln(t)}{5-t}$$

The coefficients are continuous on

$$(0, 5) \cup (5, \infty),$$

and $2 \in (0,5)$.

The maximal interval of definition for y is $(0,5)$

D

Problem 6. We consider the following equation:

$$ty'' + 2ty' - 2y = 6t$$

The fundamental solutions of the corresponding homogeneous equation are $y_1 = t$ and $y_2 = \frac{1}{t^2}$. Then the particular solution of our equation is

A. $Y = \frac{4}{3}t^2$

B. $Y = \frac{3}{5}t^3$

C. $Y = -\frac{4}{3}t^2$

D. $Y = \frac{3}{4}t^3$

E. t^5

Use $Y = -y_1 \int \frac{g y_2}{W[y_1, y_2]} + y_2 \int \frac{g y_1}{W[y_1, y_2]}$

Here this method fails because $y = c y_1$.

Problem 7. A spring is stretched 8m by a force of $\frac{9}{2}$N. A mass of 2 kg is hung from the spring and is also attached to a viscous damper that exerts a force of 3N when the velocity of the mass is 2m/s. If the mass is given an initial downward velocity of 2m/s, then its position $u(t)$ is given by

A. $u(t) = \frac{16}{3}\sin\left(\frac{3t}{8}\right)e^{-\frac{3t}{8}}$

B. $u(t) = \frac{16}{3}\sin\left(\frac{3t}{8}\right)e^{-\frac{3t}{8}} - \frac{1}{3}\cos\left(\frac{3t}{8}\right)e^{-\frac{3t}{8}}$

C. $u(t) = \frac{4}{5}\sin\left(\frac{3t}{8}\right)e^{-\frac{t}{3}}$

D. $u(t) = -\frac{1}{5}\sin\left(\frac{2t}{3}\right)e^{-\frac{t}{8}}$

E. $u(t) = \frac{1}{4}\sin\left(\frac{t}{8}\right)e^{-\frac{3t}{4}}$

$k = \frac{F_s}{L} = \frac{9/2}{8} = \frac{9}{16}$ $\qquad m = 2$

$\gamma = \frac{F_d}{v} = \frac{3}{2}$

Eq: $mu'' + \gamma u' + ku = 0$

$2u'' + \frac{3}{2}u' + \frac{9}{16}u = 0 \qquad u(0) = 0 \qquad u'(0) = 2$

$Z(r) = 2r^2 + \frac{3}{2}r + \frac{9}{16} \qquad$ roots: $-\frac{3}{8} \pm i\frac{3}{8}$

Since $u(0) = 0$, we get

$u(t) = c\sin\left(\frac{3t}{8}\right)e^{-\frac{3t}{8}}$ $\qquad$ A

Problem 8. Which of the following functions has Laplace transform equal to

$$F(s) = \frac{2s^2 + 5s + 11}{(s+1)(s^2+2s+5)}$$

A. $f(t) = 3e^{-t} + \frac{1}{2}e^{-2t}\sin(t)$

B. $f(t) = 2e^{-t} + \frac{1}{2}e^{-t}\sin(2t)$

C. $f(t) = e^{3t} + \frac{1}{3}e^{-t}\cos(2t)$

D. $f(t) = 3e^{-t} + 2e^{-3t}\sin(t)$

E. $f(t) = \frac{1}{4}e^{t} + e^{t}\cos(3t)$

The factor s^2+2s+5 has complex roots. We can write

$$F(s) = \frac{2}{s+1} + \frac{1}{s^2+2s+5}$$

$$= \frac{2}{s+1} + \frac{1}{2}\,\frac{2}{(s+1)^2+4}$$

and

$$f(t) = 2e^{-t} + \frac{1}{2}e^{-t}\sin(2t)$$

B

Problem 9. Let y be the solution to the initial value problem

$$y'' + 2y' + y = \delta(t-3), \qquad y(0) = 0, \quad y'(0) = 0.$$

What is $y(4)$?

A. $-e^2$
B. $-e$
C. e^{-1}
D. e^{-2}
E. e

$Y = \mathcal{L}(y)$ satisfies

$$(s^2+2s+1)Y = e^{-3s}$$

$$Y = \frac{e^{-3s}}{(s+1)^2}$$

Hence

$$y = u_3(t)\, f(t-3)$$

with $f(t) := \mathcal{L}^{-1}\left(\frac{1}{(s+1)^2}\right) = t e^{-t}$

We get

$$y(t) = u_3(t)\,(t-3)\, e^{-(t-3)}$$

$$\Rightarrow y(4) = e^{-1}$$

C

Problem 10. Let y be the solution to the following differential equation

$$y'' + 9y = u_\pi(t)\,\sin(t-\pi), \qquad y(0)=0, \quad y'(0)=0.$$

The solution y is given by

A. $u_\pi(t)\left[\frac{1}{4}\sin(5(t-\pi)) - \frac{1}{12}\sin(3(t-\pi))\right]$

B. $u_\pi(t)\left[\sin(t-\pi) + \sin(3(t-\pi))\right]$

C. $u_\pi(t)\left[\frac{1}{2}\cos(t-\pi) - \frac{1}{8}\sin(t-\pi)\right]$

D. $u_\pi(t)\left[\frac{1}{16}\sin(t-\pi) + \frac{1}{24}\cos(3(t-\pi))\right]$

E. $u_\pi(t)\left[\frac{1}{8}\sin(t-\pi) - \frac{1}{24}\sin(3(t-\pi))\right]$

The function Y satisfies

$$(s^2+9)Y = e^{-\pi s}\frac{1}{s^2+1} \iff Y = e^{-\pi s}\frac{1}{(s^2+1)(s^2+9)}$$

we get

$$Y = e^{-\pi s}\left(\frac{1}{8}\frac{1}{s^2+1} - \frac{1}{8}\frac{1}{s^2+9}\right)$$

$$= e^{-\pi s}\left(\frac{1}{8}\frac{1}{s^2+1} - \frac{1}{24}\frac{3}{s^2+9}\right)$$

This yields solution E