Square Function Estimates for Singular Integrals and Applications to Partial Differential Equations

Svetlana Mayboroda and Marius Mitrea*

Abstract

The purpose of the present paper is to continue the program of study of elliptic boundary value problems on Lipschitz domains with boundary data in quasi-Banach Besov spaces $B_s^{p,p}(\partial\Omega)$, initiated in [**13**]. Introducing a modified square function which is well-adapted for handling data with a fractional amount of smoothness, we establish the well-posedness of the Dirichlet and Neumann boundary problems for the Laplacian in Lipschitz domains, for a range of indices which includes values of p less than 1. An important ingredient in this regard is establishing suitable square-function estimates for singular integral of potential type.

1 Introduction

Let Ω denote a bounded domain in $\mathbb{R}^n$ with connected boundary locally representable as the graph of a Lipschitz function (referred to as "Lipschitz domain" in the sequel). The goal is to study the Dirichlet boundary-value problem for the Laplacian

$$\begin{cases} \Delta u = 0 \text{ in } \Omega, \\ u\Big|_{\partial\Omega} = f \end{cases} \tag{1.1}$$

with a special emphasis on the well-posedness of the problem at hand for data in specific smoothness spaces.

In 1995 D. Jerison and C. Kenig [**11**] undertook a thorough study of (1.1) on Sobolev-Besov scales. One of their main results, proved by means of subtle estimates for the harmonic measure associated with Ω, is that

$$\|u\|_{B^{p,p}_{s+\frac{1}{p}}(\Omega)} \leq C(\partial\Omega, p, s)\|f\|_{B_s^{p,p}(\partial\Omega)}, \tag{1.2}$$

provided the integrability exponent p, and the order of differentiability s, satisfy one of the following sets of conditions:

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$$\tfrac{2}{1+\varepsilon} < p < \tfrac{2}{1-\varepsilon} \text{ and } 0 < s < 1; \quad 1 \le p < \tfrac{2}{1+\varepsilon} \quad \text{and} \quad \tfrac{2}{p} - 1 - \varepsilon < s < 1;$$
$$\tfrac{2}{1-\varepsilon} < p \le \infty \text{ and } 0 < s < \tfrac{2}{p} + \varepsilon, \tag{1.3}$$

where $\varepsilon = \varepsilon(\Omega) > 0$. While more detailed definitions are to be given later, let us mention now that throughout the paper $B_s^{p,p}$ will denote the Besov class of functions.

There are important antecedents of these results in the literature. For boundary data in $L^p(\partial\Omega)$ and $L_1^p(\partial\Omega)$, B. Dahlberg, C. Kenig and G. Verchota ([**6**], [**8**], [**22**]) have used the non-tangential maximal operator to measure the size/smoothness of solutions. The latter is defined as

$$\mathcal{N}(u)(x) := \|u\|_{L^\infty(\gamma(x))}, \qquad x \in \partial\Omega, \tag{1.4}$$

where

$$\gamma(x) := \{y \in \Omega;\ |x - y| < \kappa \, \mathrm{dist}\,(y, \partial\Omega)\}, \tag{1.5}$$

with $\kappa = \kappa(\Omega) > 1$ fixed, is a cone-like non-tangential approach region with vertex at x. More precisely, they have derived the sharp estimates

$$\|\mathcal{N}u\|_{L^p(\partial\Omega)} \le C(\partial\Omega, p)\|f\|_{L^p(\partial\Omega)}, \qquad 2 - \varepsilon < p < \infty, \tag{1.6}$$
$$\|\mathcal{N}(\nabla u)\|_{L^p(\partial\Omega)} \le C(\partial\Omega, p)\|f\|_{L_1^p(\partial\Omega)}, \qquad 1 < p < 2 + \varepsilon, \tag{1.7}$$

where $\varepsilon = \varepsilon(\partial\Omega) > 0$.

Another result of capital importance is equivalence of the non-tangential maximal operator and the classical area integral

$$\mathcal{S}(u)(x) := \left(\int_{\gamma(x)} \mathrm{dist}\,(y, \partial\Omega)^{2-n} |\nabla u(y)|^2 \, dy \right)^{\frac{1}{2}}, \qquad x \in \partial\Omega, \tag{1.8}$$

often referred to as the *square-function* of u in the literature, for harmonic functions in Ω. More specifically,

$$\|\mathcal{N}u\|_{L^p(\partial\Omega)} \approx \|\mathcal{S}u\|_{L^p(\partial\Omega)}, \qquad 0 < p < \infty, \tag{1.9}$$

uniformly for u harmonic in Ω (the "$\le$" inequality requires that u is appropriately normalized). This striking theorem, due to B. Dahlberg [**7**], extends the ground-breaking work of D. Burkholder, R. Gundy and M. Silverstein [**3**], [**2**], and C. Fefferman and E. Stein [**9**], in the case when $\Omega = \mathbb{R}^n_+$. In particular,

$$\|\mathcal{S}u\|_{L^p(\partial\Omega)} \le C\|f\|_{L^p(\partial\Omega)}, \qquad 2 - \varepsilon < p < \infty, \tag{1.10}$$
$$\|\mathcal{S}(\nabla u)\|_{L^p(\partial\Omega)} \le C\|f\|_{L_1^p(\partial\Omega)}, \qquad 1 < p < 2 + \varepsilon, \tag{1.11}$$

for solutions of (1.1).

Let us point out that although the range (1.3) is sharp in the class of Lipschitz domains if one insists that $1 \leq p \leq \infty$, there are suitable versions of (1.2) which continue to hold for certain pairs of indices (s,p) with $p<1$. In our earlier work **[13]**, we have produced such an extension of the main result in **[11]** based on a modification of the non-tangential maximal operator (1.4), i.e.

$$\mathcal{N}_s^q(u)(x) := \|u\|_{B^{q,q}_{s+\frac{n}{q}}(\gamma(x))}, \qquad x \in \partial\Omega. \tag{1.12}$$

One attractive feature of this approach is its flexibility. For example, this allowed us to consider boundary data from certain Besov spaces with $p<1$ and to treat the case of Neumann boundary conditions. Furthermore, this is, in principle, applicable to the case of systems as well.

The goal of the present paper is to continue this work by examining the role of the square-function in this context. The main question which we address here is that of the effectiveness of objects like (1.8) for measuring the size/smoothness of solutions of (1.1) for $f \in B^{p,p}_s(\partial\Omega)$.

We are able to carry out this program by introducing an appropriate analogue of the area integral (1.8). Specifically, for each $0 \leq s \leq 1$ we define the *modified square-function*

$$\mathcal{S}_s^q(u)(x) := \left(\int_{\gamma(x)} \delta(y)^{q-n-qs} |\nabla u(y)|^q \, dy\right)^{\frac{1}{q}}, \qquad x \in \partial\Omega, \; 0<q<\infty, \tag{1.13}$$

where δ denotes the distance to the boundary. Note that this definition agrees with that of the classical square function in the case $s=0$, $q=2$. As customary, the appropriate version for $q=\infty$ is given by

$$\mathcal{S}_s^\infty(u)(x) := \sup_{y \in \gamma(x)} \delta(y)^{1-s} |\nabla u(y)|, \qquad x \in \partial\Omega. \tag{1.14}$$

Anticipating notation to be introduced later, our main result reads as follows:

Theorem 1.1 *For a bounded Lipschitz domain Ω in $\mathbb{R}^n$, consider the following Dirichlet boundary-value problem:*

$$\begin{cases} \Delta u = 0 \;\; in \; \Omega, \\ u\Big|_{\partial\Omega} = f \in B^{p,p}_s(\partial\Omega), \\ \mathcal{S}_s^q(u) \in L^p(\partial\Omega), \end{cases} \tag{1.15}$$

where

$$\tfrac{n-1}{n} < p \leq q \leq \infty, \quad (n-1)(\tfrac{1}{p}-1)_+ < s < 1. \tag{1.16}$$

Then there exists $\varepsilon = \varepsilon(\Omega) > 0$ such that (1.15) is well-posed whenever the indices satisfy (1.16) and one of the following conditions:

$$\tfrac{2}{1+\varepsilon} < p < \tfrac{2}{1-\varepsilon} \ \textit{and}\ 0 < s < 1; \quad \tfrac{2}{2+\varepsilon} < p < \tfrac{2}{1+\varepsilon} \quad \textit{and} \quad \tfrac{2}{p} - 1 - \varepsilon < s < 1;$$
$$\tfrac{2}{1-\varepsilon} < p \leq \infty \ \textit{and}\ 0 < s < \tfrac{2}{p} + \varepsilon. \tag{1.17}$$

The solution has the integral representation formula

$$u = \mathcal{D}[(\tfrac{1}{2}I + K)^{-1} f] \quad \textit{in } \Omega, \tag{1.18}$$

and satisfies the estimate

$$\|\mathcal{S}_s^q(u)\|_{L^p(\partial\Omega)} \leq C(\partial\Omega, p, q, s)\|f\|_{B_s^{p,p}(\partial\Omega)}. \tag{1.19}$$

A similar well-posedness result is valid for data in the Sobolev space $L_s^p(\partial\Omega)$ for a suitable range of indices; cf. the discussion in §5.

Here and elsewhere, $\mathcal{D}$ denotes the double layer potential operator and K stands for its (principal-value) boundary version; cf. the discussion in §2.

As it will be shown later, the fact that $\mathcal{S}_s^q(u) \in L^p(\partial\Omega)$ entails the existence of the nontangential trace

$$u\Big|_{\partial\Omega}(x) = \lim_{y \in \gamma(x)} u(y) \tag{1.20}$$

at almost every boundary point x. Since, heuristically, multiplying by δ^α amounts to (fractional) integration of order α, our main estimate, (1.19), can be viewed as the natural intermediate analogue of (1.10) and (1.11).

In the proof of Theorem 1.1, the following square-function estimate for singular integral operators (patented after the harmonic double layer (2.14)) plays a crucial role.

Theorem 1.2 *Let Ω be a Lipschitz domain in $\mathbb{R}^n$ and consider the integral operator*

$$Tf(x) = \int_{\partial\Omega} k(x,y) f(y) d\sigma_y, \qquad x \in \Omega, \tag{1.21}$$

whose kernel satisfies

$$|\nabla_x k(x,y)| \leq C|x-y|^{-n}, \qquad x \in \Omega,\ y \in \partial\Omega. \tag{1.22}$$

Also, assume that

$$T1 = const. \tag{1.23}$$

Then

$$\|\mathcal{S}_s^q(Tf)\|_{L^p(\partial\Omega)} \leq C\|f\|_{B_s^{p,p}(\partial\Omega)} \tag{1.24}$$

provided

$$\tfrac{n-1}{n} < p \leq q \leq \infty, \qquad (n-1)(\tfrac{1}{p} - 1)_+ < s < 1. \tag{1.25}$$

In turn, the proof of the above theorem, occupying the bulk of Section 4, rests on atomic estimates and interpolation.

It should be noted that our methods are flexible enough to allow us to treat an analogue of Theorem 1.1 for the Neumann boundary-value problem; the details are presented in §6. Let us also point out that an appropriate analogue of the classical g-function can also be used to measure size/smoothness in the present context. We briefly elaborate on this point in the last part of §5. Future plans include dealing with the case of the Lamé system of elastostatics and with parabolic PDE's in a similar setting.

Throughout the paper, $A \approx B$ signifies that the quotient A/B is bounded away from zero and infinity, by finite, positive constants which are *independent* of the relevant parameters in A, B. Also, if $D \subset \mathbb{R}^n$ is a closed set, we denote by δ_D the distance function to D.

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2 Background

In this section we would like to sketch the basic prerequisites, mostly concerning function spaces and interpolation. Let us start by recalling the definition of the Besov class.

Assume that $s \in \mathbb{R}$ and $0 < q \leq \infty$, $0 < p \leq \infty$. Then the Besov scale is

$$B_s^{p,q}(\mathbb{R}^n) := \Big\{ f \in \mathcal{S}'; \quad \|f\|_{B_s^{p,q}(\mathbb{R}^n)} := \Big(\sum_{j=0}^{\infty} \|2^{sj} \mathcal{F}^{-1}(\phi_j \mathcal{F} f)\|_{L^p(\mathbb{R}^n)}^q \Big)^{1/q} < \infty \Big\}, \tag{2.1}$$

where $\mathcal{F}$ denotes the Fourier transform, $\mathcal{S}'$ stands for the class of tempered distributions in $\mathbb{R}^n$ and $\{\phi_j\}_{j=0}^{\infty}$ is a suitable dyadic partition of unity (cf. [**18**]). As is well-known, the special case $p = q = \infty$, $0 < s < 1$ corresponds to the classical space of Hölder continuous functions.

In the sequel, we shall also make use of the Besov spaces defined on the Lipschitz domain Ω itself. Those are understood as restrictions to Ω of distributions from $B_s^{p,q}(\mathbb{R}^n)$. For more information and references we refer to [**18**], [**20**], [**1**], [**16**], [**11**].

The spaces (2.1) – with $n-1$ in place of n – can then be naturally transported from $\mathbb{R}^{n-1}$ to the boundary of a Lipschitz domain $\Omega \subset \mathbb{R}^n$, yielding $B_s^{p,q}(\partial\Omega)$, via a partition of unity and pull-back (i.e., the method of local charts – cf. [**21**]). The applicability of this procedure is restricted to $\frac{n-1}{n} < p, q \leq \infty$ and $(n-1)(\frac{1}{p} - 1)_+ < s < 1$, where $(a)_+ := \max\{a, 0\}$.

The Besov spaces $B_s^{p,p}(\partial\Omega)$ admit various atomic decompositions, much as in the case of their Euclidean counterparts, discussed in [**10**]. Here we highlight a case of paramount importance for the work at hand. Assume that Ω is the unbounded domain in $\mathbb{R}^n$ lying above the graph of a real-valued Lipschitz function.

Call $S_r(x) := B_r(x) \cap \partial\Omega$ a *surface ball* if $x \in \partial\Omega$ and $r > 0$. Also, fix $\frac{n-1}{n} < p < \infty$ and $(n-1)(\frac{1}{p} - 1)_+ < s < 1$. A function $a : \partial\Omega \to \mathbb{R}$ is called a $B_s^{p,p}(\partial\Omega)$-atom if

$$\exists\, S_r - \text{surface ball with } supp\,(a) \subseteq S_r, \quad r^{-1}\|a\|_{L^\infty(\partial\Omega)} + \|\nabla_{tan} a\|_{L^\infty(\partial\Omega)} \le r^{s-\frac{n-1}{p}-1}. \quad (2.2)$$

Hereafter, if ν is the outward unit normal to $\partial\Omega$ (defined a.e. with respect to the surface measure $d\sigma$ on $\partial\Omega$), we let $\nabla_{tan} := \nabla - \nu\partial_\nu$ denote the tangential gradient on $\partial\Omega$. Then

$$\|f\|_{B^{p,p}_s(\partial\Omega)} \approx \inf\Big\{ \Big(\sum_j |\mu_j|^p\Big)^{1/p};\ f = \sum_j \mu_j a_j,\ a_j \text{ are } B^{p,p}_s(\partial\Omega) \text{ atoms}, \{\mu_j\}_j \in \ell^p \Big\}. \quad (2.3)$$

Similarly, there are atomic decompositions for $B^{p,p}_{-s}(\partial\Omega)$, though the concept of atom changes. Concretely, let us assume that $(n-1)/n < p < \infty$ and $(n-1)(\frac{1}{p}-1)_+ < 1-s < 1$, and call $a : \partial\Omega \to \mathbb{R}$ an atom for $B^{p,p}_{-s}(\partial\Omega)$ if

$$\exists\, S_r - \text{surface ball with } supp\,(a) \subseteq S_r,\ \|a\|_{L^\infty(\partial\Omega)} \le r^{-s-\frac{n-1}{p}},\ \int_{\partial\Omega} a\, d\sigma = 0. \quad (2.4)$$

Then the Euclidean results from **[10]** lifted to $\partial\Omega$ give

$$\|f\|_{B^{p,p}_{-s}(\partial\Omega)} \approx \inf\Big\{ \Big(\sum_j |\mu_j|^p\Big)^{1/p};\ f = \sum_j \mu_j a_j,\ a_j \text{ are } B^{p,p}_{-s}(\partial\Omega) \text{ atoms}, \{\mu_j\}_j \in \ell^p \Big\}. \quad (2.5)$$

As is customary, Sobolev spaces on a Lipschitz domain $\Omega \subset \mathbb{R}^n$ are defined by

$$L^p_s(\Omega) := \Big\{ (I-\Delta)^{s/2} f|_\Omega;\ f \in L^p(\mathbb{R}^n) \Big\} \quad (2.6)$$

for $1 < p < \infty$, $s \ge 0$. Once again, their boundary counterparts, $L^p_s(\partial\Omega)$, can be naturally transported from $\mathbb{R}^{n-1}$ via a partition of unity and pull-back. In particular, $L^p_1(\partial\Omega)$ corresponds to the Sobolev class of functions from $L^p(\partial\Omega)$ whose tangential gradients belong to $L^p(\partial\Omega)$.

We continue our review by listing a number of interpolation results. Hereafter $(\cdot,\cdot)_{\theta,q}$ and $[\cdot,\cdot]_\theta$ will refer to interpolation by the real and the complex method, respectively.

First, recall that the following formulae hold:

$$[L^p, L^p_1]_\theta = L^p_\theta, \quad 0<\theta<1,\ 1<p<\infty, \quad (2.7)$$

$$(L^p, L^p_1)_{\theta,q} = B^{p,q}_\theta, \quad 0<\theta<1,\ 1<p,q<\infty. \quad (2.8)$$

Next,

$$[L^{p_0}_{s_0}, L^{p_1}_{s_1}]_\theta = L^{p^*}_{s^*}, \quad 0<\theta<1,\ 1<p_0,\, p_1<\infty,\ s_0 \neq s_1 \quad (2.9)$$

$$[B^{p_0,p_0}_s, B^{p_1,p_1}_s]_\theta = B^{p^*,p^*}_s, \quad 0<\theta<1,\ 0<p_0,p_1<\infty, \quad (2.10)$$

with $s^* = (1-\theta)s_0 + \theta s_1$, $\frac{1}{p^*} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. The details on the assertions (2.7)-(2.9) can be found in [**1**], while (2.10) is a special case of the interpolation theorem for Triebel-Lizorkin spaces in [**15**]. Being established on $\mathbb{R}^n$, the appropriate versions of (2.7)-(2.10) continue to hold on the boundary of a Lipschitz domain.

As far as vector-valued L^p spaces are concerned, we state the following result. Assume that A_0 and A_1 are Banach spaces and that $1 \leq p_0, p_1 < \infty$, $0 < \theta < 1$. Then

$$[L^{p_0}(A_0), L^{p_1}(A_1)]_\theta = L^{p^*}([A_0, A_1]_\theta), \tag{2.11}$$

where $\frac{1}{p^*} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.

Recall that by $L^p(A) = L^p(\partial\Omega; A)$ we denote the space of all strongly measurable functions f such that

$$\int_{\partial\Omega} \|f(x)\|_A^p \, d\sigma(x) < +\infty, \tag{2.12}$$

for $1 \leq p < \infty$ where, as before, $d\sigma$ denotes the canonical surface measure on $\partial\Omega$.

Going further, we record the criteria of membership to the Besov spaces for harmonic functions. Fix $\Omega \subset \mathbb{R}^n$ Lipschitz and hereafter set $\delta := \delta_{\partial\Omega}$ for the distance to $\partial\Omega$.

Theorem 2.1 *(cf.* [**11**]*) Suppose u is a function defined in the Lipschitz domain Ω. For $0 < \alpha < 1$, a nonnegative integer k, and $1 \leq p \leq \infty$, consider the following statements:*

(i) u belongs to $B^{p,p}_{k+\alpha}(\Omega)$;

(ii) $\delta^{1-\alpha}|\nabla^{k+1}u| + |\nabla^k u| + |u|$ belongs to $L^p(\Omega)$.

Then (ii) $\Rightarrow$ (i) for any function u. Moreover, the converse implication is also true – so that (i) $\Leftrightarrow$ (ii) – if u is harmonic in Ω.

We conclude this section by recalling the single and double layer potential operators defined by

$$\mathcal{S}f(x) := \int_{\partial\Omega} E(x-y)\, f(y)\, d\sigma_y, \qquad x \in \Omega, \tag{2.13}$$

and

$$\mathcal{D}f(x) := \int_{\partial\Omega} \partial_{\nu(y)}[E(x-y)]\, f(y)\, d\sigma_y, \qquad x \in \Omega, \tag{2.14}$$

respectively. Here $E(x)$ stands for the canonical radial fundamental solution for the Laplacian in $\mathbb{R}^n$, $n \geq 2$, and ν is the outward unit normal to $\partial\Omega$. As is well-known, $\mathcal{D}1 = 1$.

Regarding the principal-value operator

$$Kf(x) := \text{p.v.} \int_{\partial\Omega} \partial_{\nu(y)}[E(x-y)]\, f(y)\, d\sigma_y, \qquad x \in \partial\Omega, \tag{2.15}$$

with adjoint K^*, it is well-known that

$$\partial_\nu \mathcal{S}\Big|_{\partial\Omega} = -\tfrac{1}{2}I + K^* \quad \text{and} \quad \mathcal{D}\Big|_{\partial\Omega} = \tfrac{1}{2}I + K. \tag{2.16}$$

We refer the reader to **[8]**, **[22]** for more details, as well as relevant references, on these matters.

3 Preliminary results

Our first result amounts to saying that the membership of the modified square function to L^p does not depend on the particular choice of the family of non-tangential cones.

Lemma 3.1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and consider two families of non-tangential approach regions*

$$\{\gamma(x)\}_{x\in\partial\Omega}, \quad \{\gamma'(x)\}_{x\in\partial\Omega}, \text{ such that } \gamma'(x) \subseteq \gamma(x),\ \forall\, x \in \partial\Omega. \tag{3.1}$$

These are defined as in (1.5) corresponding to two choices of the parameter κ.

Next, denote by $\mathcal{S}^q_{s,\gamma}$, $\mathcal{S}^q_{s,\gamma'}$ the modified square-function (1.13) associated with (3.1), and fix $0 < p \leq q \leq \infty$, $0 \leq s \leq 1$. Then, for each $u \in C^1_{loc}(\Omega)$,

$$\|\mathcal{S}^q_{s,\gamma'}(u)\|_{L^p(\partial\Omega)} \approx \|\mathcal{S}^q_{s,\gamma}(u)\|_{L^p(\partial\Omega)}, \tag{3.2}$$

with constants depending only on $p, q, \partial\Omega$, and the geometric characteristics of the families (3.1).

Proof. This follows along the lines of similar results discussed in §3 of **[4]**, or §3 of **[12]**, with minor alterations. □

Proposition 3.2 *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and fix $\mathcal{K}$ a compact subset of Ω. Then for every harmonic function u in Ω,*

$$\|\mathcal{S}^q_s(u)\|_{L^p(\partial\Omega)} + \sup_{x\in\mathcal{K}} |u(x)| \approx \|\mathcal{N}^q_s(u)\|_{L^p(\partial\Omega)}, \tag{3.3}$$

if, in addition to (1.25), $q > \frac{n}{1-s}$. Also,

$$\|\mathcal{S}^\infty_0(u)\|_{L^p(\partial\Omega)} \leq C\|\mathcal{N}(u)\|_{L^p(\partial\Omega)}, \quad 1 < p < \infty. \tag{3.4}$$

Proof. Recall the Besov-based non-tangential maximal function (1.12). Thanks to Theorem 2.1, for each $q > \frac{n}{1-s}$ and u harmonic in Ω, we may write

$$\begin{aligned} \mathcal{N}^q_s(u)(x) &= \|u\|_{B^{q,q}_{s+\frac{n}{q}}(\gamma(x))} \\ &\approx \left(\int_{\gamma(x)} \mathrm{dist}(y, \partial\gamma(x))^{q-qs-n} |\nabla u(y)|^q\, dy\right)^{1/q} + \sup_{x\in\mathcal{K}} |u(x)|, \\ &\leq C\, \mathcal{S}^q_s(u)(x) + \sup_{x\in\mathcal{K}} |u(x)|, \end{aligned} \tag{3.5}$$

uniformly for $x \in \partial\Omega$. Here we have used the fact that $\operatorname{dist}(y, \partial\gamma(x)) \leq C\delta(y)$ for every $y \in \gamma(x)$.

Turning to the opposite inequality, choose a second family of cones γ' so that (3.1) holds. In particular,

$$\operatorname{dist}(y, \partial\gamma'(x)) \leq C\delta(y) \leq C \operatorname{dist}(y, \partial\gamma(x)), \tag{3.6}$$

uniformly for $x \in \partial\Omega$ and $y \in \gamma'(x)$. Going further, for every $x \in \partial\Omega$, we have

$$\begin{aligned} \mathcal{S}^q_{s,\gamma'}(u)(x) &= \left(\int_{\gamma'(x)} \delta(y)^{q-qs-n} |\nabla u(y)|^q \, dy\right)^{1/q} \\ &\leq C \left(\int_{\gamma(x)} \operatorname{dist}(y, \partial\gamma(x))^{q-qs-n} |\nabla u(y)|^q \, dy\right)^{1/q} \leq C\mathcal{N}^q_s(u)(x), \end{aligned} \tag{3.7}$$

thanks to our assumption on q, s and Theorem 2.1. With this in hand, Lemma 3.1 may now be used to finish the proof of (3.3).

Concerning the inequality (3.4), standard interior estimates for harmonic functions allows us to write, at every point $x \in \partial\Omega$,

$$\mathcal{S}^\infty_{0,\gamma'}(u)(x) = \sup_{y\in\gamma'(x)} \delta(y)|\nabla u(y)| \leq C \sup_{y\in\gamma'(x)} \left(\frac{\delta(y)}{\delta_{\partial\gamma}(y)} \sup_{z\in B(y,c\delta_{\partial\gamma}(y))} |u(z)|\right), \tag{3.8}$$

where the constant $c = c(\gamma, \gamma') < 1$ is chosen so that the ball $B(y, c\delta_{\partial\gamma}(y)) \subset \gamma(x)$ for every $y \in \gamma'(x)$. Then the last expression above does not exceed

$$C \sup_{y\in\gamma(x)} |u(y)| = C\mathcal{N}(u)(x), \tag{3.9}$$

as desired. □

Remark. It is useful to point out here that, for any function $u \in C^1(\Omega)$,

$$\mathcal{S}^\infty_1(u) = \mathcal{N}(\nabla u), \tag{3.10}$$

which follows by unraveling definitions.

As dictated by the representation formula (1.18), at the heart of our strategy for treating the problem (1.15) is establishing the estimate (1.19) when $u = \mathcal{D}f$. This, among other things, is the object of §4 below.

4 Square-function estimates on Besov scales

We debut with the

Proof of Theorem 1.2. First, assume $p = q$ so that

$$\|\mathcal{S}_s^q(Tf)\|_{L^q(\partial\Omega)}^q = \int_{\partial\Omega}\int_{\Omega} \chi_{\gamma(x)}(y)\, \delta(y)^{q-n-qs} |\nabla Tf(y)|^q \, dy d\sigma_x. \tag{4.1}$$

Since

$$\int_{\partial\Omega} \chi_{\gamma(x)}(y)\, d\sigma_x \approx \delta(y)^{n-1}, \tag{4.2}$$

uniformly for $y \in \Omega$, the expression on the right side of (4.1) is further equivalent to

$$\int_{\Omega} \delta(y)^{q-1-qs} |\nabla Tf(y)|^q \, dy. \tag{4.3}$$

Let us now recall the estimate

$$\|\delta^{1-s-\frac{1}{q}} |\nabla Tf|\,\|_{L^q(\Omega)} \leq C_{s,q} \|f\|_{B_s^{q,q}(\partial\Omega)}, \quad \tfrac{n-1}{n} < q \leq \infty,\ (n-1)(\tfrac{1}{q}-1)_+ < s < 1, \tag{4.4}$$

established in Theorem 3.1 of **[14]**, which is valid for any integral operator satisfying (1.22) and (1.23). Then the desired estimate (i.e. (1.24) with $p = q$) follows readily from (4.1)-(4.3). This completes the proof for $p = q$.

Turning our attention to the case when $\frac{n-1}{n} < p \leq \min\{1, q\}$ and $(n-1)(\frac{1}{p}-1)_+ < s < 1$, we utilize the atomic characterization of Besov spaces given in (2.3); thus, the goal is to analyze the action of the operator T on an individual atom.

Suppose a is a $B_s^{p,p}(\partial\Omega)$-atom with $\operatorname{supp} a \subseteq S_r(x^*)$, for some $x^* \in \partial\Omega$ and $r > 0$. We first seek a pointwise estimate on $\mathcal{S}_s^q(Tf)$ away from $\operatorname{supp} a$. Concretely, assume that $x \notin S_{10r}(x^*)$, then

$$|\nabla Ta(y)| \leq \int_{z \in S_r(x^*)} |\nabla_y k(y,z)||a(z)|\, d\sigma_z \leq C \frac{r^{s-\frac{n-1}{p}+n-1}}{|x^*-y|^n}, \tag{4.5}$$

for each $y \in \gamma(x)$, owing to the size condition imposed on atom a, and a basic geometrical observation to the effect that $|x^*-y| \approx |z-y|$ for $y \in \gamma(x)$, $x \notin S_{10r}(x^*)$ and $z \in S_r(x^*)$. Consequently, for $x \notin S_{10r}(x^*)$,

$$\mathcal{S}_s^q(Tf)(x) \leq C\, r^{s-\frac{n-1}{p}+n-1} \left(\int_{\gamma(x)} \delta(y)^{q-n-qs} \frac{1}{|y-x^*|^{qn}}\, dy \right)^{\frac{1}{q}}. \tag{4.6}$$

By locally flattening $\partial\Omega$ to $\mathbb{R}^{n-1}$ via a bi-Lipschitz change of coordinates – so that $x^* \in \partial\Omega$ corresponds to $0 \in \mathbb{R}^{n-1}$ – we see that the right side of (4.6) is bounded by

$$C r^{s-\frac{n-1}{p}+n-1} \left(\int_0^\infty \int_{\substack{|y'-x| \leq \kappa t \\ y' \in \mathbb{R}^{n-1}}} \frac{t^{q-n-qs}}{(|y'|+t)^{qn}}\, dy' dt \right)^{\frac{1}{q}}, \tag{4.7}$$

where C depends only on the Lipschitz character of Ω. In fact, since $|x| + t \leq C(|y'| + t)$ uniformly on the domain of integration, the last integral can be controlled further in terms of

$$\int_0^\infty \frac{t^{q-n-qs+n-1}}{(|x|+t)^{qn}}\,dt. \tag{4.8}$$

Introducing $t = \tau|x|$ permits us to bound (4.8) by

$$C\frac{1}{|x|^{qn+qs-q}} \int_0^\infty \frac{\tau^{q-qs-1}}{(1+\tau)^{qn}}\,d\tau \leq C\frac{1}{|x|^{qn+qs-q}}. \tag{4.9}$$

Therefore,

$$\mathcal{S}^q_s(Tf)(x) \leq Cr^{s-\frac{n-1}{p}+n-1}\frac{1}{|x-x^*|^{n+s-1}}, \qquad \forall\, x \notin S_{10r}(x^*). \tag{4.10}$$

In particular,

$$\int_{\partial\Omega\setminus S_{10r}(x^*)} |\mathcal{S}^q_s(Tf)(x)|^p\,d\sigma_x \leq C\,r^{p(s-\frac{n-1}{p}+n-1)} \int_{\partial\Omega\setminus S_{10r}(x^*)} \frac{d\sigma_x}{|x-x^*|^{p(n+s-1)}} \leq C \tag{4.11}$$

for some finite constant C, independent of the atom.

We next examine the contribution from $x \in S_{10r}(x^*)$. In this scenario, we employ a rescaling technique allowing us to invoke the estimates obtained in the case $p = q$. More specifically, Hölder's inequality gives

$$\int_{S_{10r}(x^*)} |\mathcal{S}^q_s(Ta)(x)|^p\,d\sigma_x \leq C_p\,r^{(n-1)(1-\frac{p}{q})} \left(\int_{S_{10r}(x^*)} |\mathcal{S}^q_s(Ta)(x)|^q\,d\sigma_x\right)^{\frac{p}{q}} \tag{4.12}$$

for every $0 < p \leq q$. Now let us define $\tilde{a} := r^{(n-1)(\frac{1}{p}-\frac{1}{q})}\,a$, which is easily seen to be a $B^{q,q}_s(\partial\Omega)$-atom. With this piece of notation, the right side of (4.12) simply reads

$$\left(\int_{S_{10r}(x^*)} |\mathcal{S}^q_s(T\tilde{a})(x)|^q\,d\sigma_x\right)^{\frac{p}{q}}. \tag{4.13}$$

In view of the results established for $p = q$, the last integral is bounded by the constant independent of the atom $\tilde{a} \in B^{q,q}_s(\partial\Omega)$, as desired.

Thus, at this stage,

$$\int_{\partial\Omega} |\mathcal{S}^q_s(Ta)(x)|^p\,d\sigma_x \leq C < +\infty \tag{4.14}$$

for some constant $C > 0$, independent of the atom $a \in B^{p,p}_s(\partial\Omega)$, proving (1.24) for $(n-1)/n < p \leq \min\{1, q\}$ and $(n-1)(1/p-1) < s < 1$. This completes the argument provided $q \leq 1$. Concerning the situation $q \geq 1$, the full range of indices (1.25) is covered by interpolation with the case $p = q$ (treated earlier) via the complex method. □

Corollary 4.1 *Let Ω be a Lipschitz domain in $\mathbb{R}^n$. Then for each triplet of indices s, p, q satisfying (1.25) there exists a finite constant $C = C(\Omega, s, p, q) > 0$ with the following significance.*

Whenever $u \in B^{p,p}_{s+\frac{1}{p}}(\Omega)$ is harmonic in Ω,

$$\|\mathcal{S}^q_s(u)\|_{L^p(\partial\Omega)} \leq C\|u\|_{B^{p,p}_{s+\frac{1}{p}}(\Omega)}. \tag{4.15}$$

Sketch of Proof. The key steps are as follows. First, it is possible to represent u in the form $u = \mathcal{C}(\mathrm{Tr}\, u)$ where $\mathrm{Tr}\, u \in B^{p,p}_s(\partial\Omega)$ is the trace of u on $\partial\Omega$, and $\mathcal{C}$ is a Cauchy-type singular integral operator whose properties closely mirror those of the harmonic double layer (cf. §4 in [**13**] for details in the case when $p > 1$). Then (4.15) follows from the mapping properties of $\mathcal{C}$ established in Theorem 1.2, in concert with the estimate $\|\mathrm{Tr}\, u\|_{B^{p,p}_s(\partial\Omega)} \leq C\|u\|_{B^{p,p}_{s+\frac{1}{p}}(\Omega)}$, valid for the range of indices considered here. □

Later on, when discussing the Neumann problem, it will be useful to know the mapping properties of a class of singular integrals whose kernels are patented after that of the harmonic single layer. Our main result in this regard is as follows.

Theorem 4.2 *Let Ω be a Lipschitz domain in $\mathbb{R}^n$. Consider the integral operator*

$$Rf(x) = \int_{\partial\Omega} r(x, y) f(y)\, d\sigma_y, \qquad x \in \Omega, \tag{4.16}$$

such that

$$|\nabla_x \nabla^j_y r(x, y)| \leq C|x - y|^{-(n-1+j)}, \quad j = 0, 1,\ x \in \Omega,\ y \in \partial\Omega. \tag{4.17}$$

Then

$$\|\mathcal{S}^q_{1-s}(Rf)\|_{L^p(\partial\Omega)} \leq C\|f\|_{B^{p,p}_{-s}(\partial\Omega)}, \tag{4.18}$$

whenever

$$\tfrac{n-1}{n} < p \leq q \leq \infty, \quad (n-1)(\tfrac{1}{p} - 1)_+ < 1 - s < 1. \tag{4.19}$$

Proof. We use the same general strategy as in the proof of Theorem 1.2, although certain key technical details are different. To get started, consider the case $p = q$; much as in (4.1)-(4.3), with $1 - s$ in place of s, we obtain

$$\|\mathcal{S}^q_{1-s}(Rf)\|^q_{L^q(\partial\Omega)} \approx \int_\Omega \delta(y)^{-1+qs}|\nabla Rf(y)|^q\, dy. \tag{4.20}$$

In concert with the estimate

$$\|\delta^{s-\frac{1}{q}}|\nabla Rf|\,\|_{L^q(\partial\Omega)} \leq C_{s,q}\|f\|_{B^{q,q}_{-s}(\partial\Omega)}, \quad \tfrac{n-1}{n} < q \leq \infty,\ (n-1)(\tfrac{1}{q} - 1)_+ < 1 - s < 1, \tag{4.21}$$

established in [**14**] under the assumptions (4.16)-(4.17), this readily yields (4.18) for $p = 2$.

As far as the range $\frac{n-1}{n} < p \leq \min\{1, q\}$ is concerned, matters can once again be reduced to studying the action of operator R on $B^{p,p}_{-s}(\partial\Omega)$-atom a supported in $S_r(x^*)$, $x^* \in \partial\Omega$. Away from the support, for every $x \notin S_{10r}(x^*)$ and every $y \in \gamma(x)$ one computes

$$|\nabla Ra(y)| = \Big|\int_{S_r(x^*)} [\nabla_y r(y,z) - \nabla_y r(y,x^*)] a(z)\, d\sigma_z\Big|, \tag{4.22}$$

thanks to the vanishing moment condition imposed on $B^{p,p}_{-s}(\partial\Omega)$-atom a. To continue, we estimate the expression in the brackets:

$$\int_0^1 \frac{d}{d\theta}\nabla_y r(y, \theta z + (1-\theta)x^*)\, d\theta \leq |z - x^*| \max_{\theta\in(0,1)} |\nabla_y \nabla_z r(y, \theta z + (1-\theta)x^*)|. \tag{4.23}$$

In the current scenario, $|y - \theta z - (1-\theta)x^*| \approx |y - x^*|$, $|z - x^*| \leq r$ and, therefore, (4.22) is dominated by

$$C\frac{r^{1-s+(n-1)(1-\frac{1}{p})}}{|y-x^*|^n}. \tag{4.24}$$

With this in hand, the contribution corresponding to integrating away from the support of the atom can be controlled as in (4.5)-(4.11) in the proof of Theorem 1.2, with $1-s$ in place of s.

Finally, the contribution near the support of the atom turns out to be bounded similarly to (4.12)-(4.14) by rescaling the original atom to a $B^{q,q}_{-s}$-atom, i.e. $\tilde{a} := r^{(n-1)(\frac{1}{p}-\frac{1}{q})}a$, and then invoking the $p = q$ results. □

5 Square-function estimates on Sobolev spaces

We first record and prove a general interpolation result.

Lemma 5.1 *Let Ω be a Lipschitz domain in $\mathbb{R}^n$ and assume that $0 \leq s \leq 1$, $1 < p < \infty$, $1 \leq q \leq \infty$. Then for every linear operator T mapping functions defined on $\partial\Omega$ to functions in $C^1(\Omega)$, the three-dimensional region*

$$\mathcal{O} := \{(s, 1/p, 1/q) \in [0,1]\times(0,1)\times[0,1];\ \mathcal{S}^q_s(Tf) : L^p_s(\partial\Omega) \to L^p(\partial\Omega)\} \tag{5.1}$$

is geometrically convex.

Proof. We shall only deal with the model case when Ω is the (unbounded) domain lying above the graph of a real-valued Lipschitz function; the simple adaptations to the bounded case are left to the interested reader.

Assume that the points $(s_0, 1/p_0, 1/q_0)$ and $(s_1, 1/p_1, 1/q_1)$ belong to $\mathcal{O}$. For $\Re e\, z \in [0,1]$ let us introduce

$$(\mathcal{L}_z f)(x)(y) := \delta(y+x)^{1-\left(\frac{n}{q_0}(1-z)+\frac{n}{q_1}z\right)-\left(s_0(1-z)+s_1 z\right)}\nabla Tf(y+x), \quad x \in \partial\Omega,\ y \in \Gamma, \tag{5.2}$$

where Γ is a fixed cone in $\mathbb{R}^n$ such that $\gamma(x) = x + \Gamma$ for each $x \in \partial\Omega$. Then

$$\mathcal{S}_s^q(T(\cdot)) : L_s^p(\partial\Omega) \longrightarrow L^p(\partial\Omega) \tag{5.3}$$

if and only if

$$\mathcal{L}_z : L_s^p(\partial\Omega) \longrightarrow L^p(\partial\Omega; L^q(\Gamma)). \tag{5.4}$$

Observing that $\mathcal{L}_z$ represents an analytic family of operators allows us to invoke a suitable version of Stein's interpolation theorem (cf. [**5**]) along with the complex interpolation formulae for vector-valued spaces (2.11) to complete the proof. □

We proceed further by restricting our attention to the case of the harmonic double layer.

Proposition 5.2 *Assume that Ω is a Lipschitz domain in $\mathbb{R}^n$. Then*

$$\|\mathcal{S}_s^q(\mathcal{D}f)\|_{L^p(\partial\Omega)} \leq C\|f\|_{L_s^p(\partial\Omega)} \tag{5.5}$$

provided one of the following conditions is satisfied:

$$\begin{aligned} &(i) \quad 1 < p < \infty, \ \max\left\{p, \frac{p}{p-1}\right\} < q \leq \infty \ \textit{ and } 0 < s < 1; \\ &(ii) \quad 2 \leq q \leq p < \infty \ \textit{ and } 0 < s < 1 + \frac{2}{p} - \frac{2}{q}; \\ &(iii) \quad 1 < p \leq 2, \ 2 \leq q \leq \frac{p}{p-1} \ \textit{ and } 0 < s < 3 - \frac{2}{p} - \frac{2}{q}. \end{aligned} \tag{5.6}$$

Proof. The proof is based on the classical estimates

$$\begin{aligned} \|\mathcal{S}_0^2(\mathcal{D}f)\|_{L^p(\partial\Omega)} &\leq C\|f\|_{L^p(\partial\Omega)}, \quad 1 < p < \infty, \\ \|\mathcal{N}(\mathcal{D}f)\|_{L^p(\partial\Omega)} &\leq C\|f\|_{L^p(\partial\Omega)}, \quad 1 < p < \infty, \\ \|\mathcal{N}(\nabla\mathcal{D}f)\|_{L^p(\partial\Omega)} &\leq C\|f\|_{L_1^p(\partial\Omega)}, \quad 1 < p < \infty, \end{aligned} \tag{5.7}$$

used in concert with (3.4), (3.10), and the inequality

$$\|\mathcal{S}_s^q(\mathcal{D}(f))\|_{L^q(\partial\Omega)} \leq C\|f\|_{L_s^q(\partial\Omega)}, \quad 0 < s < 1, \ \frac{n-1}{n} < q \leq \infty, \tag{5.8}$$

corresponding to (1.24) with $p = q$. The full range of indices described in the statement of the proposition is then obtained by invoking Lemma 5.1. □

To state our next result, set $a \wedge b := \min\{a, b\}$.

Corollary 5.3 *Let Ω be a Lipschitz domain in $\mathbb{R}^n$. Then*

$$\|\mathcal{S}_s^2(\mathcal{D}f)\|_{L^p(\partial\Omega)} \leq C\|f\|_{B_s^{p,p\wedge 2}(\partial\Omega)} \tag{5.9}$$

whenever

$$\tfrac{n-1}{n} < p < \infty, \qquad (n-1)(\tfrac{1}{p}-1)_+ < s < \min\{1, \tfrac{2}{p}\}. \tag{5.10}$$

Proof. The case $\frac{n-1}{n} < p \leq 2$ corresponds precisely to Theorem 1.2, whereas the case $2 < p < \infty$ is covered by Proposition 5.2, thanks to the classical embedding $B_s^{p,2}(\partial\Omega) \hookrightarrow L_s^p(\partial\Omega)$ for $0 < s < 1$, $2 \leq p < \infty$. □

We conclude this section by stating the analogue of Proposition 5.2 and Corollary 5.3 in the context of the harmonic single layer. The proofs of these results require only minor alterations compared to the proof presented in the case of the double layer, and are left to the interested reader.

Proposition 5.4 *Assume that* Ω *is a Lipschitz domain in* $\mathbb{R}^n$*. Then*

$$\|\mathbb{S}^q_{1-s}(\mathcal{S}f)\|_{L^p(\partial\Omega)} \leq C\|f\|_{L^p_{-s}(\partial\Omega)} \tag{5.11}$$

provided one of the following conditions is satisfied:

$$\begin{aligned} &(i) \quad 1 < p < \infty, \max\left\{p, \frac{p}{p-1}\right\} < q \leq \infty \ \text{ and } 0 < s < 1; \\ &(ii) \quad 2 \leq q \leq p < \infty \ \text{ and } 0 < 1-s < 1 + \frac{2}{p} - \frac{2}{q}; \\ &(iii) \quad 1 < p \leq 2,\ 2 \leq q \leq \frac{p}{p-1} \ \text{ and } 0 < 1-s < 3 - \frac{2}{p} - \frac{2}{q}. \end{aligned} \tag{5.12}$$

Corollary 5.5 *Let* Ω *be a Lipschitz domain in* $\mathbb{R}^n$*. Then*

$$\|\mathbb{S}^2_{1-s}(\mathcal{S}f)\|_{L^p(\partial\Omega)} \leq C\|f\|_{B^{p,p\wedge 2}_{-s}(\partial\Omega)} \tag{5.13}$$

whenever

$$\tfrac{n-1}{n} < p < \infty, \qquad (n-1)(\tfrac{1}{p}-1)_+ < 1-s < \min\{1, \tfrac{2}{p}\}. \tag{5.14}$$

Remark. There is also a natural modification of the classical Littlewood-Paley g-function. In the case of an unbounded domain $\Omega \subset \mathbb{R}^n$, lying above the graph of a Lipschitz function $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$, this is given by

$$g^q_s(u)(x) := \left(\int_0^\infty t^{q-1-qs}|\nabla u(x+te_n)|^q\, dt\right)^{1/q}, \quad x \in \partial\Omega. \tag{5.15}$$

The appropriate variant for the case of a bounded Lipschitz domain can be defined as

$$g^q_s(u)(x) := \left(\int_0^\eta t^{q-1-qs}|\nabla u(x+t\theta(x))|^q\, dt\right)^{1/q}, \tag{5.16}$$

where θ is some unit transversal field to $\partial\Omega$ and $\eta > 0$ is a fixed parameter, depending on the geometric characteristics of Ω.

Many of the results developed in this section have suitable versions in terms of (5.15)-(5.16). In particular, the Theorems 1.2 and 4.2 have natural analogues in this context. One result we would like to single out is as follows. For every harmonic function u defined on Lipschitz domain Ω, we have

$$\|g^q_s(u)\|_{L^p(\partial\Omega)} \leq C\|\mathcal{S}^q_s(u)\|_{L^p(\partial\Omega)}, \quad \text{if } \ 0 < p, q < \infty, \tag{5.17}$$

$$\|g^q_s(u)\|_{L^p(\partial\Omega)} \geq C\|\mathcal{S}^q_s(u)\|_{L^p(\partial\Omega)}, \quad \text{if } \ 1 < q \leq p < \infty. \tag{5.18}$$

We leave the details to the interested reader (cf. **[19]** p. 90-91 and **[12]** p. 154-155 for a related discussion).

6 Applications to partial differential equations

We are now ready to present the last details in the

Proof of Theorem 1.1. To start, fix two appropriate families of nontangential cones, γ, γ', as in (3.1) and set

$$\dot{\mathcal{N}}^{\infty}_{s,\gamma'}(u)(x) := \sup\Big\{\tfrac{|u(z_1)-u(z_2)|}{|z_1-z_2|^s};\, z_1, z_2 \in \gamma'(x)\Big\}, \quad x \in \partial\Omega. \tag{6.1}$$

In this context, we claim that there exists $C = C(q, \partial\Omega) > 0$ so that for every point $x \in \partial\Omega$ and every harmonic function u in Ω,

$$\dot{\mathcal{N}}^{\infty}_{s,\gamma'}(u)(x) \leq C\,\mathcal{S}^q_s(u)(x), \qquad \frac{n-1}{n} < q \leq \infty. \tag{6.2}$$

Indeed, by (a slight variation of) Theorem 2.1 and the Mean Value Formula for harmonic functions,

$$\begin{aligned}\dot{\mathcal{N}}^{\infty}_{s,\gamma'}(u)(x) &\leq C \sup_{z\in\gamma'(x)} \operatorname{dist}(z, \partial\gamma'(x))^{1-s}|\nabla u(z)| \\ &\leq C \sup_{z\in\gamma'(x)} \Big(\operatorname{dist}(z, \partial\gamma(x))^{q-n-qs}\int_{2|z-y|\leq \operatorname{dist}(z,\partial\gamma(x))} |\nabla u(y)|^q\,dy\Big)^{\frac{1}{q}}.\end{aligned} \tag{6.3}$$

Next, observe that $z \in \gamma'(x)$, $2|y - z| \leq \operatorname{dist}(z, \partial\gamma(x))$ entail

$$\operatorname{dist}(y, \partial\gamma(x)) \leq C\operatorname{dist}(z, \partial\gamma(x)) \leq C\operatorname{dist}(z, \partial\Omega), \tag{6.4}$$

so that, utilizing this back in (6.3), we arrive at

$$\dot{\mathcal{N}}^{\infty}_{s,\gamma'}(u)(x) \leq C\Big(\int_{\gamma(x)} \delta(y)^{q-n-qs}|\nabla u(y)|^q\,dy\Big)^{\frac{1}{q}} = C\,\mathcal{S}^q_s(u)(x), \tag{6.5}$$

as claimed.

Next, the inequality (6.2) leads to the conclusion that

$$\begin{aligned}
\mathcal{S}_s^q(u) \in L^p(\partial\Omega) \quad &\Longrightarrow \quad \dot{\mathcal{N}}_{s,\gamma'}^{\infty}(u)(x) < +\infty \text{ for a.e. } x \in \partial\Omega \\
&\Longrightarrow \quad u \in C^s(\overline{\gamma'(x)}) \text{ for a.e. } x \in \partial\Omega \\
&\Longrightarrow \quad \text{there exists } \lim_{y \in \gamma'(x)} u(y) \text{ for a.e. } x \in \partial\Omega. \qquad (6.6)
\end{aligned}$$

This justifies the claim made in connection with (1.13) in §1 to the effect that

$$\Delta u = 0 \text{ in } \Omega \;\&\; \mathcal{S}_s^q u \in L^p(\partial\Omega),\; 0 < s < 1, \Longrightarrow \exists\, u\Big|_{\partial\Omega} \in L^p(\partial\Omega). \qquad (6.7)$$

Equally important, the above reasoning shows that, for $0 < s < 1$,

$$\mathcal{S}_s^q u \in L^p(\partial\Omega) \Longrightarrow \mathcal{N}_{s,\gamma'}^{\infty}(u) \in L^p(\partial\Omega). \qquad (6.8)$$

Now, the implication

$$\Delta u = 0 \text{ in } \Omega,\; \mathcal{N}_s^{\infty}(u) \in L^p(\partial\Omega),\; u\Big|_{\partial\Omega} = 0 \Longrightarrow u \equiv 0 \text{ in } \Omega \qquad (6.9)$$

has been proved in Theorem 6.1 of **[13]**, granted that s, p are as in (1.16)-(1.17). Clearly, this takes care of the uniqueness statement in Theorem 1.1.

Finally, existence is seen by taking u as in (1.18). The issue of the invertibility of the operator $\frac{1}{2}I + K$ on the spaces $B_s^{p,p}(\partial\Omega)$ for the range of indices described in Theorem 1.1 is addressed in Theorem 1.5 of **[13]**, while the fact that $\mathcal{S}_s^q(u) \in L^p(\partial\Omega)$ follows from our results in §4. □

We conclude with a brief discussion pertaining to Neumann boundary conditions.

Theorem 6.1 *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and consider the following boundary value problem:*

$$\begin{cases} \Delta u = 0 \;\; \text{in } \Omega, \\ \partial_\nu u\Big|_{\partial\Omega} = f \in B_{-s}^{p,p}(\partial\Omega),\;\; \int_{\partial\Omega} f\, d\sigma = 0, \\ \mathcal{S}_{1-s}^q u \in L^p(\partial\Omega), \end{cases} \qquad (6.10)$$

where $\frac{n-1}{n} < p \le q \le \infty$, and $(n-1)(\frac{1}{p} - 1)_+ < 1 - s < 1$.

Then there exists $\varepsilon = \varepsilon(\Omega) > 0$ such that (6.10) is well-posed if any of the three conditions in (1.17) is satisfied with $1 - s$ in place of s.

Furthermore, the solution has the integral representation formula

$$u = \mathcal{S}[(-\tfrac{1}{2}I + K^*)^{-1} f] + const \quad \text{in } \Omega, \qquad (6.11)$$

and obeys the estimate

$$\|\mathcal{S}_{1-s}^q(u)\|_{L^p(\partial\Omega)} \le C(\partial\Omega, p, q, s)\|f\|_{B_{-s}^{p,p}(\partial\Omega)}. \qquad (6.12)$$

The main steps in the proof of Theorem 6.1 closely mirror those taken in the proof of Theorem 1.1. Indeed, the issue of existence of the solution for (6.10) is covered by the representation formula (6.11) and Theorem 1.5 in **[13]** to the effect that the operator $\frac{1}{2}I+K^*$ is invertible on the class of spaces under discussion. As far as uniqueness is concerned, the key step is to observe that, thanks to (6.8),

$$\Delta u = 0 \text{ on } \Omega, \qquad \mathcal{S}^q_{1-s}(u) \in L^p(\partial\Omega) \Longrightarrow \mathcal{N}^{\infty}_{1-s}(u) \in L^p(\partial\Omega). \tag{6.13}$$

However, it was shown in **[13]** that, under our current assumptions made on s and p,

$$\Delta u = 0 \text{ in } \Omega,\ \mathcal{N}^{\infty}_{1-s}(u) \in L^p(\partial\Omega),\ \partial_\nu u\Big|_{\partial\Omega} = 0 \Longrightarrow u \equiv const \text{ in } \Omega. \tag{6.14}$$

Taking u as in (6.11) yields existence, given that the operator $-\frac{1}{2}I + K^*$ is invertible on the subspace of $B^{p,p}_{-s}(\partial\Omega)$, consisting of functions with vanishing moment, for the range of indices in question (**[13]**). Note that the square-function estimates in the last part of §4 guarantee that (6.12) holds. □

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Svitlana Mayboroda
Department of Mathematics
University of Missouri at Columbia
Columbia, MO 65211, USA
e-mail: *svitlana@math.missouri.edu*

Marius Mitrea
Department of Mathematics
University of Missouri at Columbia
Columbia, MO 65211, USA
e-mail: *marius@math.missouri.edu*