ON LATTICES IN SEMI-STABLE REPRESENTATIONS: A PROOF OF A CONJECTURE OF BREUIL

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Abstract. For $p \geq 3$ an odd prime and a nonnegative integer $r \leq p-2$, we prove a conjecture of Breuil on lattices in semi-stable representations, that is, the anti-equivalence of categories between the category of strongly divisible lattices of weight $\leq r$ and the category of Galois stable $\mathbb{Z}_p$-lattices in semi-stable p-adic Galois representations with Hodge-Tate weights in $\{0, \dots, r\}$.

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1. Introduction

Let k be a perfect field of characteristic $p > 2$, $W(k)$ its ring of Witt vectors, $K_0 = W(k)[\frac{1}{p}]$, K/K_0 a finite totally ramified extension and $e = e(K/K_0)$ the absolute ramification index. We are interested in understanding semi-stable p-adic Galois representations of $G := \mathrm{Gal}(\bar{K}/K)$. An important result in this direction is proved by Colmez and Fontaine [CF00]: semi-stable p-adic Galois representations are classified by weakly admissible filtered (φ, N)-modules. Since G is compact, any continuous representation $\rho : G \to \mathrm{GL}_n(\mathbb{Q}_p)$ admits a G-stable $\mathbb{Z}_p$-lattice. It is thus natural to ask whether there also exists a corresponding integral structure on the side of filtered (φ, N)-modules. Fontaine and Laffaille [FL82] first attacked this question by defining $W(k)$-lattices in filtered (φ, N)-modules. Unfortunately, their theory only works for the case $e = 1$, $N = 0$ and Hodge-Tate weights in $\{0, \dots, p-2\}$. In the late 1990s, Breuil introduced the theory of filtered (φ, N)-modules over S to study semi-stable Galois representations ([Bre97], [Bre98b], [Bre99a]), where S is the p-adic completion of divided power envelop of $W(k)[u]$ with respect to the ideal $(E(u))$, and $E(u)$ is the Eisenstein polynomial for a fixed uniformizer π of K. Breuil proved that the knowledge of filtered (φ, N)-modules over S is equivalent to that of filtered (φ, N)-modules (See Theorem 2.2.1 for the precise statement). Furthermore, it turns out that there are integral structures, strongly divisible lattices, which naturally live inside filtered (φ, N)-modules over S. These structures allow for arbitrary ramification of K/K_0. For a strongly divisible lattice $\mathcal{M}$, Breuil constructed a G-stable $\mathbb{Z}_p$-lattice $T_{\mathrm{st}}(\mathcal{M})$ in a semi-stable Galois representation and raised the following conjecture (the main conjecture in [Bre02]):

Conjecture 1.0.1. *Fix a nonnegative integer $r \leq p-2$, the functor T_{st} establishes an anti-equivalence of categories between the category of strongly divisible lattices of weight $\leq r$ and the category of G-stable $\mathbb{Z}_p$-lattices in semi-stable representations of G with Hodge-Tate weights in $\{0, \dots, r\}$.*

If $r \leq 1$, the conjecture has been proved by Breuil in [Bre00] and [Bre02]. The case $e = 1$ was shown by Fontaine and Laffaille in [FL82] for crystalline representations. In [Bre99a], Breuil proved that there at least exists a strongly divisible lattice in the side of filtered (φ, N)-modules over S if $er < p-1$. Based on this result, Breuil [Bre99c] proved the case $e = 1$ for general semi-stable representations and Caruso [Car05] proved the Conjecture for $er < p-1$. Their ideas involve a weak version of Conjecture 1.0.1, see the end of §2.3 for details. In [Fal99], Faltings proved that the restriction of T_{st} to the subcategory of *filtered free* strongly divisible lattices is *fully faithful*.

In this paper, we give a complete proof for the above conjecture by using results of Kisin ([Kis05]). Let $K_\infty = \cup_{n \geq 1} K(\sqrt[p^n]{\pi})$, $G_\infty = \mathrm{Gal}(\bar{K}/K_\infty)$ and $\mathfrak{S} = W(k)[\![u]\!]$. We equip $\mathfrak{S}$ with the endomorphism φ which acts via Frobenius on $W(k)$, and sends u to u^p. Let $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ denote the category of finite free $\mathfrak{S}$-modules $\mathfrak{M}$ equipped with a φ-semi-linear map $\varphi_{\mathfrak{M}} : \mathfrak{M} \to \mathfrak{M}$ such that the cokernel of $\mathfrak{S}$-linear map $1 \otimes \varphi_{\mathfrak{M}} : \mathfrak{S} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} \to \mathfrak{M}$ is killed by $E(u)^r$. In [Kis05], Kisin proved that any G_∞-stable $\mathbb{Z}_p$-lattice T in a semi-stable Galois representation comes from an object $(\mathfrak{M}, \varphi)$ in $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$. Using the functor $\mathfrak{M} \rightsquigarrow S \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$ provided by Breuil, Kisin's theory allows us to construct "quasi-strongly divisible lattices", i.e, strongly divisible lattices without considering monodromy, to establish an anti-equivalence between the category of quasi-strongly divisible lattices and the category of G_∞-stable $\mathbb{Z}_p$-lattices in semi-stable Galois representations. Furthermore, we prove that a quasi-strongly divisible lattice is strongly divisible if and only if the corresponding G_∞-stable $\mathbb{Z}_p$-lattice is G-stable (see Theorem 3.5.4 for the more precise statement). Conjecture 1.0.1 then follows.

The paper proceeds as follows. In §2, after briefly reviewing the theory of semi-stable p-adic Galois representations, filtered (φ, N)-modules over S and definition of (quasi-)strongly divisible lattices, we are then able to give a precise statement of our main theorem. §3 is devoted to review Kisin's theory from [Kis05], which allows us to construct quasi-strongly divisible lattices and establishes an anti-equivalence between the category of quasi-strongly divisible lattices and the category of G_∞-stable $\mathbb{Z}_p$-lattices in semi-stable Galois representations; and the full faithfulness of T_{st} follows from this. In the next two sections, we prove that a quasi-strongly divisible lattice is strongly divisible if and only if the corresponding G_∞-stable $\mathbb{Z}_p$-lattice is G-stable. The idea is to use an extended version of a Falting's theorem (Theorem 5, [Fal99]). The proof of such a theorem (Theorem 4.3.4) mainly depends on the construction of the Cartier dual for quasi-strongly divisible lattices from [Car05], which we discuss in §4. In the last section, we combine our previous preparations to prove the essential surjectivity of T_{st}.

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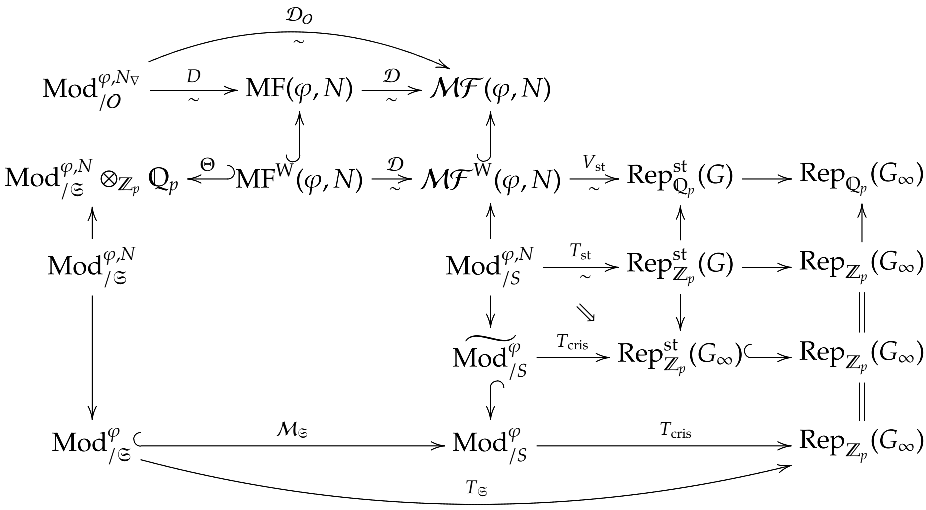
2. Preliminary and the Main Result

This paper discusses lots of categories and functors. However, we may summarize their relations and our main results as the following diagram:



Here is a general explanation of the above diagram:

• Injection arrows $\hookrightarrow$ symbolize fully faithful functors. The notations $\mathrm{Rep}^{\mathrm{st}}$ symbolize the categories of semi-stable representations with Hodge-Tate weights in $\{0, \dots, r\}$.

• The first column is about Kisin's theory on φ-modules over $\mathfrak{S}$. The second column is about classical modules in Fontaine's theory and the third about Breuil's theory on S-modules. These three theories can be connected by auxiliary categories in the first row (see §3.2). The last two columns are about the Galois sides. Note that representations of G_∞ (e.g., G_∞-stable $\mathbb{Z}_p$-lattices inside semi-stable representations) can be more conveniently described by Kisin's theory (see §3.1 and §3.4).

• The second row is about the theory over $\mathbb{Q}_p$ whereas the third row is about the theory over $\mathbb{Z}_p$, which also is the key result of this paper. Many important inputs depend on the last two rows where are about theories on $\mathbb{Z}_p$-representations of G_∞(see §3.3 and §3.4).

2.1. **Semi-stable Galois representations and weakly admissible modules.** Fix an odd prime p. Recall that a p-adic representation is a continuous linear representation of $G := \mathrm{Gal}(\bar{K}/K)$ on a finite dimensional $\mathbb{Q}_p$-vector space V and a p-adic representation V of G is called *semi-stable* ([Fon94b]) if:

$$\dim_{K_0}(B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^G = \dim_{\mathbb{Q}_p} V, \tag{2.1.1}$$

where B_{st} is the period ring constructed by Fontaine, see for example [Fon94a] or §2.2 for the construction.

In [CF00] and [Fon94b], Fontaine and Colmez gives an alternative description of semi-stable p-adic representations. Recall that a filtered (φ, N)-module is a finite dimensional K_0-vector space D endowed with:

(1) a Frobenius semi-linear injection: $\varphi : D \to D$.
(2) a linear map $N : D \to D$ such that $N\varphi = p\varphi N$.
(3) a decreasing filtration $(\mathrm{Fil}^i D_K)_{i\in\mathbb{Z}}$ on $D_K := K \otimes_{K_0} D$ by K-vector spaces such that $\mathrm{Fil}^i D_K = D_K$ for $i \ll 0$ and $\mathrm{Fil}^i D_K = 0$ for $i \gg 0$.

If D is a one dimensional (φ, N)-module, and $v \in D$ is a basis vector, then $\varphi(v) = \alpha v$ for some $\alpha \in K_0$. We write $t_N(D)$ for the p-adic valuation of α (p-adic valuation of α does not depends on choice of v) and $t_H(D)$ the unique integer i such that $\mathrm{gr}^i D_K$ is non-zero. If D has dimension $d > 1$, then we write $t_N(D) = t_N(\wedge^d D)$ and $t_H(D) = t_H(\wedge^d D)$. Recall that a filtered (φ, N)-module is called *weakly admissible* if $t_H(D) = t_N(D)$ and for any (φ, N)-submodule $D' \subset D$, $t_H(D') \leq t_N(D')$, where $D'_K \subset D_K$ is equipped with the induced filtration.

The aforementioned result of Colmez and Fontaine [CF00] is that the functor

$$D_{\mathrm{st},*} : \quad V \to (B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^G$$

establishes an equivalence of categories between the category of semi-stable p-adic representations of G and the category of weakly admissible filtered (φ, N)-modules.

In the sequel, we will instead use the contravariant functor $D_{\mathrm{st}}(V) := D_{\mathrm{st},*}(V^\vee)$, where $V^\vee$ is the dual representation of V. The advantage of this is that the Hodge-Tate weights of V is exactly the $i \in \mathbb{Z}$ such that $\mathrm{gr}^i D_{\mathrm{st}}(V)_K \neq 0$. A quasi-inverse to D_{st} is then given by :

$$V_{\mathrm{st}}(D) := \mathrm{Hom}_{\varphi,N}(D, B_{\mathrm{st}}) \cap \mathrm{Hom}_{\mathrm{Fil}^\cdot}(D_K, K \otimes_{K_0} B_{\mathrm{st}}). \tag{2.1.2}$$

Convention 2.1.1. Here we use a little different notations from those in [Bre02] and [CF00]. D_{st} here is D^*_{st} in [Bre02] and [CF00]; V_{st} here is V^*_{st} in [Bre02] and [CF00]. Also we will use T_{st} to denote T^*_{st} in [Bre02] and [Bre99a] later. The reason for using such notations is that we will always use *contravariant* functors instead of covariant functors in this paper. Removing "$*$" from the superscript looks more neat and convenient.

A filtered (φ, N)-module is called *positive* if $\mathrm{Fil}^0 D = D$. In this paper, we only consider positive filtered (φ, N)-modules. We denote the category of positive filtered (φ, N)-modules by $\mathrm{MF}(\varphi, N)$ and the category of positive weakly admissible filtered (φ, N)-modules by $\mathrm{MF}^{\mathrm{w}}(\varphi, N)$.

2.2. **Breuil's theory on filtered (φ, N)-modules over S.** Throughout the paper we will fix a uniformiser $\pi \in \mathcal{O}_K$, and $E(u) \in W(k)[u]$ the Eisenstein polynomial of π. We denote by S the p-adic completion of the divided power envelope of $W(k)[u]$ with respect to $\mathrm{Ker}(s)$, where $s : W(k)[u] \to \mathcal{O}_K$ is the canonical surjection by sending u to π. For any positive integer i, let $\mathrm{Fil}^i S \subset S$ be the p-adic closure of the ideal generated by the divided powers $\gamma_j(u) = \frac{E(u)^j}{j!}$ for all $j \geq i$. There is a unique map $\varphi : S \to S$ which extends the Frobenius on $W(k)$ and satisfies $\varphi(u) = u^p$. We define a continuous $W(k)$-linear derivation $N : S \to S$ such that $N(u) = -u$. It is easy to check that $N\varphi = p\varphi N$ and $\varphi(\mathrm{Fil}^i S) \subset p^i S$ for $0 \leq i \leq p-1$, and we write $\varphi_i = p^{-i}\varphi|_{\mathrm{Fil}^i S}$ and $c_1 = \varphi_1(E(u))$. Note that c_1 is a unit in S. Finally, we put $S_{K_0} := S \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and $\mathrm{Fil}^i S_{K_0} := \mathrm{Fil}^i S \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

Let $\mathcal{MF}(\varphi, N)$ be a category whose objects are finite free S_{K_0}-modules $\mathcal{D}$ with:

• a $\varphi_{S_{K_0}}$-semi-linear morphism $\varphi_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}$ such that the determinant of $\varphi_{\mathcal{D}}$ is invertible in S_{K_0} (the invertibility of the determinant does not depend on the choice of basis).

• a decreasing filtration over $\mathcal{D}$ of S_{K_0}-modules: $\mathrm{Fil}^i(\mathcal{D})$, $i \in \mathbb{Z}$, such that $\mathrm{Fil}^0(\mathcal{D}) = \mathcal{D}$ and that $\mathrm{Fil}^i S_{K_0} \mathrm{Fil}^j(\mathcal{D}) \subset \mathrm{Fil}^{i+j}(\mathcal{D})$.

• a K_0-linear map (monodromy) $N : \mathcal{D} \to \mathcal{D}$ such that

(1) for all $f \in S_{K_0}$ and $m \in \mathcal{D}$, $N(fm) = N(f)m + fN(m)$.
(2) $N\varphi = p\varphi N$,
(3) $N(\mathrm{Fil}^i\mathcal{D}) \subset \mathrm{Fil}^{i-1}(\mathcal{D})$.

Let $D \in \mathrm{MF}(\varphi, N)$ be a filtered (φ, N)-module. We can associate an object $\mathcal{D} \in \mathcal{MF}(\varphi, N)$ by the following:

$$\mathcal{D} := S \otimes_{W(k)} D \tag{2.2.1}$$

and • $\varphi := \varphi_S \otimes \varphi_D : \mathcal{D} \to \mathcal{D}$.

• $N := N \otimes \mathrm{Id} + \mathrm{Id} \otimes N : \mathcal{D} \to \mathcal{D}$

• $\mathrm{Fil}^0(\mathcal{D}) := \mathcal{D}$ and by induction:

$$\mathrm{Fil}^{i+1}\mathcal{D} := \{x \in \mathcal{D} | N(x) \in \mathrm{Fil}^i\mathcal{D} \text{ and } f_\pi(x) \in \mathrm{Fil}^{i+1} D_K\}$$

where $f_\pi : \mathcal{D} \twoheadrightarrow D_K$ is defined by $\lambda \otimes x \mapsto s(\lambda)x$.

For a $\mathcal{D} \in \mathcal{MF}(\varphi, N)$, Breuil associated a $\mathbb{Q}_p[G]$-module $V_{\mathrm{st}}(\mathcal{D})$. Several period rings have to be defined before we can describe this functor. Let $R = \varprojlim \mathcal{O}_{\bar{K}}/p$ where the transition maps are given by Frobenius. By the universal property of

Witt vectors $W(R)$ of R, there is a unique surjective map $\theta : W(R) \to \widehat{O_{\bar{K}}}$ to the p-adic completion $\widehat{O_{\bar{K}}}$, which lifts the projection $R \to O_{\bar{K}}/p = \widehat{O_{\bar{K}}}/p$ onto the first factor in the inverse limit. We denote by A_{cris} the p-adic completion of the divided power envelope of $W(R)$ with respect to the $\mathrm{Ker}(\theta)$, and write $B^+_{\mathrm{cris}} := A_{\mathrm{cris}}[1/p]$.

For each $n \geq 0$, fix $\pi_n \in \bar{K}$ a p^n-th root of π such that $\pi^p_{n+1} = \pi_n$. Write $\underline{\pi} = (\pi_n)_{n\geq 0} \in R$, and let $[\underline{\pi}] \in W(R)$ be the Teichmüller representation. We embed the $W(k)$-algebra $W(k)[u]$ into $W(R)$ by $u \mapsto [\underline{\pi}]$. Since $\theta([\underline{\pi}]) = \pi$ this embedding extends to an embedding $S \hookrightarrow A_{\mathrm{cris}}$, and $\theta|_S$ is the map $s : S \to O_K$ sending u to π. The embedding is compatible with Frobenius endomorphisms. As usual,we denote by B^+_{st} the ring obtained by formally adjoining the element "$\log[\underline{\pi}]$" to B^+_{cris}, and by B^+_{dR} the $\mathrm{Ker}(\theta)$-adic completion of $W(R)[1/p]$. Choose a generator t of $\mathbb{Z}_p(1) \subset A_{\mathrm{cris}}$. Such t can be constructed by $t := \log([\underline{\epsilon}])$ for $\underline{\epsilon} = (\epsilon_i)_{i\geq 0} \in R$, where ϵ_i is a primitive p^i-th root of unity such that $\epsilon^p_{i+1} = \epsilon_i$. We denote $B^+_{\mathrm{st}}[1/t]$ by B_{st}.

Let $\widehat{A_{\mathrm{st}}}$ be the p-adic completion of the P.D. polynomial algebra $A_{\mathrm{cris}}\langle X\rangle$. We endow $\widehat{A_{\mathrm{st}}}$ with a continuous G-action, a Frobenius φ, a monodromy operator N and positive filtration Fil^i as the following:

For any $g \in G$, let $\underline{\epsilon}(g) = \frac{g([\underline{\pi}])}{[\underline{\pi}]} \in A_{\mathrm{cris}}$. We extend the natural G-action and Frobenius on A_{cris} to $\widehat{A_{\mathrm{st}}}$ by putting $g(X) = \underline{\epsilon}(g)X + \underline{\epsilon}(g) - 1$ and $\varphi(X) = (1 + X)^p - 1$. We define a monodromy operator N on $\widehat{A_{\mathrm{st}}}$ to be a unique A_{cris}-linear derivation such that $N(X) = 1 + X$. For any $i \geq 0$, we define

$$\mathrm{Fil}^i\widehat{A_{\mathrm{st}}} = \{\sum_{j=0}^{\infty} a_j\gamma_j(X),\ a_j \in A_{\mathrm{cris}}, \lim_{j\to\infty} a_j = 0, a_j \in \mathrm{Fil}^{i-j}A_{\mathrm{cris}}, 0 \leq j \leq i\}.$$

Finally, by §4.2 in [Bre97], we have an isomorphism $S \xrightarrow{\sim} (\widehat{A_{\mathrm{st}}})^G$ compatible with all structures given by $u \mapsto [\underline{\pi}](1 + X)^{-1}$. Therefore, $\widehat{A_{\mathrm{st}}}$ is an S-algebra.

For any $\mathcal{D} \in \mathcal{MF}(\varphi, N)$, one can associate a $\mathbb{Q}_p[G]$-module

$$V_{\mathrm{st}}(\mathcal{D}) := \mathrm{Hom}_{S,\mathrm{Fil}^{\cdot},\varphi,N}(\mathcal{D}, \widehat{A_{\mathrm{st}}}[1/p]).$$

The following theorem is one of main results in [Bre97]:

Theorem 2.2.1 (Breuil)**.** *The functor $\mathcal{D} : D \to S \otimes_{W(k)} D$ defined in (and below) (2.2.1) induces an equivalence between the category* $\mathrm{MF}(\varphi, N)$ *and* $\mathcal{MF}(\varphi, N)$ *and there is a natural isomorphism $V_{\mathrm{st}}(D) \simeq V_{\mathrm{st}}(\mathcal{D})$ as $\mathbb{Q}_p[G]$-modules.*

From now on, we always identify $V_{\mathrm{st}}(D)$ with $V_{\mathrm{st}}(\mathcal{D})$ as the same Galois representations, and denote $\mathcal{MF}^{\mathrm{w}}(\varphi, N)$ the essential image of $\mathcal{D}$ restricted to $\mathrm{MF}^{\mathrm{w}}(\varphi, N)$.

2.3. **The Main Theorem.** Theorem 2.2.1 shows that the knowledge of filtered (φ, N)-modules over S is equivalent to that of filtered (φ, N)-modules. It turns out that integral structures can be more conveniently defined inside filtered (φ, N)-modules over S. However, when working on integral p-adic Hodge theory via S-modules, the following technical restriction has to be always assumed.

Assumption 2.3.1. Fix a positive integer $r \leq p - 2$. The filtration on the weakly admissible filtered (φ, N)-module D is such that $\mathrm{Fil}^0 D_K = D_K$ and $\mathrm{Fil}^{r+1} D_K = 0$. Equivalently, the Hodge-Tate weights of the semi-stable p-adic Galois representation under consideration are always contained in $\{0, \ldots, r\}$.

Remark 2.3.2. (1) Conjecture 1.0.1 has been proved for $r = 0$ in §3.1, [Bre02]. So we only consider the case $r > 0$ from now on ($r = 0$ will cause a little trouble only in the end).
(2) Up to the twist of the (φ, N)-module of a power of the cyclotomic character, all modules whose filtration length does not exceed r satisfy the above assumption.

Following §2.2 in [Bre02], we define the integral structures inside $\mathcal{D}$ to correspond to the Galois stable $\mathbb{Z}_p$-lattices.

Definition 2.3.3. Let D be a weakly admissible filtered (φ, N)-module satisfying Assumption 2.3.1 and $\mathcal{D} := \mathcal{D}(D) \in \mathcal{MF}^{\mathrm{w}}(\varphi, N)$. A *quasi-strongly divisible lattice of weight r* in $\mathcal{D}$ is an S-submodule $\mathcal{M}$ of $\mathcal{D}$ such that:

(1) $\mathcal{M}$ is S-finite free and $\mathcal{M}[\frac{1}{p}] \xrightarrow{\sim} \mathcal{D}$
(2) $\mathcal{M}$ is stable under φ, i.e., $\varphi(\mathcal{M}) \subset \mathcal{M}$.
(3) $\varphi(\mathrm{Fil}^r\mathcal{M}) \subset p^r\mathcal{M}$ where $\mathrm{Fil}^r\mathcal{M} := \mathcal{M} \cap \mathrm{Fil}^r\mathcal{D}$.

A *strongly divisible lattice of weight r* in $\mathcal{D}$ is a quasi-strongly divisible lattice $\mathcal{M}$ in $\mathcal{D}$ such that $N(\mathcal{M}) \subset \mathcal{M}$.

It will be more convenient and explicit to describe the category of (quasi-)strongly divisible lattices by projective limits of torsion objects. Let $'\mathrm{Mod}^{\varphi,N}_{/S}$ denote the category whose objects are 4-tuples $(\mathcal{M}, \mathrm{Fil}^r\mathcal{M}, \varphi_r, N)$, consisting of

(1) an S-module $\mathcal{M}$
(2) an S-submodule $\mathrm{Fil}^r\mathcal{M} \subset \mathcal{M}$ containing $\mathrm{Fil}^r S \cdot \mathcal{M}$.
(3) a φ-semi-linear map $\varphi_r : \mathrm{Fil}^r\mathcal{M} \to \mathcal{M}$ such that for all $s \in \mathrm{Fil}^r S$ and $x \in \mathcal{M}$ we have $\varphi_r(sx) = (c_1)^{-r}\varphi_r(s)\varphi_r(E(u)^r x)$.
(4) a $W(k)$-linear morphism $N : \mathcal{M} \to \mathcal{M}$ such that :
   (a) for all $s \in S$ and $x \in \mathcal{M}$, $N(sx) = N(s)x + sN(x)$.
   (b) $E(u)N(\mathrm{Fil}^r\mathcal{M}) \subset \mathrm{Fil}^r\mathcal{M}$.
   (c) the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Fil}^r\mathcal{M} & \xrightarrow{\varphi_r} & \mathcal{M} \\ \downarrow{\scriptstyle E(u)N} & & \downarrow{\scriptstyle c_1 N} \\ \mathrm{Fil}^r\mathcal{M} & \xrightarrow{\varphi_r} & \mathcal{M} \end{array} \tag{2.3.1}$$

Morphisms are given by S-linear maps preserving Fil^r's and commuting with φ_r and N. A sequence is defined to be *short exact* if it is short exact as a sequence of S-module, and induces a short exact sequence on Fil^r's.

We denote by $'\mathrm{Mod}^{\varphi}_{/S}$ the category which forgets the operation N in the definition of $'\mathrm{Mod}^{\varphi,N}_{/S}$. Objects in $'\mathrm{Mod}^{\varphi}_{/S}$ are called *filtered φ-module over S*. Let $\mathrm{Mod\,FI}^{\varphi,N}_{/S}$ (resp. $\mathrm{Mod\,FI}^{\varphi}_{/S}$) be the full subcategory of $'\mathrm{Mod}^{\varphi,N}_{/S}$ (resp. $'\mathrm{Mod}^{\varphi}_{/S}$) consisting of objects such that

(1) as an S-module $\mathcal{M}$ is isomorphic to $\oplus_{i\in I} S/p^{n_i}S$, where I is a finite set and n_i is a positive number.
(2) $\varphi_r(\mathcal{M})$ generates $\mathcal{M}$ over S.

Finally we denote by $\mathrm{Mod}_{/S}^{\varphi,N}$ (resp. $\mathrm{Mod}_{/S}^{\varphi}$) the full subcategory of $'\mathrm{Mod}_{/S}^{\varphi,N}$ (resp. $'\mathrm{Mod}_{/S}^{\varphi}$) such that $\mathcal{M}$ is a finite free S-module and for all n,

$$(\mathcal{M}_n, \mathrm{Fil}^r\mathcal{M}_n, \varphi_r, N) \in \mathrm{Mod}\ \mathrm{FI}_{/S}^{\varphi,N} \text{ (resp. } (\mathcal{M}_n, \mathrm{Fil}^r\mathcal{M}_n, \varphi_r) \in \mathrm{Mod}\ \mathrm{FI}_{/S}^{\varphi}),$$

where $\mathcal{M}_n = \mathcal{M}/p^n\mathcal{M}$, $\mathrm{Fil}^r\mathcal{M}_n = \mathrm{Fil}^r\mathcal{M}/p^n\mathrm{Fil}^r\mathcal{M}$, and φ_r, N are induced by modulo p^n.

Note that $\widehat{A_{\mathrm{st}}} \in {}'\mathrm{Mod}_{/S}^{\varphi,N}$. For any $\mathcal{M} \in \mathrm{Mod}_{/S}^{\varphi,N}$, define

$$T_{\mathrm{st}}(\mathcal{M}) := \mathrm{Hom}_{'\mathrm{Mod}_{/S}^{\varphi,N}}(\mathcal{M}, \widehat{A_{\mathrm{st}}}).$$

Proposition 2.3.4 (Breuil)**.** (1) *If $\mathcal{M}$ is a quasi-strongly divisible lattice in $\mathcal{D}$ with $\mathcal{D} \in \mathcal{MF}^{\mathrm{w}}(\varphi, N)$, then $(\mathcal{M}, \mathrm{Fil}^r\mathcal{M}, \varphi_r)$ is in $\mathrm{Mod}_{/S}^{\varphi}$ where $\varphi_r := \varphi/p^r$.*

(2) *The category of strongly divisible lattices of weight r is just $\mathrm{Mod}_{/S}^{\varphi,N}$. In particular, for any $\mathcal{M} \in \mathrm{Mod}_{/S}^{\varphi,N}$, there exists a $D \in \mathrm{MF}^{\mathrm{w}}(\varphi, N)$ such that $\mathcal{D}(D) \simeq \mathcal{M} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ as filtered (φ, N)-modules over S. Furthermore, $T_{\mathrm{st}}(\mathcal{M})$ is a G-stable $\mathbb{Z}_p$-lattice in $V_{\mathrm{st}}(D)$.*

Proof. (1) is a Proposition 2.1.3 in [Bre99a] and Theorem 2.2.3 in [Bre02] □

From now on, we use $\mathrm{Mod}_{/S}^{\varphi,N}$ to denote the category of strongly divisible lattices of weight r and regard $\widetilde{\mathrm{Mod}}_{/S}^{\varphi}$ as a full subcategory of $\mathrm{Mod}_{/S}^{\varphi}$, where $\widetilde{\mathrm{Mod}}_{/S}^{\varphi}$ denote the category of quasi-strongly divisible lattices. Now we can state our Main Theorem:

Theorem 2.3.5 (Main Theorem)**.** *If $0 \le r \le p-2$, the functor $\mathcal{M} \to T_{\mathrm{st}}(\mathcal{M})$ establishes an anti-equivalence of categories between the category of strongly divisible lattices of weight r and the category of G-stable $\mathbb{Z}_p$-lattices in semi-stable p-adic Galois representations with Hodge-Tate weights in $\{0, \ldots, r\}$.*

Remark 2.3.6. In fact, there exists a weak version of Conjecture 1.0.1: Fix a $\mathcal{D}$ inside $\mathcal{MF}^{\mathrm{w}}(\varphi, N)$. Consider the restriction of the functor T_{st}, namely,

$$T_{\mathrm{st}|\mathcal{D}}: \{\text{strongly divisible lattices in } \mathcal{D}\} \to \{G\text{-stable } \mathbb{Z}_p\text{-lattices in } V_{\mathrm{st}}(\mathcal{D})\}.$$

The weak version claims that all functors $T_{\mathrm{st}|\mathcal{D}}$ are equivalences. It is obvious that Conjecture 1.0.1 implies the weak one. On the other hand, from the weak version, one can deduce the essentially surjectivity of T_{st}. Therefore if the full faithfulness of T_{st} has been known, then the weak version and the strong version are equivalent. [Car05] and [Bre98a] used this ideal to prove some special cases of Conjecture 1.0.1.

3. Construction of Quasi-Strongly Divisible Lattices

Let T be a G-stable $\mathbb{Z}_p$-lattice in a semi-stable Galois representation V with Hodge-Tate weights in $\{0, \ldots, r\}$. In this section, we will use the theory from [Kis05] to prove that there exists a quasi-strongly divisible lattice $\mathcal{M} \in \mathrm{Mod}_{/S}^{\varphi}$ to correspond to $T|_{G_\infty}$. As we will see later, $\mathcal{M}$ provides the ambient module for the strongly divisible lattice corresponding to T.

3.1. (φ, N_∇)**-modules.** We equip $K_0[\![u]\!]$ with the endomorphism $\varphi : K_0[\![u]\!] \to K_0[\![u]\!]$ which acts via the Frobenius on K_0, and sends u to u^p. Suppose that $I \subset [0,1)$ is a subinterval. We set $\mathcal{O}_I$ the subring of $K_0[\![u]\!]$ whose elements converge for all $x \in \bar{K}$ such that $|x| \in I$. Put $\mathcal{O} = \mathcal{O}_{[0,1)}$. By Lemma 2.1 in [Bre97], S can be identified as the subring of $K_0[\![u]\!]$ whose elements have the following form

$$(3.1.1) \qquad \sum_{n=0}^{\infty} w_i \frac{u^i}{q(i)!},\ w_i \in W(k),\ \lim_{i\to\infty} w_i = 0,$$

where $q(i)$ is the quotient in the Euclidean division of i by e. Therefore, for any real number μ satisfying $p^{-\frac{1}{(p-1)e}} < \mu \leq 1$, we have natural inclusions $\mathfrak{S}[1/p] \hookrightarrow \mathcal{O}_{[0,\mu)} \hookrightarrow S_{K_0}$ compatible with Frobenius. Set $c_0 = E(0)/p \in K_0$ and $\lambda = \prod_{n=0}^{\infty} \varphi^n(E(u)/pc_0) \in \mathcal{O}$.

We define a derivation $N_\nabla := -u\lambda \frac{d}{du} : \mathcal{O} \to \mathcal{O}$ and denote by the same symbol the induced derivation $\mathcal{O}_I \to \mathcal{O}_I$, for each $I \subset [0,1)$.

By a φ-module over $\mathcal{O}$ we mean a finite free $\mathcal{O}$-module M, equipped with a φ-semi-linear, injective map $\varphi : M \to M$. A (φ, N_∇)*-module over* $\mathcal{O}$ is a φ-module M over $\mathcal{O}$, together with a differential operator N_∇^M over N_∇. That is, for any $f \in \mathcal{O}$ and $m \in M$, we have

$$N_\nabla^M(fm) = N_\nabla(f)m + fN_\nabla^M(m).$$

φ and N_∇^M are required to satisfy the relation $N_\nabla^M \varphi = (1/c_0)E(u)\varphi N_\nabla^M$. We will usually write N_∇ for N_∇^M if this will cause no confusion. The category of (φ, N_∇)-modules over $\mathcal{O}$ has a natural structure of a Tannakian category. We denote by $\mathrm{Mod}_{/\mathcal{O}}^{\varphi, N_\nabla}$ the category of (φ, N_∇)-modules M of height r, in the sense that the cokernel of $1 \otimes \varphi : \varphi^* M \to M$ is killed by $E(u)^r$ for our fixed positive integer r, where $\varphi^* M := \mathcal{O} \otimes_{\varphi, \mathcal{O}} M$.

In §1.2 of [Kis05], Kisin constructed a functor $D : \mathrm{Mod}_{/\mathcal{O}}^{\varphi, N_\nabla} \to \mathrm{MF}(\varphi, N)$. Let M be an object in $\mathrm{Mod}_{/\mathcal{O}}^{\varphi, N_\nabla}$. Define the underlying K_0-vector space of $D(M)$ is M/uM, and the operator φ and N are induced by φ, N_∇ on M. The construction of filtration on $D(M)$ is somewhat not strait forward. First we define a decreasing filtration on $\varphi^* M$ by

$$\mathrm{Fil}^i \varphi^* M = \{x \in \varphi^* M | 1 \otimes \varphi(x) \in E(u)^i M\}.$$

Fix any fixed real number μ such that $p^{-\frac{1}{e}} < \mu < p^{-\frac{1}{pe}}$. Lemma 1.2.6 in [Kis05] showed that there exists a unique $\mathcal{O}_{[0,\mu)}$-linear, φ-equivariant isomorphism

$$(3.1.2) \qquad \xi : D(M) \otimes_{K_0} \mathcal{O}_{[0,\mu)} \xrightarrow{\sim} \varphi^* M \otimes_{\mathcal{O}} \mathcal{O}_{[0,\mu)}.$$

The required filtration on $D(M)_K$ is defined to be the image filtration under the composite

$$D(M) \otimes_{K_0} \mathcal{O}_{[0,\mu)} \to D(M) \otimes_{K_0} \mathcal{O}/E(u)\mathcal{O} \xrightarrow{\sim} D(M) \otimes_{K_0} K = D(M)_K.$$

Theorem 1.2.8 in [Kis05] shows that the functor D induces an exact equivalence between the category $\mathrm{Mod}_{/\mathcal{O}}^{\varphi, N_\nabla}$ and $\mathrm{MF}(\varphi, N)$.

3.2. **A functor from** $\mathrm{Mod}_{/\mathcal{O}}^{\varphi,N_\nabla}$ **to** $\mathcal{MF}(\varphi,N)$**.** Combining the functor D in §3.1 with the functor $\mathcal{D}$ in §2.2 together, we obtain a functor $\mathcal{D}\circ D$ from $\mathrm{Mod}_{/\mathcal{O}}^{\varphi,N_\nabla}$ to $\mathcal{MF}(\varphi,N)$. But it will be convenient to give another description of $\mathcal{D}\circ D$ for later use.

Let M be an object in $\mathrm{Mod}_{/\mathcal{O}}^{\varphi,N_\nabla}$. Define $\mathcal{D}_{\mathcal{O}}(M)=S_{K_0}\otimes_{\varphi,\mathcal{O}}M$, a $\varphi_{S_{K_0}}$-semi-linear endomorphism $\varphi_{\mathcal{D}_{\mathcal{O}}(M)}:=\varphi_{S_{K_0}}\otimes\varphi_M$ (as usual, we will drop the subscript of $\varphi_{\mathcal{D}_{\mathcal{O}}(M)}$ if no confusion will arise) and decreasing filtration on $\mathcal{D}_{\mathcal{O}}(M)$ by

$$\mathrm{Fil}^i(\mathcal{D}_{\mathcal{O}}(M)):=\{m\in\mathcal{D}_{\mathcal{O}}(M)|(1\otimes\varphi)(m)\in\mathrm{Fil}^iS_{K_0}\otimes_{\mathcal{O}}M\}. \tag{3.2.1}$$

Note that $\varphi(\lambda)$ is a unit in S_{K_0}, we can define N on $\mathcal{D}_{\mathcal{O}}(M)$ by

$$N:=N\otimes 1+\frac{p}{\varphi(\lambda)}1\otimes N_\nabla.$$

We can naturally extend N_∇ from $\mathcal{O}$ to S_{K_0}. Note that for any $f\in S_{K_0}$ we have $N(\varphi(f))=\frac{p}{\varphi(\lambda)}\varphi(N_\nabla(f))$. Thus it is easy to check that N is a well-defined derivation of $\mathcal{D}_{\mathcal{O}}(M)$ over the derivation N of S_{K_0} defined by $N(u)=-u\frac{d}{du}$.

Proposition 3.2.1. *N is well defined on $\mathcal{D}_{\mathcal{O}}(M)$ and $(\mathcal{D}_{\mathcal{O}}(M),\varphi,\mathrm{Fil}^i,N)$ is an object in $\mathcal{MF}(\varphi,N)$.*

Proof. Let $\mathcal{D}=\mathcal{D}_{\mathcal{O}}(M)$. We check that Frobenius, filtration and monodromy defined on $\mathcal{D}$ satisfy the required properties listed in §2.2.

Since $E(u)^r$ kills the cokernel of $1\otimes\varphi:\mathcal{O}\otimes_{\varphi,\mathcal{O}}M\to M$, we see that the determinant of φ_M is a divisor of $E(u)^{rd}$, where d is the $\mathcal{O}$-rank of M. Thus the determinant of $\varphi_{\mathcal{D}}$ is a divisor of $\varphi(E(u))^{rd}=p^{rd}c_1^{rd}$, therefore is invertible in S_{K_0}. Using (3.2.1), one easily checks that $\mathrm{Fil}^iS_{K_0}\cdot\mathrm{Fil}^j\mathcal{D}\subset\mathrm{Fil}^{i+j}\mathcal{D}$. Now it suffices to check that the monodromy N satisfies the required properties.

To see $N\varphi=p\varphi N$, for any $s\in S_{K_0}$ and $m\in M$, we have

$$\begin{aligned}
N\varphi(s\otimes m) &= N(\varphi_{S_{K_0}}(s)\otimes\varphi_M(m))\\
&= N(\varphi_{S_{K_0}}(s))\otimes\varphi_M(m)+\frac{p}{\varphi(\lambda)}\varphi_{S_{K_0}}(s)\otimes N_\nabla(\varphi_M(m))\\
&= p\varphi_{S_{K_0}}(N(s))\otimes\varphi_M(m)+\frac{p}{\varphi(\lambda)}\frac{\varphi(E(u))}{\varphi(c_0)}\varphi_{S_{K_0}}(s)\otimes\varphi_M(N_\nabla(m))\\
&= p\varphi_{\mathcal{D}}(N(s)\otimes m+\frac{p}{\varphi(\lambda)}s\otimes N_\nabla(m))\\
&= p\varphi(N(s\otimes m)).
\end{aligned}$$

To check that $N(\mathrm{Fil}^i\mathcal{D})\subset\mathrm{Fil}^{i-1}\mathcal{D}$, note that

$$N_\nabla(E(u)^i)=-uiE(u)^{i-1}E'(u)\lambda=E(u)^i(-uiE'(u)\frac{\varphi(\lambda)}{pc_0}).$$

Thus $N_\nabla(\mathrm{Fil}^iS_{K_0}\otimes_{\mathcal{O}}M)\subset\mathrm{Fil}^iS_{K_0}\otimes_{\mathcal{O}}M$. Now let $x=\sum_i s_i\otimes m_i\in\mathrm{Fil}^i\mathcal{D}$. We claim that

$$E(u)(1\otimes\varphi_M)(N(x))=\frac{c_0p}{\varphi(\lambda)}N_\nabla((1\otimes\varphi_M)(x)) \tag{3.2.2}$$

In fact, since $E(u)N = \frac{c_0 p}{\varphi(\lambda)} N_\nabla$ and $N_\nabla \varphi = \frac{E(u)}{c_0} \varphi N_\nabla$, we have

$$\begin{aligned} E(u)(1 \otimes \varphi_M)(N(x)) &= E(u)(\sum_i N(s_i) \otimes \varphi_M(m_i) + \frac{p}{\varphi(\lambda)} s_i \otimes \varphi_M(N_\nabla(m_i))) \\ &= \frac{c_0 p}{\varphi(\lambda)}(\sum_i N_\nabla(s_i) \otimes \varphi_M(m_i) + s_i \otimes N_\nabla(\varphi_M(m_i))) \\ &= \frac{c_0 p}{\varphi(\lambda)} N_\nabla(\sum_i s_i \otimes \varphi_M(m_i)) \end{aligned}$$

This proves the claim (3.2.2). Finally, to prove $N(x) \in \mathrm{Fil}^{i-1}\mathcal{D}$, it suffices to show that $(1 \otimes \varphi_M)(N(x)) \in \mathrm{Fil}^{i-1} S_{K_0} \otimes_{\mathcal{O}} M$. But (3.2.2) has shown us that

$$E(u)(1 \otimes \varphi_M)(N(x)) \in \mathrm{Fil}^i S_{K_0} \otimes_{\mathcal{O}} M.$$

Then we reduce our proof to the following lemma: □

Lemma 3.2.2. *Let* $x \in S$ (resp. A_{cris}). *If* $E(u)^j x \in \mathrm{Fil}^{j+i} S$ (resp. $E([\underline{\pi}])^j x \in \mathrm{Fil}^{j+i} A_{\mathrm{cris}}$) *then* $x \in \mathrm{Fil}^i S$ (resp. $x \in \mathrm{Fil}^i A_{\mathrm{cris}}$).

Proof. We have a natural embedding $S \overset{u \mapsto [\underline{\pi}]}{\hookrightarrow} A_{\mathrm{cris}} \hookrightarrow B^+_{\mathrm{dR}}$ with respect to filtration. By definition, $\mathrm{Fil}^n B^+_{\mathrm{dR}} = E([\underline{\pi}])^n B^+_{\mathrm{dR}}$ for all $n \geq 0$. Thus, if $E([\underline{\pi}])^j x \in \mathrm{Fil}^{i+j} B^+_{\mathrm{dR}}$ then $x \in \mathrm{Fil}^i B^+_{\mathrm{dR}}$, as required. □

Corollary 3.2.3. *The following equivalences of category commute:*

$$\begin{array}{ccc} \mathrm{MF}(\varphi, N) & \xrightarrow{\mathcal{D}} & \mathcal{MF}(\varphi, N) \\ {\scriptstyle D}\uparrow & \nearrow {\scriptstyle \mathcal{D}_{\mathcal{O}}} & \\ \mathrm{Mod}_{/\mathcal{O}}^{\varphi, N_\nabla} & & \end{array}$$

Proof. Let $M \in \mathrm{Mod}_{/\mathcal{O}}^{\varphi, N_\nabla}$ and $\mathcal{D} = \mathcal{D}_{\mathcal{O}}(M)$. Proposition 3.2.1 has shown that $\mathcal{D}_{\mathcal{O}}(M) \in \mathcal{MF}(\varphi, N)$. By Theorem 2.2.1, there exists a unique $D \in \mathrm{MF}(\varphi, N)$ such that $\mathcal{D}_{\mathcal{O}}(M) = \mathcal{D}(D)$. It suffices to check that $D \simeq D(M)$. There exists an isomorphism $i_S : S_{K_0} \otimes_{\varphi, \mathcal{O}} M \simeq D \otimes_{K_0} S_{K_0}$ in $\mathcal{MF}(\varphi, N)$. Modulo u both sides, we get a K_0-linear isomorphism $i : D(M) \simeq D$. It is obvious that i is compatible with φ and N structures on both sides. To see that i is compatible with filtration, recall that the filtration on $D(M)$ depend on the construction of the unique $\mathcal{O}_{[0,\mu)}$-linear, φ-equivariant morphism ξ in (3.1.2):

$$\xi : D(M) \otimes_{K_0} \mathcal{O}_{[0,\mu)} \xrightarrow{\sim} \varphi^* M \otimes_{\mathcal{O}} \mathcal{O}_{[0,\mu)}$$

where μ is any fixed real number such that $p^{-\frac{1}{e}} < \mu < p^{-\frac{1}{pe}}$. Choose μ such that $p^{-\frac{1}{(p-1)e}} < \mu < p^{-\frac{1}{pe}}$. By (3.1.1), $\mathcal{O}_{[0,\mu)}$ is a subring of S_{K_0}. Then we have an isomorphism

$$\varphi^* M \otimes_{\mathcal{O}} \mathcal{O}_{[0,\mu)} \otimes S_{K_0} \simeq M \otimes_{\mathcal{O}, \varphi} S_{K_0} = \mathcal{D}_{\mathcal{O}}(M).$$

So $\xi \otimes_{\mathcal{O}_{[0,\mu)}} S_{K_0}$ and i_S induce an S_{K_0}-linear, filtration compatible isomorphism

$$(D(M) \otimes_{K_0} \mathcal{O}_{[0,\mu]}) \otimes S_{K_0} \simeq D \otimes_{K_0} S_{K_0}.$$

Both sides define filtration on $D(M)$ and D by modulo $E(u)$ respectively. Therefore, filtration on $D(M)$ and D coincides. □

3.3. **Finite φ-modules of finite height and finite $\mathbb{Z}_p$-representations of G_∞.** Recall that $\mathfrak{S} = W(k)[\![u]\!]$ with the endomorphism $\varphi : \mathfrak{S} \to \mathfrak{S}$ which acts on $W(k)$ via Frobenius and send u to u^p. In this subsection, we first recall the theory in [Fon90] on finite φ-modules over $\mathfrak{S}$ of finite height and associated finite $\mathbb{Z}_p$-representations of G_∞. Then we study the relations between the finite φ-module over $\mathfrak{S}$ of finite height and filtered φ-modules over S, and their associated finite representations of G_∞. These results have been essentially done in [Bre98c] and §1.1 in [Kis04].

Denote by $'\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ the category of $\mathfrak{S}$-modules $\mathfrak{M}$ equipped with a φ-semi-linear map $\varphi_{\mathfrak{M}} : \mathfrak{M} \to \mathfrak{M}$ such that the cokernel of the $\mathfrak{S}$-linear map: $1 \otimes \varphi_{\mathfrak{M}} : \mathfrak{S} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} \to \mathfrak{M}$ is killed by $E(u)^r$. (We always drop subscript $\mathfrak{M}$ of $\varphi_{\mathfrak{M}}$ if no confusion will arise.) We give $'\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ the structure of exact category induced by that on the abelian category of $\mathfrak{S}$-modules. We denote by $\mathrm{Mod\ FI}^{\varphi}_{/\mathfrak{S}}$ the full category of $'\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ consisting of those $\mathfrak{M}$ such that as an $\mathfrak{S}$-module $\mathfrak{M}$ is isomorphic to $\oplus_{i \in I} \mathfrak{S}/p^{n_i}\mathfrak{S}$, where I is a finite set and n_i is a positive integer. Finally we denote by $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ the full subcategory of $'\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ consisting of those $\mathfrak{M}$ which are $\mathfrak{S}$-finite free.

Recall that $[\underline{\pi}] \in W(R)$ constructed in §2.2. We embed $\mathfrak{S} \hookrightarrow W(R)$ by $u \mapsto [\underline{\pi}]$. This embedding is compatible with Frobenius endomorphisms. Denote by $\mathcal{O}_{\mathcal{E}}$ the p-adic completion of $\mathfrak{S}[\frac{1}{u}]$. Then $\mathcal{O}_{\mathcal{E}}$ is a discrete valuation ring with the residue field the Laurent series ring $k(\!(u)\!)$. We write $\mathcal{E}$ for the field of fractions of $\mathcal{O}_{\mathcal{E}}$. If $\mathrm{Fr}R$ denotes the field of fractions of R, then the inclusion $\mathfrak{S} \hookrightarrow W(R)$ extends to $\mathcal{O}_{\mathcal{E}} \hookrightarrow W(\mathrm{Fr}R)$. Let $\mathcal{E}^{\mathrm{ur}} \subset W(\mathrm{Fr}R)[1/p]$ denote the maximal unramified extension of $\mathcal{E}$ contained in $W(\mathrm{Fr}R)[1/p]$, and $\mathcal{O}^{\mathrm{ur}}$ its ring of integers. Since $\mathrm{Fr}R$ is easily seen to be algebraically closed, the residue field $\mathcal{O}^{\mathrm{ur}}/p\mathcal{O}^{\mathrm{ur}}$ is the separable closure of $k(\!(u)\!)$. We denote by $\widehat{\mathcal{E}^{\mathrm{ur}}}$ the p-adic completion of $\mathcal{E}^{\mathrm{ur}}$, and by $\widehat{\mathcal{O}^{\mathrm{ur}}}$ its ring of integers. $\widehat{\mathcal{E}^{\mathrm{ur}}}$ is also equal to the closure of $\mathcal{E}^{\mathrm{ur}}$ in $W(\mathrm{Fr}R)$. We write $\mathfrak{S}^{\mathrm{ur}} = \widehat{\mathcal{O}^{\mathrm{ur}}} \cap W(R) \subset W(\mathrm{Fr}R)$. We regard all these rings as subrings of $W(\mathrm{Fr}R)[1/p]$.

Recall $K_\infty = \bigcup_{n \geq 0} K(\pi_n)$ and $G_\infty = \mathrm{Gal}(\bar{K}/K_\infty)$. G_∞ naturally acts on $\mathfrak{S}^{\mathrm{ur}}$ and $\widehat{\mathcal{O}^{\mathrm{ur}}}$ and fixes the subring $\mathfrak{S} \subset W(R)$. Denote $\mathrm{Rep}_{\mathbb{Z}_p}(G_\infty)$ the category of continuous finite $\mathbb{Z}_p$-representations of G_∞. For an $\mathfrak{M} \in \mathrm{Mod\ FI}^{\varphi}_{/\mathfrak{S}}$, one can associate a finite $\mathbb{Z}_p$-representation of G_∞ by (B 1.8, [Fon90]):

$$T_{\mathfrak{S}} : \ \mathfrak{M} \to \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathfrak{S}^{\mathrm{ur}}[1/p]/\mathfrak{S}^{\mathrm{ur}}).$$

In §B.1.8.4 [Fon90] and §A.1.2 [Fon90], Fontaine has proved that the functor $T_{\mathfrak{S}} : \mathrm{Mod\ FI}^{\varphi}_{/\mathfrak{S}} \to \mathrm{Rep}_{\mathbb{Z}_p}(G_\infty)$ is an *exact* functor. If $\mathfrak{M} \simeq \bigoplus_{i=1}^{m} \mathfrak{S}/p^{n_i}\mathfrak{S}$ as finite $\mathfrak{S}$-modules, then $T_{\mathfrak{S}}(\mathfrak{M}) \simeq \bigoplus_{i=1}^{m} \mathbb{Z}/p^{n_i}\mathbb{Z}$ as finite $\mathbb{Z}_p$-modules. As the consequence, if $\mathfrak{M} \in \mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ is a finite free $\mathfrak{S}$-module with rank d, define

$$T_{\mathfrak{S}}(\mathfrak{M}) = \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathfrak{S}^{\mathrm{ur}}),$$

then $T_{\mathfrak{S}}(\mathfrak{M})$ is a continuous finite free $\mathbb{Z}_p$-representation of G_∞ with $\mathbb{Z}_p$-rank d.

As in [Bre98c] or §1.1 [Kis04], we define a functor $\mathcal{M}_{\mathfrak{S}} : {'\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}} \to {'\mathrm{Mod}^{\varphi}_{/S}}$ as follows: we have a map of $W(k)$-algebra $\mathfrak{S} \to S$ given by $u \mapsto u$, so we regard S as an $\mathfrak{S}$-algebra. We will denote by φ the map $\mathfrak{S} \hookrightarrow S$ obtained by composing this map with φ on $\mathfrak{S}$. Given an $\mathfrak{M} \in {}'\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$, set $\mathcal{M} = \mathcal{M}_{\mathfrak{S}}(\mathfrak{M}) := S \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$.

One has the map $1 \otimes \varphi : S \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \to S \otimes_{\mathfrak{S}} \mathfrak{M}$. Set

$$\mathrm{Fil}^r \mathcal{M} = \{y \in \mathcal{M} | (1 \otimes \varphi)(y) \in \mathrm{Fil}^r S \otimes_{\mathfrak{S}} \mathfrak{M} \subset S \otimes_{\mathfrak{S}} \mathfrak{M}\}$$

and define $\varphi_r : \mathrm{Fil}^r \mathcal{M} \to \mathcal{M}$ as the composite

$$\mathrm{Fil}^r \mathcal{M} \xrightarrow{1 \otimes \varphi} \mathrm{Fil}^r S \otimes_{\mathfrak{S}} \mathfrak{M} \xrightarrow{\varphi_r \otimes 1} S \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} = \mathcal{M}.$$

This gives $\mathcal{M}$ the structure of an object in ${}'\mathrm{Mod}^{\varphi}_{/S}$. We have the following result similar to Lemma 2.2.1 in [Bre98c] and Proposition 1.1.11 in [Kis04].

Proposition 3.3.1 (Breuil, Kisin). *The functor $\mathcal{M}_{\mathfrak{S}} : {}'\mathrm{Mod}^{\varphi}_{/\mathfrak{S}} \to {}'\mathrm{Mod}^{\varphi}_{/S}$ defined above induces an exact and fully faithful functor $\mathcal{M}_{\mathfrak{S}} : \mathrm{Mod}\ \mathrm{FI}^{\varphi}_{/\mathfrak{S}} \to \mathrm{Mod}\ \mathrm{FI}^{\varphi}_{/S}$. This functor is an equivalence of categories between the full subcategories consisting of objects killed by p.*

Proof. Lemma 2.2.1 in [Bre98c] and Proposition 1.1.11 in [Kis04] proved the case $r = 1$. The idea of proof can be easily extended for $0 \leq r \leq p - 2$. In particular, the equivalence of subcategories consisting of p-torsion objects is again (almost) *verbatim* the proof of Theorem 4.1.1 in [Bre99a]. □

Corollary 3.3.2. *The functor $\mathcal{M}_{\mathfrak{S}} : {}'\mathrm{Mod}^{\varphi}_{/\mathfrak{S}} \to {}'\mathrm{Mod}^{\varphi}_{/S}$ induces an exact and fully faithful functor $\mathcal{M}_{\mathfrak{S}} : \mathrm{Mod}^{\varphi}_{/\mathfrak{S}} \to \mathrm{Mod}^{\varphi}_{/S}$.*

Remark 3.3.3. In fact, the functor $\mathcal{M}_{\mathfrak{S}}$ can be proved to be an equivalence ([CL06]).

Note that A_{cris} is an object in ${}'\mathrm{Mod}^{\varphi}_{/S}$ by defining $\varphi_r := \varphi/p^r$ on $\mathrm{Fil}^r A_{\mathrm{cris}}$. For any $\mathcal{M} \in \mathrm{Mod}^{\varphi}_{/S}$, one can define a finite free continuous $\mathbb{Z}_p$-representation of G_∞:

$$T_{\mathrm{cris}} : \mathcal{M} \to \mathrm{Hom}_{{}'\mathrm{Mod}^{\varphi}_{/S}}(\mathcal{M}, A_{\mathrm{cris}}) \tag{3.3.1}$$

as in §2.3.1 in [Bre99a]. Let $\mathfrak{M} \in \mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ and $\mathcal{M} = \mathcal{M}_{\mathfrak{S}}(\mathfrak{M}) \in \mathrm{Mod}^{\varphi}_{/S}$. For any $f \in T_{\mathfrak{S}}(\mathfrak{M}) = \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathfrak{S}^{\mathrm{ur}})$, consider the natural embedding $\iota : \mathfrak{S}^{\mathrm{ur}} \hookrightarrow A_{\mathrm{cris}}$. It is easy to check that $\varphi(\iota \circ f) \in T_{\mathrm{cris}}(\mathcal{M}) = \mathrm{Hom}_{{}'\mathrm{Mod}^{\varphi}_{/S}}(\mathcal{M}, A_{\mathrm{cris}})$. Therefore, we get a natural map $\mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathfrak{S}^{\mathrm{ur}}) \to \mathrm{Hom}_{{}'\mathrm{Mod}^{\varphi}_{/S}}(\mathcal{M}_{\mathfrak{S}}(\mathfrak{M}), A_{\mathrm{cris}})$.

Lemma 3.3.4. *The natural map $T_{\mathfrak{S}}(\mathfrak{M}) \to T_{\mathrm{cris}}(\mathcal{M}_{\mathfrak{S}}(\mathfrak{M}))$ defined above is an isomorphism of finite free $\mathbb{Z}_p$-representations of G_∞.*

Proof. This is the consequence of the fact that for any $\mathfrak{M} \in \mathrm{Mod}\ \mathrm{FI}^{\varphi}_{/\mathfrak{S}}$, the natural map

$$\mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathfrak{S}^{\mathrm{ur}}[1/p]/\mathfrak{S}^{\mathrm{ur}}) \to \mathrm{Hom}_{{}'\mathrm{Mod}^{\varphi}_{/S}}(\mathcal{M}_{\mathfrak{S}}(\mathfrak{M}), A_{\mathrm{cris}}[1/p]/A_{\mathrm{cris}}) \tag{3.3.2}$$

is an isomorphism of finite $\mathbb{Z}_p[G_\infty]$-modules. Note that the left hand side of (3.3.2) is an exact functor on $\mathrm{Mod}\ \mathrm{FI}^{\varphi}_{/\mathfrak{S}}$. The right hand side is also an exact functor from the fact that $\mathrm{Ext}^1_{{}'\mathrm{Mod}^{\varphi}_{/S}}(\mathcal{M}, A_{\mathrm{cris}}[1/p]/A_{\mathrm{cris}}) = 0$ for any $\mathcal{M} \in \mathrm{Mod}\ \mathrm{FI}^{\varphi}_{/S}$ (Lemma 2.3.1.3 in [Bre99a]). Thus by the standard *dévissage*, it suffices to prove (3.3.2) for the case that p kills $\mathfrak{M}$, and this is Proposition 4.2.1 in [Bre99b]. □

3.4. **G_∞-stable $\mathbb{Z}_p$-lattices in a semi-stable Galois representation.** A (φ, N)-module over $\mathfrak{S}$ is a finite free φ-module $\mathfrak{M} \in \mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$, equipped with a linear endomorphism $N : \mathfrak{M}/u\mathfrak{M} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to \mathfrak{M}/u\mathfrak{M} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ such that $N\varphi = p\varphi N$. We denote by $\mathrm{Mod}^{\varphi,N}_{/\mathfrak{S}}$ the category of (φ, N)-module over $\mathfrak{S}$, and by $\mathrm{Mod}^{\varphi,N}_{/\mathfrak{S}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ the associated isogeny category [1]. The following theorem is one of main results (cf. Corollary 1.3.15) in [Kis05].

Theorem 3.4.1 (Kisin). *There exists a fully faithful $\otimes$-functor Θ from the category of positive weakly admissible filtered (φ, N)-modules $\mathrm{MF}^{\mathrm{w}}(\varphi, N)$ to $\mathrm{Mod}^{\varphi,N}_{/\mathfrak{S}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.*

Let $\mathfrak{M} \in \mathrm{Mod}^{\varphi,N}_{/\mathfrak{S}}$ and $M = \mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}$. Then there exists a $D \in \mathrm{MF}^{\mathrm{w}}(\varphi, N)$ such that $\mathfrak{M} = \Theta(D)$ if and only if there exists a differential operator N_∇ on M such that $(M, \varphi, N_\nabla) \in \mathrm{Mod}^{\varphi,N_\nabla}_{/\mathcal{O}}$, $D(M) \simeq D$ in $\mathrm{MF}(\varphi, N)$ and $N_\nabla \mod u = N$ on $\mathfrak{M}/u\mathfrak{M} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Such N_∇ (if exists) is necessarily unique.

Remark 3.4.2.
1. The above theorem is valid without any restriction of the maximal Hodge-Tate weight. But here we only consider the case that Hodge-Tate weights in $\{0, \dots, r\}$ with $r \leq p-2$.
2. The second paragraph of the above theorem is not the same as that of Corollary 1.3.15 in [Kis05]. But they are equivalent (See Lemma 1.3.10 and Lemma 1.3.13 in [Kis05]), and our description of Theorem 3.4.1 will be more convenient.

Furthermore, Kisin proved (cf. Proposition 2.1.5 in [Kis05]) that there exists a canonical bijection (without restriction of maximal Hodge-Tate weights)

$$\eta : \; T_{\mathfrak{S}}(\mathfrak{M}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\sim} V_{\mathrm{st}}(D) \tag{3.4.1}$$

which is compatible with the action of G_∞ on the two sides. For our purpose to connect strongly divisible lattices, we reconstruct (3.4.1) in a little different way.

Let $D \in \mathrm{MF}^{\mathrm{w}}(\varphi, N)$ be a weakly admissible filtered (φ, N)-module under our Assumption 2.3.1, $\mathfrak{M} = \Theta(D)$ and $(M, \varphi, N_\nabla) \in \mathrm{Mod}^{\varphi,N_\nabla}_{/\mathcal{O}}$ as in the Theorem 3.4.1. Let $\mathcal{D} = \mathcal{D}(D)$ (Recall $\mathcal{D}(D) := S \otimes_{W(k)} D$ in §2.2). By Corollary 3.2.3, we have $\mathcal{D} = S_{K_0} \otimes_{\varphi,\mathcal{O}} M = S_{K_0} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} = \mathcal{M}_{\mathfrak{S}}(\mathfrak{M}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, where $\mathcal{M}_{\mathfrak{S}}$ is the functor defined in Corollary 3.3.2. Then we have a natural map of $\mathbb{Z}_p[G_\infty]$-modules

$$\mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathfrak{S}^{\mathrm{ur}}) \xrightarrow{\sim} \mathrm{Hom}_{'\mathrm{Mod}^{\varphi}_{/S}}(\mathcal{M}_{\mathfrak{S}}(\mathfrak{M}), A_{\mathrm{cris}}) \hookrightarrow \mathrm{Hom}_{'\mathrm{Mod}^{\varphi}_{/S}}(\mathcal{D}, B^+_{\mathrm{cris}}). \tag{3.4.2}$$

The first map is an isomorphism by Lemma 3.3.4. Recall that

$$V_{\mathrm{st}}(\mathcal{D}) = \mathrm{Hom}_{'\mathrm{Mod}^{\varphi,N}_{/S}}(\mathcal{D}, \widehat{A_{\mathrm{st}}}[1/p]).$$

The canonical projection $\widehat{A_{\mathrm{st}}} \to A_{\mathrm{cris}}$ defined by sending $\gamma_i(X)$ to 0 induces a natural map:

$$\mathrm{Hom}_{'\mathrm{Mod}^{\varphi,N}_{/S}}(\mathcal{D}, \widehat{A_{\mathrm{st}}}[1/p]) \to \mathrm{Hom}_{'\mathrm{Mod}^{\varphi}_{/S}}(\mathcal{D}, B^+_{\mathrm{cris}}). \tag{3.4.3}$$

We claim that the above map is an bijection. Let us accept the claim and postpone the proof in Lemma 3.4.3. Recall that Theorem 2.2.1 has shown that there exists

[1] Recall that if C is an additive category, then the associated isogeny category $\mathcal{D}$ has same objects and $\mathrm{Hom}_{\mathcal{D}}(A, B) = \mathrm{Hom}_C \otimes_{\mathbb{Z}} \mathbb{Q}$ for all objects A and B.

a canonical isomorphism $V_{\mathrm{st}}(\mathcal{D}) \simeq V_{\mathrm{st}}(D)$ as $\mathbb{Q}_p$-representations of G. Therefore, combining (3.4.2) and (3.4.3), we have a natural injection

$$\eta : T_{\mathfrak{S}}(\mathfrak{M}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \hookrightarrow V_{\mathrm{st}}(D)$$

of $\mathbb{Q}_p[G_\infty]$-modules and thus $\dim_{\mathbb{Q}_p}(V_{\mathrm{st}}(D)) \geq \mathrm{rank}_{\mathfrak{S}}(\mathfrak{M}) = \dim_{K_0}(D)$. But an elementary argument (Prop. 4.5, [CF00]) showed that weak admissibility of D implies that $\dim_{\mathbb{Q}_p}(V_{\mathrm{st}}(D))$ has to be $\dim_{K_0}(D)$. Hence the map η is a bijection.

Lemma 3.4.3. *The natural map defined in* (3.4.3) *is a bijection.*

Proof. We basically follow the idea of Lemma 2.3.1.1 in [Bre99a]. For any $f \in \mathrm{Hom}_{'\mathrm{Mod}^{\varphi,N}_{/S}}(\mathcal{D}, \widehat{A_{\mathrm{st}}}[1/p])$, let f_0 be its image of the map in (3.4.3). For any $x \in D$ where $\mathcal{D} = D \otimes_{W(k)} S$, since $N^i(x) = 0$ for i enough big, we can easily check that

$$f(x) = \sum_{i=0}^{\infty} f_0(N^i(x))\gamma_i(\log(1+X)), \tag{3.4.4}$$

where $\gamma_i(x) = \frac{x^i}{i!}$ is the standard divided power. So if $f_0 = 0$, we have $f = 0$ because D generates $\mathcal{D}$. Thus (3.4.3) is injective. To prove the surjectivity, let $f_0 \in \mathrm{Hom}_{'\mathrm{Mod}^{\varphi}_{/S}}(\mathcal{D}, B^+_{\mathrm{cris}})$. For any $y \in \mathcal{D}$, define

$$f(y) = \sum_{i=0}^{\infty} f_0(N^i(y))\gamma_i(\log(1+X)).$$

To see that f is well defined, note that $f(y)$ converges in $B^+_{\mathrm{cris}}[\![X]\!]$, and if $x \in D$ then $f(x)$ converges in $\widehat{A_{\mathrm{st}}}[1/p]$ because $N^i(x) = 0$ for i enough big. By a standard computation, we can easily check that $f : \mathcal{D} \to B^+_{\mathrm{cris}}[\![X]\!]$ is S-linear. Therefore $f : \mathcal{D} \to \widehat{A_{\mathrm{st}}}[1/p]$ is well defined. It suffices to check that f preserves Frobenius, monodromy and filtration. Since f_0 preserves all these structures, it is a strait forward calculation to check that f preserves Frobenius, monodromy and filtration, combining with the facts that $\varphi(\log(1+X)) = p\log(1+X)$, $N(\log(1+X)) = 1$, $N^j(\mathrm{Fil}^i\mathcal{D}) \subset \mathrm{Fil}^{i-j}\mathcal{D}$ and $\log(1+X) \in \mathrm{Fil}^1\widehat{A_{\mathrm{st}}}$. □

Remark 3.4.4.
(1) Let $V_{\mathrm{cris}}(\mathcal{D}) := \mathrm{Hom}_{'\mathrm{Mod}^{\varphi}_{/S}}(\mathcal{D}, B^+_{\mathrm{cris}})$. The above lemma gives a natural transformation which makes the following diagram commutative:

$$\begin{array}{ccc} \mathcal{MF}(\varphi, N) & \xrightarrow{V_{\mathrm{cris}}} & \mathrm{Rep}_{\mathbb{Q}_p}(G_\infty) \\ \Big\| & \nwarrow & \uparrow \\ \mathcal{MF}(\varphi, N) & \xrightarrow{V_{\mathrm{st}}} & \mathrm{Rep}_{\mathbb{Q}_p}(G) \end{array}$$

(2) From the above proof, we see that the lemma is always valid without any restriction of the maximal Hodge-Tate weight.

One advantage of using (φ, N)-module over $\mathfrak{S}$ is that we can classify all G_∞-stable $\mathbb{Z}_p$-lattices inside the Galois representation.

Lemma 3.4.5 (Kisin).
(1) *Let V be a semi-stable representation with Hodge-Tate weights in $\{0, \ldots, r\}$. For any G_∞-stable $\mathbb{Z}_p$-lattice $T \subset V$, there always exists an $\mathfrak{N} \in \mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ such that $T_{\mathfrak{S}}(\mathfrak{N}) \simeq T$.*

(2) *The functor* $T_{\mathfrak{S}} : \mathrm{Mod}^{\varphi}_{/\mathfrak{S}} \to \mathrm{Rep}_{\mathbb{Z}_p}(G_\infty)$ *is fully faithful.*

Proof. These are easy consequences of Lemma (2.1.15) and Proposition (2.1.12) in [Kis05]. Remark that the lemma is valid without restriction of r. □

Recall that $\widetilde{\mathrm{Mod}}^{\varphi}_{/S}$ denote the category of quasi-strongly divisible lattices of weight r. Let $\mathcal{M} \in \widetilde{\mathrm{Mod}}^{\varphi}_{/S}$ be a quasi-strongly divisible lattice. By Definition 2.3.3, there exists a $\mathcal{D} \in \mathcal{MF}^{\mathrm{w}}(\varphi, N)$ such that $\mathcal{M} \subset \mathcal{D}$ and $\mathcal{D} \simeq \mathcal{D}(D)$ with D weakly admissible. Let $V := V_{\mathrm{st}}(\mathcal{D})$ be the semi-stable Galois representation. Then we can associate a G_∞-stable $\mathbb{Z}_p$-lattice in V as the following:

$$\mathcal{M} \mapsto T_{\mathrm{cris}}(\mathcal{M}) = \mathrm{Hom}_{'\mathrm{Mod}^{\varphi}_{/S}}(\mathcal{M}, A_{\mathrm{cris}}) \hookrightarrow \mathrm{Hom}_{'\mathrm{Mod}^{\varphi}_{/S}}(\mathcal{D}, B^{+}_{\mathrm{cris}}) \simeq V_{\mathrm{st}}(\mathcal{D}) = V.$$

Recall that the isomorphism $V_{\mathrm{st}}(\mathcal{D}) \xrightarrow{\sim} \mathrm{Hom}_{'\mathrm{Mod}^{\varphi}_{/S}}(\mathcal{D}, B^{+}_{\mathrm{cris}})$ has been established in Lemma 3.4.3. Therefore T_{cris} induces a functor from $\widetilde{\mathrm{Mod}}^{\varphi}_{/S}$ to $\mathrm{Rep}^{\mathrm{st}}_{\mathbb{Z}_p}(G_\infty)$, where $\mathrm{Rep}^{\mathrm{st}}_{\mathbb{Z}_p}(G_\infty)$ denotes the category of G_∞-stable $\mathbb{Z}_p$-lattices in semi-stable Galois representations with Hodge-Tate weights in $\{0, \ldots, r\}$.

Proposition 3.4.6. *The functor* T_{cris} *induces an anti-equivalence between* $\widetilde{\mathrm{Mod}}^{\varphi}_{/S}$ *and* $\mathrm{Rep}^{\mathrm{st}}_{\mathbb{Z}_p}(G_\infty)$.

Proof. We first prove the essential surjectivity of the functor. Let $\mathfrak{M} = \Theta(D)$ as in Theorem 3.4.1 and $\mathcal{D} = \mathcal{D}(D)$. By corollary 3.2.3 and Theorem 3.4.1, we see that $\mathcal{D} = \mathfrak{M} \otimes_{\mathfrak{S},\varphi} S_{K_0}$. Suppose that $T \subset V$ is a G_∞-stable $\mathbb{Z}_p$-lattice. Then by lemma 3.4.5, there exists an $\mathfrak{N} \in \mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$, such that $T \simeq T_{\mathfrak{S}}(\mathfrak{N})$. We claim that $\mathfrak{M} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq \mathfrak{N} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. In fact, since $T_{\mathfrak{S}}(\mathfrak{M})$ and $T_{\mathfrak{S}}(\mathfrak{N})$ are G_∞-stable $\mathbb{Z}_p$-lattices in V, there exist G_∞-equivariant maps $f : T_{\mathfrak{S}}(\mathfrak{M}) \to T_{\mathfrak{S}}(\mathfrak{N})$ and $g : T_{\mathfrak{S}}(\mathfrak{N}) \to T_{\mathfrak{S}}(\mathfrak{M})$ such that $f \circ g = p^n \mathrm{Id}$. By full faithfullness of $T_{\mathfrak{S}}$, there exists $F : \mathfrak{N} \to \mathfrak{M}$ and $G : \mathfrak{M} \to \mathfrak{N}$ such that $F \circ G = p^n \mathrm{Id}$. Hence the claim follows. Now put $\mathcal{N} = \mathcal{M}_{\mathfrak{S}}(\mathfrak{N})$. We see that $\mathcal{N}$ is a quasi-strongly divisible lattice in $\mathcal{D}$, and by Lemma 3.3.4, $T_{\mathrm{cris}}(\mathcal{N}) = T$. This proves that the functor is essential surjective. Let $\mathcal{M}, \mathcal{N} \in \widetilde{\mathrm{Mod}}^{\varphi}_{/S}$ and $f : T_{\mathrm{cris}}(\mathcal{N}) \to T_{\mathrm{cris}}(\mathcal{M})$ a morphism of $\mathbb{Z}_p[G_\infty]$-module. From the above proof, there exist $\mathfrak{M}, \mathfrak{N} \in \mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ such that $T_{\mathfrak{S}}(\mathfrak{M}) = T_{\mathrm{cris}}(\mathcal{M})$ and $T_{\mathfrak{S}}(\mathfrak{N}) = T_{\mathrm{cris}}(\mathcal{N})$. Since $T_{\mathfrak{S}}$ is fully faithful (Lemma 3.4.5 (2)), there exists $\mathfrak{f} : \mathfrak{M} \to \mathfrak{N}$ a morphism in $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ such that $T_{\mathfrak{S}}(\mathfrak{f}) = f$. Then by Lemma 3.3.2 and Lemma 3.3.4, we have $T_{\mathrm{cris}}(\mathcal{M}_{\mathfrak{S}}(\mathfrak{f})) = f$. It suffices to show that $\mathcal{M} = \mathcal{M}_{\mathfrak{S}}(\mathfrak{M})$ and $\mathcal{N} = \mathcal{M}_{\mathfrak{S}}(\mathfrak{N})$. Therefore, we reduce the proof to the following □

Lemma 3.4.7. *Let* $\mathcal{M}$, $\mathcal{M}'$ *be two quasi-strongly lattices contained in* $\mathcal{D}$. *If* $T_{cris}(\mathcal{M}) = T_{cris}(\mathcal{M}')$ *then* $\mathcal{M} = \mathcal{M}'$.

We postpone our proof of the Lemma after Lemma 5.3.1.

We may summarize our discussion in this subsection into the follow commutative diagram:

$$\begin{array}{ccccc} & & \overset{T_{\mathfrak{S}}}{\curvearrowright} & & \\ \mathrm{Mod}^{\varphi}_{/\mathfrak{S}} & \overset{\mathcal{M}_{\mathfrak{S}}}{\hookrightarrow} & \mathrm{Mod}^{\varphi}_{/S} & \xrightarrow{T_{\mathrm{cris}}} & \mathrm{Rep}_{\mathbb{Z}_p}(G_\infty) \\ & & \uparrow & & \uparrow \\ & & \widetilde{\mathrm{Mod}}^{\varphi}_{/S} & \xrightarrow[\sim]{T_{\mathrm{cris}}} & \mathrm{Rep}^{\mathrm{st}}_{\mathbb{Z}_p}(G_\infty) \end{array}$$

3.5. **Fully faithfulness of T_{st}.** Now suppose that T is a *G-stable* $\mathbb{Z}_p$-lattice in a semi-stable Galois representation V. By Proposition 3.4.6, there exists a quasi-strongly divisible lattice $\mathcal{M}$ in $\mathcal{D}$ such that $T_{\mathrm{cris}}(\mathcal{M}) = T|_{G_\infty}$ and there exists an $\mathfrak{M} \in \mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ such that $\mathcal{M} = \mathcal{M}_{\mathfrak{S}}(\mathfrak{M})$.

Proposition 3.5.1. *Notations as the above. If $N(\mathcal{M}) \subset \mathcal{M}$, then $(\mathcal{M}, \varphi, \mathrm{Fil}^r\mathcal{M}, N)$ is a strongly divisible lattice in $\mathcal{D}$ and $T_{\mathrm{st}}(\mathcal{M}) = T$.*

Proof. $\mathcal{M}$ is clearly a strongly divisible lattice in $\mathcal{D}$. It suffices to prove that $T_{\mathrm{st}}(\mathcal{M}) = T$. By Proposition 2.3.4,

$$T_{\mathrm{st}}(\mathcal{M}) = \mathrm{Hom}_{'\mathrm{Mod}^{\varphi,N}_{/S}}(\mathcal{M}, \widehat{A_{\mathrm{st}}}) \subset V_{\mathrm{st}}(\mathcal{D}) \simeq V_{\mathrm{st}}(D) = V$$

is a G-stable $\mathbb{Z}_p$-lattice. As in (3.4.3), the canonical projection $\widehat{A_{\mathrm{st}}} \to A_{\mathrm{cris}}$ defined by sending $\gamma_i(X) \to 0$ induces a natural map

$$T_{\mathrm{st}}(\mathcal{M}) = \mathrm{Hom}_{'\mathrm{Mod}^{\varphi,N}_{/S}}(\mathcal{M}, \widehat{A_{\mathrm{st}}}) \to \mathrm{Hom}_{'\mathrm{Mod}^{\varphi}_{/S}}(\mathcal{M}, A_{\mathrm{cris}}) = T_{\mathrm{cris}}(\mathcal{M}). \tag{3.5.1}$$

Then we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Hom}_{'\mathrm{Mod}^{\varphi,N}_{/S}}(\mathcal{M}, \widehat{A_{\mathrm{st}}}) & \hookrightarrow & \mathrm{Hom}_{'\mathrm{Mod}^{\varphi,N}_{/S}}(\mathcal{D}, \widehat{A_{\mathrm{st}}}[1/p]) \\ \downarrow{\scriptstyle (3.5.1)} & & \wr\downarrow{\scriptstyle (3.4.3)} \\ \mathrm{Hom}_{'\mathrm{Mod}^{\varphi}_{/S}}(\mathcal{M}, A_{\mathrm{cris}}) & \hookrightarrow & \mathrm{Hom}_{'\mathrm{Mod}^{\varphi}_{/S}}(\mathcal{D}, B^+_{\mathrm{cris}}) \\ \| & & \| \\ T & \hookrightarrow & V \end{array}$$

Thus it suffices to show that (3.5.1) is an isomorphism of $\mathbb{Z}_p$-modules. This has been proved in §2.3.1, [Bre99a]. □

Corollary 3.5.2. *The functor T_{st} in the Main Conjecture 1.0.1 is fully faithful.*

Proof. Let $\mathcal{M}$, $\mathcal{M}'$ be strongly divisible lattices, $\mathcal{D} = \mathcal{M} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, $\mathcal{D}' = \mathcal{M}' \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and $T_{\mathrm{st}}(\mathcal{M})$, $T_{\mathrm{st}}(\mathcal{M}')$ G-stable $\mathbb{Z}_p$-lattices in $V_{\mathrm{st}}(\mathcal{D})$, $V_{\mathrm{st}}(\mathcal{D}')$ respectively. Suppose that $f : T_{\mathrm{st}}(\mathcal{M}) \to T_{\mathrm{st}}(\mathcal{M}')$ is a morphism of $\mathbb{Z}_p[G]$-modules. Tensoring by $\mathbb{Q}_p$, there exists an $\mathfrak{f} : \mathcal{D}' \to \mathcal{D}$ such that $V_{\mathrm{st}}(\mathfrak{f}) = f \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. It suffices to show that $\mathfrak{f}(\mathcal{M}') \subset \mathcal{M}$. Select an n such that $p^n\mathfrak{f}(\mathcal{M}') \subset \mathcal{M}$. Then $\mathfrak{g} := p^n\mathfrak{f}$ is a morphism of strongly divisible lattices and $T_{\mathrm{st}}(\mathfrak{g}) = p^n f$. Note that (3.5.1) is an isomorphism of $\mathbb{Z}_p[G_\infty]$-modules. So if $\mathfrak{g}$ is regarded as a morphism of quasi-strongly divisible lattices, we have $T_{\mathrm{cris}}(\mathfrak{g}) = T_{\mathrm{st}}(\mathfrak{g}) = p^n f$. On the other hand, by Proposition 3.4.6, T_{cris} is fully faithful, there exists a morphism $\mathfrak{g}' : \mathcal{M}' \to \mathcal{M}$ in $\mathrm{Mod}^{\varphi}_{/S}$ such that $T_{\mathrm{cris}}(\mathfrak{g}') = f$. Therefore $p^n\mathfrak{g}' = \mathfrak{g} = p^n\mathfrak{f}$. Then $\mathfrak{f} = \mathfrak{g}'$ and $\mathfrak{f}(\mathcal{M}') = \mathfrak{g}'(\mathcal{M}') \subset \mathcal{M}$. □

Also we reduce the proof of the essential surjectivity of T_{st} to the following

Lemma 3.5.3. *Notations as the above, If T is G-stable then $N(\mathcal{M}) \subset \mathcal{M}$.*

We will devote the next two sections to prove this Lemma. Combining with Proposition 3.5.1, Corollary 3.5.2 and Proposition 3.4.6, we prove the Main Theorem (Theorem 2.3.5) and the following

Theorem 3.5.4. *The functor T_{cris} induces an anti-equivalence between the category of quasi-strongly divisible lattices of weight r and the category of G_∞-stable $\mathbb{Z}_p$-lattices inside semi-stable Galois representations with Hodge-Tate weights in $\{0,\dots,r\}$. Furthermore, a quasi-strongly divisible lattice $\mathcal{M}$ is strongly divisible if and only if $T_{\mathrm{cris}}(\mathcal{M})$ is G-stable.*

4. Cartier Dual and a Theorem to Connect $\mathcal{M}$ with $T_{\mathrm{cris}}(\mathcal{M})$

In this section, we extend a theorem of Faltings (cf. Theorem 5, [Fal99]) to a more general setting to connect filtered φ-modules over S with their associated $\mathbb{Z}_p$-representations of G_∞. This theorem is one of technical keys to prove Lemma 3.5.3. For this purpose, we need more explicit structure of $\mathrm{Fil}^r\mathcal{M}$ and a notion of Cartier dual for $\mathcal{M} \in \mathrm{Mod}^{\varphi}_{/S}$. Luckily, such Cartier dual has been available from the thesis of Caruso [Car05]. In the following two section, we always regard $W(k)[u]$ and S as subrings of A_{cris} via $u \mapsto [\underline{\pi}]$, and denote the identity matrix by I.

4.1. **Structure of filtration of quasi-strongly divisible lattice.**

Lemma 4.1.1. *Let A be a $d\times d$ matrix with coefficients in $W(k)[u]$. Suppose that there exist matrices B' and C with coefficients in S and Fil^pS respectively such that $AB' = E(u)^rI + C$. Then*

1. *There exists a matrix B with coefficients in S such that $AB = E(u)^rI$.*
2. *Let $a_i \in A_{\mathrm{cris}}$ for $i = 1,\dots,d$. If $(a_1,\dots,a_d)A$ is in $\mathrm{Fil}^rA_{\mathrm{cris}}$, then there exists $b_i \in A_{\mathrm{cris}}$ and $c_i \in \mathrm{Fil}^pA_{\mathrm{cris}}$ for $i = 1,\dots,d$ such that*
$$(a_1,\dots,a_d) = (b_1,\dots,b_d)B + (c_1,\dots,c_d).$$

Proof. Note that for any $f \in S$, we can always write $f = f_0 + f_1$ with $f_0 \in W(k)[u]$ and $f_1 \in \mathrm{Fil}^pS$. So $B' = B_0 + B_1$ with B_0's coefficients in $W(k)[u]$ and B_1's coefficients in Fil^pS. Therefore, $E(u)^rI = AB_0 + C_1$ with C_1's coefficients in $W(k)[u] \cap \mathrm{Fil}^pS = E(u)^pW(k)[u]$. Thus $C_1 = E(u)^pC_2$ with C_2's coefficients in $W(k)[u]$. Now we have $E(u)^rI = AB_0 + E(u)^pC_2$. Since $E(u)^n \to 0$ p-adically in S when $n \to \infty$, $I - E(u)^{p-r}C_2$ is invertible. Thus

$$E(u)^rI = AB_0(I - E(u)^{p-r}C_2)^{-1}. \tag{4.1.1}$$

Let $B = B_0(I - E(u)^{p-r}C_2)^{-1}$ and we settle (1).

For (2), write $(a_1,\dots,a_d) = (b'_1,\dots,b'_d)+(c_1,\dots,c_d)$ with $b'_i \in W(R)$ and $c_i \in \mathrm{Fil}^pA_{\mathrm{cris}}$ for $i = 1,\dots,d$. It suffices to prove that there exists $b_i \in A_{\mathrm{cris}}$ such that $(b'_1,\dots,b'_d) = (b_1,\dots,b_d)B$. Note that

$$(a_1,\dots,a_d)A = (b'_1,\dots,b'_d)A + (c_1,\dots,c_d)A \in \mathrm{Fil}^rA_{\mathrm{cris}}.$$

Then $(b'_1, \dots, b'_d)A \in \mathrm{Fil}^r A_{\mathrm{cris}} \cap W(R) = E(u)^r W(R)$. So there exists $b_i \in W(R)$ such that $(b'_1, \dots, b'_d)A = E(u)^r(b_1, \dots, b_d)$. Multiplying by B on both sides, we get $(b'_1, \dots, b'_d)AB = E(u)^r(b_1, \dots, b_d)B$. Finally, $(b'_1, \dots, b'_d) = (b_1, \dots, b_d)B$ as required. □

Proposition 4.1.2. *Let $\mathcal{M} \in \mathrm{Mod}^{\varphi}_{/S}$. There exists $\alpha_1, \dots, \alpha_d \in \mathrm{Fil}^r\mathcal{M}$ such that*

(1) $\mathrm{Fil}^r\mathcal{M} = \bigoplus_{i=1}^{d} S\alpha_i + (\mathrm{Fil}^p S)\mathcal{M}$.

(2) *$E(u)^r\mathcal{M} \subseteq \bigoplus_{i=1}^{d} S\alpha_i$ and $(\varphi_r(\alpha_1), \dots, \varphi_r(\alpha_d))$ is a basis of $\mathcal{M}$.*

Proof. Considering $\mathcal{M}/p\mathcal{M}$, by Proposition 2.2.1.3 in [Bre99a], $\mathcal{M}/p\mathcal{M}$ has a "base adaptée", i.e., there exists a basis $(e_1, \dots, e_d)$ of $\mathcal{M}$ and $\alpha_1, \dots, \alpha_d \in \mathrm{Fil}^r\mathcal{M}$ such that

$$\mathrm{Fil}^r\mathcal{M}/p\mathrm{Fil}^r\mathcal{M} = \bigoplus_{i=1}^{d} S_1\bar{\alpha}_i + \mathrm{Fil}^p S_1(\mathcal{M}/p\mathcal{M}) \tag{4.1.2}$$

such that $(\bar{\alpha}_1, \dots, \bar{\alpha}_d) = (u^{r_1}\bar{e}_1, \dots, u^{r_d}\bar{e}_d)$ with $0 \leq r_i \leq er$, where $S_1 = S/pS$ and $\bar{\alpha}_i$, $\bar{e}_i$ is the image of α_i, e_i in $\mathcal{M}/p\mathcal{M}$ respectively. Let $\tilde{\mathcal{M}} = \bigoplus_{i=1}^{d} S\alpha_i + (\mathrm{Fil}^p S)\mathcal{M}$. Then $\tilde{\mathcal{M}} \subset \mathrm{Fil}^r\mathcal{M}$. We claim that the natural map

$$f : \tilde{\mathcal{M}}/\mathrm{Fil}^p S\mathcal{M} \to \mathrm{Fil}^r\mathcal{M}/\mathrm{Fil}^p S\mathcal{M}$$

is surjective. To see the claim, note that $S/\mathrm{Fil}^p S \xrightarrow{\sim} W(k)[u]/(E(u)^p)$ is Noetherian. By Nakayama's lemma, it suffices to show that $f \mod p$ is a surjection. Note that

$$\mathrm{Fil}^r\mathcal{M}/\mathrm{Fil}^p S\mathcal{M} \mod p = (\mathrm{Fil}^r\mathcal{M})_1/(\mathrm{Fil}^p S\mathcal{M})_1$$

where $(\mathrm{Fil}^r\mathcal{M})_1 = \mathrm{Fil}^r\mathcal{M}/p\mathrm{Fil}^r\mathcal{M}$ and $(\mathrm{Fil}^p S\mathcal{M})_1 = \mathrm{Fil}^p S\mathcal{M}/p\mathrm{Fil}^p S\mathcal{M}$. By (4.1.2), we see that $f \mod p$ is surjective and thus prove the claim. Then

$$\mathrm{Fil}^r\mathcal{M} = \tilde{\mathcal{M}} = \bigoplus_{i=1}^{d} S\alpha_i + (\mathrm{Fil}^p S)\mathcal{M}. \tag{4.1.3}$$

Let $(\alpha_1, \dots, \alpha_d) = (e_1, \dots, e_d)A$ where A is a $d \times d$ matrix with coefficients in S. Write $A = A_0 + A_1$ with A_0's coefficients in $W(k)[u]$ and A_1's coefficients in $\mathrm{Fil}^p S$. Replacing $(\alpha_1, \dots, \alpha_d)$ by $(e_1, \dots, e_d)A_0$, we can always assume that A's coefficients are in $W(k)[u]$. By (4.1.3), there exists $d \times d$ matrices B', C with coefficients in S, $\mathrm{Fil}^p S$ respectively such that $E(u)^r I = AB' + C$. Then by Lemma 4.1.1, there exists a B with coefficients in S such that $AB = E(u)^r I$. Therefore $E(u)^r\mathcal{M} \subset \bigoplus_{i=1}^{d} S\alpha_i$. Since $\varphi_r(\mathrm{Fil}^r\mathcal{M})$ generates $\mathcal{M}$ and one always has $p|\varphi_r(\mathrm{Fil}^p S)$, we see that $(\varphi_r(\alpha_1), \dots, \varphi_r(\alpha_d))$ is a basis of $\mathcal{M}$. □

Let $\mathcal{D} \in \mathcal{MF}(\varphi, N)$ be a filtered (φ, N)-module over S. Following §3 in [Bre97], we define

$$\mathrm{Fil}^r(A_{\mathrm{cris}} \otimes_S \mathcal{D}) = \sum_{i=0}^{r} \mathrm{Im}(\mathrm{Fil}^{r-i}A_{\mathrm{cris}} \otimes_S \mathrm{Fil}^i\mathcal{D}), \tag{4.1.4}$$

where $\mathrm{Im}(\mathrm{Fil}^{r-i}A_{\mathrm{cris}} \otimes_S \mathrm{Fil}^i\mathcal{D})$ is the image of $\mathrm{Fil}^{r-i}A_{\mathrm{cris}} \otimes_S \mathrm{Fil}^i\mathcal{D}$ in $A_{\mathrm{cris}} \otimes_S \mathcal{D}$. We also define $\mathrm{Fil}^r(A_{\mathrm{cris}} \otimes_S \mathcal{M}) = \mathrm{Fil}^r(A_{\mathrm{cris}} \otimes_S \mathcal{D}) \cap (A_{\mathrm{cris}} \otimes_S \mathcal{M})$.

Corollary 4.1.3. *Notations as Proposition 4.1.2,*

$$\mathrm{Fil}^r(A_{\mathrm{cris}} \otimes_S \mathcal{M}) = \bigoplus_{i=1}^{d} A_{\mathrm{cris}} \otimes \alpha_i + \mathrm{Fil}^p A_{\mathrm{cris}} \otimes_S \mathcal{M}.$$

Proof. Since we always have $\mathrm{Fil}^{r-i} S \cdot \mathrm{Fil}^i \mathcal{D} \subset \mathrm{Fil}^r \mathcal{D}$, it is easy to see that

$$\mathrm{Fil}^r(A_{\mathrm{cris}} \otimes_S \mathcal{D}) = A_{\mathrm{cris}} \otimes_S \mathrm{Fil}^r \mathcal{D}.$$

Then the corollary follows the fact that $\mathrm{Fil}^i \mathcal{M} = \mathrm{Fil}^i \mathcal{D} \cap \mathcal{M}$. □

By the above corollary, we can $\varphi_{A_{\mathrm{cris}}}$-semi-linearly extend φ_r of $\mathcal{M}$ to

$$\varphi_r : \mathrm{Fil}^r(A_{\mathrm{cris}} \otimes_S \mathcal{M}) \to A_{\mathrm{cris}} \otimes_S \mathcal{M}$$

and we see that $(A_{\mathrm{cris}} \otimes_S \mathcal{M}, \mathrm{Fil}^r(A_{\mathrm{cris}} \otimes_S \mathcal{M}), \varphi_r)$ is an object in $'\mathrm{Mod}^{\varphi}_{/S}$.

4.2. **Cartier dual on** $\mathrm{Mod}^{\varphi}_{/S}$**.** In this subsection, we recall the construction of Cartier dual on $\mathrm{Mod}^{\varphi}_{/S}$ from [Car05]. Let $\mathcal{M} \in \mathrm{Mod}^{\varphi}_{/S}$. Define $\mathcal{M}^* := \mathrm{Hom}_S(\mathcal{M}, S)$,

$$\mathrm{Fil}^r \mathcal{M}^* := \{f \in \mathcal{M}^* | f(\mathrm{Fil}^r \mathcal{M}) \subset \mathrm{Fil}^r S\}$$

and

$$\varphi_r : \mathrm{Fil}^r \mathcal{M}^* \to \mathcal{M}^*, \text{ for all } x \in \mathrm{Fil}^r \mathcal{M}, \varphi_r(f)(\varphi_r(x)) = \varphi_r(f(x)).$$

Note that $\varphi_r(f)$ is well defined because $\varphi_r(\mathrm{Fil}^r \mathcal{M})$ generates $\mathcal{M}$.

Theorem 4.2.1 (Caruso)**.** *The functor* $\mathcal{M} \to \mathcal{M}^*$ *induces an exact anti-equivalence on* $\mathrm{Mod}^{\varphi}_{/S}$ *and* $(\mathcal{M}^*)^* = \mathcal{M}$.

Proof. Proposition V 3.3.1 in [Car05] proved the theorem on the category of *strongly divisible lattices*. But the same proof also works on $\mathrm{Mod}^{\varphi}_{/S}$ if we ignore monodromy. □

Example 4.2.2. Let S^* be the Cartier dual of S. Then S^* is the S-rank-1 quasi-strongly divisible lattice with $\mathrm{Fil}^r S^* = S$ and $\varphi_r(1) = 1$.

4.3. **Application to Galois representations.** Let $\mathcal{M} \in \mathrm{Mod}^{\varphi}_{/S}$ and $\mathcal{M}^*$ its Cartier dual. The canonical perfect pairing $\mathcal{M} \times \mathcal{M}^* \to S$ in the construction of Cartier dual is compatible with filtration and Frobenius on both sides. Taking Cartier dual on both sides, note that $(\mathcal{M}^*)^* \simeq \mathcal{M}$ by Theorem 4.2.1, we have a map

$$i : S^* \to \mathcal{M}^* \times (\mathcal{M}^*)^* \simeq \mathcal{M}^* \times \mathcal{M}$$

and i induces a pairing

$$\tilde{i} : \mathrm{Hom}_{'\mathrm{Mod}^{\varphi}_{/S}}(\mathcal{M}, A_{\mathrm{cris}}) \times \mathrm{Hom}_{'\mathrm{Mod}^{\varphi}_{/S}}(\mathcal{M}^*, A_{\mathrm{cris}}) \to \mathrm{Hom}_S(S^*, A_{\mathrm{cris}}). \tag{4.3.1}$$

Lemma 4.3.1. *The above pairing induces a perfect paring of* $\mathbb{Z}_p$*-representations of* G_∞*:*

$$T_{\mathrm{cris}}(\mathcal{M}) \times T_{\mathrm{cris}}(\mathcal{M}^*) \to T_{\mathrm{cris}}(S^*) = \mathbb{Z}_p(r). \tag{4.3.2}$$

Proof. This has been essentially proved in Chapter 5, §4 of [Car05]. The proof consists of two steps. The first step is to check the image of $\tilde{i}$ is in $\mathrm{Hom}_{'\mathrm{Mod}^{\varphi}_{/S}}(S^*, A_{\mathrm{cris}})$. This is basically a direct check by the definition of Cartier dual. The proof of the perfectness of the paring (4.3.2) is non-trivial. It suffices to show that the pairing is perfect by modulo p, and the proof is contained in the proof of Theorem V.

4.3.1 in [Car05]. (Although the hypotheses of Theorem 4.3.1 require $er < p-1$, the statement is always valid for any e if we only consider the paring induced by filtered φ-modules over S killed by p, as explained in Caruso's remark in the end of proof.) □

We use A^*_{cris} to denote A_{cris} with non-canonical filtration $\mathrm{Fil}^r A^*_{\mathrm{cris}} = A_{\mathrm{cris}}$ and Frobenius $\varphi_r(1) = 1$.

Lemma 4.3.2. *There are natural isomorphisms of $\mathbb{Z}_p[G_\infty]$-modules:*

$$\mathrm{Hom}_{A_{\mathrm{cris}},\mathrm{Fil}^r,\varphi}(A^*_{\mathrm{cris}}, A_{\mathrm{cris}} \otimes_S \mathcal{M}^*) \simeq \mathrm{Fil}^r(A_{\mathrm{cris}} \otimes_S \mathcal{M}^*)^{\varphi_r=1} \simeq \mathrm{Hom}_{'\mathrm{Mod}^{\varphi}_{/S}}(\mathcal{M}, A_{\mathrm{cris}}).$$

Proof. While the first isomorphism is totally trivial to check, the second isomorphism needs some arguments. Let $\alpha_1, \dots, \alpha_d \in \mathrm{Fil}^r\mathcal{M}$ constructed in Proposition 4.1.2, $(e_1, \dots, e_d) = (\varphi_r(\alpha_1), \dots, \varphi_r(\alpha_d))$ a basis of $\mathcal{M}$ and $(e_1^*, \dots, e_d^*)$ the dual basis. Write $(\alpha_1, \dots, \alpha_d) = (e_1, \dots, e_d)A$ where A is a d by d matrix with coefficients in S. By the argument after formula (4.1.3), we may assume that all A's coefficients are in $W(k)[u]$. By Lemma 4.1.1, there exists a matrix B with coefficients in S such that $AB = BA = E(u)^r I$. Put $(\alpha_1^*, \dots, \alpha_d^*) = (e_1^*, \dots, e_d^*)B^t$ (Here t means transpose). It is easy to check that $\alpha_i^* \in \mathrm{Fil}^r\mathcal{M}^*$ for $i = 1, \dots, d$.

Forgetting filtration and Frobenius structure for a while, since $\mathcal{M}$ is S-finite free, we can identify $A_{\mathrm{cris}} \otimes_S \mathcal{M}^*$ with $\mathrm{Hom}_S(\mathcal{M}, A_{\mathrm{cris}})$ by sending $\sum\limits_{i=1}^d a_i \otimes e_i^*$ to $\sum\limits_{i=1}^d a_i e_i^*$. For any $f \in \mathrm{Fil}^r(A_{\mathrm{cris}} \otimes_S \mathcal{M}^*) = A_{\mathrm{cris}} \otimes_S \mathrm{Fil}^r\mathcal{M}^*$ (Lemma 4.1.3), write $f = \sum_i a_i \otimes f_i$ with $a_i \in A_{\mathrm{cris}}$ and $f_i \in \mathrm{Fil}^r\mathcal{M}^*$. Then for any $x \in \mathrm{Fil}^r\mathcal{M}$, $f(x) = \sum_i a_i f_i(x) \in \mathrm{Fil}^r S \cdot A_{\mathrm{cris}} \subset \mathrm{Fil}^r A_{\mathrm{cris}}$. That is, f is a map from $\mathcal{M}$ to A_{cris} preserving filtration. On the other hand, let f be an S-linear map from $\mathcal{M}$ to A_{cris} preserving filtration. Then $f(\alpha_i) \in \mathrm{Fil}^r A_{\mathrm{cris}}$ for all $i = 1, \dots, d$. Denote $a_i = f(e_i)$, $i = 1, \dots, d$. We have $(a_1, \dots, a_d)A \in \mathrm{Fil}^r A_{\mathrm{cris}}$ where A is the matrix constructed in the first paragraph. By Lemma 4.1.1, we have

$$(a_1, \dots, a_d) = (b_1, \dots, b_d)B + (c_1, \dots, c_d)$$

with $b_i \in A_{\mathrm{cris}}$ and $c_i \in \mathrm{Fil}^p A_{\mathrm{cris}}$ for $i = 1, \dots, d$. So we have

$$f = \sum_{i=1}^d a_i e_i^* = \sum_{i=1}^d b_i \alpha_i^* + \sum_{i=1}^d c_i e_i^* \in \mathrm{Fil}^r(A_{\mathrm{cris}} \otimes_S \mathcal{M}^*).$$

Therefore, we have $f \in \mathrm{Hom}_S(\mathcal{M}, A_{\mathrm{cris}})$ preserves filtration if and only of $f \in \mathrm{Fil}^r(A_{\mathrm{cris}} \otimes_S \mathcal{M}^*)$. Now suppose that $f \in \mathrm{Hom}_S(\mathcal{M}, A_{\mathrm{cris}})$ also preserves Frobenius, that is, $f(\varphi_r(x)) = \varphi_r(f(x))$ for all $x \in \mathrm{Fil}^r\mathcal{M}$. Then

$$\varphi_r(f)(e_i) = \varphi_r(f)(\varphi_r(\alpha_i)) = \varphi_r(f(\alpha_i)) = f(\varphi_r(\alpha_i)) = f(e_i), \ \forall i = 1, \dots, d.$$

Therefore, $\varphi_r(f) = f$. On the other hand, if $f \in (A_{\mathrm{cris}} \otimes_S \mathcal{M}^*)^{\varphi_r=1}$, reversing the above argument shows that $f \in \mathrm{Hom}_{'\mathrm{Mod}^{\varphi}_{/S}}(\mathcal{M}, A_{\mathrm{cris}})$. □

By the above Lemma, we get

$$T_{\mathrm{cris}}(\mathcal{M}) \simeq \mathrm{Fil}^r(A_{\mathrm{cris}} \otimes_S \mathcal{M}^*)^{\varphi_r=1} \hookrightarrow A_{\mathrm{cris}} \otimes_S \mathcal{M}^*. \tag{4.3.3}$$

So we also have $T_{\mathrm{cris}}(\mathcal{M}^*) \hookrightarrow A_{\mathrm{cris}} \otimes_S \mathcal{M}$.

Recall that t is a generator of $\mathbb{Z}_p(1) = (\mathrm{Fil}^1 A_{\mathrm{cris}})^{\varphi_1=1} \subset A_{\mathrm{cris}}$.

Corollary 4.3.3. *The following diagram commutes*

$$\begin{array}{ccc} T_{\mathrm{cris}}(\mathcal{M}) \times T_{\mathrm{cris}}(\mathcal{M}^*) & \hookrightarrow & A_{\mathrm{cris}} \otimes_S \mathcal{M}^* \times A_{\mathrm{cris}} \otimes_S \mathcal{M} \\ \downarrow{\scriptstyle (4.3.2)} & & \downarrow \\ \mathbb{Z}_p(r) & \xrightarrow{1 \mapsto t^r} & A_{\mathrm{cris}} \end{array} \tag{4.3.4}$$

where the top row is induced by (4.3.3) *and the right column is induced by the canonical pairing* $\mathcal{M} \times \mathcal{M}^* \to S$.

Proof. This follows the fact that (4.3.2) is induced by taking dual of the canonical pairing $\mathcal{M} \times \mathcal{M}^* \to S$. □

Now we can construct the following theorem to compare $\mathcal{M} \otimes_S A_{\mathrm{cris}}$ with $T^{\vee}_{\mathrm{cris}}(\mathcal{M}) \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}}$.

Theorem 4.3.4. *There exist* A_{cris}*-linear injections*

$$\iota^* : T^{\vee}_{\mathrm{cris}}(\mathcal{M})(r) \otimes_{\mathbb{Z}_p} A^*_{\mathrm{cris}} \to A_{\mathrm{cris}} \otimes_S \mathcal{M}, \quad \iota : A_{\mathrm{cris}} \otimes_S \mathcal{M} \to T^{\vee}_{\mathrm{cris}}(\mathcal{M}) \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}}$$

such that ι *and* ι^* *are compatible with* G_∞*-actions, Frobenius and filtration. Furthermore,* $\iota \circ \iota^* = \mathrm{Id} \otimes t^r$.

Remark 4.3.5. (1) Suppose that $\mathcal{M}$ is further a strongly divisible lattice. Let $\mathcal{D} = \mathcal{M} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and $D \in \mathrm{MF}^{\mathrm{w}}(\varphi, N)$ such that $\mathcal{D} = \mathcal{D}(D)$. In [Bre97], Breuil extended the classical isomorphism $D \otimes_{K_0} B_{\mathrm{st}} \simeq V^{\vee}_{\mathrm{st}}(D) \otimes_{\mathbb{Q}_p} B_{\mathrm{st}}$ to the $\widehat{B_{\mathrm{st}}}$-version: $\iota_S : \mathcal{D} \otimes_S \widehat{B_{\mathrm{st}}} \simeq V^{\vee}_{\mathrm{st}}(\mathcal{D}) \otimes_{\mathbb{Q}_p} \widehat{B_{\mathrm{st}}}$ where $\widehat{B_{\mathrm{st}}} := \widehat{A_{\mathrm{st}}}[1/p, 1/t]$. Note that B_{st} is a $\widehat{B_{\mathrm{st}}}$-algebra after modulo X. It is not hard to see that $\iota_S \otimes_{\widehat{B_{\mathrm{st}}}} B_{\mathrm{st}} \simeq \iota \otimes_{A_{\mathrm{cris}}} B_{\mathrm{st}}$. Therefore, ι may be seen as an integral version of ι_S.

(2) There exists a geometric interpretation of the above theorem. Conjecturally, the log-crystalline cohomology of certain varieties satisfies the axioms of strongly divisible modules, but it is not conjectured that any strongly divisible module can be seen as a log-crystalline cohomology group. See §4 in [Bre02] for the exposé of this direction.

(3) If $\mathcal{M}$ comes from an $\mathfrak{M} \in \mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$, i.e., $\mathcal{M} = \mathcal{M}_{\mathfrak{S}}(\mathfrak{M})$, then we have a similar result as the above theorem without restriction of r. See §5.3, [Liu06] for details.

Proof. We use the same idea as the proof of Theorem 5 (ii) in [Fal99]. First an easy computation shows that $T_{\mathrm{cris}}(\mathcal{M}) = \mathrm{Hom}_{A_{\mathrm{cris}}, \mathrm{Fil}^r, \varphi}(A_{\mathrm{cris}} \otimes_S \mathcal{M}, A_{\mathrm{cris}})$. Then we get a map:

$$\tilde{\iota} : \ T_{\mathrm{cris}}(\mathcal{M}) \times A_{\mathrm{cris}} \otimes_S \mathcal{M} \to A_{\mathrm{cris}} \tag{4.3.5}$$

Therefore, we get a natural map

$$\iota : A_{\mathrm{cris}} \otimes_S \mathcal{M} \to T^{\vee}_{\mathrm{cris}}(\mathcal{M}) \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}},$$

and it is easy to check that ι preserves G_∞-actions, Frobenius and filtration. On the other hand, by (4.3.3) and Lemma 4.3.1, we get

$$\iota^* : \ T_{\mathrm{cris}}(\mathcal{M}^*) \otimes_{\mathbb{Z}_p} A^*_{\mathrm{cris}} = T^{\vee}_{\mathrm{cris}}(\mathcal{M})(r) \otimes_{\mathbb{Z}_p} A^*_{\mathrm{cris}} \hookrightarrow A_{\mathrm{cris}} \otimes_S \mathcal{M},$$

and Lemma 4.3.2 shows that the above map is compatible with G_∞-actions, Frobenius and filtration. Combining ι^* with (4.3.5), it suffices to show the following diagram commutes:

$$\begin{array}{ccc} T_{\mathrm{cris}}(\mathcal{M}) \times T_{\mathrm{cris}}(\mathcal{M}^*) \otimes_{\mathbb{Z}_p} A^*_{\mathrm{cris}} & \xrightarrow{\mathrm{Id}\times\iota^*} & T_{\mathrm{cris}}(\mathcal{M}) \times A_{\mathrm{cris}} \otimes_S \mathcal{M} \\ \Big\downarrow{\scriptstyle (4.3.2)\otimes\mathrm{Id}} & & \Big\downarrow{\scriptstyle (4.3.5)} \\ \mathbb{Z}_p(r) \otimes_{\mathbb{Z}_p} A^*_{\mathrm{cris}} & \xrightarrow{1\mapsto t^r} & A_{\mathrm{cris}} \end{array}$$

Note that we have an injection $T_{\mathrm{cris}}(\mathcal{M}) \hookrightarrow A_{\mathrm{cris}} \otimes_S \mathcal{M}^*$ by (4.3.3). So the commutativity of the above diagram follows the commutativity of diagram (4.3.4), and this is proved in Corollary 4.3.3. □

Let $\alpha_1, \dots, \alpha_d \in \mathrm{Fil}^r \mathcal{M}$ as in Proposition 4.1.2 and $e_1, \dots, e_d \in \mathcal{M}$ a basis of $\mathcal{M}$. Let $\mathfrak{e}_1, \dots, \mathfrak{e}_d$ be a basis of $T^\vee_{\mathrm{cris}}(\mathcal{M})$. By Theorem 4.3.4, we have

$$\iota(\alpha_1, \dots, \alpha_d) = (\mathfrak{e}_d, \dots, \mathfrak{e}_d)C,$$

where C is a $d \times d$-matrix with coefficients in $\mathrm{Fil}^r A_{\mathrm{cris}}$.

Lemma 4.3.6. *There exists a $d\times d$-matrix C' with coefficients in A_{cris} such that coefficients of $C'C - t^r I$ are all in $\mathrm{Fil}^p A_{\mathrm{cris}}$.*

Proof. Forgetting G_∞-actions, Frobenius and filtration structures, we may identify $T^\vee_{\mathrm{cris}}(\mathcal{M}) \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}}$ with $T^\vee_{\mathrm{cris}}(\mathcal{M})(r) \otimes_{\mathbb{Z}_p} A^*_{\mathrm{cris}}$ as finite free A_{cris}-modules. In particular, we regard $(\mathfrak{e}_1, \dots, \mathfrak{e}_d)$ as a basis of $T^\vee_{\mathrm{cris}}(\mathcal{M})(r)$. Then $\iota^* \circ \iota$ makes sense and $\iota^* \circ \iota = t^r \otimes \mathrm{Id}$ by Theorem 4.3.4. Therefore, we get

$$t^r(\alpha_1, \dots, \alpha_d) = \iota^* \circ \iota(\alpha_1, \dots, \alpha_d) = \iota^*(\mathfrak{e}_1, \dots, \mathfrak{e}_d)C \tag{4.3.6}$$

Note that $\mathrm{Fil}^r A^*_{\mathrm{cris}} = A_{\mathrm{cris}}$, so $(\mathfrak{e}_1, \dots, \mathfrak{e}_d) \in \mathrm{Fil}^r(T^\vee_{\mathrm{cris}}(\mathcal{M})(r) \otimes_{\mathbb{Z}_p} A^*_{\mathrm{cris}})$, and then $\iota^*(\mathfrak{e}_1, \dots \mathfrak{e}_d)$ is in $\mathrm{Fil}^r(\mathcal{M} \otimes_S A_{\mathrm{cris}})$. By Corollary 4.1.3, we have

$$\iota^*(\mathfrak{e}_1, \dots, \mathfrak{e}_d) = (\alpha_1, \dots, \alpha_d)C' + (e_1, \dots, e_d)D \tag{4.3.7}$$

where $e_1, \dots, e_d$ is a basis of $\mathcal{M}$, C' and D are $d\times d$-matrices with coefficients in A_{cris} and $\mathrm{Fil}^p A_{\mathrm{cris}}$ respectively. Write $(\alpha_1, \dots \alpha_d) = (e_1, \dots, e_d)A$ with A a $d \times d$-matrix. Combining (4.3.6) and (4.3.7), we have

$$t^r A = AC'C + DC.$$

By Lemma 4.1.2, there exists a $d \times d$-matrix B with coefficients in S such that $AB = BA = E(u)^r I$, we get $E(u)^r(t^r I - C'C) = BDC$. Note that the coefficients of C and D are in $\mathrm{Fil}^r A_{\mathrm{cris}}$ and $\mathrm{Fil}^p A_{\mathrm{cris}}$ respectively. Thus the coefficients of $E(u)^r(t^r I - C'C)$ are in $\mathrm{Fil}^{r+p} A_{\mathrm{cris}}$. By Lemma 3.2.2, the coefficients of $C'C - t^r I$ are all in the $\mathrm{Fil}^p A_{\mathrm{cris}}$. □

5. The Proof of Lemma 3.5.3

In this section, we will show how to recover monodromy N on $\mathcal{M}$ by the G-action on T and then prove Lemma 3.5.3. Recall that T is a G-stable $\mathbb{Z}_p$-lattice in a semi-stable p-adic Galois representation V, $\mathcal{M} = \mathcal{M}_{\mathfrak{S}}(\mathfrak{M})$ the quasi-strongly divisible lattice such that $T_{\mathrm{cris}}(\mathcal{M}) = T|_{G_\infty}$ (Proposition 3.4.6) and $\mathcal{D} := \mathcal{M} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \in \mathcal{MF}^{\mathrm{w}}(\varphi, N)$ to correspond V. We first construct a G-action on $A_{\mathrm{cris}} \otimes_S \mathcal{D}$ by using N on $\mathcal{D}$.

5.1. **G-action on $A_{\mathrm{cris}} \otimes_S \mathcal{D}$.** We already have a natural semi-linear G_∞-action on $A_{\mathrm{cris}} \otimes_S \mathcal{D}$ induced from the G_∞-action on A_{cris}. We extend this to a G-action by using N on $\mathcal{D}$. For any $\sigma \in G$, recall $\underline{\epsilon}(\sigma) = \frac{\sigma([\underline{\pi}])}{[\underline{\pi}]}$. For any $a \otimes x \in A_{\mathrm{cris}} \otimes_S \mathcal{D}$, define

$$\sigma(a \otimes x) = \sum_{i=0}^{\infty} \sigma(a)\gamma_i(-\log(\underline{\epsilon}(\sigma))) \otimes N^i(x). \tag{5.1.1}$$

where $\gamma_i(x) = \frac{x^i}{i!}$ is the standard divided power. Note that if $\sigma \in G_\infty$, then $\log(\underline{\epsilon}(\sigma)) = 0$ and $\sigma(a \otimes x) = \sigma(a) \otimes x$. Thus G-action defined above (if it is well defined) is compatible with the natural G_∞-action on $A_{\mathrm{cris}} \otimes_S \mathcal{D}$.

Lemma 5.1.1. *The above action is well defined A_{cris}-semi-linear G-action on $A_{\mathrm{cris}} \otimes_S \mathcal{D}$ and compatible with Frobenius and filtration.*

Proof. In fact, this result has been explicitly or non-explicitly used in several papers, e.g., §4 in [Fal99]. To see the series in the right side of (5.1.1) converges, note that $\mathcal{D} = D \otimes_{W(k)} S$ and N is nilpotent on D. It suffices to show that $\gamma_i(-\log(\underline{\epsilon}(\sigma))) \to 0$ when $i \to \infty$. This is a well-known result. See for example, §5.2.4 in [Fon94a].

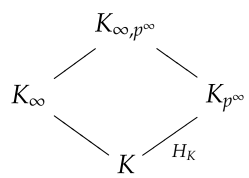
For any $f(u) \in S$, $x \in \mathcal{D}$ and $\sigma, \tau \in G$, we need to check that

(1) $\sigma(1 \otimes f(u)x) = \sigma(f([\underline{\pi}]) \otimes x) = f(\sigma([\underline{\pi}])) \otimes \sigma(x)$
(2) $\sigma(\tau(1 \otimes x)) = (\sigma \circ \tau)(1 \otimes x)$.
(3) the G-action preserves filtration and commutes with φ.

It is fairly standard direct calculations to check these equations combining with facts that $\mathrm{Fil}^1 S \cdot N(\mathrm{Fil}^i \mathcal{D}) \subset \mathrm{Fil}^i \mathcal{D}$, $\log(\underline{\epsilon}(\sigma)) \in \mathrm{Fil}^1 A_{\mathrm{cris}}$ and $N\varphi = p\varphi N$ in $\mathcal{D}$. □

One the other hand, given the G-action on $A_{\mathrm{cris}} \otimes_S \mathcal{D}$ defined via (5.1.1), we want to define a certain logarithm of the G-action to recover N. (We should be careful at this point because the G-action is not linear). A technical result is needed to define such a logarithm.

For any field extension F over $\mathbb{Q}_p$, denote $F_{p^\infty} = \bigcup_{n=1}^{\infty} F(\zeta_{p^n})$ with ζ_{p^n} a p^n-th primitive root of unity. Thus $K_{\infty,p^\infty} = \bigcup_{n=1}^{\infty} K(\sqrt[p^n]{\pi}, \zeta_{p^n})$ is Galois. So we have the following field extensions



Let $H_K = \mathrm{Gal}(K_{p^\infty}/K) \subset \mathrm{Gal}(\mathbb{Q}_{p,p^\infty}/\mathbb{Q}_p) \simeq \mathbb{Z}_p^\times$. So H_K may be identified as a closed subgroup of $\mathbb{Z}_p^\times$.

Lemma 5.1.2. (1) $K_{p^\infty} \cap K_\infty = K$.
(2) $\mathrm{Gal}(K_{\infty,p^\infty}/K_\infty) \simeq H_K$ *and* $\mathrm{Gal}(K_{\infty,p^\infty}/K_{p^\infty}) \simeq \mathbb{Z}_p(1)$.
(3) $\mathrm{Gal}(K_{\infty,p^\infty}/K) = \mathrm{Gal}(K_{\infty,p^\infty}/K_{p^\infty}) \rtimes \mathrm{Gal}(K_{\infty,p^\infty}/K_\infty) \simeq \mathbb{Z}_p(1) \rtimes H_K$. *$H_K$ acts on $\mathbb{Z}_p(1)$ by the cyclotomic character.*

Proof. We only need to prove (1). For any $n \geq 0$, let $F_n = K(\pi_n) \cap K_{p^\infty}$ and denote $K(\pi_n)$ by K_n. We prove that $F_n = K$ by an induction on n. The case $n = 0$ is trivial.

Now suppose that $F_n = K$ and $F_{n+1} \neq K$. We first show that $\zeta_p \in K$. Note that

$$[F_{n+1} \cdot K_n : K_n] \mid [K_{n+1} : K_n] = p \text{ and } F_{n+1} \cdot K_n \neq K_n,$$

we have $[F_{n+1} \cdot K_n : K_n] = p$ and $F_{n+1} \cdot K_n = K_{n+1}$. Moreover, since $K \subset F_{n+1} \cap K_n \subset F_n = K$, K_{n+1}/K_n is Galois and hence $\mathrm{Gal}(K_{n+1}/K_n) \simeq \mathrm{Gal}(F_{n+1}/K)$. Let $\sigma \in \mathrm{Gal}(K_{n+1}/K_n)$ be a nontrivial element, then $\sigma(\pi_{n+1})/\pi_{n+1} \in K_{n+1}$ is nontrivial p-th root of unity. So $\zeta_p \in K_{n+1}$. Note that

$$[K_n(\zeta_p) : K_n] \leq p - 1 \text{ and } [K_n(\zeta_p) : K_n] | [K_{n+1} : K_n] = p,$$

we have $K_n(\zeta_p) = K_n$ and $\zeta_p \in K_n$. By the induction that $F_n = K$, $\zeta_p \in K$.

Now $\mathrm{Gal}(K_{p^\infty}/K)$ is a closed subgroup of $\mathrm{Gal}(\mathbb{Q}_{p,p^\infty}/\mathbb{Q}_p(\zeta_p)) \simeq 1 + p\mathbb{Z}_p$ (Note that this fails if $p = 2$). Since $[F_{n+1} : K] = p$, there must exist an m such that $\zeta_{p^m} \in K$, $\zeta_{p^{m+1}} \notin K$ and $F_{n+1} = K(\zeta_{p^{m+1}})$. In particular, $\mathrm{Gal}(K_{n+1}/K_n) \simeq \mathrm{Gal}(K(\zeta_{p^{m+1}})/K(\zeta_{p^m})) \simeq \mathbb{Z}/p\mathbb{Z}$. Choose $\sigma \in \mathrm{Gal}(K_{n+1}/K_n)$ such that $\sigma(\zeta_{p^{m+1}}) = \zeta_p \zeta_{p^{m+1}}$. Then $\sigma(\pi_{n+1}) = \zeta_p^b \pi_{n+1}$ for some $b \in (\mathbb{Z}/p\mathbb{Z})^\times$. Write $\zeta_{p^{m+1}} = \sum_{i=0}^{p-1} a_i \pi_{n+1}^i$ with $a_i \in \mathcal{O}_{K_n}$. Then

$$\zeta_p \zeta_{p^{m+1}} = \sigma(\zeta_{p^{m+1}}) = \sigma(\sum_{i=0}^{p-1} a_i \pi_{n+1}^i) = \sum_{i=0}^{p-1} a_i \zeta_p^{bi} \pi_{n+1}^i.$$

Thus we have $a_0 = \zeta_p a_0$ and $a_0 = 0$. Then $\zeta_{p^{m+1}}$ is not a unit. Contradiction. Therefore F_{n+1} has to be K. □

Remark 5.1.3. The above Lemma fails if $p = 2$ in general. For example, let $K = \mathbb{Q}_2$ and $\pi = 2$. Then $\mathbb{Q}_2(\sqrt{2}) \subset \mathbb{Q}_2(\zeta_8)$.

Fix a topological generator τ of $\mathrm{Gal}(K_{\infty,p^\infty}/K_{p^\infty})$, the above Lemma shows that $-\log(\underline{\epsilon}(\tau))$ is a generator of $(\mathrm{Fil}^1 A_{\mathrm{cris}})^{\varphi_1=1}$. So from now on, we fix $t := -\log(\underline{\epsilon}(\tau))$. Note that τ acts trivially on $\underline{\epsilon}(\tau)$, thus on t. Therefore, for any $n \geq 0$ and $x \in \mathcal{D}$, an easy induction on n shows that

$$(5.1.2) \qquad (\tau - 1)^n(x) = \sum_{m=n}^{\infty} \left(\sum_{i_1+\cdots+i_n=m, i_j \geq 1} \frac{m!}{i_1! \cdots i_n!} \right) \gamma_m(t) \otimes N^m(x)$$

In particular, $(\tau - 1)^n(x) \in \mathrm{Fil}^n B^+_{\mathrm{cris}} \otimes_S \mathcal{D}$ and $\frac{(\tau-1)^n}{n}(x) \to 0$ p-adically as $n \to \infty$ (in fact, it is easy to show that $\gamma_n(t)/n \to 0$ p-adically, see §5.2.4, [Fon94a]). So we can define

$$(5.1.3) \qquad \log(\tau)(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(\tau - 1)^n}{n}(x).$$

and a direct computation shows that

$$(5.1.4) \qquad \log(\tau)(x) = t \otimes N(x).$$

5.2. **A $\mathbb{Q}_p$-version of Theorem 4.3.4.** Let $D \in \mathrm{MF}^{\mathrm{w}}(\varphi, N)$ be a weakly admissible filtered (φ, N)-module and $\mathcal{D} = \mathcal{D}(D) := D \otimes_{W(k)} S \in \mathcal{MF}^{\mathrm{w}}(\varphi, N)$. By Lemma 3.4.3, the map

$$(5.2.1) \qquad V_{\mathrm{st}}(\mathcal{D}) = \mathrm{Hom}_{'\mathrm{Mod}^{\varphi,N}_{/S}}(\mathcal{D}, \widehat{A_{\mathrm{st}}}[1/p]) \to \mathrm{Hom}_{'\mathrm{Mod}^{\varphi}_{/S}}(\mathcal{D}, B^+_{\mathrm{cris}}).$$

induced by the canonical projection $\widehat{A_{\mathrm{st}}} \to A_{\mathrm{cris}}$ defined by sending $\gamma_i(X) \to 0$ is an isomorphism compatible with G_∞-action. On the other hand,

$$\mathrm{Hom}_{'\mathrm{Mod}^{\varphi}_{/S}}(\mathcal{D}, B^+_{\mathrm{cris}}) \simeq \mathrm{Hom}_{A_{\mathrm{cris}},\mathrm{Fil}^{\cdot},\varphi}(A_{\mathrm{cris}} \otimes_S \mathcal{D}, B^+_{\mathrm{cris}}). \tag{5.2.2}$$

By Lemma 5.1.1, we have a natural G-action on $A_{\mathrm{cris}} \otimes_S \mathcal{D}$ via (5.1.1). So there exists a G-action on the right side of (5.2.2) defined by

$$\sigma(f)(x) = \sigma(f(\sigma^{-1}(x))) \text{ for any } x \in A_{\mathrm{cris}} \otimes \mathcal{D}.$$

Combining (5.2.1) with (5.2.2) together, we have

Lemma 5.2.1. *The map*

$$V_{\mathrm{st}}(\mathcal{D}) = \mathrm{Hom}_{'\mathrm{Mod}^{\varphi,N}_{/S}}(\mathcal{D}, \widehat{A_{\mathrm{st}}}[1/p]) \to \mathrm{Hom}_{A_{\mathrm{cris}},\mathrm{Fil}^{\cdot},\varphi}(A_{\mathrm{cris}} \otimes_S \mathcal{D}, B^+_{\mathrm{cris}})$$

induced by (5.2.1) *and* (5.2.2) *is a G-equivariant isomorphism.*

Proof. Lemma 3.4.3 has proved the above map is a $\mathbb{Q}_p$-linear bijection. So we only need to check the G-equivariance. For any $f \in \mathrm{Hom}_{'\mathrm{Mod}^{\varphi,N}_{/S}}(\mathcal{D}, \widehat{A_{\mathrm{st}}}[1/p])$, let $f_0 \in \mathrm{Hom}_{'\mathrm{Mod}^{\varphi}_{/S}}(\mathcal{D}, B^+_{\mathrm{cris}})$ be its image of the map defined in (5.2.1). It suffices to check for any $x \in D$, $\sigma \in G$, $\sigma(f)_0(x) = \sigma(f_0(\sigma^{-1}(x)))$. Using (3.4.4) and the fact that $\sigma(X) = \underline{\epsilon}(\sigma)X + \underline{\epsilon}(\sigma) - 1$, we have:

$$\begin{aligned} \sigma(f(x)) &= \sum_{i\geq 0} \sigma(f_0(N^i(x)))\gamma_i(\log(1+\sigma(X))) \\ &= \sum_{i\geq 0} \sigma(f_0(N^i(x))) \sum_{j=0}^{i} \gamma_{i-j}(\log(\underline{\epsilon}(\sigma)))\gamma_j(\log(1+X)) \end{aligned}$$

Modulo X, then we get

$$\begin{aligned} \sigma(f)_0(x) &= \sum_{j\geq 0} \sigma(f_0(N^j(x)))\gamma_j(\log(\underline{\epsilon}(\sigma))) \\ &= \sigma(f_0(\sum_{j\geq 0} \gamma_j(\log(\sigma^{-1}\underline{\epsilon}(\sigma))) \otimes N^j(x))) \\ &= \sigma(f_0(\sigma^{-1}(x))). \end{aligned}$$

□

Corollary 5.2.2. *The B^+_{cris}-linear injections:*

$$\iota \otimes_{\mathbb{Z}_p} \mathbb{Q}_p : \; A_{\mathrm{cris}} \otimes_S \mathcal{D} \to V^\vee_{\mathrm{st}}(\mathcal{D}) \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}},$$

$$\iota^* \otimes_{\mathbb{Z}_p} \mathbb{Q}_p : \; V^\vee_{\mathrm{st}}(\mathcal{D})(r) \otimes_{\mathbb{Z}_p} A^*_{\mathrm{cris}} \to A_{\mathrm{cris}} \otimes_S \mathcal{D}.$$

are compatible with G-actions, where ι and ι^* are constructed in Theorem 4.3.4.

5.3. **Proof of the Main Theorem.** Using notations in §3.5 and Lemma 3.5.3. Recall that T is a G-stable $\mathbb{Z}_p$-lattice in a semi-stable p-adic Galois representation V, and $\mathcal{M}$ the quasi-strongly divisible lattice such that $T_{\mathrm{cris}}(\mathcal{M}) = T|_{G_\infty}$ (Proposition 3.4.6). Also recall that τ is the fixed topological generator of $\mathrm{Gal}(K_{\infty,p^\infty}/K_{p^\infty})$ discussed in §5.1. We will use Lemma 4.3.6 and Corollary 5.2.2 to prove N is stable on $\mathcal{M}$ by two steps. The first step is to show that $A_{\mathrm{cris}} \otimes_S \mathcal{M}$ is G-stable in $A_{\mathrm{cris}} \otimes_S \mathcal{D}$. More generally, we have the following:

Lemma 5.3.1. *Notations as in Theorem 4.3.4. Let $\mathcal{M}, \mathcal{M}' \in \mathrm{Mod}^{\varphi}_{/S}$. Suppose that we have the following commutative diagram:*

$$\begin{array}{ccc} A_{\mathrm{cris}} \otimes_S \mathcal{M}' & \xrightarrow{\iota_{\mathcal{M}'}} & T^\vee_{\mathrm{cris}}(\mathcal{M}') \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}} \\ \downarrow \mathfrak{f} & & \downarrow f \\ A_{\mathrm{cris}} \otimes_S \mathcal{M} & \xrightarrow{\iota_{\mathcal{M}}} & T^\vee_{\mathrm{cris}}(\mathcal{M}) \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}} \end{array} \tag{5.3.1}$$

where $\mathfrak{f}$ and f are A_{cris}-linear or τ-semi-linear morphisms compatible with Frobenius and filtration. If $p|f$ then $p|\mathfrak{f}$.

Proof. We only prove the case that $\mathfrak{f}$ and f are A_{cris}-linear. The proof for τ-semi-linear case is totally the same.

Let d' be the S-rank of $\mathcal{M}'$, $\alpha'_1, \dots, \alpha'_{d'} \in \mathrm{Fil}^r \mathcal{M}'$ such that $\varphi_r(\alpha'_1), \dots, \varphi_r(\alpha'_{d'})$ is a basis of $\mathcal{M}'$. Since $\mathfrak{f}$ preserves filtration, $\mathfrak{f}(\alpha'_1, \dots, \alpha'_{d'}) \in [\mathrm{Fil}^r(A_{\mathrm{cris}} \otimes_S \mathcal{M})]^d$. By Corollary 4.1.3, we have

$$\mathrm{Fil}^r(A_{\mathrm{cris}} \otimes_S \mathcal{M}) = \bigoplus_{i=1}^{d} A_{\mathrm{cris}} \otimes \alpha_i + \mathrm{Fil}^p A_{\mathrm{cris}} \otimes_S \mathcal{M}. \tag{5.3.2}$$

with $(e_1, \dots, e_d) = (\varphi_r(\alpha_1), \dots, \varphi_r(\alpha_d))$ a basis of $\mathcal{M}$. Therefore there exist $d \times d'$-matrices X, W with coefficients in A_{cris}, $\mathrm{Fil}^p A_{\mathrm{cris}}$ respectively such that

$$\mathfrak{f}(\alpha'_1, \dots, \alpha'_{d'}) = (\alpha_1, \dots, \alpha_d)X + (e_1, \dots, e_d)W. \tag{5.3.3}$$

We claim that coefficients of X are in $\mathrm{Fil}^1 A_{\mathrm{cris}} + pA_{\mathrm{cris}}$.

To see the claim, applying $\iota_{\mathcal{M}}$ on the both sides of (5.3.3), we have

$$\iota_{\mathcal{M}} \circ \mathfrak{f}(\alpha'_1, \dots, \alpha'_{d'}) = \iota_{\mathcal{M}}(\alpha_1, \dots, \alpha_d)X + \iota_{\mathcal{M}}(e_1, \dots, e_d)W = (\mathfrak{e}_1, \dots, \mathfrak{e}_d)(CX + W'),$$

where $\mathfrak{e}_1, \dots, \mathfrak{e}_d$ is a basis of $T^\vee_{\mathrm{cris}}(\mathcal{M})$ as in Lemma 4.3.6 and C, W' are matrices with coefficients in A_{cris}, $\mathrm{Fil}^p A_{\mathrm{cris}}$ respectively such that $\iota_{\mathcal{M}}(\alpha_1, \dots, \alpha_d) = (\mathfrak{e}_1, \dots, \mathfrak{e}_d)C$ and $\iota_{\mathcal{M}}(e_1, \dots, e_d)W = (\mathfrak{e}_1, \dots, \mathfrak{e}_d)W'$. On the other hand, since diagram (5.3.1) is commutative and $p|\mathfrak{f}$, all the coefficients of $CX + W'$ are in pA_{cris}. By Lemma 4.3.6, there exists a matrix C' such that the coefficients of $C'C - t^r I$ are in $\mathrm{Fil}^p A_{\mathrm{cris}}$. Thus coefficients of $t^r X$ are in $\mathrm{Fil}^p A_{\mathrm{cris}} + pA_{\mathrm{cris}}$. To show the claim, it suffices to show that if $x \in A_{\mathrm{cris}}$ and $t^r x \in pA_{\mathrm{cris}} + \mathrm{Fil}^p A_{\mathrm{cris}}$ then $x \in \mathrm{Fil}^1 A_{\mathrm{cris}} + pA_{\mathrm{cris}}$. Recall $R = \varprojlim \mathcal{O}_{\bar{K}}/p$ constructed in §2.2. For any $(a_i)_{i \geq 0} \in R$ with $a_i \in \mathcal{O}_K/p$, let $\hat{a}_i \in \mathcal{O}_K$ be a lift of a_i, then $a^{(0)} = \lim_{n\to\infty} (\hat{a}_n)^{p^n}$ is well defined and independent of the choice of $\hat{a}_i$. We define the valuation on R by $v_R((a_i)_{i\geq 0}) = v(a^{(0)})$ where $v(\cdot)$ is the standard valuation of $\mathcal{O}_{\bar{K}}$ (§1.2.2 and §1.2.3, [Fon94a]). Let $\mathrm{Fil}^i R$ be the image of $\mathrm{Fil}^i(W(R))$ under the reduction $\bmod p$. We see that $\mathrm{Fil}^1 R = \{x \in R | v_R(x) \geq 1\}$ and $A_{\mathrm{cris}}/(pA_{\mathrm{cris}} + \mathrm{Fil}^p A_{\mathrm{cris}}) \simeq R/\mathrm{Fil}^p R$. Let $\bar{x}$ and $\bar{t}$ be the image of x and t in $R/\mathrm{Fil}^p R$

respectively. Note that $v_R(\bar{t}) = v_R(\frac{\tau(\pi)}{\pi} - 1) = \frac{p}{p-1}$. Since $\bar{t}^r\bar{x} \in \mathrm{Fil}^p R$, $v_R(\bar{t}^r\bar{x}) \geq p$. But $v_R(\bar{t}^r) = \frac{rp}{p-1} < p-1$ because $r \leq p-2$. Therefore, $v_R(\bar{x}) \geq 1$ and $x \in \mathrm{Fil}^1 A_{\mathrm{cris}} \mod p$.

Now since $\mathfrak{f}$ is compatible with Frobenius, by (5.3.3) we have

$$\begin{aligned} \mathfrak{f}((\varphi_r(\alpha'_1), \ldots, \varphi_r(\alpha'_{d'}))) &= \varphi_r((\alpha_1, \ldots, \alpha_d)X + (e_1, \ldots e_d)W) \\ &= (e_1, \ldots, e_d)\varphi(X) + \varphi(e_1, \ldots, e_d)\varphi_r(W) \end{aligned}$$

Since coefficients of X are in $\mathrm{Fil}^1 A_{\mathrm{cris}} + pA_{\mathrm{cris}}$, we have $p|\varphi(X)$. Note that $p|\varphi_r(W)$ because W's coefficients are in $\mathrm{Fil}^p A_{\mathrm{cris}}$. Finally, since $\varphi_r(\alpha'_1), \ldots, \varphi_r(\alpha'_{d'})$ is a basis of $\mathcal{M}'$, we get $p|\mathfrak{f}$. □

Proof of Lemma 3.4.7. It suffices to prove that $\mathcal{M}' \subset \mathcal{M}$. Choose a smallest integer n such that $p^n\mathcal{M}' \subset \mathcal{M}$. Then $p^n : \mathcal{M}' \to \mathcal{M}$ is a morphism in $\mathrm{Mod}^{\varphi}_{/S}$. Use Lemma 5.3.1 for $\mathfrak{f} = p^n$ and $f = p^n$. Then we see that n has to be 0. □

Combining Theorem 4.3.4 with Corollary 5.2.2, we have the following commutative diagram:

(5.3.4)
$$\begin{array}{ccc} A_{\mathrm{cris}} \otimes_S \mathcal{D} & \xrightarrow{\iota \otimes_{\mathbb{Z}_p} \mathbb{Q}_p} & V^{\vee}_{\mathrm{st}}(\mathcal{D}) \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}} \\ \uparrow & & \uparrow \\ A_{\mathrm{cris}} \otimes_S \mathcal{M} & \xrightarrow{\iota} & T^{\vee}_{\mathrm{cris}}(\mathcal{M}) \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}} \end{array}$$

where the top row map is compatible with G-action and the bottom row map is compatible with G_∞-action. We claim that $A_{\mathrm{cris}} \otimes_S \mathcal{M}$ is stable under G. To check this, it suffices to check $A_{\mathrm{cris}} \otimes_S \mathcal{M}$ is stable under τ. Since $T^\vee = T^\vee_{\mathrm{cris}}(\mathcal{M})$ is a G-stable $\mathbb{Z}_p$-lattice, we see that $T^\vee \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}}$ is stable under τ. Choose n such that $p^n\tau(A_{\mathrm{cris}} \otimes_S \mathcal{M}) \subseteq A_{\mathrm{cris}} \otimes_S \mathcal{M}$. Now using Lemma 5.3.1 for $\mathfrak{f} = p^n\tau$ on $A_{\mathrm{cris}} \otimes_S \mathcal{M}$ and $f = p^n\tau$ on $T^\vee_{\mathrm{cris}}(\mathcal{M}) \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}}$, we have $\tau(A_{\mathrm{cris}} \otimes_S \mathcal{M}) \subseteq A_{\mathrm{cris}} \otimes_S \mathcal{M}$.

Now we are ready to show that $\mathcal{M}$ is stable under N. By (5.1.4), for any $x \in \mathcal{M}$, $t \otimes N(x) = \log(\tau)(x)$. We claim that $t \otimes N(\mathcal{M}) \subset A_{\mathrm{cris}} \otimes_S \mathcal{M}$ by proving that $\log(\tau)(\mathcal{M}) \subset A_{\mathrm{cris}} \otimes_S \mathcal{M}$. It suffices to show that $\frac{(\tau-1)^n}{n}(\mathcal{M}) \subset A_{\mathrm{cris}} \otimes_S \mathcal{M}$ for all $n \geq p$. Let $(\alpha_1, \ldots, \alpha_d) \in \mathrm{Fil}^r\mathcal{M}$ constructed in Proposition 4.1.2, $(e_1, \ldots, e_d) = (\varphi_r(\alpha_1), \ldots \varphi_r(\alpha_d))$ a basis of $\mathcal{M}$. Using (5.1.2), we see that

$$(\tau - 1)^n(\alpha_1, \ldots, \alpha_d) \in [\mathrm{Fil}^n B^+_{\mathrm{cris}}(A_{\mathrm{cris}} \otimes_S \mathcal{M})]^d.$$

Since $\tau(\mathcal{M}) \subset (A_{\mathrm{cris}} \otimes_S \mathcal{M})$, we get

$$(\tau - 1)^n(\alpha_1, \ldots, \alpha_d) \in [\mathrm{Fil}^n A_{\mathrm{cris}}(A_{\mathrm{cris}} \otimes_S \mathcal{M})]^d.$$

Therefore,

$$(\tau - 1)^n(e_1, \ldots, e_d) = \varphi_r((\tau - 1)^n(\alpha_1, \ldots, \alpha_d)) \in [\varphi_r(\mathrm{Fil}^n A_{\mathrm{cris}}) \cdot \varphi(A_{\mathrm{cris}} \otimes_S \mathcal{M})]^d.$$

Now it suffices to check that for any $n \geq p$ and $x \in \mathrm{Fil}^n A_{\mathrm{cris}}$, $\varphi_r(x)/n \in A_{\mathrm{cris}}$. We can further reduce the problem to check if $\frac{\varphi(E(u)^m)}{p^r nm!} \in S$ for all $m \geq n \geq p$. Note that $c_1 = \varphi(E(u))/p$ is a unit in S. So it is equivalent to show that $\frac{p^{m-r}}{nm!} \in \mathbb{Z}_p$ for all $m \geq n \geq p$ and we include the computation in the lemma below. Thus we prove the claim that $t \otimes N(x) \in A_{\mathrm{cris}} \otimes_S \mathcal{M}$.

Lemma 5.3.2. *If $m \geq n \geq p > 2$ and $r < p - 1$, then $m - r - v_p(nm!) \geq 0$.*

Proof. Since $n \geq p$, $v_p(n) \leq \frac{n}{p} \leq \frac{m}{p}$. Hence

$$d = m - v_p(nm!) \geq m - \frac{m}{p-1} - \frac{m}{p} = \frac{m(p^2-3p+1)}{p(p-1)} \geq \frac{p^2-3p+1}{p-1} = p-2-\frac{1}{p-1}.$$

Since d is an integer, it follows that $d \geq p-2 \geq r$. □

Finally, suppose that we have

$$N((e_1,\ldots,e_d)) = (e_1,\ldots,e_d)W$$

with coefficients of W in S_{K_0}. Select the smallest number n such that all coefficients of p^nW are in S. Then $p^nN(\mathcal{M}) \subset \mathcal{M}$. Since $E(u)N(\mathrm{Fil}^r\mathcal{D}) \subset \mathrm{Fil}^r\mathcal{D}$, we have

(5.3.5) $$E(u)p^nN((\alpha_1,\ldots,\alpha_d)) = (\alpha_1,\ldots,\alpha_d)X + (e_1,\ldots,e_d)Y$$

with coefficients of X, Y in S, Fil^pS respectively. On the other hand, note that $t \otimes N(\mathcal{M}) \subset A_{\mathrm{cris}} \otimes_S \mathcal{M}$ and $t \otimes N(\mathrm{Fil}^r\mathcal{M}) \subset \mathrm{Fil}^r(A_{\mathrm{cris}} \otimes_S \mathcal{M})$, we have

(5.3.6) $$tN((\alpha_1,\ldots,\alpha_d)) = (\alpha_1,\ldots,\alpha_d)X' + (e_1,\ldots,e_d)Y'$$

with coefficients of X', Y' in A_{cris}, $\mathrm{Fil}^pA_{\mathrm{cris}}$ respectively. Combining (5.3.5) with (5.3.6), we have

$$A(tX - E(u)p^nX') = tY - E(u)p^nY'$$

where $(\alpha_1,\ldots,\alpha_d) = (e_1,\ldots,e_d)A$. By Lemma 4.1.2, there exists a $d\times d$-matrix B with coefficients in S such that $BA = AB = E(u)^rI$, we have

$$E(u)^r(tX - E(u)p^nX') = tBY - E(u)p^nBY'$$

Note that the right hand side is in $\mathrm{Fil}^1A_{\mathrm{cris}} \cdot \mathrm{Fil}^pA_{\mathrm{cris}}$. By Lemma 3.2.2, we get $E(u)^{r-1}(tX - E(u)p^nX') \in \mathrm{Fil}^pA_{\mathrm{cris}}$. Modulo $\mathrm{Fil}^pA_{\mathrm{cris}} + pA_{\mathrm{cris}}$ both sides, we get the coefficients of $E(u)^{r-1}tX$ are in $\mathrm{Fil}^pA_{\mathrm{cris}} + pA_{\mathrm{cris}}$ (here we may assume that $n \geq 1$). An almost the same argument as in the proof of Lemma 5.3.1 shows that the coefficients of X are in $\mathrm{Fil}^1S + pS$.

Now consider

$$\begin{aligned} c_1p^nN((e_1,\ldots,e_d)) &= c_1p^nN(\varphi_r(\alpha_1),\ldots,\varphi_d(\alpha_d)) \\ &= p^n\varphi_r(E(u)N((\alpha_1,\ldots,\alpha_d))) \\ &= \varphi_r((\alpha_1,\ldots,\alpha_d))\varphi(X) + \varphi((e_1,\ldots,e_d))\varphi_r(Y) \end{aligned}$$

But $p|\varphi(X)$ and $p|\varphi_r(Y)$ in A_{cris}. This contradicts to the selection of n unless $n = 0$. That is, W has all its coefficients in S and then $N(\mathcal{M}) \subset \mathcal{M}$.

References

[Bre97] Christophe Breuil, *Représentations p-adiques semi-stables et transversalité de Griffiths*, Math. Ann. **307** (1997), no. 2, 191–224.

[Bre98a] ———, *Cohomologie étale de p-torsion et cohomologie cristalline en réduction semi-stable*, Duke Math. J. **95** (1998), no. 3, 523–620.

[Bre98b] ———, *Construction de représentations p-adiques semi-stables*, Ann. Sci. École Norm. Sup. (4) **31** (1998), no. 3, 281–327.

[Bre98c] ———, *Schémas en groupes et corps des normes*, Unpublished (1998).

[Bre99a] ———, *Représentations semi-stables et modules fortement divisibles*, Invent. Math. **136** (1999), no. 1, 89–122.

[Bre99b] ———, *Une application de corps des normes*, Compositio Math. **117** (1999), no. 2, 189–203.

[Bre99c] ______, *Une remarque sur les représentations locales p-adiques et les congruences entre formes modulaires de Hilbert*, Bull. Soc. Math. France **127** (1999), no. 3, 459–472.

[Bre00] ______, *Groupes p-divisibles, groupes finis et modules filtrés*, Ann. of Math. (2) **152** (2000), no. 2, 489–549.

[Bre02] ______, *Integral p-adic Hodge theory*, Algebraic geometry 2000, Azumino (Hotaka), Adv. Stud. Pure Math., vol. 36, Math. Soc. Japan, Tokyo, 2002, pp. 51–80.

[Car05] Xavier Caruso, *Conjecture de l'inértié modre de Serre*, Thesis, Université de Paris-Sud 11 (2005).

[CF00] Pierre Colmez and Jean-Marc Fontaine, *Construction des représentations p-adiques semi-stables*, Invent. Math. **140** (2000), no. 1, 1–43.

[CL06] Xavier Caruso and Tong Liu, *Quasi-crystalline representations*, In preparation (2006).

[Fal99] Gerd Faltings, *Integral crystalline cohomology over very ramified valuation rings*, J. Amer. Math. Soc. **12** (1999), no. 1, 117–144.

[FL82] Jean-Marc Fontaine and Guy Laffaille, *Construction de représentations p-adiques*, Ann. Sci. École Norm. Sup. (4) **15** (1982), no. 4, 547–608 (1983).

[Fon90] Jean-Marc Fontaine, *Représentations p-adiques des corps locaux. I*, The Grothendieck Festschrift, Vol. II, Progr. Math., vol. 87, Birkhäuser Boston, Boston, MA, 1990, pp. 249–309.

[Fon94a] ______, *Le corps des périodes p-adiques*, Astérisque (1994), no. 223, 59–111.

[Fon94b] ______, *Représentations p-adiques semi-stables*, Astérisque (1994), no. 223, 113–184, With an appendix by Pierre Colmez, Périodes p-adiques (Bures-sur-Yvette, 1988).

[Kis04] Mark Kisin, *Moduli of finite flat group schemes and modularity*, Preprint (2004).

[Kis05] ______, *Crystalline representations and F-crystals*, Preprint, To appear in Drinfeld's 50th birthday volume. (2005).

[Liu06] Tong Liu, *Torsion Galois representations and a Conjecture of Fontaine*, In preparation (2006).

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