A NOTE ON POTENTIAL DIAGONALIZABILITY OF CRYSTALLINE REPRESENTATIONS

HUI GAO, TONG LIU

Abstract. Let $K_0/\mathbb{Q}_p$ be a finite unramified extension and G_{K_0} denote the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/K_0)$. We show that all crystalline representations of G_{K_0} with Hodge-Tate weights $\subseteq \{0, \cdots, p-1\}$ are potentially diagonalizable.

1

Contents

1. Introduction

Let p be a prime, K a finite extension over $\mathbb{Q}_p$ and G_K denote the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$. In [BLGGT10] §1.4, *potential diagonalizability* is defined for a potential crystalline representation of G_K. Since the potential diagonalizability is the local condition at p for a global Galois representation in the automorphic lifting theorems proved in [BLGGT10] (cf. Theorem **A, B, C**), it is quite interesting to investigate what kind of potential crystalline representations are indeed potentially diagonalizable. Let K_0 be a finite unramified extension of $\mathbb{Q}_p$. By using Fontaine-Laffaille's theory, Lemma 1.4.3 (2) in [BLGGT10] proved that any crystalline representation of G_{K_0} with Hodge-Tate weights in $\{0, \dots, p-2\}$ is potentially diagonalizable.

In this short note, we show that the idea in [BLGGT10] can be extended to prove the potential diagonalizability of crystalline representations of G_{K_0} if Hodge-Tate weights are in $\{0, \dots, p-1\}$. Let $\rho : G_{K_0} \to \mathrm{GL}_d(\overline{\mathbb{Q}}_p)$ be a crystalline representation with Hodge-Tate weights in $\{0, \dots, p-1\}$. To prove the potential diagonalizability of ρ, we first reduce to the case that ρ is irreducible. Then ρ is *nilpotent* (see definition in §2.2). Note that Fontaine-Laffaille's theory can be

1991 *Mathematics Subject Classification.* Primary 14F30,14L05.
Key words and phrases. p-adic Galois representations, Crystalline representations, nilpotency, Potential Diagonalizability, Fontaine-Laffaille Data .
[1]The second author is partially supported by NSF grant DMS-0901360.

extended to nilpotent representations. Hence we can follow the similar idea in [BLGGT10] to conclude the potential diagonalizability of ρ.

Acknowledgement: It is pleasure to thank David Geraghty and Toby Gee for very useful conversations and correspondence.

Notations

Throughout this note, K is always a finite extension of $\mathbb{Q}_p$ with the absolute Galois group $G_K := \mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$. Let K_0 be a finite unramified extension of $\mathbb{Q}_p$ with the residue field k. We denote $W(k)$ its ring of integers and $\mathrm{Frob}_{W(k)}$ the arithmetic Frobenius on $W(k)$. If E is a finite extension of $\mathbb{Q}_p$ then we write $\mathcal{O}_E$ the ring of integers, ϖ its uniformizer and $\mathbb{F} = \mathcal{O}/\varpi\mathcal{O}$ its residue field. If A is a local ring, we denote $\mathfrak{m}_A$ the maximal ideal of A. Let $\rho : G_K \to \mathrm{GL}_d(A)$ be a continuous representation with the ambient space $M = \oplus_{i=1}^d A$. We always denote ρ^* the dual representation induced by $\mathrm{Hom}_A(M, A)$. Let $\rho : G_K \to \mathrm{GL}_d(\overline{\mathbb{Q}}_p)$ be a de Rham representation of G_K. Then $D_{\mathrm{dR}}(\rho^*)$ is a graded $K \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$-module. For any embedding $\tau : K \to \overline{\mathbb{Q}}_p$, we define the set of τ-Hodge-Tate weights

$$\mathrm{HT}_\tau(\rho) := \{i \in \mathbb{Z} | \mathrm{gr}^i(D_{\mathrm{dR}}(\rho^*)) \otimes_{K\otimes_{\mathbb{Q}_p}\overline{\mathbb{Q}}_p} (K \otimes_{K,\tau} \overline{\mathbb{Q}}_p) \neq 0\}.$$

In particular, if ϵ denotes the p-adic cyclotomic character then $\mathrm{HT}_\tau(\epsilon) = \{1\}$ (here our convention is slightly different from that in [BLGGT10]).

2. Definitions and Preliminary

2.1. **Potential Diagonalizability.** We recall the definition of potential diagonalizability from [BLGGT10]. Given two continuous representations $\rho_1, \rho_2 : G_K \to \mathrm{GL}_d(\mathcal{O}_{\overline{\mathbb{Q}}_p})$, we say that ρ_1 *connects to* ρ_2, denoted by $\rho_1 \sim \rho_2$, if:

- the two reductions $\bar\rho_i := \rho_i \bmod \mathfrak{m}_{\mathcal{O}_{\overline{\mathbb{Q}}_p}}$ are equivalent to each other;
- both ρ_1 and ρ_2 are potentially crystalline;
- for each embedding $\tau : K \hookrightarrow \overline{\mathbb{Q}}_p$, we have $\mathrm{HT}_\tau(\rho_1) = \mathrm{HT}_\tau(\rho_2)$;
- ρ_1 and ρ_2 define points on the same irreducible component of the scheme $\mathrm{Spec}(R^\square_{\bar\rho_1,\{\mathrm{HT}_\tau(\rho_1)\},K'\text{-cris}}[\frac{1}{p}])$ for some sufficiently large field extension K'/K. Here $R^\square_{\bar\rho_1,\{\mathrm{HT}_\tau(\rho_1)\},K'\text{-cris}}$ is the quotient of the framed universal deformation ring $R^\square_{\bar\rho_1}$ corresponding to liftings ρ with $\mathrm{HT}_\tau(\rho) = \mathrm{HT}_\tau(\rho_1)$ for all τ and with $\rho |_{G_{K'}}$ crystalline. The existence of $R^\square_{\bar\rho_1,\{\mathrm{HT}_\tau(\rho_1)\},K'\text{-cris}}$ is the main result of [Kis08].

A representation $\rho : G_K \to \mathrm{GL}_d(\mathcal{O}_{\overline{\mathbb{Q}}_p})$ is called *diagonalizable* if it is crystalline and connects to a sum of crystalline characters $\chi_1 \oplus \cdots \oplus \chi_d$. It is called *potentially diagonalizable* if $\rho |_{G_{K'}}$ is diagonalizable for some finite extension K'/K.

Remark 2.1.1. By Lemma 1.4.1 of [BLGGT10], the potential diagonalizability is well defined for a representation $\rho : G_K \to \mathrm{GL}_d(\overline{\mathbb{Q}}_p)$ because for any two G_K-stable $\mathcal{O}_{\overline{\mathbb{Q}}_p}$-lattices L and L', L is potentially diagonalizable if and only if L' is potentially diagonalizable.

Lemma 2.1.2. *Suppose $\rho : G_K \to \mathrm{GL}_d(\overline{\mathbb{Q}}_p)$ is potentially crystalline. Let Fil^i be a G_K-invariant filtration on ρ. Then ρ is potentially diagonalizable if and only if $\oplus_i \mathrm{gr}^i\rho$ is potentially diagonalizable.*

Proof. We can always choose a G_K-stable $\mathcal{O}_{\overline{\mathbb{Q}}_p}$-lattice M inside the ambient space of ρ such that $\mathrm{Fil}^i\rho \cap M$ is the $\mathcal{O}_{\overline{\mathbb{Q}}_p}$-summand of M and the reduction $\bar{M}$ is semi-simple. Then the lemma follows (7) on page 20 of [BLGGT10]. □

2.2. **Nilpotency and Fontaine-Laffaille Data.** Let $\mathcal{O} := \mathcal{O}_E$ for a finite extension E over $\mathbb{Q}_p$ and $W(k)_{\mathcal{O}} := W(k) \otimes_{\mathbb{Z}_p} \mathcal{O}$. Following [FL82], let $\mathcal{MF}_{\mathcal{O}}$ denote the category of finite $W(k)_{\mathcal{O}}$-modules M with

- a decreasing filtration $\mathrm{Fil}^i M$ by $W(k)_{\mathcal{O}}$-submodules which are $W(k)$-direct summands with $\mathrm{Fil}^0 M = M$ and $\mathrm{Fil}^p M = \{0\}$;
- $\mathrm{Frob}_{W(k)} \otimes 1$-semi-linear maps $\varphi_i : \mathrm{Fil}^i M \to M$ with $\varphi_i |_{\mathrm{Fil}^{i+1} M} = p\varphi_{i+1}$ and $\sum_i \varphi_i(\mathrm{Fil}^i M) = M$.

The morphisms in $\mathcal{MF}_{\mathcal{O}}$ are $W(k)_{\mathcal{O}}$-linear morphisms that compatible with φ_i and Fil^i structures. We denote $\mathcal{MF}_{\mathcal{O},\mathrm{tor}}$ the full sub-category of $\mathcal{MF}_{\mathcal{O}}$ consisting objects which are killed by some p-power, and denote $\mathcal{MF}_{\mathcal{O},\mathrm{fr}}$ the full category of $\mathcal{MF}_{\mathcal{O}}$ whose objects are finite $W(k)_{\mathcal{O}}$-free. Obviously, if $M \in \mathcal{MF}_{\mathcal{O},\mathrm{fr}}$ then $M/\varpi^m M$ is in $\mathcal{MF}_{\mathcal{O},\mathrm{tor}}$ for all m.

It turns out that the category $\mathcal{MF}_{\mathcal{O},\mathrm{tor}}$ is abelian (see §1.10 in [FL82]). An object M in $\mathcal{MF}_{\mathcal{O},\mathrm{tor}}$ is called *nilpotent* if there is no nontrivial subobject $M' \subset M$ such that $\mathrm{Fil}^1 M' = \{0\}$. Denote the full subcategory of nilpotent objects by $\mathcal{MF}^{\mathrm{n}}_{\mathcal{O},\mathrm{tor}}$. An object $M \in \mathcal{MF}_{\mathcal{O},\mathrm{fr}}$ is called *nilpotent* if $M/\varpi^m M$ is nilpotent for all m. Denote by $\mathcal{MF}^{\mathrm{n}}_{\mathcal{O},\mathrm{fr}}$ the full subcategory of $\mathcal{MF}_{\mathcal{O},\mathrm{fr}}$ whose objects are nilpotent.

Let $\mathrm{Rep}_{\mathcal{O}}(G_{K_0})$ be the category of finite $\mathcal{O}$-modules with continuous $\mathcal{O}$-linear G_{K_0}-actions. We define a functor T^*_{cris} from the category $\mathcal{MF}^{\mathrm{n}}_{\mathcal{O},\mathrm{tor}}$ (resp. $\mathcal{MF}^{\mathrm{n}}_{\mathcal{O},\mathrm{fr}}$) to $\mathrm{Rep}_{\mathcal{O}}(G_{K_0})$:

$$T^*_{\mathrm{cris}}(M) := \mathrm{Hom}_{W(k),\varphi_i,\mathrm{Fil}^i}(M, A_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} (\mathbb{Q}_p/\mathbb{Z}_p)) \text{ if } M \in \mathcal{MF}_{\mathcal{O},\mathrm{tor}},$$

and

$$T^*_{\mathrm{cris}}(M) := \mathrm{Hom}_{W(k),\varphi_i,\mathrm{Fil}^i}(M, A_{\mathrm{cris}}) \text{ if } M \in \mathcal{MF}_{\mathcal{O},\mathrm{fr}}.$$

Let $\mathrm{Rep}^{[0,p-1]}_{E,\mathrm{cris}}(G_{K_0})$ denote the category of continuous E-linear G_{K_0}-representations on finite dimensional E-vector spaces V such that V are crystalline with Hodge-Tate weights in $\{0,\dots,p-1\}$. An object $V \in \mathrm{Rep}^{[0,p-1]}_{E,\mathrm{cris}}(G_{K_0})$ is called *nilpotent* if V does not admit nontrivial unramified quotient [2]. We denote by $\mathrm{Rep}^{[0,p-1],\mathrm{n}}_{\mathcal{O},\mathrm{cris}}(G_{K_0})$ the category of G_{K_0}-stable $\mathcal{O}$-lattices in nilpotent representations in $\mathrm{Rep}^{[0,p-1]}_{E,\mathrm{cris}}(G_{K_0})$.

We gather the following useful results from [FL82] and [Laf80].

Theorem 2.2.1. (1) *The contravariant functor T^*_{cris} from $\mathcal{MF}^{\mathrm{n}}_{\mathcal{O},\mathrm{tor}}$ to $\mathrm{Rep}_{\mathcal{O}}(G_{K_0})$ is exact and fully faithful.*
(2) *An object $M \in \mathcal{MF}_{\mathcal{O},\mathrm{fr}}$ is nilpotent if and only if $M/\varpi M$ is nilpotent.*
(3) *The essential image of T^*_{cris} from $\mathcal{MF}^{\mathrm{n}}_{\mathcal{O},\mathrm{tor}}$ is closed under taking sub-objects and quotients.*
(4) *Let V be a crystalline representation of G_{K_0}. V is nilpotent if and only if $V|_{G_{K'}}$ is nilpotent for any unramified extension K'/K_0.*
(5) *T^*_{cris} induces an anti-equivalence between the category $\mathcal{MF}^{\mathrm{n}}_{\mathcal{O},\mathrm{fr}}$ and the category $\mathrm{Rep}^{[0,p-1],\mathrm{n}}_{\mathcal{O},\mathrm{cris}}(G_{K_0})$.*

[2] It is easy to check that V admits a nontrivial unramified quotient as $\mathbb{Q}_p$-representations if and only if V admits a nontrivial unramified quotient as E-representations. See the proof of Theorem 2.2.1 (4).

Proof. (1) and (2) follow from Theorem 6.1 in [FL82]. Note that $\underline{\mathrm{U}}_S$ in [FL82] is just T^*_{cris} here. To prove (3), we may assume that $\mathcal{O} = \mathbb{Z}_p$ and it suffices to check that T^*_{cris} sends simple objects in $\mathcal{MF}^{\mathrm{n}}_{\mathcal{O},\mathrm{tor}}$ to simple objects in $\mathrm{Rep}_{\mathcal{O}}(G_{K_0})$ (see Property 6.4.2 in [Car06]). And this is proved in [FL82], §6.13 (a). (4) is clear because V is nilpotent if and only if $(V^*)^{I_{K_0}} = \{0\}$ where I_{K_0} is the inertia subgroup of G_{K_0}. (5) has been essentially proved in [FL82] and [Laf80] but has not been recorded in literature. So we sketch the proof here. First, by (1), it is clear that $T^*_{\mathrm{cris}}(M)$ is a continuous $\mathcal{O}$-linear G_{K_0}-representation on a finite free $\mathcal{O}$-module T. Using the totally same proof for the case when $\mathrm{Fil}^{p-1}M = \{0\}$ in [FL82] (using Theorem in [FL82] §0.6 and (1)), we has $\mathrm{rank}_{\mathcal{O}}(T) = \mathrm{rank}_{W(k)_{\mathcal{O}}} M = d$. It is easy to see that M is a $W(k)$-lattice in $D_{\mathrm{cris}}(V^*)$ where $V = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T$. Hence V is crystalline with Hodge-Tate weights in $\{0, \dots, p-1\}$. To see V is nilpotent, note that V has a unramified quotient $\tilde{V}$ is equivalent to that there exists an $M' \subset M$ such that $M' \cap \mathrm{Fil}^1 M = \{0\}$ and M/M' has no torsion (just let $M' := D_{\mathrm{cris}}(\tilde{V}^*) \cap M$). So M is nilpotent implies that V is nilpotent. Hence by (1), T^*_{cris} is an exact, fully faithful functor from $\mathcal{MF}^{\mathrm{n}}_{\mathcal{O},\mathrm{fr}}$ to $\mathrm{Rep}^{[0,p-1],\mathrm{n}}_{\mathcal{O},\mathrm{cris}}(G_{K_0})$. To prove the essential surjectiveness of T^*_{cris}, it suffices to assume that $\mathcal{O} = \mathbb{Z}_p$ (indeed, suppose that T is an object in $\mathrm{Rep}^{[0,p-1],\mathrm{n}}_{\mathcal{O},\mathrm{cris}}(G_{K_0})$ with $d = \mathrm{rank}_{\mathcal{O}} T$. Let $V = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T$ and $D = D_{\mathrm{cris}}(V^*)$. It is well-known that D is a finite free $E \otimes_{\mathbb{Q}_p} K_0$-module with rank d. If there exists an $M \in \mathcal{MF}^{\mathrm{n}}_{\mathbb{Z}_p,\mathrm{fr}}$ such that $T^*_{\mathrm{cris}}(M) \simeq T$ as $\mathbb{Z}_p[G]$-modules. By the full faithfulness of T^*_{cris}, M is naturally a $W(k)_{\mathcal{O}}$-module. Since D is $E \otimes_{\mathbb{Q}_p} K_0$-free, it is standard to show that M is finite $W(k)_{\mathcal{O}}$-free by computing $\mathcal{O}_i$-rank of M_i, where $M_i := M \otimes_{W(k)_{\mathcal{O}}} \mathcal{O}_i$ and $W(k)_{\mathcal{O}} \simeq \prod_i \mathcal{O}_i$). Now suppose that T is an object in $\mathrm{Rep}^{[0,p-1],\mathrm{n}}_{\mathbb{Z}_p,\mathrm{cris}}$. Let $V = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T$ and $D = D_{\mathrm{cris}}(V^*)$. By [Laf80] §3.2, there always exists a $W(k)$-lattice $M \in \mathcal{MF}_{\mathbb{Z}_p,\mathrm{fr}}$ inside D. We has to show that M is nilpotent. Suppose that $\bar{M} := M/pM$ is not nilpotent. Then there exists $N \subset \bar{M}$ such that $\mathrm{Fil}^1 N = \{0\}$. Consequently $\varphi_0(\mathrm{Fil}^0 N) = \varphi_0(N) = N$. By Fitting lemma, $\bigcap_m (\varphi_0)^m(\bar{M}) \neq \{0\}$. So $\bigcap_m (\varphi_0)^m(M) \neq \{0\}$ which means that $V = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T^*_{\mathrm{cris}}(M)$ must has a unramified quotient. This contradicts that V is nilpotent and hence M has to be nilpotent by (2). It remains to show that other G_{K_0}-stable $\mathbb{Z}_p$-lattices $L' \subset V$ is also given by an object $M' \in \mathcal{MF}^{\mathrm{n}}_{\mathbb{Z}_p,\mathrm{fr}}$. Let $L := T^*_{\mathrm{cris}}(M)$. Without loss of generality, we can assume that $L' \subset L$. So for sufficient large m, we have the exact sequence $0 \to f_m(L') \to L/p^m L \to L/L' \to 0$ where f_m is the map $L' \hookrightarrow L \twoheadrightarrow L/p^m L$. It easy to check that $L/p^m L \simeq T^*_{\mathrm{cris}}(M/p^m M)$. By (3), there exists an object $M'_m \in \mathcal{MF}^{\mathrm{n}}_{\mathcal{O},\mathrm{tor}}$ such that $T^*_{\mathrm{cris}}(M'_m) \simeq f_m(L')$. Finally $M' = \varprojlim_m M'_m$ is the required object in $\mathcal{MF}^{\mathrm{n}}_{\mathcal{O},\mathrm{fr}}$. □

Contravariant functors like T^*_{cris} are not convenient for deformation theory. So we define a covariant variant for T^*_{cris}. Define $T_{\mathrm{cris}}(M) := (T^*_{\mathrm{cris}}(M))^*(p-1)$, more precisely,

$$T_{\mathrm{cris}}(M) := \mathrm{Hom}_{\mathcal{O}}(T^*_{\mathrm{cris}}(M), E/\mathcal{O})(p-1) \text{ if } M \in \mathcal{MF}^{\mathrm{n}}_{\mathcal{O},\mathrm{tor}},$$

and

$$T_{\mathrm{cris}}(M) := \mathrm{Hom}_{\mathcal{O}}(T^*_{\mathrm{cris}}(M), \mathcal{O})(p-1) \text{ if } M \in \mathcal{MF}^{\mathrm{n}}_{\mathcal{O},\mathrm{fr}}.$$

Let $\rho : G_{K_0} \to \mathrm{GL}_d(\mathcal{O})$ be a continuous representation such that there exists an $M \in \mathcal{MF}^{\mathrm{n}}_{\mathcal{O},\mathrm{fr}}$ satisfying $T_{\mathrm{cris}}(M) = \rho$. Then $T_{\mathrm{cris}}(\bar{M}) = \bar{\rho} := \rho \mod \varpi\mathcal{O}$ where $\bar{M} := M/\varpi M$. Let $\mathcal{C}^f_{\mathcal{O}}$ denote the category of Artinian local $\mathcal{O}$-algebras for which the structure map $\mathcal{O} \to R$ induces an isomorphism on residue fields. Define a deformation functor $D^{\mathrm{n}}_{\mathrm{cris}}(R) := \{$lifts $\tilde{\rho} : G_{K_0} \to \mathrm{GL}_d(R)$ of $\bar{\rho}$ such that there exists an $M \in \mathcal{MF}^{\mathrm{n}}_{\mathcal{O},\mathrm{tor}}$ satisfying $T_{\mathrm{cris}}(M) \simeq \tilde{\rho}\}$.

Proposition 2.2.2. *Assume that $W(k) \subset \mathcal{O}$. Then $D^{\mathrm{n}}_{\mathrm{cris}}$ is pro-represented by a formally smooth $\mathcal{O}$-algebra $R^{\mathrm{n}}_{\bar{\rho},\mathrm{cris}}$.*

Proof. By (1) and (3) in Theorem 2.2.1, $D^{\mathrm{n}}_{\mathrm{cris}}$ is a sub-functor of the framed Galois deformation functor of $\bar{\rho}$ and pro-represented by an $\mathcal{O}$-algebra $R^{\mathrm{n}}_{\bar{\rho},\mathrm{cris}}$. The formal smoothness of $R^{\mathrm{n}}_{\bar{\rho},\mathrm{cris}}$ is totally the same as that in Lemma 2.4.1 in [CHT08]. Indeed, suppose that R is an object of $\mathcal{C}^f_{\mathcal{O}}$ and I is an ideal of R with $\mathfrak{m}_R I = (0)$. To prove the formal smoothness of $R^{\mathrm{n}}_{\bar{\rho},\mathrm{cris}}$, we have to show that any lift in $D^{\mathrm{n}}_{\mathrm{cris}}(R/I)$ admits a lift in $D^{\mathrm{n}}_{\mathrm{cris}}(R)$. Then this is equivalent to lift the corresponding $N \in \mathcal{MF}_{\mathcal{O},\mathrm{tor}}$ (note that any lift N of $\bar{M}$ will be automatically in $\mathcal{MF}^{\mathrm{n}}_{\mathcal{O},\mathrm{tor}}$ by Theorem 2.2.1 (3)). And this is just the same proof in Lemma 2.4.1 in [CHT08]. Note that the proof did not use the restrictions (assumed for §2.4.1 loc. cit.) that $\mathrm{Fil}^{p-1}M = \{0\}$ and $\dim_k(\mathrm{gr}^i \mathbf{G}^{-1}_{\tilde{v}}(\bar{r}|_{G_{F_{\tilde{v}}}}) \otimes_{\mathcal{O}_{F_{\tilde{v}}},\tilde{\tau}} \mathcal{O} \leq 1$.

□

3. The Main Theorem and Its Proof

Theorem 3.0.3 (**Main Theorem**). *Suppose $\rho : G_{K_0} \to \mathrm{GL}_d(\overline{\mathbb{Q}}_p)$ is a crystalline representation, and for each $\tau : K_0 \hookrightarrow \overline{\mathbb{Q}}_p$, the Hodge-Tate numbers $\mathrm{HT}_\tau(\rho) \subseteq \{a_\tau, \ldots, a_\tau + p - 1\}$, then ρ is potentially diagonalizable.*

Proof. We may assume that ρ factors through $\mathrm{GL}_d(\mathcal{O})$ for a sufficient large $\mathcal{O}$. By Lemma 2.1.2, we can assume that ρ is irreducible and hence $\rho^*(p-1)$ is nilpotent. Just as in the proof of Lemma 1.4.3 of [BLGGT10], we can assume $a_\tau = 0$ for all τ. Then we can choose an unramified extension K', such that $\bar{\rho}\ |_{G_{K'}}$ has a $G_{K'}$-invariant filtration with 1-dimensional graded pieces. By Theorem 2.2.1 (4), $\rho^*(p-1)$ is still nilpotent when restricted to $G_{K'}$. Without loss of generality, we can assume that $K_0 = K'$. Now there exists an $M \in \mathcal{MF}^{\mathrm{n}}_{\mathcal{O},\mathrm{fr}}$ such that $T_{\mathrm{cris}}(M) \simeq \rho$. Then $\bar{M} := M/\varpi M$ is nilpotent and $T_{\mathrm{cris}}(\bar{M}) \simeq \bar{\rho}$. Note that $\bar{M}$ has filtration with rank-1 $\mathbb{F} \otimes_{\mathbb{Z}/p\mathbb{Z}} k$-graded pieces to correspond to the filtration of $\bar{\rho}$. Now by Lemma 1.4.2 of [BLGGT10], we lift $\bar{M}$ to $M' \in \mathcal{MF}_{\mathcal{O},\mathrm{fr}}$ which has filtration with rank-1 $W(k)_{\mathcal{O}}$-graded pieces (note the proof of Lemma 1.4.2 did not use the restriction that $\mathrm{HT}_\tau(\rho) \subseteq \{0, \ldots, p-2\}$). Hence M' is nilpotent by Theorem 2.2.1 (3). Then $\rho' = T_{\mathrm{cris}}(M')$ is crystalline and has a G_{K_0}-invariant filtration with 1-dimensional graded pieces by Theorem 2.2.1 (5). Then part 1 of Lemma 1.4.3 of [BLGGT10] implies that ρ' is potentially diagonalizable. Now it suffices to show that ρ connects to ρ'. But it is obvious that $R^{\mathrm{n}}_{\bar{\rho},\mathrm{cris}}$ is a quotient $R^{\square}_{\bar{\rho},\{\mathrm{HT}_\tau(\rho)\},K\text{-cris}}$. By Proposition 2.2.2, we see that ρ and ρ' must be in the same connected component of $\mathrm{Spec}(R^{\square}_{\bar{\rho},\{\mathrm{HT}_\tau(\rho)\},K\text{-cris}}[\frac{1}{p}])$. Hence $\rho \sim \rho'$ and ρ is potentially diagonalizable.

□

References

[BLGGT10] Thomas Barnet-Lamb, Toby Gee, David Geraghty, and Richard Taylor. Potential automorphy and change of weight. preprint, 2010.

[Car06] Xavier Caruso. Représentations semi-stables de torsion dans le case $er < p - 1$. *J. Reine Angew. Math.*, 594:35–92, 2006.

[CHT08] Laurent Clozel, Michael Harris, and Richard Taylor. Automorphy for some l-adic lifts of automorphic mod l Galois representations. *Publ. Math. Inst. Hautes Études Sci.*, (108):1–181, 2008. With Appendix A, summarizing unpublished work of Russ Mann, and Appendix B by Marie-France Vignéras.

[FL82] Jean-Marc Fontaine and Guy Laffaille. Construction de représentations p-adiques. *Ann. Sci. École Norm. Sup. (4)*, 15(4):547–608 (1983), 1982.

[Kis08] Mark Kisin. Potentially semi-stable deformation rings. *J. Amer. Math. Soc.*, 21(2):513–546, 2008.

[Laf80] Guy Laffaille. Groupes p-divisibles et modules filtrés: le cas peu ramifié. *Bull. Soc. Math. France*, 108(2):187–206, 1980.

E-mail address: gaoh@math.purdue.edu, tongliu@math.purdue.edu

Department of Mathematics, Purdue University, 150 N. University Street, West Lafayette, IN 47907, USA