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Compatibility of Kisin modules for different uniformizers

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Abstract. Let p be a prime and T a lattice inside a semi-stable representation V. We prove that Kisin modules associated to T by selecting different uniformizers are isomorphic after tensoring a subring in $W(R)$. As consequences, we show that several lattices inside the filtered (φ, N)-module of V constructed from Kisin modules are independent on the choice of uniformizers. Finally, we use a similar strategy to show that the Wach module can be recovered from the $(\varphi, \hat{G})$-module associated to T when V is crystalline and the base field is unramified.

1. Introduction

Let k be a perfect field of characteristic p, $W(k)$ its ring of Witt vectors, $K_0 = W(k)[\frac{1}{p}]$, K/K_0 a finite totally ramified extension, $G_K := \mathrm{Gal}(\overline{K}/K)$.

To understand the p-adic Hodge structure of G_K-stable $\mathbb{Z}_p$-lattices in semi-stable representations, the method of Kisin modules is powerful. Recall the definition of Kisin modules in the following: We *fix* a uniformiser $\pi \in K$ with Eisenstein polynomial $E(u)$. Put

$$\mathfrak{S} := W(k)[\![u]\!];$$

$\mathfrak{S}$ is equipped with a Frobenius endomorphism φ via $u \mapsto u^p$ and the natural Frobenius on $W(k)$. A *Kisin module of height r* is a finite free $\mathfrak{S}$-module $\mathfrak{M}$ with φ-semi-linear endomorphism $\varphi_{\mathfrak{M}} : \mathfrak{M} \to \mathfrak{M}$ such that $E(u)^r \mathfrak{M} \subset \langle \varphi_{\mathfrak{M}}(\mathfrak{M}) \rangle$, where $\langle \varphi_{\mathfrak{M}}(\mathfrak{M}) \rangle$ is the $\mathfrak{S}$-submodule of $\mathfrak{M}$ generated by $\varphi_{\mathfrak{M}}(\mathfrak{M})$. By the result of Kisin [9], for any G_K-stable $\mathbb{Z}_p$-lattice T inside a semi-stable representation V with Hodge–Tate weights in $\{0, \dots, r\}$, there exists a unique Kisin module $\mathfrak{M}(T)$ of height r attached to T (see Section 2.1 for more precise meaning of this sentence).

It is obvious that the construction of Kisin modules depends on the choice of the uniformizer π. If we choose another uniformizer π' of K, then we get another $\mathfrak{M}'(T)$. A natural question is: what is the relationship between $\mathfrak{M}(T)$ and $\mathfrak{M}'(T)$?

It turns out that each choice of uniformizer π determines an embedding $\mathfrak{S} \hookrightarrow W(R)$ via $u \mapsto [\underline{\pi}]$ (see Section 2.1 for details of the definition of $W(R)$ and $[\underline{\pi}]$). We denote by $\mathfrak{S}_{\underline{\pi}}$

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and $\mathfrak{S}_{\underline{\pi}'}$ the image of embedding determined by $\underline{\pi}$ and $\underline{\pi}'$, respectively. By the main result of [12], there exists a G_K-action on $W(R) \otimes_{\varphi, \mathfrak{S}_{\underline{\pi}}} \mathfrak{M}(T)$ which commutes with $\varphi_{\mathfrak{M}}$. In this paper, we prove the following:

Theorem 1.0.1. *There exists a $W(R)$-linear isomorphism*

$$W(R) \otimes_{\varphi, \mathfrak{S}_{\underline{\pi}}} \mathfrak{M}(T) \simeq W(R) \otimes_{\varphi, \mathfrak{S}_{\underline{\pi}'}} \mathfrak{M}'(T)$$

compatible with φ-actions and G_K-actions on the both sides.

In fact, $W(R)$ in the above isomorphism can be replaced by a much smaller ring $\tilde{\mathfrak{S}}_{\underline{\pi},\underline{\pi}'}$, and an even smaller ring $\mathfrak{S}_{\underline{\pi},\underline{\pi}'}$ when V is crystalline. See Theorem 2.2.1 for more details. It turns out that we can extend Theorem 1.0.1 to discuss the relation between Kisin modules and Wach modules ([2]). Assume that $K = K_0$ is unramified and let T be a G_K-stable $\mathbb{Z}_p$-lattice inside a crystalline representation. Then we can attach the Wach module $\mathfrak{N}(T)$ and the Kisin module $\mathfrak{M}(T)$ to T. Let ζ_{p^n} be a primitive p^n-th root of unity. Set $K_{p^\infty} := \bigcup_{n=1}^{\infty} K(\zeta_{p^n})$ and $H_{p^\infty} := \mathrm{Gal}(\overline{K}/K_{p^\infty})$. The following theorem describes a direct relation between the Kisin module and the Wach module.

Theorem 1.0.2. *We have $\mathfrak{N}(T) \simeq (\hat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}(T))^{H_{p^\infty}}$.*

Here $\hat{\mathcal{R}} \subset W(R)$ is a subring constructed in [12, Section 2.2] and it was proved that $\hat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} \subset W(R) \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$ is G_K-stable ([12, Section 2.2], also see Section 2.1).

Theorem 1.0.1 can be used to understand lattices in the filtered (φ, N)-modules attached to semi-stable representations of G_K. More precisely, let V be a de Rham representation of G_K and $T \subset V$ a G_K-stable $\mathbb{Z}_p$-lattice. It is well known ([1]) that V is semi-stable over a finite extension K'/K. By using the Kisin module attached to $T|_{G_{K'}}$, we can construct various lattices in either $D_{\mathrm{st},K'}(V) := (V^\vee \otimes_{\mathbb{Q}_p} B_{\mathrm{st}})^{G_{K'}}$ or $D_{\mathrm{dR}}(V) := (V^\vee \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{G_K}$, where $V^\vee$ denotes the dual of V. One consequence of Theorem 1.0.1 is that the constructions of such lattices are independent of the choice of π. In the end, we also discuss several lattices (inside the filtered (φ, N)-module) whose constructions are independent of Kisin's theory. But they are useful to discuss the p-adic Hodge properties for (p-adic completion of) the direct limit of de Rham representations. In particular, we hope these will be useful to understand those representations discussed in [6] and [5].

The arrangement of this paper is as follows: In Section 2, we setup notations and summarize the facts needed for the proof of Theorem 1.0.1 and give a more precise version of Theorem 1.0.1. We give the proof of Theorems 1.0.1–1.0.2 in Section 3. We also show that the compatibility of Kisin modules when base changes (Theorem 3.2.1). Section 4 is devoted to discuss various lattices inside filtered (φ, N)-modules attached to potentially semi-stable representations. We show that two types of lattices constructed from Kisin's theory do not depend on the choice of uniformizers and they are compatible with base change. In Section 4.3, we show that several lattices (constructed without using Kisin' theory) may help us to understand the p-adic completion of direct limit of de Rham representations. In particular, we hope that our strategy is useful to those representations studied in [6]. The last section is the erratum of [13].

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2. Preliminary and main results

2.1. Kisin modules and $(\varphi, \hat{G})$-modules. We set up notations and recall some facts on (integral) p-adic Hodge theory in this subsection. We fix a nonnegative integer r throughout the paper. Let V be a semi-stable representation of G_K with Hodge–Tate weights in $\{0, \dots, r\}$. Write $V^\vee$ for the dual of V. By the well-known theorem of Fontaine and Colmez, the functor

$$V \mapsto D_{\mathrm{st}}(V) := (V^\vee \otimes_{\mathbb{Q}_p} B_{\mathrm{st}})^{G_K}$$

induces an *anti-equivalence* between the category of semi-stable representations with Hodge–Tate weights in $\{0, \dots, r\}$ and the category of weakly admissible filtered (φ, N)-modules $(D, \varphi, N, \{\mathrm{Fil}^i D_K\})$ with $\mathrm{Fil}^0 D_K = D_K$ and $\mathrm{Fil}^{r+1} D_K = \{0\}$. Here $D_K := K \otimes_{K_0} D$ as usual. The readers should be careful that we use the *contravariant* version of D_{st}, which were denoted by D^*_{st} in many papers. But the current version of D_{st} is more convenient for integral theory.

Let $R = \varprojlim \mathcal{O}_{\overline{K}}/p$, where the transition maps are given by Frobenius. By the universal property of the Witt vectors $W(R)$ of R, there is a unique surjective projection map

$$\theta : W(R) \to \hat{\mathcal{O}}_{\overline{K}}$$

to the p-adic completion $\hat{\mathcal{O}}_{\overline{K}}$ of $\mathcal{O}_{\overline{K}}$, which lifts the projection $R \to \mathcal{O}_{\overline{K}}/p$ onto the first factor in the inverse limit. We denote by A_{cris} the p-adic completion of the divided power envelope of $W(R)$ with respect to $\mathrm{Ker}(\theta)$. As usual, we write $B^+_{\mathrm{cris}} = A_{\mathrm{cris}}[\frac{1}{p}]$ and denote by B^+_{dR} the $\mathrm{Ker}(\theta)$-adic completion of $W(R)[\frac{1}{p}]$. For any subring $A \subset B^+_{\mathrm{dR}}$, we define a filtration on A by $\mathrm{Fil}^i A = A \cap (\mathrm{Ker}(\theta))^i B^+_{\mathrm{dR}}$.

Now select a uniformizer π of K. Let $E(u) \in W(k)[u]$ be the Eisenstein polynomial of π. Let $\pi_n \in \overline{K}$ be a p^n-th root of π such that $(\pi_{n+1})^p = \pi_n$; write $\underline{\pi} = (\pi_n)_{n \geq 0} \in R$ and let $[\underline{\pi}] \in W(R)$ be the Techmüller representative. We embed the $W(k)$-algebra $W(k)[u]$ into $W(R) \subset A_{\mathrm{cris}}$ by the map $u \mapsto [\underline{\pi}]$. Recall $\mathfrak{S} = W(k)[\![u]\!]$. This embedding extends to the embedding $\mathfrak{S} \hookrightarrow W(R)$ which are compatible with Frobenious endomorphisms.

We denote by S the p-adic completion of the divided power envelope of $W(k)[u]$ with respect to the ideal generated by $E(u)$. Write $S_{K_0} := S[\frac{1}{p}]$. There is a unique map (Frobenius) $\varphi_S : S \to S$ which extends the Frobenius on $\mathfrak{S}$. We write N_S for the K_0-linear derivation on S_{K_0} such that $N_S(u) = -u$. Let $\mathrm{Fil}^n S \subset S$ be the p-adic completion of the ideal generated by

$$\gamma_i(E(u)) := \frac{E(u)^i}{i!}$$

with $i \geq n$. One can show that the embedding $W(k)[u] \to W(R)$ via $u \mapsto [\underline{\pi}]$ extends to the embedding $S \hookrightarrow A_{\mathrm{cris}}$ compatible with Frobenius φ and filtration (note that $E([\underline{\pi}])$ is a generator of $\mathrm{Fil}^1 W(R)$). We set $B^+_{\mathrm{st}} := B^+_{\mathrm{cris}}[\mathfrak{u}] \subset B^+_{\mathrm{dR}}$ with $\mathfrak{u} := \log([\underline{\pi}])$.

Let

$$K_\infty := \bigcup_{n=0}^{\infty} K(\pi_n)$$

and $\hat{K}$ its Galois closure over K. Then $\hat{K} = \bigcup_{n=1}^{\infty} K_\infty(\zeta_{p^n})$ with ζ_{p^n} a primitive p^n-th root of unity. Write

$$G_\infty := \mathrm{Gal}(\overline{K}/K_\infty), \quad K_{p^\infty} := \bigcup_{n=1}^{\infty} K(\zeta_{p^n}), \quad G_{p^\infty} := \mathrm{Gal}(\hat{K}/K_{p^\infty}),$$
$$H_K := \mathrm{Gal}(\hat{K}/K_\infty), \qquad \hat{G} := \mathrm{Gal}(\hat{K}/K).$$

For any $g \in G_K$,

$$\underline{\epsilon}(g) := \frac{g(\underline{\pi})}{\underline{\pi}}$$

is a cocycle with value in R. Set $\underline{\epsilon} := (\zeta_{p^i})_{i \geq 0} \in R$ and $t := -\log([\underline{\epsilon}]) \in A_{\mathrm{cris}}$ as usual.

As a subring of A_{cris}, S is not stable under the action of G_K, though S is fixed by G_∞. Define a subring inside B^+_{cris}:

$$\mathcal{R}_{K_0} := \left\{ x = \sum_{i=0}^{\infty} f_i t^{\{i\}} : f_i \in S_{K_0} \text{ and } f_i \to 0 \text{ as } i \to +\infty \right\},$$

where $t^{\{i\}} = \frac{t^i}{p^{\tilde{q}(i)}\tilde{q}(i)!}$ and $\tilde{q}(i)$ satisfies $i = \tilde{q}(i)(p-1) + r(i)$ with $0 \leq r(i) < p-1$. Define

$$\hat{\mathcal{R}} := W(R) \cap \mathcal{R}_{K_0}.$$

One can show that $\mathcal{R}_{K_0}$ and $\hat{\mathcal{R}}$ are stable under the G_K-action and the G_K-action factors through $\hat{G}$ ([12, Section 2.2]). Let I_+R be the maximal ideal of R and $I_+\hat{\mathcal{R}} = W(I_+R) \cap \hat{\mathcal{R}}$. By [12, Lemma 2.2.1], one has $\hat{\mathcal{R}}/I_+\hat{\mathcal{R}} \simeq \mathfrak{S}/u\mathfrak{S} = W(k)$.

Recall that a *Kisin module of height r* is a finite free $\mathfrak{S}$-module $\mathfrak{M}$ with φ-semi-linear endomorphism $\varphi_{\mathfrak{M}} : \mathfrak{M} \to \mathfrak{M}$ such that

$$E(u)^r \mathfrak{M} \subset \langle \varphi_{\mathfrak{M}}(\mathfrak{M}) \rangle,$$

where $\langle \varphi_{\mathfrak{M}}(\mathfrak{M}) \rangle$ is the $\mathfrak{S}$-submodule of $\mathfrak{M}$ generated by $\varphi_{\mathfrak{M}}(\mathfrak{M})$. A morphism between two Kisin modules is just an $\mathfrak{S}$-linear map compatible with Frobenius. As a subring of A_{cris} via $u \to [\underline{\pi}]$, $\mathfrak{S}$ and S are not stable under the action of G_K, but stable under G_∞. This allows us to define a functor $T_{\mathfrak{S}}$ from the category of Kisin modules to the category of finite free $\mathbb{Z}_p$-representations of G_∞ via the following formula:

$$T_{\mathfrak{S}}(\mathfrak{M}) := \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, W(R)).$$

See [10, Section 2.2] for more details on $T_{\mathfrak{S}}$. In particular, by [10, Proposition 2.2.1], we can change $\mathfrak{S}^{\mathrm{ur}}$ to $W(R)$ in the definition of $T_{\mathfrak{S}}$.

Let us review the theory of $(\varphi, \hat{G})$-modules, which is a variation of that of Kisin modules. Following [12], a *finite free $(\varphi, \hat{G})$-module of height r* is a triple $(\mathfrak{M}, \varphi, \hat{G})$, where

(1) $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ is a finite free Kisin module of height r,

(2) $\hat{G}$ is a $\hat{\mathcal{R}}$-semi-linear $\hat{G}$-action on $\hat{\mathfrak{M}} := \hat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$,

(3) $\hat{G}$ commutes with $\varphi_{\hat{\mathfrak{M}}}$ on $\hat{\mathfrak{M}}$, i.e., for any $g \in \hat{G}$, $g\varphi_{\hat{\mathfrak{M}}} = \varphi_{\hat{\mathfrak{M}}} g$,

(4) regard $\mathfrak{M}$ as a $\varphi(\mathfrak{S})$-submodule in $\hat{\mathfrak{M}}$, then $\mathfrak{M} \subset \hat{\mathfrak{M}}^{H_K}$,

(5) $\hat{G}$ acts on $W(k)$-module $M := \hat{\mathfrak{M}}/I_+\hat{\mathcal{R}}\hat{\mathfrak{M}} \simeq \mathfrak{M}/u\mathfrak{M}$ trivially.

A morphism between two finite free $(\varphi, \hat{G})$-modules is a morphism of Kisin modules that commutes with $\hat{G}$-action on $\hat{\mathfrak{M}}$'s. For a finite free $(\varphi, \hat{G})$-module $\hat{\mathfrak{M}} = (\mathfrak{M}, \varphi, \hat{G})$, we can associate a $\mathbb{Z}_p[G_K]$-module:

$$\hat{T}(\hat{\mathfrak{M}}) := \mathrm{Hom}_{\hat{\mathcal{R}},\varphi}(\hat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}, W(R)), \tag{2.1.1}$$

where G_K acts on $\hat{T}(\hat{\mathfrak{M}})$ via $g(f)(x) = g(f(g^{-1}(x)))$ for any $g \in G_K$ and $f \in \hat{T}(\hat{\mathfrak{M}})$.

By [10, Example 2.3.5], there exists an element $\mathfrak{t} \in W(R)$ such that $\mathfrak{t} \bmod p \neq 0$ and $\varphi(\mathfrak{t}) = c_0^{-1}E(u)\mathfrak{t}$, where $c_0 p$ is the constant term of $E(u)$. Such a $\mathfrak{t}$ is unique up to $\mathbb{Z}_p^\times$. The following theorem summarizes the main results in [12].

Theorem 2.1.1 ([12]). *The following statements hold:*

(1) $\hat{T}$ *induces an anti-equivalence between the category of finite free* $(\varphi, \hat{G})$*-modules of height* r *and the category of* G_K*-stable* $\mathbb{Z}_p$*-lattices in semi-stable representations of* G_K *with Hodge–Tate weights in* $\{0, \dots, r\}$.

(2) $\hat{T}$ *induces a natural* $W(R)$*-linear injection*

$$\hat{\iota} : W(R) \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} \to \hat{T}^\vee(\hat{\mathfrak{M}}) \otimes_{\mathbb{Z}_p} W(R) \tag{2.1.2}$$

such that $\hat{\iota}$ *is compatible with Frobenius and* G_K*-actions on both sides. Moreover,*

$$(\varphi(\mathrm{t}))^r (\hat{T}^\vee(\hat{\mathfrak{M}}) \otimes_{\mathbb{Z}_p} W(R)) \subset \hat{\iota}(W(R) \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}).$$

(3) *There exists a natural isomorphism* $T_{\mathfrak{S}}(\mathfrak{M}) \xrightarrow{\sim} \hat{T}(\hat{\mathfrak{M}})$ *of* $\mathbb{Z}_p[G_\infty]$*-modules.*

2.2. A refinement of Theorem 1.0.1. The theory described by Theorem 2.1.1 depends on the choice of uniformizer π in K. Fix a G_K-stable $\mathbb{Z}_p$-lattice T inside a semi-stable representation V; if we select another uniformizer π', then we obtain $\mathfrak{M}'$ and $\hat{\iota}'$ in (2.1.2). As indicated in the introduction, one main goal of this paper is to understand the relation between $\mathfrak{M}$ and $\mathfrak{M}'$. Let $\mathfrak{S}_{\underline{\pi}}$ (resp. $S_{\underline{\pi}}$) denote the image of embedding $\mathfrak{S} \hookrightarrow W(R)$ (resp. $S \hookrightarrow A_{\mathrm{cris}}$) via $u \mapsto [\underline{\pi}]$. Write $\underline{\pi}' = \underline{\nu}\underline{\pi}$ with $\underline{\nu} = (\nu_n)_{n \geq 0} \in R$. Note that ν_0 is a unit. So $\log([\underline{\nu}]) \in B^+_{\mathrm{cris}}$.

We denote by $\mathfrak{S}_{\underline{\pi}'}$ and $S_{\underline{\pi}'}$ the subrings of $W(R)$ and A_{cris}, respectively, via $u \mapsto [\underline{\pi}']$. Let $\tilde{S}_{\underline{\pi},\underline{\pi}'}$ be the smallest ring inside B^+_{cris} containing $S_{\underline{\pi}}[\frac{1}{p}]$, $S_{\underline{\pi}'}[\frac{1}{p}]$ and $\log([\underline{\nu}])$. Set

$$\tilde{\mathfrak{S}}_{\underline{\pi},\underline{\pi}'} := W(R) \cap \tilde{S}_{\underline{\pi},\underline{\pi}'}.$$

Similarly, let $S_{\underline{\pi},\underline{\pi}'}$ be the smallest ring inside B^+_{cris} containing $S_{\underline{\pi}}[\frac{1}{p}]$ and $S_{\underline{\pi}'}[\frac{1}{p}]$ and set

$$\mathfrak{S}_{\underline{\pi},\underline{\pi}'} := W(R) \cap S_{\underline{\pi},\underline{\pi}'}.$$

Theorem 2.2.1. *Notations as above, we have*

$$\hat{\iota}(\tilde{\mathfrak{S}}_{\underline{\pi},\underline{\pi}'} \otimes_{\varphi, \mathfrak{S}_{\underline{\pi}}} \mathfrak{M}) = \hat{\iota}'(\tilde{\mathfrak{S}}_{\underline{\pi},\underline{\pi}'} \otimes_{\varphi, \mathfrak{S}_{\underline{\pi}'}} \mathfrak{M}')$$

as submodules of $T^\vee \otimes_{\mathbb{Z}_p} W(R)$.

If V *is crystalline, then* $\tilde{\mathfrak{S}}_{\underline{\pi},\underline{\pi}'}$ *in the above equation can be replaced by* $\mathfrak{S}_{\underline{\pi},\underline{\pi}'}$.

Remark 2.2.2. (1) Let $\nu = \pi/\pi'$. If $\nu \in W(k)^\times$, then we can arrange π'_n so that

$$[\underline{\pi}] = [\bar{\nu}][\underline{\pi}']$$

with $\bar{\nu} = \nu \bmod p \in k^\times$. Hence $\mathfrak{S}_{\underline{\pi},\underline{\pi}'} = \mathfrak{S}_{\underline{\pi}} = \mathfrak{S}_{\underline{\pi}'}$.

(2) If $\nu \notin W(k)^\times$, then the situation could be more complicated. So far we do not have a good description for $\tilde{\mathfrak{S}}_{\underline{\pi},\underline{\pi}'}$, even for $\mathfrak{S}_{\underline{\pi},\underline{\pi}'}$. We warn the readers that $\mathfrak{S}_{\underline{\pi},\underline{\pi}'}$ may be larger than the smallest ring containing $\mathfrak{S}_{\underline{\pi}}$ and $\mathfrak{S}_{\underline{\pi}'}$. For example, let $\tilde{E}(u)$ be the Eisenstein polynomial of π'. Then $\tilde{E}([\underline{\pi}'])/E([\underline{\pi}])$ is a unit in $W(R)$, because $\mathrm{Fil}^1 W(R)$ is a principal ideal and $E([\underline{\pi}])$ and $\tilde{E}([\underline{\pi}'])$ are generators of $\mathrm{Fil}^1 W(R)$. Hence

$$x = \varphi(\tilde{E}([\underline{\pi}'])/E([\underline{\pi}])) = \tilde{E}([\underline{\pi}']^p)/E([\underline{\pi}]^p) \in W(R).$$

But $E([\underline{\pi}]^p)/p$ is a unit in $S_{\underline{\pi}}$. Therefore, $x \in \mathfrak{S}_{\underline{\pi},\underline{\pi}'}$. In general, x is not in the smallest ring containing $\mathfrak{S}_{\underline{\pi}}$ and $\mathfrak{S}_{\underline{\pi}'}$. See the following example.

Example 2.2.3. Let $K = \mathbb{Q}_p(\zeta_p)$. Let $\pi = \zeta_p - 1$ and $\pi' = \zeta_p \pi$. We can choose $\underline{\pi}$ and $\underline{\pi}'$ such that $\underline{\pi}' = \underline{\pi}\underline{\epsilon}'$ with $\underline{\epsilon}'^p = \underline{\epsilon}$. Then the smallest ring $\tilde{\mathfrak{S}}$ containing $\mathfrak{S}_{\underline{\pi}}$ and $\mathfrak{S}'_{\underline{\pi}}$ is inside $W(k)[\![[\underline{\pi}], [\underline{\epsilon}'] - 1]\!]$. If x is in $\tilde{\mathfrak{S}}$, then

$$\varphi(x) \in W(k)[\![[\underline{\pi}], [\underline{\epsilon}] - 1]\!] \subset \mathcal{R}_{K_0} \cap W(R) = \hat{\mathcal{R}}.$$

On the other hand, because $\varphi^2(E([\underline{\pi}]))/p$ is a unit in $S_{\underline{\pi}}$, we can write $\varphi(x)$ as a series in $K_0[\![[\pi], [\underline{\epsilon}] - 1]\!]$. It is easy to see that this series is not in $W(k)[\![[\underline{\pi}], [\underline{\epsilon}] - 1]\!]$. But by [10, Lemma 7.1.2], for any $y \in \mathcal{R}_{K_0}$, there is only one way to expand y in a series in $K_0[\![[\underline{\pi}], [\underline{\epsilon}] - 1]\!]$. So $\varphi(x)$ is not in $W(k)[\![[\underline{\pi}], [\underline{\epsilon}] - 1]\!]$. Contradiction and $\varphi(x) \notin \tilde{\mathfrak{S}}$.

Notations 2.2.4. We will reserve φ and N to denote Frobenius action and monodromy action on many different rings and modules. To distinguish them, we sometime add subscripts to indicate over which those structures are defined. For example, $\varphi_{\mathfrak{M}}$ is the Frobenius defined on $\mathfrak{M}$. We always drop these subscripts if no confusions arise. As we have indicated as before, Kisin's theory (and its related theory, like the theories of Breuil modules and $(\varphi, \hat{G})$-modules, which will be used below) depends on the choice of the unformizer π, or more precisely, depends on the choice of π_n and hence the embedding $\mathfrak{S} \hookrightarrow W(R)$ via $u \mapsto [\underline{\pi}]$. We add subscripts $\underline{\pi}$ to subrings in $W(R)$ to denote subrings (like $\mathfrak{S}, S$) whose embeddings to B^+_{dR} depends on the embedding $\mathfrak{S} \hookrightarrow W(R)$ via $u \mapsto [\underline{\pi}]$. But we always drop subscripts when we just discuss the general theory where the embedding $\mathfrak{S} \hookrightarrow W(R)$ via $u \mapsto [\underline{\pi}]$ is always fixed. Finally, $\gamma_i(x)$, $\mathrm{M}_{d \times d}(A)$ and Id denote the standard divided power $\frac{x^i}{i!}$, the ring of $d \times d$-matrices with coefficients in ring A and the identity map, respectively; $V^\vee$ denotes the dual of a representation V.

2.3. Some facts on the theory of Breuil modules. We will use extensively the theory of Breuil modules, which we review in this subsection. Following [3], a *filtered φ-module over* $S[\frac{1}{p}]$ is a finite free $S[\frac{1}{p}]$-module $\mathcal{D}$ with

(1) a φ_S-semi-linear morphism $\varphi_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}$ such that the determinant of $\varphi_{\mathcal{D}}$ is invertible in $S[\frac{1}{p}]$,

(2) a decreasing filtration over $\mathcal{D}$ of S_{K_0}-modules $\{\mathrm{Fil}^i(\mathcal{D})\}_{i \in \mathbb{Z}}$ with $\mathrm{Fil}^0(\mathcal{D}) = \mathcal{D}$ and $\mathrm{Fil}^i S_{K_0} \cdot \mathrm{Fil}^j(\mathcal{D}) \subset \mathrm{Fil}^{i+j}(\mathcal{D})$.

Similarly, we define *filtered φ-modules over S* by changing $S[\frac{1}{p}]$ to S everywhere in the above definition, but we still require that the determinant of φ is in $S[\frac{1}{p}]$.

A *Breuil module* is a filtered φ-module $\mathcal{D}$ over $S[\frac{1}{p}]$ with the following extra monodromy structure: a K_0-linear map (monodromy) $N_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}$ such that

(1) for all $f \in S_{K_0}$ and $m \in \mathcal{D}$, $N_{\mathcal{D}}(fm) = N_S(f)m + fN_{\mathcal{D}}(m)$,

(2) $N_{\mathcal{D}}\varphi = p\varphi N_{\mathcal{D}}$,

(3) $N_{\mathcal{D}}(\mathrm{Fil}^i \mathcal{D}) \subset \mathrm{Fil}^{i-1} \mathcal{D}$.

A filtered (φ, N)-module D is called *positive* if $\mathrm{Fil}^0 D_K = D_K$. It turns out that the category of positive filtered (φ, N)-modules and the category of Breuil modules are equivalent. More precisely, for any positive filtered (φ, N)-module $(D, \varphi, N, \mathrm{Fil}^i D_K)$, we can associate a Breuil module $\mathcal{D}$ by defining

$$\mathcal{D} = S \otimes_{W(k)} D, \quad \varphi_{\mathcal{D}} := \varphi_S \otimes \varphi_D, \quad N_{\mathcal{D}} := N_S \otimes \mathrm{Id} + \mathrm{Id} \otimes N_D.$$

Define $\mathrm{Fil}^0 \mathcal{D} := \mathcal{D}$ and by induction

$$\mathrm{Fil}^{i+1} \mathcal{D} := \{x \in \mathcal{D} : N(x) \in \mathrm{Fil}^i \mathcal{D} \text{ and } f_\pi(x) \in \mathrm{Fil}^{i+1} D_K\},$$

where $f_\pi : \mathcal{D} \twoheadrightarrow D_K$ is defined by $s(u) \otimes x \mapsto s(\pi)x$.

In [3, Section 6], Breuil proved the above functor $\mathcal{D} : D \to S \otimes_{W(k)} D$ is an equivalence of categories. Furthermore, D and $\mathcal{D}(D)$ give rise to the same Galois representations ([4, Proposition 4.1.1.2]), namely, there is a natural isomorphism

$$\mathrm{Hom}_{W(k),\varphi,N,\mathrm{Fil}^i}(D, B_{\mathrm{st}}^+) \simeq \mathrm{Hom}_{S,\varphi,N,\mathrm{Fil}^i}(\mathcal{D}(D), \hat{B}_{\mathrm{st}}^+)$$

as $\mathbb{Q}_p[G_K]$-modules. Here $\hat{B}_{\mathrm{st}}^+$ is the period ring defined in [3].

Remark 2.3.1. In the theory of Kisin and Breuil modules, we use implicitly or explicitly the above isomorphism to connect Galois representations associated to filtered (φ, N)-modules with those of Breuil modules or Kisin modules. To make the above isomorphism, one set the monodromy N on B_{st}^+ via $N(\mathfrak{u}) = 1$ (see [4, Section 3.1.1]). So strictly speaking, the monodromy structure on B_{st}^+ may depend on the choice of uniformizer π. On the other hand, pick another uniformizer π' of K. We have $\pi = \nu\pi'$ with ν a unit in $\mathcal{O}_K$. Hence $\mathfrak{u} = \mathfrak{u}' + \beta$ with $\mathfrak{u}' = \log([\underline{\pi}'])$ and β in B_{cris}^+. So $N(\mathfrak{u}') = 1$ if and only $N(\mathfrak{u}) = 1$. This shows that the monodromy structure on B_{st}^+ is unique when we declare $N(\mathfrak{u}) = 1$ and it does not depend on the choice of *uniformizers* in $\mathcal{O}_K$.

One can naturally extend Frobenius from $\mathcal{D}$ to $A_{\mathrm{cris}} \otimes_S \mathcal{D}$ via $\varphi := \varphi_{A_{\mathrm{cris}}} \otimes \varphi_{\mathcal{D}}$. We define a semi-linear G_K-action on $A_{\mathrm{cris}} \otimes_S \mathcal{D}$ via

$$\sigma(a \otimes x) = \sum_{i=0}^{\infty} \sigma(a)\gamma_i(-\log([\underline{\epsilon}(\sigma)])) \otimes N^i(x) \tag{2.3.1}$$

for $\sigma \in G_K$, $x \in \mathcal{D}$ and $a \in A_{\mathrm{cris}}$. This G_K-action commutes with φ on $A_{\mathrm{cris}} \otimes_S \mathcal{D}$ (see [11, Lemma 5.1.1]).

Given a Kisin module $\mathfrak{M}$, one can define a filtered φ-module $\mathcal{M}_{\mathfrak{S}}(\mathfrak{M})$ over S as follows. Set $\mathcal{M} := \mathcal{M}_{\mathfrak{S}}(\mathfrak{M}) = S \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ and extend Frobenius $\varphi_{\mathfrak{M}}$ to $\mathcal{M}$ by $\varphi_{\mathcal{M}} := \varphi_S \otimes \varphi_{\mathfrak{M}}$. Define a filtration on $\mathcal{M}$ via

$$\mathrm{Fil}^i \mathcal{M} := \{x \in \mathcal{M} : 1 \otimes \varphi_{\mathfrak{M}}(x) \in \mathrm{Fil}^i S \otimes_{\mathfrak{S}} \mathfrak{M}\}, \tag{2.3.2}$$

where $1 \otimes \varphi_{\mathfrak{M}} : \mathcal{M} = S \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \to S \otimes_{\mathfrak{S}} \mathfrak{M}$ is an S-linear map.

Now let V be a semi-stable representation of G_K with Hodge–Tate weights in $\{0, \dots, r\}$, $T \subset V$ a G_K-stable $\mathbb{Z}_p$-lattice inside V, $D = D_{\mathrm{st}}(V)$ the filtered (φ, N)-module attached to V and $(\mathfrak{M}, \varphi, \hat{G})$ the $(\varphi, \hat{G})$-module attached to T via Theorem 2.1.1. Let $\mathcal{D} = \mathcal{D}(D)$ be the Breuil module and $\mathcal{M} = \mathcal{M}_{\mathfrak{S}}(\mathfrak{M})$. The following theorem summarize the relations between Breuil modules, filtered (φ, N)-modules and $(\varphi, \hat{G})$-modules (Kisin modules):

Theorem 2.3.2. *Notations as above, the following statements hold:*

(1) *There exists a natural isomorphism* $\alpha : \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathcal{M}_{\mathfrak{S}}(\mathfrak{M}) \simeq \mathcal{D}$ *as filtered* φ*-modules over* $S[\frac{1}{p}]$.

(2) *There exists a natural injection*

$$\iota : A_{\mathrm{cris}} \otimes_S \mathcal{D} \to V^\vee \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}} \tag{2.3.3}$$

which is compatible with Frobenius φ and G_K-actions on both sides.

(3) *The isomorphism α induces the commutative diagram*

$$\begin{tikzcd} A_{\mathrm{cris}} \otimes_S \mathcal{D} \arrow[r, "\iota"] & V^\vee \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}} \\ W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \arrow[r, "\hat{\iota}"] \arrow[u, hook] & T^\vee \otimes_{\mathbb{Z}_p} W(R), \arrow[u, hook] \end{tikzcd} \tag{2.3.4}$$

where the top map is equation (2.3.3) *and the bottom map is equation* (2.1.2). *The left vertical arrow is induced by α restricted to $\mathfrak{S} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ and the right arrow is induced by the injection $T^\vee \hookrightarrow V^\vee$.*

Proof. Part (1) follows from the compatibility between Kisin modules and Breuil modules. See [11, Section 3.4]. Part (2) is proved in [11, Section 5.2]. The key point is that $\mathrm{Hom}_{A_{\mathrm{cris}},\varphi,\mathrm{Fil}^i}(A_{\mathrm{cris}} \otimes_S \mathcal{D}, B^+_{\mathrm{cris}})$ is canonically isomorphic to V as $\mathbb{Q}_p[G_K]$-modules. The proof of part (3) relies on the construction of $(\varphi, \hat{G})$-modules. See [10, Theorem 5.4.2] and [12, Proposition 3.1.3]. $\square$

3. The proof of the main theorems

We will prove Theorem 2.2.1, Theorem 3.2.1 and Theorem 1.0.2 in this section. Our strategy is almost the same as that in [12, Section 3.2].

3.1. The proof of Theorem 2.2.1. To prove Theorem 2.2.1, we first show that the injection ι in equation (2.3.3) does not depend on the choices of uniformizer. More precisely, let $\mathcal{D}'$ denote the Breuil module attached to V and ι' the injection in equation (2.3.3) for the choice of uniformizer π'. We claim:

Lemma 3.1.1. *There exists an A_{cris}-linear isomorphism*

$$\beta : A_{\mathrm{cris}} \otimes_{S_{\underline{\pi}}} \mathcal{D} \to A_{\mathrm{cris}} \otimes_{S_{\underline{\pi}'}} \mathcal{D}'$$

which is compatible with G_K-actions and Frobenius such that the following diagram commutes:

$$\begin{tikzcd} A_{\mathrm{cris}} \otimes_{S_{\underline{\pi}}} \mathcal{D} \arrow[r, "\iota"] & V^\vee \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}} \\ A_{\mathrm{cris}} \otimes_{S_{\underline{\pi}'}} \mathcal{D}' \arrow[r, "\iota'"] \arrow[u, "\beta", "\wr"'] & V^\vee \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}}. \arrow[u, equal] \end{tikzcd} \tag{3.1.1}$$

Proof. Let $I_+S = S \cap uK_0[\![u]\!]$ and $D := \mathcal{D}/I_+S\mathcal{D}$. Then D is a finite-dimensional K_0-vector space with Frobenius φ and monodromy N on D induced from that on $\mathcal{D}$. It was shown in [3, Proposition 6.2.1.1] that there exists a unique (φ, N)-equivariant section

$$s : D \hookrightarrow \mathcal{D}$$

and that

$$\mathcal{D} = S \otimes_{W(k)} s(D)$$

as S-modules. By [13, Proposition 2.6],

$$s(D) \subset V^\vee \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}} \subset V^\vee \otimes_{\mathbb{Q}_p} B^+_{\mathrm{st}}$$

has the following relation with $D_{\mathrm{st}}(V) = (V^\vee \otimes_{\mathbb{Q}_p} B^+_{\mathrm{st}})^{G_K}$: There exists a (necessarily unique) isomorphism $i : D_{\mathrm{st}}(V) \to D$ compatible with φ and N such that the following diagram commutes:

$$\text{(3.1.2)} \qquad \begin{array}{ccc} D_{\mathrm{st}}(V) & \hookrightarrow & V^\vee \otimes_{\mathbb{Q}_p} B^+_{\mathrm{st}} \\ \wr\downarrow i & & \downarrow \bmod \mathfrak{u} \\ s(D) & \xhookrightarrow{\iota} & V^\vee \otimes_{\mathbb{Q}_p} B^+_{\mathrm{cris}}, \end{array}$$

where $\mathfrak{u} = \log([\underline{\pi}]) \in B^+_{\mathrm{st}}$, and the inverse of i is given by $y \mapsto \sum_{n=0}^\infty N^n(y) \otimes \gamma_n(\mathfrak{u})$. If we fix a K_0-basis $\tilde e_1, \dots, \tilde e_d$ of $D_{\mathrm{st}}(V)$, then by the above diagram, we obtain a basis $e_1, \dots, e_d$ of $s(D)$ by modulo $\mathfrak{u}$ to $\tilde e_1, \dots, \tilde e_d$, and

$$(e_1, \dots, e_d) = (\tilde e_1, \dots, \tilde e_d) \sum_{n=0}^\infty \gamma_n(-\mathfrak{u})(\bar N)^n,$$

where $\bar N \in \mathrm{M}_{d\times d}(K_0)$ is the matrix such that $N(\tilde e_1, \dots, \tilde e_d) = (\tilde e_1, \dots, \tilde e_d)\bar N$.

Now by changing to another uniformizer π', we get $s'(D')$ injects to $V^\vee \otimes_{\mathbb{Q}_p} B^+_{\mathrm{cris}}$. Modulo $\tilde e_1, \dots, \tilde e_d$ by $\mathfrak{u}' = \log([\underline{\pi}'])$, we get the basis $e'_1, \dots, e'_d$ of $s'(D')$ and

$$(e'_1, \dots, e'_d) = (\tilde e_1, \dots, \tilde e_d) \sum_{n=0}^\infty \gamma_n(-\mathfrak{u}')(\bar N)^n.$$

Write $\underline{\pi} = \underline{\nu}\underline{\pi}'$ with $\underline{\nu} = (\nu_n)_{n\geq 0} \in R$. Since ν_0 is a unit in $\mathcal{O}_K$, $\log([\underline{\nu}])$ is in B^+_{cris}. Now we get

$$\text{(3.1.3)} \qquad (e_1, \dots, e_d) = (e'_1, \dots, e'_d) \sum_{n=0}^\infty \gamma_n(-\log([\underline{\nu}]))(\bar N)^n.$$

We remark the sum in the right side of the above equation is indeed a finite sum because $\bar N^n = 0$ if n is large enough. Now the lemma follows from the facts that $s(D) \otimes_{W(k)} S \simeq \mathcal{D}$ as S-modules and that the matrix $\sum_{n=0}^\infty \gamma_n(-\log([\underline{\nu}]))(\bar N)^n$ has coefficients in B^+_{cris}. □

Corollary 3.1.2. *Let $\hat e_1, \dots, \hat e_d$ be an $S_{\underline{\pi}}[\frac{1}{p}]$-basis of $\mathcal{D}$ and $\hat e'_1, \dots, \hat e'_d$ an $S_{\underline{\pi}'}[\frac{1}{p}]$-basis of $\mathcal{D}'$. Then*

$$(\hat e'_1, \dots, \hat e'_d) = (\hat e_1, \dots, \hat e_d) X$$

with an invertible matrix X whose entries are in $\tilde S_{\underline{\pi},\underline{\pi}'}$. If V is crystalline, then X has entries in $S_{\underline{\pi},\underline{\pi}'}$.

Now we are ready to prove Theorem 2.2.1. Let $\hat{e}_1, \dots, \hat{e}_d$ be an $\mathfrak{S}$-basis of $\mathfrak{M}$, and $\hat{e}'_1, \dots, \hat{e}'_d$ an $\mathfrak{S}$-basis of $\mathfrak{M}'$, respectively. Regarding $\mathfrak{M}$ as an $\varphi(\mathfrak{S})$-submodule of $\mathcal{D}$ via the isomorphism $\alpha : \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathcal{M}_{\mathfrak{S}}(\mathfrak{M}) \simeq \mathcal{D}$ by Theorem 2.3.2 (1), we can regard $\{\hat{e}_i\}$ as an $S_{\underline{\pi}}[\frac{1}{p}]$-basis of $\mathcal{D}$. Similarly, $\{\hat{e}'_i\}$ is an $S'_{\underline{\pi}}[\frac{1}{p}]$ basis of $\mathcal{D}'$. So by the above corollary, we may write

$$(\hat{e}'_1, \dots, \hat{e}'_d) = (\hat{e}_1, \dots, \hat{e}_d)X$$

with X having entries in $\tilde{S}_{\underline{\pi},\underline{\pi}'}$, and in $S_{\underline{\pi},\underline{\pi}'}$ if V is crystalline.

Now to prove Theorem 2.2.1, it suffices to show that X has entries in $W(R)$. Define an ideal

$$I^{[1]}W(R) := \{x \in W(R) : \varphi^n(x) \in \mathrm{Fil}^1 W(R) \text{ for all } n \geq 0\}.$$

By [7, Proposition 5.1.3], $I^{[1]}W(R)$ is a principal ideal. We record the following useful lemma:

Lemma 3.1.3. *Let a be a generator of $I^{[1]}W(R)$ and $x \in B^+_{\mathrm{cris}}$. If $ax \in W(R)$, then $x \in W(R)$.*

Proof. See [12, proof of Lemma 3.2.2]. Note that $\varphi(\mathfrak{t})$ is also proved to be a generater of $I^{[1]}W(R)$ there. □

Note that the construction of $\mathfrak{t}$ also depends on the choice of $\underline{\pi}$. So we denote $\mathfrak{t}'$ for the choice of $\underline{\pi}'$. By Theorem 2.1.1 (2), we have $\hat{\iota}(\hat{e}_i) \in T^\vee \otimes_{\mathbb{Z}_p} W(R)$ and then $(\varphi(\mathfrak{t}'))^r \hat{\iota}(\hat{e}_i)$ is in $\hat{\iota}'(W(R) \otimes_{\varphi,\mathfrak{S}_{\underline{\pi}'}} \mathfrak{M}')$. Then parts (2)–(3) of Theorem 2.3.2 imply that $(\varphi(\mathfrak{t}'))^r X$ has entries in $W(R)$. Then X must has entries in $W(R)$ by the above lemma. This completes the proof of Theorem 2.2.1.

3.2. Compatibility of basis change. Assume that T is a G_K-stable $\mathbb{Z}_p$-lattice in semi-stable representation V of G_K with $(\mathfrak{M}, \varphi, \hat{G})$ the corresponding $(\varphi, \hat{G})$-module via the fixed uniformizer π. Let K' be a finite extension of K and $(\mathfrak{M}', \varphi, \hat{G})$ the $(\varphi, \hat{G})$-module corresponding to $T|_{G_{K'}}$ via the fixed uniformizer π' of $\mathcal{O}_{K'}$. We would like to compare $\mathfrak{M}$ and $\mathfrak{M}'$.

Let k' be the residue field of $\mathcal{O}_{K'}$ and let $K'_0 := W(k')[\frac{1}{p}]$. Suppose that $\underline{\pi} = \underline{\nu}\underline{\pi}'^m$, where $\underline{\nu} = (\nu_n)_{n \geq 0} \in R$ with $\nu_0 \in \mathcal{O}^\times_{K'}$ a unit. Let $\tilde{S}_{\underline{\pi},\underline{\pi}'} \subset B^+_{\mathrm{cris}}$ be the smallest K_0-algebra containing $S_{\underline{\pi}}[\frac{1}{p}]$, $S_{\underline{\pi}'}[\frac{1}{p}]$ and $\log([\underline{\nu}])$, and $\tilde{\mathfrak{S}}_{\underline{\pi},\underline{\pi}'} = W(R) \cap \tilde{S}_{\underline{\pi},\underline{\pi}'}$. Further, let $S_{\underline{\pi},\underline{\pi}'} \subset B^+_{\mathrm{cris}}$ be the smallest K_0-algebra containing $S_{\underline{\pi}}[\frac{1}{p}]$, $S_{\underline{\pi}'}[\frac{1}{p}]$, and $\mathfrak{S}_{\underline{\pi},\underline{\pi}'} = W(R) \cap S_{\underline{\pi},\underline{\pi}'}$. The following result is very similar to Theorem 2.2.1.

Theorem 3.2.1. *Notations as above, we have*

$$\hat{\iota}(\tilde{\mathfrak{S}}_{\underline{\pi},\underline{\pi}'} \otimes_{\varphi,\mathfrak{S}_{\underline{\pi}}} \mathfrak{M}) = \hat{\iota}'(\tilde{\mathfrak{S}}_{\underline{\pi},\underline{\pi}'} \otimes_{\varphi,\mathfrak{S}_{\underline{\pi}'}} \mathfrak{M}')$$

as submodules of $T^\vee \otimes_{\mathbb{Z}_p} W(R)$.

If V is crystalline, then $\tilde{\mathfrak{S}}_{\underline{\pi},\underline{\pi}'}$ in the above equation can be replaced by $\mathfrak{S}_{\underline{\pi},\underline{\pi}'}$.

Proof. Here we provide a similar proof to that of Theorem 2.2.1. We first reproduce Lemma 3.1.1. We claim that there exists an A_{cris}-linear isomorphism

$$\beta : A_{\mathrm{cris}} \otimes_{S_{\underline{\pi}}} \mathcal{D} \to A_{\mathrm{cris}} \otimes_{S_{\underline{\pi}'}} \mathcal{D}'$$

which is compatible with $G_{K'}$-action, Frobenius such that the following diagram commutes:

$$\begin{array}{ccc} A_{\mathrm{cris}} \otimes_{S_{\underline{\pi}}} \mathcal{D} & \xrightarrow{\iota} & V^{\vee} \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}} \\ \beta \Big\uparrow \wr & & \Big\| \\ A_{\mathrm{cris}} \otimes_{S_{\underline{\pi}'}} \mathcal{D}' & \xrightarrow{\iota'} & V^{\vee} \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}}. \end{array}$$

The only difference is that β is only $G_{K'}$-equivariant. To prove the claim, we use almost the same proof as that of Lemma 3.1.1 but with extra care on the monodromy structure of B^+_{st}. Write $V' := V|_{G_{K'}}$ and $D' := \mathcal{D}'/I_+S\mathcal{D}'$. We still have diagram (3.1.2) for V' and V. Fix a K_0-basis $\tilde{e}_1, \dots, \tilde{e}_d$ of $D_{\mathrm{st}}(V)$. Then $\{\tilde{e}_i\}$ is a K'_0-basis of $D_{\mathrm{st}}(V')$. Modulo $\mathfrak{u}' = \log([\underline{\pi}'])$, we have a basis $e'_1, \dots, e'_d$ of $s'(D')$ and the relation

$$(e'_1, \dots, e'_d) = (\tilde{e}_1, \dots, \tilde{e}_d) \sum_{n=0}^{\infty} \gamma_n(-\mathfrak{u}')(\bar{N}')^n,$$

where $\bar{N}' \in \mathrm{M}_{d \times d}(K'_0)$ is the matrix such that $N(\tilde{e}_1, \dots, \tilde{e}_d) = (\tilde{e}_1, \dots, \tilde{e}_d)\bar{N}'$. Note that we use the convention $N(\mathfrak{u}') = 1$ by Remark 2.3.1.

Similarly, we obtain a K_0-basis $e_1, \dots, e_d$ of $s(D)$ and

$$(e_1, \dots, e_d) = (\tilde{e}_1, \dots, \tilde{e}_d) \sum_{n=0}^{\infty} \gamma_n(-\mathfrak{u})(\bar{N})^n,$$

with $\bar{N} \in \mathrm{M}_{d \times d}(K_0)$ the matrix such that $N(\tilde{e}_1, \dots, \tilde{e}_d) = (\tilde{e}_1, \dots, \tilde{e}_d)\bar{N}$. But the convention used here is $N(\mathfrak{u}) = 1$. To find the relation between $\bar{N}$ and $\bar{N}'$, let us fix the convention $N(\mathfrak{u}) = 1$. Since $\underline{\pi} = \underline{\nu}\underline{\pi}'^m$, we have $\mathfrak{u} = m\mathfrak{u}' + \log([\underline{\nu}])$ and then $N(\mathfrak{u}') = \frac{1}{m}$. Consider the equation

$$(e'_1, \dots, e'_d) \sum_{n=0}^{\infty} \gamma_n(\mathfrak{u}')(\bar{N}')^n = (\tilde{e}_1, \dots, \tilde{e}_d) = (e_1, \dots, e_d) \sum_{n=0}^{\infty} \gamma_n(\mathfrak{u})(\bar{N})^n.$$

Taking monodromy on the both sides, we get $\bar{N}' = m\bar{N}$. So

$$\sum_{n=0}^{\infty} \gamma_n(-\mathfrak{u}')(\bar{N}')^n = \sum_{n=0}^{\infty} \gamma_n(-m\mathfrak{u}')(\bar{N})^n.$$

Hence we still obtain equation (3.1.3):

$$(e_1, \dots, e_d) = (e'_1, \dots, e'_d) \sum_{n=0}^{\infty} \gamma_n(-\log([\underline{\nu}]))\bar{N}^n.$$

The remaining arguments for the proof of the claim and the theorem are the same as those of Theorem 2.2.1. □

3.3. Comparison between Wach modules and Kisin modules. Throughout this subsection, we assume that $K = K_0$ is unramified. We have a natural embedding $W(k)[\![v]\!]$ to $W(R)$ via $v \mapsto [\underline{\epsilon}] - 1$ and denote $\mathfrak{S}_{\underline{\epsilon}} \subset W(R)$ the ring $\mathfrak{S}$ via the embedding $v \mapsto [\underline{\epsilon}] - 1$.

Note that $\Gamma := \mathrm{Gal}(K_{p^\infty}/K)$ acts on $W(k)[\![v]\!]$ naturally and commutes with φ-action. Set $q := \varphi(v)/v$. Following [2], a *Wach module of height* r is a finite free $\mathfrak{S}_{\underline{\epsilon}}$-module $\mathfrak{N}$ with the following structure:

(1) There exist semi-linear φ-action and Γ-action on $\mathfrak{N}$ such that $\varphi_{\mathfrak{N}}$ and $\Gamma_{\mathfrak{N}}$ commutes.

(2) The cokernel of linear map $1 \otimes \varphi_{\mathfrak{N}} : \mathfrak{S}_{\underline{\epsilon}} \otimes_{\varphi, \mathfrak{S}_{\underline{\epsilon}}} \mathfrak{N} \to \mathfrak{N}$ is killed by q^r.

(3) $\Gamma_{\mathfrak{N}}$ acts on $\mathfrak{N}/v\mathfrak{N}$ trivially.

For any Wach module $\mathfrak{N}$, we can attach a $\mathbb{Z}_p[G_K]$-module

$$T_{\mathrm{Wa}}(\mathfrak{N}) := \mathrm{Hom}_{\mathfrak{S}_{\underline{\epsilon}}, \varphi}(\mathfrak{N}, W(R)).$$

For any $f \in T_{\mathrm{Wa}}(\mathfrak{N})$, $g \in G_K$, g acts on f via $(g \circ f)(x) = g(f(g^{-1}x))$ for all $x \in \mathfrak{N}$, where G_K acts on $\mathfrak{N}$ via $G_K \twoheadrightarrow \Gamma$. We note that usually one attaches $\mathfrak{N}$ a representation via $\tilde{T}(\mathfrak{N}) := (\mathfrak{N} \otimes_{\mathfrak{S}_{\underline{\epsilon}}} \mathbf{A})^{\varphi=1}$ (as in [2, Section I.2]), where $\mathbf{A}$ is constructed as follows: Let $\mathcal{E}_{\underline{\epsilon}}^{\mathrm{ur}}$ be the maximal unramified extension of $\mathcal{E}_{\underline{\epsilon}}$ in $W(\mathrm{Fr}R)$, where $\mathrm{Fr}R$ is the fraction field of R and $\mathcal{E}_{\underline{\epsilon}}$ is the fraction field of the p-adic completion of $W(k)[\![v]\!][\frac{1}{v}]$. Set $\mathbf{A}$ to be the p-adic completion of the ring of integers of $\mathcal{E}_{\underline{\epsilon}}^{\mathrm{ur}}$. But it is well known that T_{Wa} is the dual of $\tilde{T}$.

Let $\mathrm{B}_{\mathrm{rig}}^{+}$ be the ring of series

$$\sum_{n=0}^{\infty} a_n v^n, \quad a_n \in K_0,$$

such that the formal series $\sum_{n=0}^{\infty} a_n X^n$ converges for any $x \in \mathfrak{m}_{\mathcal{O}_{\overline{K}}}$ (the maximal ideal of $\mathcal{O}_{\overline{K}}$). Let $\tilde{\mathrm{B}} \subset \mathcal{R}_{K_0}$ be the subring containing the sequence $\sum_{n=0}^{\infty} a_n t^{\{n\}}$. It is easy to check that $\mathrm{B}_{\mathrm{rig}}^{+} \subset \tilde{\mathrm{B}}$.

The theorem below is a summary of properties of Wach modules that we need from [2]:

Theorem 3.3.1. *The following statements hold:*

(1) *The functor T_{Wa} induces an anti-equivalence between the category of G_K-stable $\mathbb{Z}_p$-lattices in crystalline representations with Hodge–Tate weights in $\{0, \dots, r\}$ and the category of Wach modules of height r.*

(2) *Write $T := T_{\mathrm{Wa}}(\mathfrak{N})$. Then T_{Wa} induces an injection*

$$\iota_{\mathrm{Wa}} : W(R) \otimes_{\mathfrak{S}_{\underline{\epsilon}}} \mathfrak{N} \hookrightarrow T^{\vee} \otimes_{\mathbb{Z}_p} W(R)$$

with the corkernel killed by v^r.

(3) *We have*

$$D_{\mathrm{cris}}(V) = (\mathrm{B}_{\mathrm{rig}}^{+} \otimes_{\mathfrak{S}_{\underline{\epsilon}}} \mathfrak{N})^{\Gamma}$$

and $(\mathrm{B}_{\mathrm{rig}}^{+} \otimes_{\mathfrak{S}_{\underline{\epsilon}}} \mathfrak{N})/(\mathrm{B}_{\mathrm{rig}}^{+} \otimes_{\mathbb{Q}_p} D_{\mathrm{cris}}(V))$ is killed by some power of $\prod_{n=1}^{\infty} \frac{\varphi^{n-1}(q)}{p}$.

Proof. See [2, Theorem 2, Proposition II.2.1, Proposition III.2.1, Theorem III.3.1]. □

Now we can follow the similar idea of Section 3.1 to prove Theorem 1.0.2. Let $\hat{e}_1, \dots, \hat{e}_d$ be an $\mathfrak{S}_{\underline{\epsilon}}$-basis of the Wach module $\mathfrak{N}$ and $e_1, \dots, e_d$ a K_0-basis of $D_{\mathrm{cris}}(V)$. Theorem 3.3.1 (3) implies that

$$(e_1, \dots, e_d) = (\hat{e}_1, \dots, \hat{e}_d)Y$$

with Y a matrix having entries in $\mathrm{B}_{\mathrm{rig}}^{+}$. Since $\varphi^{n-1}/(q)p$ is a unit in $\tilde{\mathrm{B}}$ for $n \geq 1$, Y is an invertible matrix with $Y^{-1} \in \mathrm{M}_{d\times d}(\tilde{\mathrm{B}})$. On the other hand, if $\hat{e}'_1, \dots, \hat{e}'_d$ is an $\mathfrak{S}_{\underline{\pi}}$-basis of the Kisin module $\mathfrak{M}$, then we have seen from Section 3.1 that

$$(e_1, \dots, e_d) = (\hat{e}'_1, \dots, \hat{e}'_d)Y'$$

with Y' a matrix having entries in $S_{\underline{\pi}}[\frac{1}{p}]$. Note that both Y and Y' are invertible matrices in $\mathrm{M}_{d\times d}(\mathcal{R}_{K_0})$. Therefore

$$(\hat{e}_1, \dots, \hat{e}_d) = (\hat{e}'_1, \dots, \hat{e}'_d)X$$

with $X = Y'Y^{-1}$. On the other hand, Theorem 3.3.1 (2) implies that

$$v^r(\hat{\iota}(\hat{e}'_1, \dots, \hat{e}'_d)) \subset \iota_{\mathrm{Wa}}(W(R) \otimes_{\mathfrak{S}_{\underline{\epsilon}}} \mathfrak{N}).$$

Therefore $v^r X$ has entries in $W(R)$. It is well known that $v = [\underline{\epsilon}]-1$ is a generator of $I^{[1]}W(R)$. So Lemma 3.1.3 implies that X has entries in $W(R)$. Similarly, we can show that X^{-1} has entries in $W(R)$.

Now we conclude that

$$\hat{\iota}(\hat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}) = \iota_{\mathrm{Wa}}(\hat{\mathcal{R}} \otimes_{\mathfrak{S}_{\underline{\epsilon}}} \mathfrak{N}).$$

To prove Theorem 1.0.2, it suffices to show that $\mathfrak{S}_{\underline{\epsilon}} = (\hat{\mathcal{R}})^{H_{p^\infty}}$. Since it is easy to show that

$$\tilde{\mathrm{B}} \cap W(R) = \mathfrak{S}_{\underline{\epsilon}},$$

it suffices to check that $(\mathcal{R}_{K_0})^{H_{p^\infty}} = \tilde{\mathrm{B}}$.

Note that the G_K-actions on $\mathcal{R}_{K_0}$ factors through $\hat{G}$. We have the following results on G_{p^∞} and G_∞-invariants of $\mathcal{R}_{K_0}$:

Lemma 3.3.2. *We have* $(\mathcal{R}_{K_0})^{G_{p^\infty}} = \tilde{\mathrm{B}}$ *and* $(\mathcal{R}_{K_0})^{G_\infty} = S[\frac{1}{p}]$.

Proof. We first show that $(\mathcal{R}_{K_0})^{G_{p^\infty}} = \tilde{\mathrm{B}}$. First let $p > 2$. Since $\hat{G} \simeq G_{p^\infty} \rtimes H_K$ by [11, Lemma 5.1.2], we can pick a $\tau \in G_{p^\infty}$ such that τ is a topological generator of G_{p^∞} and $[\underline{\epsilon}(\tau)] = \exp(-t)$. For any $x \in \mathcal{R}_{K_0}$, by the definition of $\mathcal{R}_{K_0}$, we may write

$$x = \sum_{i=0}^{\infty} f_i u^i, \quad f_i \in \tilde{\mathrm{B}}.$$

It suffices to show that $f_i = 0$ for any $i > 0$. Note that τ acts on $\tilde{\mathrm{B}}$ trivially and

$$\tau(u) = u[\underline{\epsilon}(\tau)] = u\exp(-t).$$

Hence

$$\tau(x) = \sum_{i=0}^{\infty} f_i(\exp(-t))^i u^i.$$

So by [10, Lemma 7.1.2], $x \in (\mathcal{R}_{K_0})^{G_{p^\infty}}$ implies that $f_i(\exp(-t))^i = f_i$ for all i. Therefore $f_i = 0$ unless $i = 0$. If $p = 2$, then [12, Section 4.1] shows that we can pick a $\tau \in G_{p^\infty}$ such that $[\underline{\epsilon}(\tau)] = \exp(-2t)$. The remaining proof follows the same steps as before.

For the proof of the equality $(\mathcal{R}_{K_0})^{G_\infty} = S[\frac{1}{p}]$, we use the essentially the same idea. For any $x \in \mathcal{R}_{K_0}$, we can write

$$x = \sum_{j=0}^{\infty} f_j t^j, \quad f_j \in S[\tfrac{1}{p}].$$

For any $g \in G_\infty$, $g(u) = u$ and $g(t) = \chi_p(g)t$, where χ_p is the p-adic cyclotomic character. Then the statement that $(\mathcal{R}_{K_0})^{G_\infty} = S[\frac{1}{p}]$ again follows [10, Lemma 7.1.2]. □

4. Applications to de Rham representations

4.1. Various lattices in $D_{\mathrm{st}}(V)$. Let T be a G_K-stable $\mathbb{Z}_p$-lattice inside a semi-stable representation V of G_K with Hodge–Tate weights in $\{0,\dots,r\}$. By using Kisin modules or its variation, we can attach the following φ-stable $W(k)$-lattices (related to T) in $D_{\mathrm{st}}(V)$: Let $\hat{\mathfrak{M}} = (\mathfrak{M}, \varphi, \hat{G})$ be the $(\varphi, \hat{G})$-module attached to T, $\mathcal{D} = S[\frac{1}{p}] \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ and $D := \mathcal{D}/I_+ S\mathcal{D}$. Recall there exists a unique (φ, N)-equivariant section $s : D \to \mathcal{D}$. By [13, Proposition 2.6], there exists a unique isomorphism of $W(k)$-modules $i : D_{\mathrm{st}}(V) \simeq s(D)$ to make diagram (3.1.2) commutes. Now we can define

$$M_{\mathrm{st}}(T) := (i^{-1} \circ s)(\mathfrak{M}/u\mathfrak{M}) \subset D_{\mathrm{st}}(V)$$

as in [13, Section 2.3]. On the other hand, set $\mathcal{M} = \mathcal{M}_S(\mathfrak{M}) = S \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \subset \mathcal{D}$, we can define

$$\tilde{M}_{\mathrm{st}}(T) := i^{-1}(s(D) \cap \mathcal{M}).$$

Let $\varpi \in \mathcal{O}_K$ and $\underline{\varpi} = (\varpi_n) \in R$ with ϖ_n a p^n-th root of ϖ. Set $\mathfrak{v} := \log([\underline{\varpi}])$ and $A_{\mathrm{st}}^+ := A_{\mathrm{cris}}[\mathfrak{v}]$. It is obvious that $A_{\mathrm{st}}^+[\frac{1}{p}] = B_{\mathrm{st}}^+$ and the construction depends on the choice of $\mathfrak{v}$. If we define the monodromy operator N on B_{st}^+ via $N(\mathfrak{v}) = 1$, then we see that A_{st}^+ is G_K-stable, φ-stable and N-stable inside B_{st}^+. Define

$$M_{\mathrm{inv}}(T) := (T^\vee \otimes_{\mathbb{Z}_p} A_{\mathrm{st}}^+)^{G_K}.$$

If V is crystalline, then $M_{\mathrm{inv}}(T) = (T^\vee \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}})^{G_K}$ and the construction of $M_{\mathrm{inv}}(T)$ does not depend on the choice of ϖ in this case.

Remark 4.1.1. According to Remark 2.3.1, the integral theory via Kisin modules or Breuil modules uses the convention $N(\mathfrak{u}) = 1$. So if we set $N(\mathfrak{v}) = 1$ as above, then we change the monodromy setting of Breuil–Kisin theory. But luckily, the construction of M_{inv} does not depend on Breuil and Kisin's theory.

The following proposition summarizes some properties of these lattices.

Proposition 4.1.2. *The following statements hold:*

(1) $\tilde{M}_{\mathrm{st}}(T) \subset M_{\mathrm{st}}(T)$. *There exists a constant c_1 depending on $e = [K : K_0]$ and r such that* $p^{c_1} M_{\mathrm{st}}(T) \subset \tilde{M}_{\mathrm{st}}(T)$.

(2) *Assume that V is crystalline. Then $\tilde{M}_{\mathrm{st}}(T) \subset M_{\mathrm{inv}}(T)$. There exists a constant c_2 depending on e and r such that $p^{c_2} M_{\mathrm{inv}}(T) \subset \tilde{M}_{\mathrm{st}}(T)$.*

(3) *$M_{\mathrm{st}}(T)$ is N-stable inside D. So is $\tilde{M}_{\mathrm{st}}(T)$ if $p > 2$.*

(4) *$M_{\mathrm{inv}}(T)$ is φ-stable and N-stable inside $D_{\mathrm{st}}(V)$.*

(5) *The functor $M_{\mathrm{inv}} : T \mapsto M_{\mathrm{inv}}(T)$ is left exact.*

(6) *If $e = 1$, $r \le p-2$ and V is crystalline, then $M_{\mathrm{st}} = \tilde{M}_{\mathrm{st}} = M_{\mathrm{inv}}$.*

Proof. (1) Write $\tilde{M} = \tilde{M}_{\mathrm{st}}(T)$, $q : \mathcal{D} \twoheadrightarrow D$ and $M = s \circ q(\mathcal{M})$. It is easy to check that $\mathfrak{M}/u\mathfrak{M} = q(\mathcal{M})$ inside D. So it suffices to show that $\tilde{M} \subset M$. Note that $s \circ q(x) = x$ for any $x \in s(D)$. Since $\tilde{M} \subset \mathcal{M}$, we see that $\tilde{M} = s \circ q(\tilde{M}) \subset s \circ q(\mathcal{M}) = M$. To show the existence of the constant c_1, it suffices to show that there exists a constant c_1 such that $p^{c_1} M \subset \mathcal{M}$ and this has been proved in [10, Lemma 7.3.1].

(2) We regard $\mathcal{D}$ as a submodule of $V^\vee \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}}$ via the injection

$$\iota : A_{\mathrm{cris}} \otimes_S \mathcal{D} \hookrightarrow V^\vee \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}}$$

by Theorem 2.3.2 (2). It is easy to check that $\mathcal{M} \subset T^\vee \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}}$ by Theorem 2.3.2 (3). By the construction of isomorphism i in diagram (3.1.2), $s(D) = D_{\mathrm{cris}}(V)$ if V is crystalline. So we have that $\tilde{M}_{\mathrm{st}}(T) \subset M_{\mathrm{inv}}(T)$. Let $e_1, \dots, e_d$ be a $W(k)$-basis of $\tilde{M} = \tilde{M}_{\mathrm{st}}(T)$. For any $x \in M_{\mathrm{inv}}(T)$ we may write

$$x = \sum_i a_i e_i, \quad a_i \in K_0.$$

By [10, Lemma 5.3.4], $t^r x \in A_{\mathrm{cris}} \otimes_S \mathcal{M}$. By [14, proof of Proposition 2.4.1], we see that there exists a constant c_3 depending on e and r such that $p^{c_3} A_{\mathrm{cris}} \otimes_S \mathcal{M} \subset A_{\mathrm{cris}} \otimes_{W(k)} M$. By (1), we may assume that $p^{c_3} A_{\mathrm{cris}} \otimes_S \mathcal{M} \subset A_{\mathrm{cris}} \otimes_{W(k)} \tilde{M}$. That is,

$$p^{c_3} t^r x = \sum_i a_i p^{c_3} t^r e_i \in A_{\mathrm{cris}} \otimes_{W(k)} \tilde{M}.$$

So $p^{c_3} t^r a_i \in A_{\mathrm{cris}}$. Let c_4 be the largest integer depending on r such that $t^r / p^{c_4} \in A_{\mathrm{cris}}$. Then we see that $c_2 = c_3 + c_4$ is the required constant.

(3) These are consequences of [13, Proposition 2.15 and Proposition 2.13]. Note that [13, Proposition 2.13] requires $p > 2$.

Parts (4) and (5) are obvious from the construction. We note that (5) is different from statement (3) because we change the N-structure on B^+_{st} by setting $N(\mathfrak{v}) = 1$.

(6) In this situation, the G_K-stable $\mathbb{Z}_p$-lattices can be studied by Fontaine–Laffaille's theory in [8]. Let us recall that a *strongly divisible $W(k)$-lattice* $(L, \mathrm{Fil}^i L, \varphi_i)$ is a finite free $W(k)$-module L with the following structures:

- a filtration $\mathrm{Fil}^i L \subset L$ such that $\mathrm{Fil}^0 L = L$, $\mathrm{Fil}^{p-1} L = \{0\}$ and $L/\mathrm{Fil}^i L$ is torsion free,
- $\varphi_i : \mathrm{Fil}^i L \to L$ is a Frobenius semi-linear map such that $\varphi_i|_{\mathrm{Fil}^{i+1} L} = p\varphi_{i+1}$,
- $\sum_{i=0}^{p-2} \varphi_i(\mathrm{Fil}^i L) = L$.

Since $p^i | \varphi(\mathrm{Fil}^i A_{\mathrm{cris}})$ in A_{cris} for $0 \le i \le p-1$, one can define $\varphi_i := \varphi/p^i : \mathrm{Fil}^i A_{\mathrm{cris}} \to A_{\mathrm{cris}}$. By the main result in [8], there exists a strongly divisible $W(k)$-lattice $(L, \mathrm{Fil}^i L, \varphi_i)$ such that $\mathrm{Hom}_{W(k), \mathrm{Fil}^i, \varphi_i}(L, A_{\mathrm{cris}}) \simeq T$ and $L[\frac{1}{p}] = D_{\mathrm{cris}}(V)$ as filtered (φ, N)-modules (we can define φ on L by $\varphi = \varphi_0$). On the other hand, define

$$\mathcal{L} := S \otimes_{W(k)} L, \quad \mathrm{Fil}^r \mathcal{L} := \sum_{i=0}^{r} \mathrm{Fil}^i S \otimes \mathrm{Fil}^{r-i} L,$$

and a semi-linear map

$$\varphi_r := \sum_{i=0}^{r} \varphi_{i,S} \otimes \varphi_{r-i,L} : \mathrm{Fil}^r \mathcal{L} \to \mathcal{L},$$

where $\varphi_{i,S} := \varphi_S / p^i : \mathrm{Fil}^i S \to S$ and $\varphi_{r-i,L} = \varphi_{r-i} : \mathrm{Fil}^{r-i} L \to L$. It is easy to check that $(\mathcal{L}, \mathrm{Fil}^r \mathcal{L}, \varphi_r)$ is a *quasi-strongly divisible S-lattice*[1)] inside $\mathcal{D}$ in the sense of [11, Definition 2.3.3]. On the other hand, $\mathcal{M} = \mathcal{M}_{\mathfrak{S}}(\mathfrak{M})$ is also a quasi-strongly divisible S-lattice inside $\mathcal{D}$, which is the key point in [11, Section 3.4]. For any quasi-strongly divisible S-lattice $\mathcal{N}$ inside $\mathcal{D}$, [11, Proposition 3.4.6] shows that the functor

$$T_{\mathrm{cris}} : \mathcal{N} \mapsto \mathrm{Hom}_{S, \mathrm{Fil}^r, \varphi_r}(\mathcal{N}, A_{\mathrm{cris}})$$

1) Here we use "S-lattices" to distinguish strongly divisible $W(k)$-lattices in Fontaine–Laffaille's theory.

establishes an anti-equivalence between the category of quasi-strongly divisible S-lattices and the category of G_∞-stable $\mathbb{Z}_p$-lattices inside semi-stable representations with Hodge–Tate weights in $\{0,\dots,r\}$. Now we claim that $T_{\mathrm{cris}}(\mathcal{L}) \simeq T_{\mathrm{cris}}(\mathcal{M})$ as $\mathbb{Z}_p[G_\infty]$-modules and consequently $\mathcal{L} \simeq \mathcal{M}$. Indeed, it is straightforward to check that

$$T_{\mathrm{cris}}(\mathcal{L}) \simeq \mathrm{Hom}_{W(k),\mathrm{Fil}^i,\varphi_i}(L, A_{\mathrm{cris}})|_{G_\infty} \simeq T|_{G_\infty}.$$

On the other hand, combing [10, Lemma 3.3.4] and Theorem 2.1.1 (3), we see that

$$T_{\mathrm{cris}}(\mathcal{M}) \simeq T_{\mathfrak{S}}(\mathfrak{M}) \simeq \hat{T}(\hat{\mathfrak{M}})|_{G_\infty} = T|_{G_\infty}.$$

Hence $T_{\mathrm{cris}}(\mathcal{L}) \simeq T_{\mathrm{cris}}(\mathcal{M})$. In summary, there is an S-linear isomorphism $\mathcal{M} \simeq S \otimes_{W(k)} L$ compatible with φ-structures. Recall that $s : D \to \mathcal{D}$ is unique φ-equivariant section for the projection $\mathcal{D} \twoheadrightarrow D$. So we conclude that $s \circ q(\mathcal{M}) = L = M_{\mathrm{st}}(T) = \tilde{M}_{\mathrm{st}}(T)$.

It remains to show that $M_{\mathrm{inv}}(T) = M_{\mathrm{st}}(T)$. The idea is the same as the proof of existence of c_2 ($c_2 = 0$ in this case). Let $e_1,\dots,e_d$ be a $W(k)$-basis of $L = \tilde{M}_{\mathrm{st}}(T)$. For any $x \in M_{\mathrm{inv}}(T)$ we may write $x = \sum_i a_i e_i$ with $a_i \in K_0$. It was shown in [10, Lemma 5.3.4] that $t^r x \in A_{\mathrm{cris}} \otimes_S \mathcal{M} = A_{\mathrm{cris}} \otimes_{W(k)} L$. Hence $t^r x = \sum_i a_i t^r e_i \in A_{\mathrm{cris}} \otimes_{W(k)} L$. So $t^r a_i \in A_{\mathrm{cris}}$. As $r \leq p-2$, a_i has to be in $W(k)$ to make $a_i t^r \in A_{\mathrm{cris}}$. □

Remark 4.1.3. The functor M_{st} enjoys some nice properties. For example, it is useful to study torsion representations discussed in [13] and [14], and it is compatible with tensor products. But M_{st} does not have good exact properties where M_{inv} is left exact. And this is important for Section 4.3.

Example 4.1.4. Unfortunately, the functor M_{st} is not left exact as claimed in [13, Theorem 2.3] (the remaining of the theorem is still correct). Indeed, [13, Example 2.21] just serves the example that M_{st} neither left exact nor right exact. For convenience of the readers, we repeat the example here. Let $K = \mathbb{Q}_p(\pi)$ with $\pi^{p-1} = p$. Set $E(u) = u^{p-1} - p$. Let $\mathfrak{M}$ be the rank-2 Kisin module given by $\varphi(e_1) = e_1$ and $\varphi(e_2) = ue_1 + E(u)e_2$ with $\{e_i\}$ an $\mathfrak{S}$-basis of $\mathfrak{M}$. Let $\mathfrak{S}^* = \mathfrak{S} \cdot \mathfrak{e}$ be the rank-1 Kisin module with $\mathfrak{e}$ the basis and $\varphi(\mathfrak{e}) = E(u)\mathfrak{e}$. Consider the sequence of Kisin modules

$$0 \to \mathfrak{S}^* \xrightarrow{\mathfrak{i}} \mathfrak{M} \xrightarrow{\mathfrak{f}} \mathfrak{S} \to 0, \tag{4.1.1}$$

where $\mathfrak{f}$ and $\mathfrak{i}$ is induced by $\mathfrak{f}(e_1) = p$ and $\mathfrak{f}(e_2) = u$ and $\mathfrak{i}(\mathfrak{e}) = ue_1 - pe_2$. It is easy to check that the sequence is a left exact sequence of Kisin modules with height 1 and the sequence is exact after tensoring $\mathcal{O}_{\mathcal{E}}$, which is the p-adic completion of $\mathfrak{S}[\frac{1}{u}]$. As explained in [13, above Example 2.20, Lemma 2.19], [9, Theorem (0.4)] implies that the above sequence of Kisin modules can be extended naturally to a sequence of $(\varphi, \hat{G})$-modules, and $\hat{T}$ of the sequence is an *exact* sequence of G_K-stable $\mathbb{Z}_p$-lattices in crystalline representations with Hodge–Tate weights in $\{0, 1\}$:

$$0 \to \mathbb{Z}_p \to T \to \mathbb{Z}_p(1) \to 0.$$

Now modulo u to the sequence in (4.1.1), we get the sequence of $W(k)$-modules

$$0 \to W(k) \cdot \bar{\mathfrak{e}} \xrightarrow{i} M \xrightarrow{f} W(k) \to 0,$$

where $M = \mathfrak{M}/u\mathfrak{M} \simeq M_{\mathrm{st}}(T)$. We can easily check that the above sequence is not exact on M and $W(k)$. Hence the functor M_{st} is not left exact according to the construction of M_{st}.

4.2. Various lattices in $D_{\mathrm{dR}}(V)$. Let V be a de Rham representation of G_K with Hodge–Tate weights in $\{0,\dots,r\}$ and T a G_K-stable $\mathbb{Z}_p$-lattice inside V. It has been proved that V is potentially semi-stable ([1]). Let us assume that V is semi-stable over K', which is a finite and Galois over K. Let k' be the residue field of K' and $K'_0 := W(k')[\frac{1}{p}]$.

Set

$$D_{\mathrm{dR}}(V) := (V^\vee \otimes_{\mathbb{Q}_p} B^+_{\mathrm{dR}})^{G_K} \quad \text{and} \quad D_{\mathrm{st},K'} := (V^\vee \otimes_{\mathbb{Z}_p} B^+_{\mathrm{st}})^{G_{K'}}.$$

It is well known that

$$D_{\mathrm{dR},K'}(V) := (V^\vee \otimes_{\mathbb{Q}_p} B^+_{\mathrm{dR}})^{G_{K'}} = K' \otimes_{K'_0} D_{\mathrm{st},K'}(V)$$

and $D_{\mathrm{st},K'}$ has a semi-linear $\mathrm{Gal}(K'/K)$-action. Let $M_{*,K'}(T) \subset D_{\mathrm{st},K'}(V)$ denote the lattices M_{st}, $\tilde{M}_{\mathrm{st}}$ and M_{inv} constructed in Section 3.1 for $T|_{G_{K'}}$.

We define one more lattice before discussing the properties of $M_{*,K'}$. Let $(\mathfrak{M},\varphi,\hat{G})$, $\mathcal{M} \subset \mathcal{D}$ and D denote the data attached to $T|_{G_{K'}}$ as in the beginning of Section 3.1. Since $\mathcal{D} = S \otimes_{W(k)} s(D)$ via section s and D is isomorphic to $D_{\mathrm{st},K'}(V)$ via the isomorphism i in diagram (3.1.2), we may identify $\mathcal{D}/\mathrm{Fil}^1 S\mathcal{D}$ with $D_{\mathrm{dR},K'}(V)$. Set

$$M_{\mathrm{dR},K'}(T) := \mathcal{M}/\mathrm{Fil}^1 S\mathcal{M} \subset \mathcal{D}/\mathrm{Fil}^1 S\mathcal{D} \simeq D_{\mathrm{dR},K'}(V)$$

and

$$M_{\mathrm{dR}}(T) := D_{\mathrm{dR}}(V) \cap M_{\mathrm{dR},K'}(T).$$

The following proposition shows that the constructions of $M_{\mathrm{st},K'}$ and $M_{\mathrm{dR},K'}$ do not depend on the choice of uniformizer $\pi \in \mathcal{O}_{K'}$.

Proposition 4.2.1. *Notations as above, the constructions of $M_{\mathrm{st},K'}$ and $M_{\mathrm{dR},K'}$ do not depend on the choice of uniformizer $\pi \in \mathcal{O}_{K'}$. If V is potentially crystalline then $\tilde{M}_{\mathrm{st},K'}$ and $M_{\mathrm{inv},K'}$ also do not depend on the choice of uniformizer $\pi \in \mathcal{O}_{K'}$.*

Proof. Since we only use $G_{K'}$-structure in the following proof, without loss of generality, we may assume that $K = K'$. Suppose that we select another uniformizer $\pi' \in \mathcal{O}_K$ and the embedding $\mathfrak{S} \subset W(R)$ via $u \mapsto [\underline{\pi}']$. We add $'$ to all data for the chosen uniformizer π' and the embedding $\mathfrak{S} \subset W(R)$ via $u \mapsto [\underline{\pi}']$. We note that the embedding $s : D \subset \mathcal{D} \subset T^\vee \otimes_{\mathbb{Z}_p} B^+_{\mathrm{cris}}$ indeed depends on such choice because the isomorphism $i^{-1} : s(D) \simeq D_{\mathrm{st}}(V)$ is given by $y \mapsto \sum_{n=0}^\infty N^i(y) \otimes \gamma_i(\mathfrak{u})$, unless $N = 0$ or, equivalently, V is crystalline. So we label the isomorphisms $i_{\underline{\pi}} : D_{\mathrm{st}}(V) \simeq s(D)$ and $i_{\underline{\pi}'} : D_{\mathrm{st}}(V) \simeq s'(D')$ to distinguish them. Recall that $I_+R \subset R$ is the maximal ideal of R. Let $\nu : W(R) \to W(\bar{k})$ be the projection induced by modulo $W(I_+R)$. It was shown in [13, Section 2.3] that the projection ν can be extended naturally to $\nu : B^+_{\mathrm{st}} \to \tilde{K}_0$ such that $\nu(\mathfrak{u}) = 0$, where $\tilde{K}_0 = W(\bar{k})[\frac{1}{p}]$. We write $I_+ := \mathrm{Ker}(\nu)$.

From the construction of M_{st}, we have the following commutative diagram:

$$\begin{tikzcd}
D_{\mathrm{st}}(V) \arrow[rr, hook] \arrow[d, "i"', "\wr"] & & B^+_{\mathrm{st}} \otimes_S \mathcal{D} \arrow[r, two heads, "\bmod I_+"] \arrow[d, two heads, "\bmod \mathfrak{u}"] & \tilde{K}_0 \otimes_{K_0} D_{\mathrm{st}}(V) \arrow[d, equal] \\
s(D) \arrow[r, hook, "s"] & \mathcal{D} \arrow[r, hook] & B^+_{\mathrm{cris}} \otimes_S \mathcal{D} \arrow[r, two heads, "\bmod I_+"] & \tilde{K}_0 \otimes_{K_0} D \\
& \mathfrak{M} \arrow[r, hook] \arrow[u, hook] & W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \arrow[r, two heads, "\bmod I_+"] \arrow[u, hook] \arrow[ur, "\gamma"] & W(\bar{k}) \otimes_{W(k)} \mathfrak{M}/u\mathfrak{M}. \arrow[u, hook]
\end{tikzcd}$$

Here γ is just the composite of the maps

$$W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \xrightarrow{\bmod I_+} W(\bar{k}) \otimes_{W(k)} \mathfrak{M}/u\mathfrak{M}$$

and

$$W(\bar{k}) \otimes_{W(k)} \mathfrak{M}/u\mathfrak{M} \hookrightarrow \tilde{K}_0 \otimes_{K_0} D.$$

Let us write α for the composite of maps in the first row of the above diagram. It is obvious that the first row (hence α) does not depend on the choice of uniformizer π, while the second and third rows do. The above diagram and the construction of $M_{\mathrm{st}}(T) = i^{-1} \circ s(\mathfrak{M}/u\mathfrak{M})$ shows that $M_{\mathrm{st}}(T) = (\alpha^{-1} \circ \gamma)(W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M})$.

Now we select another uniformizer $\pi' \in \mathcal{O}_{K'}$ and the embedding $\mathfrak{S} \subset W(R)$ via $u \mapsto [\underline{\pi}']$. We still get the above diagram and

$$M'_{\mathrm{st}}(T) = (\alpha^{-1} \circ \gamma')(W(R) \otimes_{\varphi,\mathfrak{S}_{\underline{\pi}'}} \mathfrak{M}').$$

By Theorem 1.0.1, we have

$$W(R) \otimes_{\varphi,\mathfrak{S}_{\underline{\pi}'}} \mathfrak{M}' = W(R) \otimes_{\varphi,\mathfrak{S}_{\underline{\pi}}} \mathfrak{M}$$

as submodules of $T^\vee \otimes_{\mathbb{Z}_p} W(R)$. Hence

$$\gamma(W(R) \otimes_{\varphi,\mathfrak{S}_{\underline{\pi}}} \mathfrak{M}) = \gamma'(W(R) \otimes_{\varphi,\mathfrak{S}_{\underline{\pi}'}} \mathfrak{M}').$$

Since α is independent of the choice of π, we conclude that $M_{\mathrm{st}}(T) = M'_{\mathrm{st}}(T)$.

We use a similar idea as above to show that $M_{\mathrm{dR}}(T)$ does not depend on the choice of π. For any subring $B \subset B^+_{\mathrm{dR}}$, recall that $\mathrm{Fil}^1 B = \mathrm{Fil}^1 B^+_{\mathrm{dR}} \cap B$. For any ring $A \subset B^+_{\mathrm{st}}$ such that $W(k) \subset A$, we have a natural map

$$\theta : A \otimes_{W(k)} \mathcal{O}_K \subset B^+_{\mathrm{st}} \otimes_{K_0} K \subset B^+_{\mathrm{dR}} \to B^+_{\mathrm{dR}}/\mathrm{Fil}^1 B^+_{\mathrm{dR}} = \mathbb{C}_p$$

induced by modulo Fil^1. Now according to the construction of $M_{\mathrm{dR}}(T)$, we can modify the above diagram as follows:

$$\begin{tikzcd}
D_{\mathrm{dR}}(V) \arrow[rr, hook] \arrow[d, "i"', "\wr"] & & K \otimes_{K_0} B^+_{\mathrm{st}} \otimes_S \mathcal{D} \arrow[r, two heads, "\bmod \mathrm{Fil}^1"] \arrow[d, two heads, "\bmod \mathfrak{u}"] & \mathbb{C}_p \otimes_K D_{\mathrm{dR}}(V) \arrow[d, equal] \\
K \otimes_{K_0} s(D) \arrow[r, hook, "s_K"] & K \otimes_{K_0} \mathcal{D} \arrow[r, hook] & B^+_{\mathrm{cris}} \otimes_S (K \otimes_{K_0} \mathcal{D}) \arrow[r, two heads, "\bmod \mathrm{Fil}^1"] & \mathbb{C}_p \otimes_K (K \otimes_{K_0} D) \\
 & \mathcal{M} \arrow[r, hook] \arrow[u, hook] & A_{\mathrm{cris}} \otimes_S \mathcal{M} \arrow[r, two heads, "\bmod \mathrm{Fil}^1"] \arrow[u, hook] \arrow[ur, "\gamma_K"] & \mathcal{O}_{\mathbb{C}_p} \otimes_{\mathcal{O}_K} \mathcal{M}/\mathrm{Fil}^1 \mathcal{M}. \arrow[u, hook]
\end{tikzcd}$$

We see that the map α_K in the first row is still independent of the choice of uniformizer π and $M_{\mathrm{dR}}(T) := (\alpha_K^{-1} \circ \gamma_K)(A_{\mathrm{cris}} \otimes_S \mathcal{M})$. Now repeat the proof of that M_{st} does not depend on the choice of π, we conclude that $M_{\mathrm{dR}}(T)$ does not depend on the choice of π.

When V is crystalline (as we assume that $K' = K$), we see that $s(D) = D_{\mathrm{cris}}(V)$ which does not depend on choice of π. It is clear from the construction of $\tilde{M}_{\mathrm{st}}$ that

$$\tilde{M}_{\mathrm{st}}(T) = s(D) \cap (A_{\mathrm{cris}} \otimes_S \mathcal{M}).$$

Since $A_{\mathrm{cris}} \otimes_{\varphi,\mathfrak{S}_{\underline{\pi}'}} \mathfrak{M}' = A_{\mathrm{cris}} \otimes_{\varphi,\mathfrak{S}_{\underline{\pi}}} \mathfrak{M}$ as submodules of $T^\vee \otimes_{\mathbb{Z}_p} W(R)$ by Theorem 1.0.1, we conclude that $\tilde{M}_{\mathrm{st}}$ does not depend on the choice of uniformizer π. Finally, it is obvious from the construction that $M_{\mathrm{inv}}(T)$ does not depend on the choice of π or ϖ if V is crystalline. $\square$

We would like to discuss the formation of those functors when base changes. Let K''/K' be a finite extension, k'' the residue field of K'' and $\mathcal{O}_{K''}$ the ring of integers.

Proposition 4.2.2. *The following statements hold:*

(1) $M_{\mathrm{st},K''}(T) = W(k'') \otimes_{W(k')} M_{\mathrm{st},K'}(T)$.

(2) $M_{\mathrm{dR},K''}(T) = \mathcal{O}_{K''} \otimes_{\mathcal{O}_{K'}} M_{\mathrm{dR},K'}(T)$.

Proof. To prove (1) and (2), we use almost the same ideas as in the proof of the above proposition. By the first commutative diagram in the above proof,

$$M_{\mathrm{st},K''}(T) = (\alpha^{-1} \circ \gamma)(W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}'') \subset D_{\mathrm{st},K''}(V).$$

By Theorem 3.2.1,

$$W(R) \otimes_{\varphi,\mathfrak{S}_{\underline{\pi}''}} \mathfrak{M}'' = W(R) \otimes_{\varphi,\mathfrak{S}_{\underline{\pi}'}} \mathfrak{M}'$$

as submodules of $T^\vee \otimes_{\mathbb{Z}_p} W(R)$. Hence we have $M_{\mathrm{st},K'}(T)$ is just $(\alpha^{-1} \circ \gamma)(W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}'')$ restricted to $D_{\mathrm{st},K'}(V)$. That is,

$$M_{\mathrm{st},K'}(T) = M_{\mathrm{st},K''}(T) \cap D_{\mathrm{st},K'}(V).$$

As $M_{\mathrm{st},K'}(T)$ is a $W(k')$-lattice inside $D_{\mathrm{st},K'}(V)$ and $D_{\mathrm{st},K''}(V) = W(k'') \otimes_{W(k')} D_{\mathrm{st},K'}(V)$, we get $W(k'') \otimes_{W(k')} M_{\mathrm{st},K'}(T) \subset M_{\mathrm{st},K''}(T)$. But

$$\begin{aligned} W(\bar{k}) \otimes_{W(k')} \mathfrak{M}'/[\underline{\pi}']\mathfrak{M}' &= W(R) \otimes_{\varphi,\mathfrak{S}_{\pi'}} \mathfrak{M}' \bmod W(I_+R) \\ &= W(R) \otimes_{\varphi,\mathfrak{S}_{\pi''}} \mathfrak{M}'' \bmod W(I_+R) \\ &= W(\bar{k}) \otimes_{W(k'')} \mathfrak{M}''/[\underline{\pi}'']\mathfrak{M}''. \end{aligned}$$

Hence $M_{\mathrm{st},K'}(T)$ must generate $M_{\mathrm{st},K''}(T)$ as $W(k'')$-modules and then we conclude

$$W(k'') \otimes_{W(k')} M_{\mathrm{st},K'}(T) = M_{\mathrm{st},K''}(T).$$

The proof of (2) proceeds similarly. □

If V is semi-stable non-crystalline, then $M_{\mathrm{inv},K}$ in general does depend on the choice of ϖ. To study de Rham representations of G_K in the next subsection, we fix a uniformizer ϖ of $\mathcal{O}_K$ to define A_{st} from now. Set

$$\tilde{M}_{\mathrm{inv}}(T) := D_{\mathrm{dR}}(V) \cap (\mathcal{O}_{K'} \otimes_{W(k')} M_{\mathrm{inv},K'}(T)) = (\mathcal{O}_{K'} \otimes_{W(k')} M_{\mathrm{inv},K'}(T))^{\mathrm{Gal}(K'/K)}.$$

It is easy to see that $\tilde{M}_{\mathrm{inv}}(T)$ is an $\mathcal{O}_K$-lattice in $D_{\mathrm{dR}}(V)$ but lose the (φ, N)-action. The following lemma summarizes the useful properties of $\tilde{M}_{\mathrm{inv}}$.

Lemma 4.2.3. *The following statements hold:*

(1) *Let K''/K' be a finite extension with the residue field k''. Then*

$$M_{\mathrm{inv},K''}(T) = W(k'') \otimes_{W(k')} M_{\mathrm{inv},K'}(T).$$

(2) *The construction of $\tilde{M}_{\mathrm{inv}}$ does not depend on the choice of K'.*

(3) *The functor $\tilde{M}_{\mathrm{inv}}$ is left exact.*

(4) *Assume that $T_1^\vee \hookrightarrow T_2^\vee$ is an injection of two G_K-stable $\mathbb{Z}_p$-lattices of de Rham representations. If p^a kills the torsion part of $T_2^\vee/T_1^\vee$, then p^a kills the torsion part of $\tilde{M}_{\mathrm{inv}}(T_2)/\tilde{M}_{\mathrm{inv}}(T_1)$.*

Proof. (1) It is clear that $M_{\mathrm{inv},K''}(T)[\frac{1}{p}] \simeq W(k'') \otimes_{W(k')} M_{\mathrm{inv},K'}(T)[\frac{1}{p}]$. So we have

$$W(k'') \otimes_{W(k')} M_{\mathrm{inv},K'}(T) \subset M_{\mathrm{inv},K''}(T)$$

as lattices inside $M_{\mathrm{inv},K''}(T)[\frac{1}{p}]$. Hence to prove (1) it suffices to assume that K''/K' is Galois. Write

$$D'' := M_{\mathrm{inv},K''}(T)[\tfrac{1}{p}], \quad D' := M_{\mathrm{inv},K'}(T)[\tfrac{1}{p}], \quad K_0'' := W(k'')[\tfrac{1}{p}], \quad K_0' := W(k')[\tfrac{1}{p}].$$

We see that D'' has a semi-linear $\mathrm{Gal}(K''/K')$-action and

$$D'' \simeq K_0'' \otimes_{K_0'} D'$$

as $\mathrm{Gal}(K''/K')$-modules, where $\mathrm{Gal}(K''/K')$ acts on D' trivially. Furthermore, it is obvious that the $\mathrm{Gal}(K''/K')$-action factors through $\Gamma := \mathrm{Gal}(K_0''/K_0')$, $M_{\mathrm{inv},K''}(T) \subset D''$ is Γ-stable and $M_{\mathrm{inv},K'}(T) = M_{\mathrm{inv},K''}(T)^{\Gamma}$. Then $M_{\mathrm{inv},K''}(T) = W(k'') \otimes_{W(k')} M_{\mathrm{inv},K'}(T)$ by the étale descent.

(2) Suppose that K'' is another Galois extension of K such that V is semi-stable over K''. Let k'' be the residue field of K''. We need to show that

$$(4.2.1) \quad (\mathcal{O}_{K'} \otimes_{W(k')} M_{\mathrm{inv},K'}(T))^{\mathrm{Gal}(K'/K)} = (\mathcal{O}_{K''} \otimes_{W(k'')} M_{\mathrm{inv},K''}(T))^{\mathrm{Gal}(K''/K)}.$$

Without loss of generality, we can assume that $K' \subset K''$. As $\mathrm{Gal}(K''/K)$-module, (1) shows that

$$\mathcal{O}_{K''} \otimes_{W(k'')} M_{\mathrm{inv},K''}(T) \simeq \mathcal{O}_{K''} \otimes_{\mathcal{O}_{K'}} (\mathcal{O}_{K'} \otimes_{W(k')} M_{\mathrm{inv},K'}(T)).$$

Note that $\mathrm{Gal}(K''/K')$ acts on $(\mathcal{O}_{K'} \otimes_{W(k')} M_{\mathrm{inv},K'}(T))$ trivially, we obtain

$$\begin{aligned}(\mathcal{O}_{K''} \otimes_{W(k'')} M_{\mathrm{inv},K''}(T))^{\mathrm{Gal}(K''/K')} &= (\mathcal{O}_{K''})^{\mathrm{Gal}(K''/K')} \otimes_{W(k')} M_{\mathrm{inv},K'}(T) \\ &= \mathcal{O}_{K'} \otimes_{W(k')} M_{\mathrm{inv},K'}(T).\end{aligned}$$

Then equation (4.2.1) follows by taking $\mathrm{Gal}(K'/K)$-invariants by the both sides of the above equation.

(3) Suppose that we are given an exact sequence

$$0 \to T' \to T \to T'' \to 0.$$

Applying functor $M_{\mathrm{inv},K'}$, we obtain a left exact sequence

$$0 \to M'' \to M \xrightarrow{f} M'$$

by Proposition 4.1.2 (5). We can decompose the above sequence into two sequences

$$0 \to M'' \to M \xrightarrow{g} N \to 0 \quad \text{and} \quad N \overset{i}{\hookrightarrow} M'$$

such that the first sequence is exact. We note that N is a finite free $W(k')$-module as N is a submodule of M'. In the following, we denote $A_{K'} := \mathcal{O}_{K'} \otimes_{W(k')} A$ for a $W(k')$-module A. Since $\mathcal{O}_{K'}$ is flat over $W(k')$, we still get the exact sequence

$$0 \longrightarrow M''_{K'} \longrightarrow M_{K'} \xrightarrow{g_{K'}} N_{K'} \longrightarrow 0 \quad \text{and} \quad N_{K'} \overset{i_{K'}}{\hookrightarrow} M'_{K'}.$$

Taking $\mathrm{Gal}(K'/K)$-invariants, we obtained a left exact sequence

$$0 \longrightarrow \tilde{M}_{\mathrm{inv}}(T'') \longrightarrow \tilde{M}_{\mathrm{inv}}(T) \xrightarrow{g_{K'}} (N_{K'})^{\mathrm{Gal}(K'/K)} \quad \text{and} \quad (N_{K'})^{\mathrm{Gal}(K'/K)} \hookrightarrow \tilde{M}_{\mathrm{inv}}(T').$$

Write $f_{K'} : \tilde{M}_{\mathrm{inv}}(T) \to \tilde{M}_{\mathrm{inv}}(T')$. We can easily check that $\mathrm{Ker}(f_{K'}) = \mathrm{Ker}(g_{K'})$. Hence the sequence

$$0 \to \tilde{M}_{\mathrm{inv}}(T'') \to \tilde{M}_{\mathrm{inv}}(T) \to \tilde{M}_{\mathrm{inv}}(T')$$

is left exact.

(4) Since $\tilde{M}_{\mathrm{inv}}$ is left exact, without loss of generality, we can assume that $T_2^\vee / T_1^\vee$ is killed by p^a. It is obvious from the construction of $M_{\mathrm{inv},K'}$ to see that $M_{\mathrm{inv},K'}(T_2)/M_{\mathrm{inv},K'}(T_1)$ is killed by p^a. After tensoring $\mathcal{O}_{K'}$ and taking Galois invariants, it is trivial to check that p^a kills $\tilde{M}_{\mathrm{inv}}(T_2)/\tilde{M}_{\mathrm{inv}}(T_1)$. □

It is easy to see that

$$\tilde{M}_{\mathrm{inv}}(T) \subset (T^\vee \otimes_{\mathbb{Z}_p} A_{\mathrm{st}} \otimes_{W(\bar{k})} \mathcal{O}_{\overline{K}})^{G_K}.$$

But we do not know how to prove that

$$\tilde{M}_{\mathrm{inv}}(T) = (T^\vee \otimes_{\mathbb{Z}_p} A_{\mathrm{st}} \otimes_{W(\bar{k})} \mathcal{O}_{\overline{K}})^{G_K}.$$

4.3. The direct limit of de Rham representations. Let I be a partial order set and let $\{L_i\}_{i\in I}$ be a family of G_K-stable $\mathbb{Z}_p$-lattices in de Rham representations V_i of G_K with Hodge–Tate weights in $\{-r, \dots, 0\}$ (r is independent on i). Let

$$L := \varinjlim_{i\in I} L_i$$

be the direct limit. Fix a uniformizer $\varpi \in \mathcal{O}_K$ as the last subsection to define A_{st} and $\tilde{M}_{\mathrm{inv}}$. We define a covariant version of $\tilde{M}_{\mathrm{inv}}$ via $\tilde{M}^*_{\mathrm{inv}}(T) := \tilde{M}_{\mathrm{inv}}(T^\vee)$. Set

$$M_i = \tilde{M}^*_{\mathrm{inv}}(L_i) \quad \text{and} \quad M := \varinjlim_{i\in I} M_i.$$

Recall that L is p-adically separated if L injects into the p-adic limits $\hat{L}$ of L, or equivalently, $\bigcap_{n=1}^\infty p^n L = \{0\}$.

Proposition 4.3.1. *If L is p-adically separated, then M is p-adically separated.*

Proof. Write $f_{ij} : L_i \to L_j$ and $g_{ij} = \tilde{M}^*_{\mathrm{inv}}(f_{ij}) : M_i \to M_j$. Note that $L_j/f_{ij}(L_i)$ and $M_j/g_{ij}(M_i)$ may not be torsion free. Pick a $y \in M_i$ such that $g_i(y) \neq 0$, where $g_i : M_i \to M$ is the natural map. We need to show that $g_i(y) \notin \bigcap_{n=1}^\infty p^n M$. Suppose to the contrary that $g_i(y) \in \bigcap_{n=1}^\infty p^n M$. Then there exists a subset $J = \{j_n\} \subset I$ with $j_n < j_{n+1}$ and $y_n \in M_{j_n}$ such that $p^n y_n = g_{ij_n}(y)$. Consider the space $\mathrm{Ker}(f_{ij_n}) \subset L_i$, which is an increasing sequence of saturated finite free $\mathcal{O}_K$-modules inside L_i. So they have to be stable after deleting finite many j_n. Hence without loss of generality, we may assume that all $\mathrm{Ker}(f_{ij_n})$ are the same and then $f_{ij_n}(L_i)$ are all isomorphic. Now we decompose $f_{ij_n} : L_i \to L_{j_n}$ into

$$L_i \xrightarrow{\tilde{f}_n} f_{ij_n}(L_i) \xhookrightarrow{\alpha_n} L_{j_n}$$

and apply the functor $\tilde{M}^*_{\mathrm{inv}}$. Then we get the map

$$M_i \xrightarrow{\tilde{g}_n} \tilde{M}^*_{\mathrm{inv}}(f_{ij_n}(L_i)) \xhookrightarrow{\tilde{\alpha}_n} M_{j_n},$$

where $\tilde{g}_n = \tilde{M}^*_{\mathrm{inv}}(\tilde{f}_n)$ and $\tilde{\alpha}_n = \tilde{M}^*_{\mathrm{inv}}(\alpha_n)$. Note that $\tilde{\alpha}_n$ is an injection because $\tilde{M}_{\mathrm{inv}}$ is left exact. Since $g_{ij_n} = \tilde{\alpha}_n \circ \tilde{f}_n$, if p^{a_n} (resp. p^{b_n}) kills the torsion part of $\tilde{M}^*_{\mathrm{inv}}(f_{ij_n}(L_i))/\tilde{g}_n(M_i)$ (resp. $M_{j_n}/\tilde{\alpha}_n(\tilde{M}^*_{\mathrm{inv}}(f_{ij_n}(L_i))))$, then $p^{a_n+b_n}$ kills the torsion part of $M_{j_n}/g_{ij_n}(M_i)$. Now we claim that there exists an integer m_i such that p^{m_i} kills the torsion part of $L_{j_n}/f_{ij_n}(L_i)$ for all n. Let us first accept the claim. Then Lemma 4.2.3 (4) proves that p^{m_i} kills the torsion part of $M_{j_n}/\tilde{\alpha}_n(\tilde{M}^*_{\mathrm{inv}}(f_{ij_n}(L_i)))$. Since the exact sequences

$$0 \longrightarrow \mathrm{Ker}(f_{ij_n}) \longrightarrow L_i \xrightarrow{\tilde{f}_n} f_{ij_n}(L_i) \longrightarrow 0$$

are isomorphic for all n, we see that a_n is independent of n. Hence there exists an m'_i independent of n such that $p^{m'_i}$ kills the torsion part of $M_{j_n}/g_{ij_n}(M_i)$, and this contradicts the existence of y_n. Hence $g_i(y)$ has to be 0 and M is p-adically separated.

Now it suffices to settle the claim. Since all $f_{ij_n}(L_i)$ are isomorphic, without loss of generality, we may assume that f_{ij_n} are injective for all n. Hence we may regard L_i as a submodule of L_{j_n} via f_{ij_n}. Let $T_n = (\mathbb{Q}_p \otimes_{\mathbb{Z}_p} L_i) \cap L_{j_n}$. Then T_n are increasing finite free $\mathbb{Z}_p$-lattices inside $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} L_i$. It suffices to show that $T_m = T_{m+1}$ if m is sufficiently large. In fact, if T_n is keeping increasing its size, then it is easy to show there must be a $x \in L_i$ such that $x = p^l y_{n_l}$ for a $y_{n_l} \in T_{n_l} \subset L_{n_l}$ for $l \geq 1$. But this contradicts that L is p-adically separated. □

The above proposition is actually motivated by the situations in [5, 6]. In the following, we discuss the situation in [6, Section 7.2] and also use notations there. Fix a compact open subgroup K^p of $\mathrm{GL}_2(\hat{\mathbb{Z}}^p)$. We refer to K^p as the "tame level". Fix a finite extension E of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_E$. Write

$$\mathrm{H}^1(K^p)_{\mathcal{O}_E} := \varinjlim_{K_p} \mathrm{H}^1(Y(K^p K_p)/_{\overline{\mathbb{Q}}}, \mathcal{O}_E),$$

where the inductive limit is taken over all open subgroups K_p of $\mathrm{GL}_2(\mathbb{Z}_p)$, $Y(K^p K_p)$ is the modular curve determined by $K^p K_p$ and the cohomology is étale cohomology. By [6, Lemma 7.2.1], $\mathrm{H}^1(K^p)_{\mathcal{O}_E}$ is torsion free and p-adically separated. Set $\hat{\mathrm{H}}^1(K^p)_{\mathcal{O}_E}$ to be the p-adic completion of $\mathrm{H}^1(K^p)_{\mathcal{O}_E}$ and

$$\hat{\mathrm{H}}^1(K^p)_E := E \otimes_{\mathcal{O}_E} \hat{\mathrm{H}}^1(K^p)_{\mathcal{O}_E}.$$

Due to [6, Lemma 7.2.5], $\hat{\mathrm{H}}^1(K^p)_E$ is an admissible unitary representation of $\mathrm{GL}_2(\mathbb{Q}_p)$.

Since we are only concerned with the local properties at p, we restrict all the above Galois modules (they are $\mathbb{Z}_p[\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]$-modules) to $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ but still use the same notations. Now apply the functor $\tilde{M}^*_{\mathrm{inv}}$ to $\mathrm{H}^1(Y(K^p K_p)/_{\overline{\mathbb{Q}}}, \mathcal{O}_E)$ and set

$$M_{K_p K^p} := \tilde{M}^*_{\mathrm{inv}}(\mathrm{H}^1(Y(K^p K_p)/_{\overline{\mathbb{Q}}}, \mathcal{O}_E)).$$

By the comparison theorem, $M_{K_p K^p}$ is obviously a $\mathbb{Z}_p$-lattice in the de Rham cohomology $\mathrm{H}^1_{\mathrm{dR}}(Y(K^p K_p)_{\overline{\mathbb{Q}_p}}, E)$. Proposition 4.3.1 implies that

$$M_{K^p} := \varinjlim_{K_p} M_{K_p K^p}$$

is p-adic separated. Define $\hat{\mathrm{H}}^1_{\mathrm{dR}}(K^p)$ to be the p-adic completion of M_{K^p} and

$$\hat{\mathrm{H}}^1_{\mathrm{dR}}(K^p)_E := E \otimes_{\mathcal{O}_E} \hat{\mathrm{H}}^1_{\mathrm{dR}}(K^p).$$

By the construction of $\tilde{M}_{\mathrm{inv}}$, we easily see that $\hat{\mathrm{H}}^1_{\mathrm{dR}}(K^p)_E$ has a continuous action of $\mathrm{GL}_2(\mathbb{Q}_p)$.

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Question 4.3.2. It is natural to ask the following questions:

(1) What we can say about the $\mathrm{GL}_2(\mathbb{Q}_p)$-action on $\hat{\mathrm{H}}^1_{\mathrm{dR}}(K^p)_E$? Is it an admissible unitary representation of $\mathrm{GL}_2(\mathbb{Q}_p)$?

(2) Is there any relation between $\hat{\mathrm{H}}^1(K^p)_E$ and $\hat{\mathrm{H}}^1_{\mathrm{dR}}(K^p)_E$? Could we build a comparison theorem to compare them?

Let T be a G_K-stable $\mathbb{Z}_p$-lattice inside a de Rham representation. We may define

$$M_{\mathrm{inv}}^{W(\bar{k})}(T) := \varinjlim_{F} M_{\mathrm{inv},F}(T^\vee),$$

where F runs through all finite extensions of K. It is easy to see that $M_{\mathrm{inv}}^{W(\bar{k})}(T)$ is a finite free $W(\bar{k})$-module with a (φ, N, G_K)-action and the G_K-action factors through a finite quotient of G_K. We note that Proposition 4.3.1 is still valid after replacing $\tilde{M}_{\mathrm{inv}}$ by $M_{\mathrm{inv}}^{W(\bar{k})}$, because the proof only uses Lemma 4.2.3 (3) and (4), and it is easy to check that these are still valid for $M_{\mathrm{inv}}^{W(\bar{k})}$. If we apply the functor $M_{\mathrm{inv}}^{W(\bar{k})}$ to $\mathrm{H}^1(K^p)_{\mathcal{O}_E}$, then the direct limit of

$$M_{\mathrm{inv}}^{W(\bar{k})}(\mathrm{H}^1(Y(K^pK_p)/\overline{\mathbb{Q}}_p, \mathcal{O}_E))$$

is separated and can be completed. Denote this completion by $\hat{\mathrm{H}}^{1,W(\bar{k})}_{\mathrm{st}}(K^p)_{\mathcal{O}_E}$, which has a natural (φ, N, G_K)-action and a $\mathrm{GL}_2(\mathbb{Q}_p)$-action. It is natural ask Question 4.3.2 (1) for $\hat{\mathrm{H}}^{1,W(\bar{k})}_{\mathrm{st}}(K^p)_{\mathcal{O}_E}$ again and the following:

Question. What are relations between $\hat{\mathrm{H}}^{1,W(\bar{k})}_{\mathrm{st}}(K^p)_{\mathcal{O}_E}$, $\hat{\mathrm{H}}^1(K^p)_E$ and $\hat{\mathrm{H}}^1_{\mathrm{dR}}(K^p)_E$?

Finally, we may define

$$M_{\mathrm{st}}^{W(\bar{k})}(T) := \varinjlim_{F} M_{\mathrm{st},F}(T^\vee),$$

which is another $W(\bar{k})$-lattice inside $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} M_{\mathrm{inv}}^{W(\bar{k})}(T)$ which is (φ, N, G_K)-stable. Though the functor $M_{\mathrm{st}}^{W(\bar{k})}$ enjoys many good properties (e.g., $M_{\mathrm{st},F}(\mathrm{H}^1(Y(K^pK_p)/\overline{\mathbb{Q}}_p, \mathcal{O}_E))$ does have a geometric interpretation if $Y(K^pK_p)$ has a good reduction over F), we do not know whether the direct limit of $M_{\mathrm{st}}^{W(\bar{k})}(\mathrm{H}^1(Y(K^pK_p)/\overline{\mathbb{Q}}_p, \mathcal{O}_E)$ is p-adic separated as the functor M_{st} in general is not left exact. Hence its p-adic completion may contain few information to understand $\hat{\mathrm{H}}^1(K^p)_E$.

5. Erratum for [13]

Theorem 2.3 in [13] claimed that the functor M_{st} is left exact. Unfortunately, this is false as Example 4.1.4 explains. Given an exact sequence of lattices in semi-stable representations $0 \to T' \to T \to T'' \to 0$, [13, Lemma 2.19] showed that the associated sequence of Kisin module $0 \to \mathfrak{M}'' \to \mathfrak{M} \to \mathfrak{M}' \to 0$ is left exact. But it is not true in general that the sequence

$$0 \to \mathfrak{M}''/u\mathfrak{M}'' \to \mathfrak{M}/u\mathfrak{M} \to \mathfrak{M}'/u\mathfrak{M}' \to 0$$

is exact on $\mathfrak{M}/u\mathfrak{M}$. This is exactly the mistake ([13, sentence right before Lemma 2.19]) in the proof that M_{st} is left exact.

Except this claim, [13, Theorem 2.3] is still correct and we have not used this claim in [13] and our other papers.

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